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FULL LENGTH PAPER

Design of price mechanisms for network resource allocation via price of anarchy

Ying-Ju Chen · Jiawei Zhang

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Abstract We study the design of price mechanisms for communication network problems in which a user's utility depends on the amount of flow she sends through the network, and the congestion on each link depends on the total traffic flows over it. The price mechanisms are characterized by a set of axioms that have been adopted in the cost-sharing games, and we search for the price mechanisms that provide the minimum price of anarchy. We show that, given the non-decreasing and concave utilities of users and the convex quadratic congestion costs in each link, if the price mechanism cannot depend on utility functions, the best achievable price of anarchy is $4(3-2\sqrt{2}) \approx 31.4\%$. Thus, the popular marginal cost pricing with price of anarchy less than $1/3 \approx 33.3\%$ is nearly optimal. We also investigate the scenario in which the price mechanisms can be made contingent on the users' preference profile while such information is available.

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1 Introduction

We consider a communication network in which a set of users require services from links in the network. The utility of a user depends on the amount of flow she sends

Y.-J. Chen (✉)
Department of Industrial Engineering and Operations Research,
University of California, Berkeley, CA, USA
e-mail: chen@ieor.berkeley.edu

J. Zhang
Stern School of Business, New York University, New York, NY, USA
e-mail: jzhang@stern.nyu.edu

through the network, and the congestion on each link depends on the total traffic flows over it. Typically, in the centralized system, the system designer has full control over the flows and aims at maximizing the users' aggregate utility net the congestion cost. As such control is impossible in the decentralized system, congestion pricing, i.e., imposing surcharges on users while sending the communication flows, has been proposed to align the users' incentives. These mechanisms treat the network as a collection of scarce resources, and allocate these resources among competing streams through a price scheme that depends on the congestion cost; see [23] for a comprehensive survey. Among those congestion pricing schemes, the common price mechanism is of particular interest. In a common price mechanism, users choose communication flows via each available path, but they are charged a common unit price in each link for the rate they send through that link. Such a mechanism avoids the burden of keeping track of link flows from each user and balances the trade-off between simplicity and efficiency for network flow allocations. Moreover, it eliminates the users' incentives to split their requests of communication flows or to submit a common request and then redistribute the flows.

The proportionally-fair price mechanism proposed by [12] is an example of the common price mechanism. Specifically, under this price mechanism, the users are charged proportional to the shares of resource they use; the specific weights used in their mechanism follow from the shadow prices that capture their impacts on the networks. Despite its simplicity, Kelly et al. show that the centralized solution can be achieved when the users are price takers, i.e., they do not anticipate the consequence of price change in response to their communication flows. However, this remarkable result does not hold if price anticipation exists [34]. When the price anticipation is considered, the decentralized behavior exhibits a Nash equilibrium, in which each user chooses a communication flow profile that is her best response against other users' decisions. Johari and Tsitsiklis [11] elaborate on such strategic interactions among users and establish the efficiency guarantee under the marginal cost pricing, i.e., the price charged to a user corresponds to the incremental cost that arises from the users' flow requests by one additional unit of flow.

A natural question is whether there exists a mechanism for communication networks better than the marginal cost pricing. In fact, the network resource allocation falls into a broad category of cost-sharing games, in which users share the joint output, but incur costs individually for their contributions [35]. In the economics literature, many price mechanisms have been proposed for allocating the joint costs to align the users' incentives, whereas the marginal cost pricing is merely one of these alternatives. Other popular price mechanisms include, e.g., the Aumann-Shapley pricing, the average cost pricing, and the Ramsey pricing (see [32]).

To study the design of price mechanisms for communication networks, we adopt the axiomatic approach to confine the set of price schemes and characterize the price mechanisms that achieve the minimum price of anarchy. The price of anarchy can be informally stated as the worst case efficiency loss of equilibria under a given mechanism [14]. Its central idea is to provide a lower bound for the ratio of aggregate payoff in any Nash equilibrium over that of the centralized solution—this is denoted as the coordination ratio. Price of anarchy has been applied in various areas, including transportation problems, resource allocation of network bandwidths, communication network design, competitive dynamic pricing, supply chain management, and allocation of divisible goods; see the excellent survey by [23]. Notably, price of anarchy also coincides with the worst case ratio or the performance guarantee of the online algorithms as well as approximation algorithms for NP-hard problems. In this paper, a price mechanism is said to be better than another if it provides a smaller price of anarchy. This, however, does not imply the dominance between two mechanisms either for every instance or on average. For ease of presentation, even though price of anarchy and efficiency loss are interchangeable and are both widely used in the literature, we in the sequel use price of anarchy throughout to avoid confusion.

The axiomatic approach follows closely the literature on cost-sharing games. In this literature, the axiomatic approach is one of the main themes because it provides a normative justification in purely economic terms for mechanisms that are considered desirable. In particular, our candidate mechanisms are characterized by four axioms (rescaling, additivity, positivity, and weak consistency) proposed by [28]; these axioms are adopted and investigated in various papers, including [4,13,19], and [31]. These price mechanisms include some popular schemes with various applications such as Aumann-Shapley pricing [3], average cost sharing scheme [21], and marginal cost pricing; hence, they are natural candidates of our network allocation problem. Moreover, this allows us to compare the optimal price mechanism to the marginal cost pricing, the benchmark mechanism that has been investigated by [11]. Further, since [11] have established that the price of anarchy is at most $1/3 \approx 33.3\%$ under the marginal cost pricing, a constant performance bound is guaranteed to exist in this class of price mechanisms, thereby making our pricing design problem meaningful. In Sect. 3.1, we provide detailed illustrations of these axioms and the rationale behind those popular axioms.

We make similar assumptions on model characteristics as those in [11]. In particular, we assume that a user's utility depends only on the aggregate flow (throughput) she sends through the network, regardless of the paths she chooses; nevertheless, the congestion cost is incurred on a per link basis to represent the aggregate disutility of the users that use the same link. Users' utility functions are non-decreasing and concave, and the congestion cost of each link is a convex and quadratic function of aggregate flow. Note that the quadratic structure is equivalent to the affine marginal latency assumption in [26] for transportation networks. The affine marginal cost assumption serves as the first-order approximation when detailed descriptions of cost per link are either unavailable or unimportant, and it is still quite rich in the recent literature (see [2,6,11,26,27], and references therein). It is also worth noting that the monetary cost is substituted for latency only under this affine assumption.

We show that, given the non-decreasing and concave utilities of users and the convex quadratic congestion costs in each link, if the price mechanism cannot depend on utility functions, the best achievable price of anarchy is $4(3-2\sqrt{2}) \approx 31.4\%$. To obtain this bound, it is crucial that the congestion costs are decomposable and the users' utilities depend merely on aggregate flows. Hence, the inherent nature of communication flow problems leads to an efficient scalable price mechanism. The marginal cost pricing studied in [11] falls into this category, and that the price of anarchy is $1/3 \approx 33.3\%$ implies it is nearly optimal even when price anticipation exists. Our result may justify why the marginal cost pricing is adopted in cost-sharing games.

We also investigate the scenario in which the users' utility profile is publicly known, and show that whether efficiency can be achieved critically depends on the properties of the users' utilities.

Our paper is closely related to [11]. The main contribution of [11] is to derive the price of anarchy for the marginal cost pricing, while our paper focuses on the characterization of the optimal pricing mechanism. Furthermore, our approach to derive price of anarchy for a given mechanism is different from that in [11]. They prove that worst case scenarios can be identified through a sequence of optimization problems, and then obtain price of anarchy after solving the Nash equilibria in detail for the worst cases. On the contrary, we obtain bounds directly from subgradient inequalities that represent the equilibrium conditions. We further provide tight bounds that are contingent on the number of users. Another closely related work is a recent paper by [20], who compares the serial, average, and incremental cost sharing rules for the single-output environment. Nevertheless, he includes the prices paid by users as surplus loss and excludes the aggregate system cost–which he calls the parsimonious computation. Based on this alternate accounting, he shows that all three methods do not offer a non-vanishing efficiency guarantee, but asymptotically serial cost sharing outperforms the others. Unlike his work, our paper incorporates the network structure, adopts the conventional definition of the aggregate surplus, and addresses optimal mechanism design.

The rest of this paper is organized as follows. In Sect. 2, we present the model setting. In Sect. 3, we first introduce the four axioms, and then focus on the single-link case and characterize price mechanisms that satisfy those axioms. Section 4 analyzes the price of anarchy when price mechanisms cannot depend on users' preferences. In Sect. 5, we allow the dependence on users' preferences, and obtain the price of anarchy for single-link problems. Section 6 concludes.

2 The model

We consider a communication network in which a set of users I (where $|I| = n \geq 2$) require services from links in the network. User $i \in I$ can choose an amount of flow q_i from her position to the desired destinations through a number of paths. Let Γ_i be the set of paths that are available to user i, and define path $p \in \Gamma_i$ if path p is available to user i. Define the entire set of paths $\Gamma = \bigcup_{i \in I} \Gamma_i$. For user i, we have $q_i = \sum_{p \in \Gamma_i} q_{ip}$, where q_{ip} is the flow over path p. Let E be the set of links, and $e = 1, \dots, |E|$ denote the indices of the links in the network. A link e satisfies $e \in p$ if path p goes through e. We use y_{ie} to denote the total flow that user i sends via link e. From the flow conservation, $y_{ie} = \sum_{p: p \in \Gamma_i, e \in p} q_{ip}$, for any $i \in I$ and $e \in E$.

The congestion cost in a link depends on its aggregate traffic flow; thus, we use $C_e(Q_e)$ to represent this congestion cost, where $Q_e \equiv \sum_{i \in I} y_{ie}$ is the aggregate traffic flow on link e. In this paper, we are primarily interested in the case with quadratic cost structure, i.e., $C_e(Q_e) = b_e Q_e + (1/2) a_e Q_e^2$, where constants $b_e \geq 0, a_e > 0$. In Sect. 4.3, we investigate the scenario in which price mechanisms are extended to incorporate general continuous cost structure. We let $\mathbf{Q} \equiv (Q_1, \dots, Q_{|E|}) \in R_+^{|E|}$

represent the traffic flows in this network, where R^m_+ denotes the set of all m-dimensional nonnegative vectors and $R_+ \equiv R^1_+$.

Now we introduce the users' payoffs. User i gets gross payoff $U_i(q_i)$ while sending flow q_i, where $U_i(\cdot)$ is nonnegative, increasing, and concave. From the fact that $U_i(q_i)$ depends on user i's total traffic flow q_i, users do not discriminate against any path. When users' utilities are linear, we define $U_i(q_i) = u_i q_i$, where u_i's are nonnegative constants, and $u_1 = \max_{i \in I} u_i$ without loss of generality. In the single-link problem, we suppress the subscripts associated with the link. That is, $C(Q) = bQ + (1/2)aQ^2$, where $Q = \sum_{i \in I} q_i$.

2.1 Centralized system

The centralized solution is the allocation $\{q_{ip}\}$ that solves the optimization problem:

$$\max_{q_{ip} \geq 0} \sum_{i=1}^{n} U_i\left(\sum_{p \in \Gamma_i} q_{ip}\right) - \sum_{e=1}^{|E|} C_e\left(\sum_{i=1}^{n} \sum_{p:e \in p} q_{ip}\right).$$

Let $\{q^*_{ip}\}$'s and $\{Q^*_e\}$'s denote respectively the planned flows on paths and links, and Π^* denotes the aggregate payoff under this allocation. Recall that for any $m \in N$, a vector $\xi \in R^m$ is called a subgradient of a concave function $f : R^m \to R$ at $x \in R^m$ if

$$f(x^{'}) \leq f(x) + \xi(x^{'} - x)^T, \quad \forall x^{'} \in R^m,$$

where y^T is the transpose of vector y. With the quadratic cost structure, we assume that

Assumption 1 There exist $i \in I$ and $p \in \Gamma_i$ such that $\xi - \sum_{e \in p} b_e > 0$, where ξ is a subgradient of $U_i(\cdot)$ at 0.

Assumption 1 eliminates trivial cases where $\{q^*_{ip} = 0, \ \forall p \in \Gamma_i, \ \forall i \in I\}$ is the centralized solution. To see this, if $\{q^*_{ip} = 0, \ \forall p \in \Gamma_i, \ \forall i \in I\}$ were the centralized solution, the Karush-Kuhn-Tucker (KKT) condition would imply that sending any infinitesimal flow via any path is unprofitable, which contradicts Assumption 1.

Now we derive the centralized solution for a special case, namely the single-link problem when users possess linear utility functions. This simple case not only is illustrative but will be recalled intensively for our analysis later on. Note that in this case, Assumption 1 implies that $u_1 > b$. The central planner's problem becomes $\max_{\mathbf{q}} \left\{\sum_i u_i q_i - (bQ + (1/2)aQ^2) | \text{ s.t. } q_i \geq 0, \ \forall i \in I\right\}$. Since all quantities have the same impact on the aggregate cost $bQ + (1/2)aQ^2$, it is always optimal to allocate $q_1 = Q$ and for all other users $q_i = 0$. The centralized solution boils down to a quadratic maximization problem $\max_{q_1 \geq 0}\{u_1 q_1 - bq_1 - (1/2)aq_1^2\}$. Applying the first-order condition, the centralized solution is $q^*_1 = Q^* = (u_1 - b)/a$, which yields the aggregate payoff $\Pi^* = (u_1 - b)^2/(2a)$.

2.2 Nash equilibrium

Now we introduce the decentralized system. The price mechanism $\mathbf{P}(C,\mathbf{Q}) \equiv (P_1(C,\mathbf{Q}),\ldots,P_{|E|}(C,\mathbf{Q}))$ specifies the common price charged per link, where C denotes the aggregate cost function. Given $\mathbf{P}(C,\mathbf{Q})$, user i's net payoff is $U_i(q_i) - \sum_{p\in\Gamma_i} q_{ip} \sum_{e\in p} P_e(C,\mathbf{Q})$ (i.e., the gross payoff net the prices paid in each link). The equilibrium concept is Nash equilibrium, and we focus on pure-strategy equilibria throughout this paper. In a Nash equilibrium, each user's strategy (i.e., flows $\{q_{ip}, p \in \Gamma_i\}$) is her best response given all others' strategies [8]. Note that these flows depend on the price mechanism $\mathbf{P}(C,\mathbf{Q})$. In the sequel, we suppress this dependence for ease of presentation.

More specifically, suppose that the flow profiles $\{\overline{q}_{ip}, p\in\Gamma_i, i\in I\}$ correspond to a Nash equilibrium given $\mathbf{P}$, and denote $\left\{\mathbf{q}_i, \overline{\mathbf{q}}_{-i}\right\}$ as the flow profile while replacing user i's strategies in $\{\overline{q}_{ip}\}$'s by $\{q_{ip}\}_{p\in\Gamma_i}$. A Nash equilibrium requires that for every user i,

$$\{\overline{q}_{ip}\}_{p\in\Gamma_i} \in \operatorname{argmax}_{\{q_{ip}\geq 0\}} \left\{ U_i\left(\sum_{p\in\Gamma_i} q_{ip}\right) - \sum_{p\in\Gamma_i} q_{ip} \sum_{e\in p} P_e\left(C_e, \left\{\mathbf{q}_i, \overline{\mathbf{q}}_{-i}\right\}\right) \right\}, \tag{1}$$

where $\{\overline{q}_{ip}\}_{p\in\Gamma_i}$ is the flow profile selected by user i. Let

$$\overline{\Pi}(\{\overline{q}_{ip}\}) = \sum_{i\in I} U_i(\overline{q}_i) - \sum_{e\in E}\left[b_e \overline{Q}_e + \frac{1}{2} a_e \overline{Q}_e^2\right]$$

be the aggregate payoff, where $\overline{q}_i = \sum_{p\in\Gamma_i} \bar{q}_{ip}$, and $\bar{Q}_e = \sum_{i\in I}\sum_{p:e\in p, p\in\Gamma_i} \bar{q}_{ip}$. Note that the prices charged on users do not contribute to the aggregate payoff, but they affect users' payoffs and behavior in equilibrium. Moreover, the aggregate prices paid by users may not be equal to the aggregate cost, i.e., price mechanisms need not be budget balancing.

As we demonstrate in Sects. 3 and 4, a Nash equilibrium typically exists for the general network case; however, the uniqueness can only be guaranteed if we restrict ourselves to the single-link problem. Since there may exist multiple equilibria given a price mechanism, we denote $\bar{\Pi} = \min_{\{\bar{q}_{ip}\}} \bar{\Pi}(\{\bar{q}_{ip}\})$, and the worst case coordination ratio (efficiency) under price mechanism $\mathbf{P}$ is defined as $\min\{\bar{\Pi}/\Pi^*\}$, where the minimum is taken over all network structures, all Nash equilibria, all utility profiles, and all quadratic cost functions. Note that $\bar{\Pi}$ is the aggregate payoff of the worst equilibrium because we are unable to coordinate the users into choosing a particular equilibrium. The price of anarchy is defined as $1-\min\{\bar{\Pi}/\Pi^*\}$. Assumption 1 and $U_i(q_i)\geq 0$, for all $q_i \geq 0$, guarantee that the denominator, i.e., the social surplus under the centralized solution, is always strictly positive. Therefore, the coordination ratio is well-defined.

Since users' behavior depends crucially on the price mechanism $\mathbf{P}$, in the next section we focus on the selection of price mechanisms.

3 Price mechanisms and axiomatic approach

In this section, we first review the axiomatic approach and introduce four axioms, and then focus on single-link problems to characterize price schemes that satisfy these axioms. In the end we discuss the existence and uniqueness of Nash equilibrium for the single-link problems.

3.1 Axiomatic approach and the four axioms

The axiomatic approach of cost sharing problems was first introduced by Aumann and Shapley, where they consider values of cooperative non-atomic games, see [32] for an extensive survey. Since then, the attention of economists has been put mainly on the axiomatic characterization of existing allocation rules (e.g., [18,28]), and the development of new sharing rules by pre-selecting a set of desirable axioms, see [7] and [22]. In this paper, we will focus on the price mechanisms under which users are charged a common unit price [32]. Samet and Tauman [28] find four axioms (rescaling, weak consistency, additivity, and positivity) that characterize a family of mechanisms, which include the marginal cost pricing and the Aumann-Shapley price. The marginal cost pricing can be characterized axiomatically by strengthening the positivity axiom, and the Aumann-Shapley price is the only cost-sharing mechanism that satisfies the above four axioms. These price mechanisms have been adopted due to their simplicity.

Since these price mechanisms have been adopted and include the benchmark case (the marginal cost pricing studied in [11]), they shall be natural candidates for our network allocation problem. Recall that $P_l(C, \mathbf{Q})$ is the unit price for link l if the cost structure is $C(\mathbf{Q}) \equiv C(Q_1, \ldots, Q_m)$ and the flow profile is $\mathbf{Q}$, and $\mathbf{P}_m(C, \mathbf{Q}) \equiv (P_1(C, \mathbf{Q}), \ldots, P_m(C, \mathbf{Q}))$ is the price vector. In our network model, the aggregate cost is quadratic and separable, i.e., $C(\mathbf{Q}) = \sum_{e \in E} C_e(Q_e) = \sum_{e \in E} \left(b_e Q_e + (1/2) a_e Q_e^2\right)$. Note that these axioms are defined for general cost structure $C(\mathbf{Q}) \equiv C(Q_1, \ldots, Q_m)$ for arbitrary cost-sharing games, where the integer $m > 0$ is the number of distinct outputs and $Q_1, \ldots, Q_m$ are the aggregate amounts of these outputs.

The definitions and interpretations of these axioms are stated in the following.

Axiom 1 (RESCALING). *Let $\alpha_1, \ldots, \alpha_m$ be m positive numbers. If $C(\mathbf{Q}) = F(\alpha_1 Q_1, \ldots, \alpha_m Q_m)$, for all $\mathbf{Q} \in R_+^m$, then $P_l(C, \mathbf{Q}) = \alpha_l P_l(F, (\alpha_1 Q_1, \ldots, \alpha_m Q_m))$, for all $l \in \{1, \ldots, m\}$.*

The rescaling axiom requires that the price mechanism should be invariant to the change of scale. To illustrate, consider a simple example in which there is only one link and suppose that congestion cost is F when the flow is measured in fluid ounce. Denote C as the congestion cost when the flow is measured in quart, then $C(Q) = F(32Q)$. From our definition of the price mechanisms, the price per quart charged on this link, $P(C, Q)$, and the price per fluid ounce, $P(F, 32Q)$, should satisfy the following relation: $P(C, Q) = 32P(F, 32Q)$. Next, if the aggregate flows of each link contribute to the aggregate congestion cost in a homogeneous way, then the prices in different links should be identical.

Axiom 2 (WEAK CONSISTENCY). *If* $C(\mathbf{Q}) = G\left(\sum_{l=1}^{m} Q_l\right)$, *for all* $\mathbf{Q} \in R_+^m$, *then*

$$P_j(C, \mathbf{Q}) = P_k(C, \mathbf{Q}) = \mathbf{P}_1\left(G, \sum_{l=1}^{m} Q_l\right), \quad \forall j, k \in \{1, \ldots, m\},$$

where $\mathbf{P}_1$ *is the price defined for the single-link case.*

To elaborate on this axiom, suppose that a user splits her flow, say 15 units, along one particular link to two parts, say type-1 flows $q_1 = 10$ and type-2 flows $q_2 = 5$; further, these two types of flows are in essence identical in generating the congestion cost in this link. The user can certainly interpret the congestion cost as a two-variable function $C(q_1, q_2)$, but in fact the aggregate cost depends only on the aggregate flows $q_1 + q_2 = 15$. In such a scenario, we can find a cost function $G(q_1 + q_2) = C(q_1, q_2)$, and it is reasonable to assume that the price charged to the user according to $G(q_1 + q_2)$ should be the same under these two cost functions. Note that under our quadratic cost structure, the parameters a_e and b_e are allowed to be arbitrary. Thus, the condition of this axiom does not hold in the generic cases and it only imposes a very mild restriction to our network flow pricing problem.

The next axiom requires that if the cost can be decomposed, then we can also decompose the price scheme.

Axiom 3 (ADDITIVITY). *For any pair of continuous cost functions* (F, G) *such that* $C(\mathbf{Q}) = F(\mathbf{Q}) + G(\mathbf{Q})$, *for all* $\mathbf{Q} \in R_+^m$, *the price mechanism satisfies* $\mathbf{P}_m(C, \mathbf{Q}) = \mathbf{P}_m(F, \mathbf{Q}) + \mathbf{P}_m(G, \mathbf{Q})$.

As an illustration, let us consider the example in which the congestion cost may come from the wasted labor and the fuel. In this case, we can decompose the congestion cost into two parts, and charge the users based on their impacts on the wasted labor and the fuel. The additivity axiom requires that the aggregate price should be the sum of the prices arising from the wasted labor and the fuel.

Finally, if the aggregate cost increases as the flow profile $Q_1, \ldots, Q_m$ increases, then the price mechanism should be nonnegative at that point $Q_1, \ldots, Q_m$. Two vectors $\mathbf{Q}'$, $\mathbf{Q}$ are said to satisfy $\mathbf{Q}' \leq \mathbf{Q}$ if $Q_l' \leq Q_l$, for all $l \in \{1, \ldots, m\}$.

Axiom 4 (POSITIVITY). *Let* $\mathbf{Q} \in R_+^m$. *If for all* $\mathbf{Q}' \leq \mathbf{Q}$, $C(\mathbf{Q}')$ *is nondecreasing, i.e., all the components of the gradient* $\nabla C(\mathbf{Q}')$ *are nonnegative, then* $\mathbf{P}_m(C, \mathbf{Q}) \geq 0$.

This axiom implies that if a user's flow increases the congestion cost in any link, she should be charged a nonnegative price for this behavior. Having introduced the four axioms, we next focus on the single-link problems and characterize the price mechanisms that satisfy these axioms.

3.2 Price mechanisms and Nash equilibria in single-link problems

In the following, we define $P_1(C, Q)$ as the unit price if the cost structure is $C(\cdot)$ and the aggregate traffic flow is Q. The rescaling axiom in the single-link problems requires only a single scaling factor α, and the weak consistency axiom degenerates in this special case. As we focus on the quadratic cost function $C(Q)$ in this paper, the price mechanism is defined only for the space of quadratic functions $C(Q) = bQ + (1/2)aQ^2$ where $Q \geq 0$. According to the following lemma, the price mechanism that satisfies the axioms turns out to be linear:

Lemma 1 *In the single-link problems, a price mechanism for* $C(Q) = bQ + (1/2)aQ^2$ *satisfies the rescaling, additivity, and positivity axioms if and only if* $P_1(C, Q) = \lambda(b\beta + aQ)$, *where* $\lambda \geq 0$, $\beta \geq 0$.

Proof Let us first show that the price scheme satisfies those axioms. Consider the additivity under quadratic costs. Suppose $C(Q) = F(Q) + G(Q)$, and $C(\cdot), F(\cdot), G(\cdot)$ are all quadratic. We obtain the relation between their parameters: $b^C = b^F + b^G$ and $a^C = a^F + a^G$, where the superscripts denote the corresponding cost functions. Now our price scheme satisfies $P_1(C, Q) = \lambda(b^C\beta + a^C Q) = \lambda(b^F\beta + a^F Q) + \lambda(b^G\beta + a^G Q) = P_1(F, Q) + P_1(G, Q)$, and therefore the additivity under quadratic costs holds.

Next, we verify the rescaling axiom. Since $C(Q) = F(\alpha Q)$, we obtain that

$$b^C Q + \frac{1}{2}a^C Q^2 = b^F \alpha Q + \frac{1}{2}a^F (\alpha Q)^2, \quad \forall Q \geq 0, \tag{2}$$

where b^C, a^C, b^F, and a^F are the corresponding coefficients in the cost functions C and F. The above Eq. (2) implies that

$$b^C - \alpha b^F = \frac{1}{2}\left(\alpha^2 a^F - a^C\right) Q.$$

Thus, the only possibility is that $b^F = b^C/\alpha$ and $a^F = a^C/\alpha^2$. Therefore, we have

$$\alpha P_1(F, \alpha Q) = \alpha\lambda(b^F\beta + a^F \alpha Q) = \alpha\lambda\left(\frac{b^C}{\alpha}\beta + \frac{a^C}{\alpha^2}\alpha Q\right) = P_1(C, Q).$$

The price scheme is clearly nonnegative when $\lambda \geq 0$, $\beta \geq 0$, and therefore it satisfies the positivity axiom as well. We conclude that the price scheme satisfies all those axioms.

Now we proceed to show that if a price mechanism satisfies all the axioms, it must take the linear form as presented. Suppose $P_1(C, Q)$ satisfies those axioms. Since $C(Q)$ can be parameterized by the coefficients a, b, we define $P_1(C, Q) \equiv \Psi(\mathbf{c}, Q)$, where $\mathbf{c} \equiv (a, b)$. Given two pairs $\mathbf{c}_1 \equiv (a_1, b_1)$ and $\mathbf{c}_2 \equiv (a_2, b_2)$ and an arbitrary constant $\tau \in [0, 1]$, we obtain that

$$\begin{aligned}
&[\tau b_1 + (1-\tau)b_2]Q + \frac{1}{2}[\tau a_1 + (1-\tau)a_2]Q^2 \\
&\quad = \tau\left(b_1 Q + \frac{1}{2}a_1 Q^2\right) + (1-\tau)\left(b_2 Q + \frac{1}{2}a_2 Q^2\right).
\end{aligned}$$

Thus, from the additivity axiom, we know that

$$\Psi(\tau \mathbf{c}_1 + (1-\tau)\mathbf{c}_2, Q) = \tau \Psi(\mathbf{c}_1, Q) + (1-\tau)\Psi(\mathbf{c}_2, Q),$$

for every pair $(\mathbf{c}_1, \mathbf{c}_2)$, any $\tau \in [0, 1]$, and any aggregate flow $Q \geq 0$. Thus $P_1(\mathbf{c}, Q)$ is both concave and convex in $\mathbf{c}$, where the domain of $\mathbf{c}$ is $(0, \infty) \times [0, \infty)$, a convex set. Therefore, $\Psi(\mathbf{c}, Q)$ is linear in $\mathbf{c}$, and we can represent $\Psi(\mathbf{c}, Q)$ as $\Psi((a, b), Q) = af(Q) + bg(Q)$, with both $f(Q)$, $g(Q)$ being independent of a, b.

Now apply the rescaling axiom. Consider two convex and quadratic cost functions C, F such that $C(Q) = F(\alpha Q)$. We obtain $b^F = b^C/\alpha, a^F = a^C/\alpha^2$, and the rescaling axiom requires that

$$\alpha \left(\frac{a^C}{\alpha^2} f(\alpha Q) + \frac{b^C}{\alpha} g(\alpha Q) \right) = a^C f(Q) + b^C g(Q), \ \forall Q \geq 0,$$
$$\forall \alpha > 0, \forall a^C > 0, \ \forall b^C \geq 0.$$

Plugging in $b^C = 0$ leads to $f(Q) = f(\alpha Q)/\alpha$, for all $Q \geq 0$, for all $\alpha > 0$, which immediately implies that $f(Q)$ is linear in Q, i.e., $f(Q) = C_f Q$, for some constant C_f. Moreover, $g(\alpha Q) = g(Q)$ holds for every $Q \geq 0$ and every $\alpha > 0$; consequently, $g(Q) = C_g$ is a constant function.

In sum, $P_1(C, Q) = C_g b^C + C_f a^C Q$. This can be rewritten as $P_1(C, Q) = \lambda(b^C \beta + a^C Q)$, and $\lambda \geq 0$ and $\beta \geq 0$ are necessary so that $P_1(C, Q) \geq 0$, for all (a^C, b^C). □

The marginal cost pricing and the average cost pricing correspond to respectively $(\lambda = 1, \beta = 1)$ and $(\lambda = 1/2, \beta = 2)$. That is, they are respectively $P_1(C, Q) = b + aQ$, and $P_1(C, Q) = b + (1/2)aQ$. Note also that when the users possess linear utility functions, the centralized solution is $q_1^* = Q^* = (u_1 - b)/a$, which yields the aggregate payoff $\Pi^* = (u_1 - b)^2/(2a)$.

Lemma 1 shows that the unit price for our single-link communication flow problems has to take the linear form. This reduces the problem of finding the price mechanism that could potentially be very complicated to an optimization problem over two parameters λ and β in single-link problems.

Existence and uniqueness of Nash equilibrium. Given the price mechanism, a Nash equilibrium $\{\bar{q}_i\}$'s requires that for all $i \in I$, $\bar{q}_i \in \text{argmax}_{q_i \geq 0} \{U_i(q_i) - q_i \lambda(b\beta + a(q_i + \sum_{j \neq i} \bar{q}_j))\}$. We neglect the trivial case $\lambda = 0$, i.e., $P_1(Q) = 0$, in which all users want to send unbounded flows. Except this, the existence and uniqueness of Nash equilibrium can be established.

Lemma 2 *Suppose $\lambda > 0$. In a single-link problem with price $P_1(Q) = \lambda(b\beta + aQ)$, there exists a unique Nash equilibrium, for every pair $\lambda > 0$ and $\beta \geq 0$.*

Proof This is a direct extension of [33, Theorem 2]. We merely have to redefine the cost function $C(Q) = Q\lambda(b\beta + aQ) = \lambda b\beta Q + \lambda a Q^2$, which is strictly convex. □

The existence and uniqueness of Nash equilibrium allow us to predict the users' behavior in the decentralized system. We then move on to study the price of anarchy and design the optimal mechanisms.

4 Price of anarchy when prices are independent of users' preferences

In this section, we assume that information of users' preferences is not available. Therefore, the price mechanisms shall depend merely on the cost parameters, i.e., $\{b_e, a_e\}$'s. We first focus on the single-link problem with linear utilities and obtain the optimal price mechanism. We then propose to use the common parameters obtained in this special case for unit prices per link in the general problem, and establish lower bounds of the coordination ratios. In the end, we discuss the optimality when the price mechanisms are defined for all continuous costs. Since we do not use extensively the number of links, we replace the notation $\mathbf{P}(\mathbf{C}, \mathbf{Q})$ by $\mathbf{P}(\mathbf{Q})$.

4.1 Price of anarchy in single-link problems with linear utilities

We first consider the single-link problem ($C(Q) = bQ + (1/2)aQ^2$), and assume that utilities are all linear, i.e., $U_i(q_i) = u_i q_i$. Note that according to Lemma 1, we can focus on the design of λ and β. For ease of notation, we use $\{\bar{q}_i\}$'s to denote the equilibrium flow profiles for a given pair (λ, β), and use $\bar{\Pi}$ to denote the corresponding aggregate net payoffs of the users. Recall that the centralized solution is $q_1^* = Q^* = (u_1 - b)/a$, and the associated aggregate payoff is $\Pi^* = (u_1 - b)^2/(2a)$. The following lemma establishes a lower bound of the aggregate payoff in the Nash equilibrium for some special cases, which turns out to be useful as we derive the price of anarchy for given price mechanisms.

Lemma 3 *Consider the single-link problem with linear utilities, quadratic cost, and the corresponding price mechanism $P_1(Q) = \lambda(b\beta + aQ)$. Then for every pair $\lambda > (n-1)/[4(n+1)]$ and $\beta \geq 0$, if $\bar{q}_1 > 0$, we have*

$$\begin{aligned}\bar{\Pi} \geq\ & \frac{1}{2}a(Q^*)^2 \frac{2(n(2\lambda - 1) + 2\lambda)}{\lambda(n(4\lambda - 1) + 4\lambda + 1)} \\ & + bQ^* \frac{-2(n(\lambda - 1) + \lambda)(\beta\lambda + \lambda - 1)}{\lambda(n(4\lambda - 1) + 4\lambda + 1)} - \frac{b^2}{a} \frac{(n(\lambda + 2) - \lambda)(\beta\lambda + \lambda - 1)^2}{2\lambda(n(4\lambda - 1) + 4\lambda + 1)}.\end{aligned}$$

Proof The best response of user i requires that $(u_i - \lambda b\beta - \lambda a\bar{Q} - \lambda a\bar{q}_i)\bar{q}_i = 0$. Summing over $i \in I$, we have

$$\bar{\Pi} \equiv \sum_{i=1}^{n} u_i \bar{q}_i - \left(b\bar{Q} + \frac{1}{2}a\bar{Q}^2\right) = (\lambda\beta - 1)b\bar{Q} + \left(\lambda - \frac{1}{2}\right)a\bar{Q}^2 + \lambda a \sum_{i=1}^{n} \bar{q}_i^2.$$

Since $\bar{q}_1 > 0$, by the first-order condition for user 1,

$$\bar{q}_1 + \bar{Q} = \frac{u_1/\lambda - b}{a} - b - \frac{\beta b}{a} = \frac{1}{\lambda}Q^* + \frac{b}{a}\frac{1 - \lambda\beta}{\lambda}.$$

We can then rewrite $\sum_{i=1}^{n} \bar{q}_i^2$ as

$$\sum_{i=1}^{n} \bar{q}_i^2 = \bar{q}_1^2 + \sum_{i\geq 2}^{n} \bar{q}_i^2 \geq \bar{q}_1^2 + \frac{1}{n-1}\left(\sum_{i\geq 2}^{n} \bar{q}_i\right)^2 = \bar{q}_1^2 + \frac{1}{n-1}(\bar{Q} - \bar{q}_1)^2.$$

Using this and replacing $\bar{Q}$ by

$$\frac{1}{\lambda}Q^* + \frac{b}{a}\frac{1-\lambda\beta}{\lambda} - \bar{q}_1,$$

we have

$$\begin{aligned}
\bar{\Pi} &\geq (\lambda\beta - 1)b\bar{Q} + \left(\lambda - \frac{1}{2}\right)a\bar{Q}^2 + \lambda a\bar{q}_1^2 + \frac{\lambda a}{n-1}(\bar{Q} - \bar{q}_1)^2 \\
&= (\lambda\beta - 1)b\left(\frac{1}{\lambda}Q^* + \frac{b}{a}\frac{1-\lambda\beta}{\lambda} - \bar{q}_1\right) + \left(\lambda - \frac{1}{2}\right)a\left(\frac{1}{\lambda}Q^* + \frac{b}{a}\frac{1-\lambda\beta}{\lambda} - \bar{q}_1\right)^2 \\
&\quad + \lambda a\bar{q}_1^2 + \frac{\lambda a}{n-1}\left(\frac{1}{\lambda}Q^* + \frac{b}{a}\frac{1-\lambda\beta}{\lambda} - 2\bar{q}_1\right)^2.
\end{aligned}$$

We can rearrange the above formula according to the descending order of $\bar{q}_1$. The coefficient of $\bar{q}_1^2$ is

$$a\left(\frac{2n+2}{n-1}\lambda - \frac{1}{2}\right),$$

and the coefficient of $\bar{q}_1$ is

$$-\frac{a((2n+2)\lambda - n + 1)}{\lambda(n-1)}Q^* + (\lambda\beta - 1)b\left(-1 + \frac{1}{\lambda}\frac{(2n+2)\lambda - n + 1}{n-1}\right).$$

Applying the inequality

$$Ay^2 + By \geq -\frac{B^2}{4A}$$

provided that $A > 0$, we have

$$\begin{aligned}
\bar{\Pi} &\geq (\lambda\beta - 1)b\left(\frac{1}{\lambda}Q^* + \frac{b}{a}\frac{1-\lambda\beta}{\lambda}\right) + a\left(\lambda - \frac{1}{2} + \frac{\lambda}{n-1}\right)\left(\frac{1}{\lambda}Q^* + \frac{b}{a}\frac{1-\lambda\beta}{\lambda}\right)^2 \\
&\quad - \frac{\left(-\frac{a((2n+2)\lambda-n+1)}{\lambda(n-1)}Q^* + (\lambda\beta - 1)b(-1 + \frac{1}{\lambda}\frac{(2n+2)\lambda-n+1}{n-1})\right)^2}{4a\left(\frac{2n+2}{n-1}\lambda - \frac{1}{2}\right)},
\end{aligned}$$

if

$$\frac{2n+2}{n-1}\lambda - \frac{1}{2} > 0.$$

We can rearrange it by descending order of Q^*. Hence if $(2n+2)\lambda/(n-1)-1/2 > 0$,

$$\bar{\Pi} \geq \frac{1}{2}a(Q^*)^2 \frac{2(n(2\lambda-1)+2\lambda)}{\lambda(n(4\lambda-1)+4\lambda+1)} \\ +bQ^* \frac{-2(n(\lambda-1)+\lambda)(\beta\lambda+\lambda-1)}{\lambda(n(4\lambda-1)+4\lambda+1} - \frac{b^2}{a}\frac{(n(\lambda+2)-\lambda)(\beta\lambda+\lambda-1)^2}{2\lambda(n(4\lambda-1)+4\lambda+1}.$$

□

The lower bound obtained in the above lemma will be used to derive the price of anarchy for given price mechanisms. We first show that a tight lower bound can be obtained in the special case of $\lambda\beta = 1$. We later verify that the optimal price mechanism belongs to this group and it is the only family of price mechanisms that provide strictly positive bounds of coordination ratios.

Theorem 1 *Consider the single-link problem with linear utilities and quadratic cost. Suppose Assumption 1 holds. When $P(Q) = b + \lambda a Q$ where $\lambda \in [n/(2n+2), 1]$, the coordination ratio is at least*

$$\frac{2(n(2\lambda-1)+2\lambda)}{\lambda(n(4\lambda-1)+4\lambda+1)}.$$

Proof In a Nash equilibrium $\{\bar{q}_i\}$'s, the best response of user i requires that $(u_i - \lambda b\beta - \lambda a\bar{Q} - \lambda a\bar{q}_i)\bar{q}_i = 0$. When $P(Q) = b + \lambda a Q$, $\{\bar{q}_i = 0, \forall i \in I\}$ cannot be an equilibrium since if this were the case, the KKT condition for user 1 would require $u_1 - b \leq 0$, a contradiction to Assumption 1. If in equilibrium, $\bar{q}_1 = 0$ but we can find a user $i \neq 1$ such that $\bar{q}_i > 0$, then $u_1 - b - \lambda a\bar{Q} \geq u_i - b - \lambda a\bar{Q} > u_i - b - \lambda a\bar{Q} - \lambda a\bar{q}_i = 0$, and hence $\bar{q}_1 = 0$ cannot be user 1's best response. Therefore, $\bar{q}_1 > 0$.

Recall the formula in Lemma 3. The price scheme $P(Q) = b + \lambda a Q$ implies that $\lambda\beta = 1$, in which case the last two terms vanish, and the coefficient of $(1/2)aQ^{*2} = \Pi^*$ becomes the lower bound of the desired ratio. Note that within the range $\lambda \in [n/(2n+2), 1]$, $\lambda \geq 0$ and $\beta \geq 1$ under condition $\lambda\beta = 1$, and the bound is nonnegative as well. □

We next show that these bounds are all tight, in the sense that the worse case works for arbitrary quadratic costs.

Lemma 4 *Consider the single-link problem with linear utilities and quadratic costs. Suppose Assumption 1 holds and $P(Q) = b + \lambda a Q$ where $\lambda \in [n/(2n+2), 1]$, the coordination ratio is at most* $2(n(2\lambda-1)+2\lambda)[\lambda(n(4\lambda-1)+4\lambda+1)]$.

Proof Let

$$\frac{u_i - b}{u_1 - b} = \frac{2n\lambda + 2\lambda + 1}{n(4\lambda-1)+4\lambda+1} \equiv \theta, \ \forall i \geq 2,$$

where $\theta \in [1/2, 1]$ if $\lambda \in [n/(2n+2), 1]$. Consider the following allocations:

$$\bar{q}_1 = \frac{1}{\lambda}\left(1 - \frac{(n-1)\theta + 1}{n+1}\right) Q^*, \quad \bar{q}_i = \frac{1}{\lambda}\left(\theta - \frac{(n-1)\theta + 1}{n+1}\right) Q^*, i \geq 2.$$

Summing over $i \in I$ to obtain $\bar{Q}$, we obtain

$$\bar{q}_1 + \bar{Q} = \frac{1}{\lambda} Q^*, \quad \bar{q}_i + \bar{Q} = \frac{1}{\lambda}\theta Q^*, \; i \geq 2.$$

It can then be verified that $\bar{q}_i > 0$ and the KKT conditions are satisfied. Thus, $\{\bar{q}_i\}$'s constitute a Nash equilibrium.

Since $\bar{\Pi} = \sum_{i \in I}(u_i - b)\bar{q}_i - (1/2)a\bar{Q}^2$, the ratio of aggregate payoff is

$$\frac{\bar{\Pi}}{\Pi^*} = \frac{2}{\lambda}\left(1 - \frac{(n-1)\theta + 1}{n+1}\right) + \frac{2(n-1)}{\lambda}\theta\left(\theta - \frac{(n-1)\theta + 1}{n+1}\right) - \frac{1}{\lambda^2}\left(\frac{(n-1)\theta + 1}{n+1}\right)^2,$$

where the right-hand side can be simplified to $2(n(2\lambda-1)+2\lambda)[\lambda(n(4\lambda-1)+4\lambda+1)]$ while plugging in

$$\theta = \frac{2n\lambda + 2\lambda + 1}{n(4\lambda - 1) + 4\lambda + 1}.$$

Thus, the price of anarchy in this case is upper bounded as mentioned in the lemma. □

When λ, β satisfy the conditions of Theorem 1, the bound is decreasing in the number of users. Therefore, the coordination problem deteriorates as the population becomes large. However, when

$$\frac{n(1+\delta)}{2n+2} \leq \lambda \leq 1,$$

where $\delta > 0$, a constant limit always exists for arbitrary quadratic costs. The marginal cost pricing corresponds to $\lambda = 1$ and $\beta = 1$, in which case the bound becomes $(2n+4)/(3n+5)$, and it converges to $2/3$ as obtained in [11]. However, our result refines that of [11], since we provide tight bounds for any number of users (we in Sect. 4.2 show that this bound works for general network problems with concave utilities). From Lemma 4, when users' utilities are linear, the worst case occurs when

$$\frac{u_i - b}{u_1 - b} = \frac{2n+3}{3n+5},$$

for all $i \geq 2$. This is obtained as we plug in $\lambda = 1$ in

$$\frac{u_i - b}{u_1 - b} = \frac{2n\lambda + 2\lambda + 1}{n(4\lambda - 1) + 4\lambda + 1},$$

and it is the worst case because the bound is attained. The average cost pricing ($\lambda = 1/2$, $\beta = 2$) leads to the bound $4/(n+3)$, which vanishes as n approaches ∞. Under quadratic costs, the average cost pricing coincides with the Aumann-Shapley price, the Shapley-Shubik price and the nucleolus rule [30]. Nevertheless, this sharing rule performs extremely poorly when we take into consideration the non-cooperative strategic interaction among users.

Now we characterize the optimal price mechanism and provide the bounds of coordination ratios.

Theorem 2 *Consider the single-link problem with linear utilities and quadratic cost. Suppose that Assumption 1 holds and the price mechanisms depend merely on the cost structures. Let*

$$\lambda^* = \frac{2n + 2n^2 + \sqrt{2n(1+n)^3}}{4(1+n)^2}, \quad \text{and} \quad \beta^* = \frac{1}{\lambda^*},$$

the tight bound of coordination ratio is

$$\eta_n \equiv \frac{4(1+n)^2\sqrt{2n(1+n)^3}}{(1 + 2n + n^2 + \sqrt{2n(1+n)^3})(2n + 2n^2 + \sqrt{2n(1+n)^3})},$$

with $4(3 - 2\sqrt{2}) \approx 0.686$ *being its limit. Moreover, for all* $\lambda > 0$, $\beta \geq 1$ *that differ from the above pair, the price of anarchy is strictly less than* η_n.

Proof We first consider the price parameters

$$\lambda = \frac{2n + 2n^2 + \sqrt{2}\sqrt{n(1+n)^3}}{4(1+n)^2}, \quad \text{and} \quad \beta = \frac{4(1+n)^2}{2n + 2n^2 + \sqrt{2}\sqrt{n(1+n)^3}}.$$

Note that such a choice satisfies the conditions in Theorem 1, and hence we obtain a tight bound given this price mechanism.

We shall prove that for any other choice, the bound of coordination ratio cannot be further improved. We first consider the region discussed in Theorem 1. The ratio $2(n(2\lambda - 1) + 2\lambda)/[\lambda(n(4\lambda - 1) + 4\lambda + 1)]$ does not depend on β, is unimodal in λ, and achieves the maximum when

$$\lambda = \frac{2n + 2n^2 + \sqrt{2}\sqrt{n(1+n)^3}}{4(1+n)^2} \in \left[\frac{n}{2n+2}, 1\right].$$

Hence the lower bound in this region is always less than η_n from Lemma 4.

Now we consider the case $\lambda\beta - 1 = 0$ but $\lambda \notin [n/(2n+2), 1]$, and construct examples to obtain the upper bound. When $\lambda \in (0, 1/4]$, we choose

$$\theta \equiv \frac{u_i - b}{u_1 - b} \in \left(0, \frac{1}{2}\right], \quad \text{for all } i \geq 2.$$

It can be verified that for all users other than user 1, $\bar{q}_i = 0$ in equilibrium. Applying the first-order condition on $\bar{q}_1$, we have $\bar{q}_1 = (u_1 - b)/(2\lambda a)$, and hence the aggregate payoff becomes

$$\bar{\Pi} = \left(2\lambda - \frac{1}{2}\right) a \frac{(u_1 - b)^2}{4\lambda^2 a^2} = \frac{1}{\lambda}\left(1 - \frac{1}{4\lambda}\right)\Pi^*.$$

This gives us a non-positive ratio as $\lambda \in (0, \frac{1}{4}]$. On the other hand, if $\lambda \in (1/4, n/(2n+2)]$, we set $u_i = u_1$, for all $i \geq 2$. By symmetry the equilibrium quantity $\bar{q}_i = \bar{q}_j$, for all $i, j \in I$, and hence the first-order condition yields

$$\bar{q}_i = \frac{1}{\lambda(n+1)} Q^*.$$

The corresponding ratio of aggregate payoff is

$$\frac{n(2\lambda + n(2\lambda - 1))}{\lambda^2(n+1)^2} \equiv A(\lambda).$$

Differentiating $A(\lambda)$, we obtain

$$A^{'}(\lambda) = \frac{2n(n - \lambda(n+1))}{\lambda^3(n+1)^2} \geq \frac{2n}{\lambda^3(n+1)^2}\left(n - \frac{n}{2n+2}(n+1)\right) = \frac{n^2}{\lambda^3(n+1)^2} > 0,$$

where the first inequality follows from $\lambda \leq n/(2n+2)$. Thus, $A(\lambda)$ is increasing in λ. If we fix $u_i = u_1$, for all $i \geq 2$, the upper bound of coordination ratio when $\lambda \in (1/4, n/(2n+2)]$ is less than $A(n/(2n+2)) = 0$, i.e., no positive bound can be obtained.

If $\lambda\beta - 1 \equiv \gamma > 0$, we can let $u_1 \in (b, \lambda\beta b)$. KKT conditions suggest that $\bar{q}_i = 0$, for all $i \in I$, which makes the aggregate payoff 0 and hence no strictly positive bound can be obtained.

Finally, suppose that $\lambda\beta - 1 \equiv \gamma < 0$. Define $D = b/(u_1 - b)$ and let $u_i = u_1$, for all $i \geq 2$. From the first-order condition and symmetry among users,

$$\bar{q}_i = \frac{1}{n+1}\left(\frac{1}{\lambda} - \gamma D\right) Q^*.$$

Note that $1/\lambda - \gamma D > 0$ for all $D > 0$, since $\gamma < 0$. The aggregate payoff becomes

$$\bar{\Pi} = \sum_{i \in I}(u_i - b)\bar{q}_i - \frac{1}{2}a\bar{Q}^2 = \Pi^*\left(\frac{2n}{n+1}\left(\frac{1}{\lambda} - \gamma D\right) - \left(\frac{n}{n+1}\left(\frac{1}{\lambda} - \gamma D\right)\right)^2\right).$$

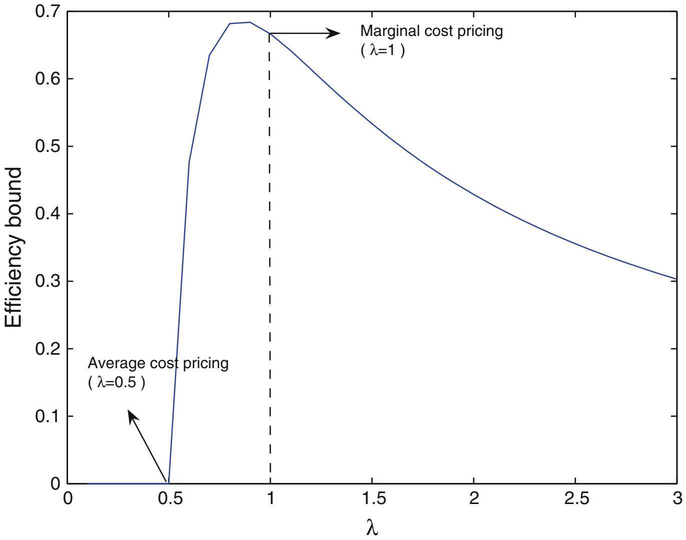

Fig. 1 Efficiency bounds for given λ's when $\lambda\beta = 1$ as $n \to \infty$

The ratio can be expressed as a quadratic function of D where the coefficient of D^2 is $-n^2\gamma^2/(n+1)^2 < 0$, and hence it approaches $-\infty$ while letting D approach ∞. Thus for any fixed constant M, there exists D such that $\bar{\Pi}/\Pi^* < -M$ if $\gamma < 0$. Following a similar argument, we can show that when $\gamma > 0$ and $0 < \lambda < 1/2$, as $n \to \infty$ and $D \to 0$, the ratio will be in the neighborhood of $(2\lambda - 1)/\lambda < 0$. □

The above theorem shows that our proposed price mechanism indeed achieves the smallest price of anarchy if the central planner has no access to users' preferences. The optimal pricing scheme is

$$P(Q) = b + \frac{2n + 2n^2 + \sqrt{2}\sqrt{n(1+n)^3}}{4(1+n)^2} aQ,$$

which has never been proposed elsewhere. The worst case occurs when users possess linear utilities, and except the highest one they are identical. When the optimal price mechanism is used, it is achieved when $(u_i - b)/(u_1 - b) \to \sqrt{2}/2$ as n approaches ∞ (according to Lemma 4). Such a relation among users' preferences is nontrivial as well.

Figure 1 shows the bounds of coordination ratios for given λ's, where the bounds are achieved when the number of users approaches infinity. When $\lambda < 1/2$, the price mechanism provides no guarantee of price of anarchy. When λ is above 1/2, which corresponds to the average cost pricing, the bound increases and attains its optimum at around $0.833 \approx (2 + \sqrt{2})/4$. It then decreases as λ becomes large. Figure 2 compares the bounds of coordination ratios with respect to the number of users under the marginal cost pricing, the average cost pricing, and our optimal price mechanism. As the population becomes large, the bound deteriorates due to the increasing difficulty

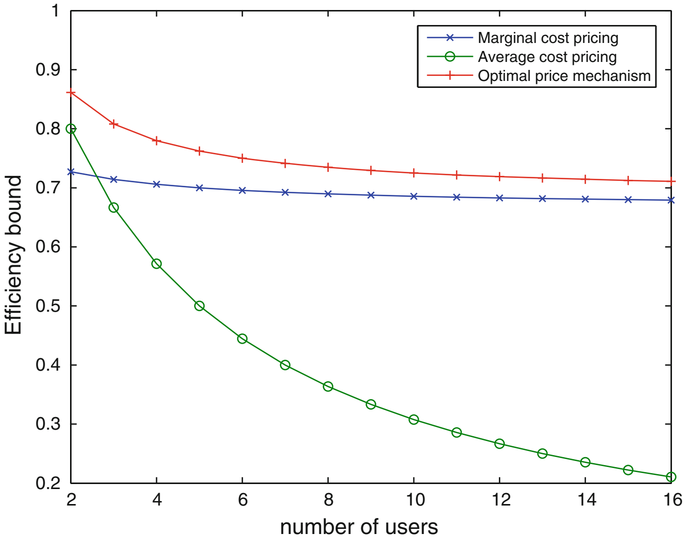

Fig. 2 Efficiency bounds versus n

of coordination among users, regardless of the price mechanisms used for congestion control. We further find that the marginal cost pricing outperforms the average cost pricing except when there are only two users, and the gap between the optimal mechanism and the marginal cost pricing becomes narrower when the system scale becomes large. The limiting bounds are respectively $4(3-2\sqrt{2}) \approx 68.6\%$ and $2/3 \approx 66.7\%$. Thus, the marginal cost pricing not only achieves social optimality when the market is perfectly competitive (i.e., when users are price takers), but also provides a nearly optimal, although suboptimal, guarantee of coordination ratio in any large-scale communication networks with price anticipation.

Finally, since we change the price faced by the users, when $\lambda\beta \neq 1$, the cost in equilibrium actually exceeds the aggregate utility of the users. Although a user can always ensure herself a zero payoff by setting $q_i = 0$, the price mechanism may induce the users to request too much and ultimately the cost outweighs the aggregate utility. Thus, the ratio could be negative and is not bounded below. Inappropriate selections of price mechanisms may cause disasters.

4.2 Extension to general network problems with concave utilities

We now consider the general network structure when users possess concave utilities. We propose to adopt the linear unit prices for each link in the network structure, and use the common parameters λ^*, β^* for all unit prices. That is, we choose $P_e(\mathbf{Q}) = \lambda^*(b_e\beta^* + a_e Q_e)$, where

$$\lambda^* = \frac{2n + 2n^2 + \sqrt{2}\sqrt{n(1+n)^3}}{4(1+n)^2}, \quad \text{and} \quad \beta^* = \frac{1}{\lambda^*}.$$

Note that a slight generalization of Lemma 1 shows that this price mechanism satisfies the four axioms.

Given the price mechanisms, the existence of Nash equilibrium follows from an argument similar to [11]. Furthermore, we can follow the procedure of [10] and [34] to establish lower bounds of coordination ratios in the general network. Note that even though [11] focus primarily on the marginal cost pricing, their proof technique remains valid for our more general price mechanisms (with appropriate modifications). Thus, since in Sect. 4.1 we have proved that the proposed price scheme is the unique mechanism that provides the maximum achievable bound of coordination ratio for the single-link problems, it is suboptimal to adopt any mechanism that is properly different from our proposed price scheme in the single-link problem. We need not characterize the set of price mechanisms that satisfy the four axioms in the general network structure, nor do we modify the rescaling axiom to make it compatible to the flow conservation. This simple price scheme outperforms all other mechanisms that satisfy the four axioms in terms of price of anarchy. We summarize the results in the following corollary.

Corollary 1 *In the general network problems with concave utilities and quadratic cost, suppose that Assumption 1 holds and the price mechanisms depend merely on the cost structures. Then the tight bound of coordination ratio is η_n. Moreover, for all pairs of $\lambda > 0$ and $\beta \geq 1$ that differ from the above pair, the price of anarchy is strictly less than η_n.*

4.3 Optimality when price mechanisms are extended to incorporate continuous costs

We now briefly discuss the case when price mechanisms are extended to incorporate continuous costs. The purpose of introducing the continuous costs is to demonstrate the flexibility of the family of price mechanisms that follow from the four axioms. Since we focus only on the quadratic cost structures in our flow allocation problem, we are mainly interested in the corresponding unit price $P_1(C, Q)$ for $C(Q) = bQ + (1/2)aQ^2$. The following lemma shows that the linearity continues to hold, but the parameter β can take value from $[1, \infty)$ (cf. in Lemma 1, the region of β is $[0, \infty)$).

Lemma 5 *Suppose the price mechanism is defined for all continuous cost functions. In the single-link problem, the price mechanism satisfies the rescaling, additivity, and positivity axioms only if there exist constants $\lambda \geq 0$ and $\beta \geq 1$ such that the unit price for $C(Q) = bQ + (1/2)aQ^2$ is $P_1(C, Q) = \lambda(b\beta + aQ)$, for all Q. Conversely, if such λ and β exist for quadratic costs, then there exists a mechanism that satisfies the above axioms for all continuous functions and coincides with $P_1(C, Q) = \lambda(b\beta + aQ)$ when applied to the cost $C(Q) = bQ + (1/2)aQ^2$.*

Proof When the number of outputs $m = 1$, [28, Propositions 1 and 2] show that for a general cost function $\tilde{C}(Q)$, if a price mechanism satisfies the rescaling, additivity, and positivity axioms, then there exists a nonnegative measure μ such that

$$P_1(\tilde{C}, Q) = \int_0^1 \frac{\partial \tilde{C}}{\partial Q}(tQ)d\mu(t).$$

This price mechanism is defined for all continuous cost functions $\tilde{C}(\cdot)$. In particular, when $C(Q) = bQ + (1/2)aQ^2$, the corresponding price mechanism translates to

$$P_1(C, Q) = \int_0^1 \frac{\partial C}{\partial Q}(tQ)d\mu(t) = b\int_0^1 d\mu(t) + aQ\int_0^1 td\mu(t).$$

Thus, if we define

$$\lambda = \int_0^1 td\mu(t), \quad \text{and} \quad \beta = \frac{\int_0^1 d\mu(t)}{\int_0^1 td\mu(t)},$$

the price mechanism for this particular cost can be expressed as $P_1(C, Q) = \lambda(b\beta + aQ)$. The facts $\lambda \geq 0$ and $\beta \geq 1$ follow from that μ is nonnegative and

$$\int_0^1 d\mu(t) \geq \int_0^1 td\mu(t).$$

On the other hand, suppose that there exist nonnegative constants λ and β such that $P_1(C, Q) = \lambda(b\beta + aQ)$. We shall extend this unit price to a price mechanism that is defined for all continuous cost functions. Let us define the measure μ as such: $\mu(0) = \lambda\beta, \mu(1) = \lambda$, and $\mu(x) = 0$,for all $x \notin \{0, 1\}$. With this choice, μ is nonnegative and

$$P_1(C, Q) = b\int_0^1 d\mu(t) + aQ\int_0^1 td\mu(t).$$

For general continuous cost $\tilde{C}$, we propose to use the same measure and define

$$P_1(\tilde{C}, Q) = \int_0^1 \frac{\partial \tilde{C}}{\partial Q}(tQ)d\mu(t).$$

Since this price is a linear functional, the rescaling and additivity axioms hold, and the positivity follows from the nonnegativity of measure μ. □

Lemma 5 reduces the problem of finding the price mechanism to an optimization problem over two parameters λ and β in single-link problems as well. Under this price mechanism, the existence and uniqueness of Nash equilibrium is guaranteed as Lemma 2 applies equally well to this alternative scenario.

Note that Lemma 5 guarantees that if we find a mechanism that achieves the optimal price of anarchy, we can construct an extended mechanism defined for all continuous functions. Moreover, the price parameters now satisfy $\lambda \geq 0, \beta \geq 1$. Therefore, Lemma 2 continues to apply here, and we can extend our proposed $P_e(\mathbf{Q}) = b_e + \lambda^* a_e Q_e$, to a price mechanism for continuous costs. Under this price mechanism

$$\beta^* = \frac{4(1+n)^2}{2n + 2n^2 + \sqrt{2}\sqrt{n(1+n)^3}} \geq 1,$$

and hence it remains feasible even if the price mechanism is defined for all continuous cost functions. In the single-link problem, since $P(Q) = b + \lambda^* a Q$ achieves the minimum price of anarchy among all prices with $\lambda \geq 0, \beta \geq 0$, it remains optimal when the search is confined within $\lambda \geq 0, \beta \geq 1$. This bound goes through in the general network problem with concave utilities; hence, our proposed price mechanism is still optimal for the quadratic cost functions, even if price mechanisms are defined for all continuous costs. Note that this does not imply that our price mechanism remains optimal for other cost structures. Our point here is that we can easily extend our proposed price mechanism to incorporate general cost structures.

5 Price of anarchy when information of users' preferences is available for single-link problems

In this section, we assume that the system designer knows the users' preference profile, and the price mechanism can therefore depend on the users' preferences. There are several examples of price mechanisms that depend on the users' preferences, including Ramsey pricing [32], traffic tolls on peak-hour congestion and gasoline [24], and externality-reducing taxation [29]. This informational requirement is not stricter than that needed when the system optimal routing/transmission is implementable, as assumed in the centralized communication networks for decades [12]. Furthermore, this assumption is adopted in a number of economics papers in different contexts, including, e.g., the optimal taxation design [25] and the monopoly pricing problem [9]. The utility profile may be known to the public when either the planner has statistical knowledge of the society [25] or when the monopoly seller has access to general market research [9]. Our purpose, in this section, is to demonstrate that information acquisition regarding user preferences may lead to significant efficiency improvement for the communication networks. Note also that the system efficiency may still suffer from the impact of hidden actions because the users can self-select their communication flows and they are free to split or combine their flows.

We focus on the single-link problem, and again use the notation $P(Q) \equiv P_1(C, Q)$, and adopt $P(Q) = \lambda(b\beta + aQ)$. However, we use $\bar{q}_i(\lambda, \beta) \equiv \bar{q}_i(P)$ to denote the equilibrium flows. From Lemma 2, $\bar{q}_i(\lambda, \beta)$ is uniquely determined by $\lambda > 0, \beta \geq 1$.

This makes the mechanism design problem tractable, as opposed to the network case. In this scenario, the system designer knows the utilities $\{U_i\}$'s but not whom they belong to. Thus, the system designer will use this information to design the price mechanism.

5.1 Linear utilities

We first consider the case with linear utilities, i.e., $U_i(q_i) = u_i q_i$, and show that system optimality is achieved for any instance $\{u_i\}$'s.

Theorem 3 *Consider the single-link problem with quadratic cost and suppose users' preferences are accessible. With linear utilities, there exist a continuum of (λ, β)'s such that $\bar{\Pi}(\lambda, \beta) = \Pi^*$.*

Proof Let

$$\Pi_i(q_i, \bar{q}_{-i}(\lambda, \beta)) = u_i q_i - \lambda \left[b\beta + a(q_i + \sum_{j \neq i} \bar{q}_j(\lambda, \beta)) \right] q_i$$

be user i's payoff while sending q_i, assuming that other users follow the equilibrium strategies. The derivative of user i's net payoff with respect to q_i is

$$\begin{aligned} \frac{\partial \Pi_i(q_i, \bar{q}_{-i}(\lambda, \beta))}{\partial q_i} &= u_i - \lambda b\beta - \lambda a(q_i + \sum_{j \neq i} \bar{q}_j(\lambda, \beta)) - \lambda a q_i \\ &= a\left[\frac{u_i - b}{a} - \frac{b}{a}(\lambda\beta - 1)\right] - \lambda a \left(\sum_{j \neq i} \bar{q}_j(\lambda, \beta) + 2q_i\right) \\ &= aQ^* \left[\frac{u_i - b}{u_1 - b} - \frac{b}{aQ^*}(\lambda\beta - 1)\right] - \lambda a \left(\sum_{j \neq i} \bar{q}_j(\lambda, \beta) + 2q_i\right). \end{aligned}$$

Let us first assume that $u_1 > u_2 \geq \cdots \geq u_n$. Define

$$\theta = \frac{u_2 - b}{u_1 - b} \in [0, 1).$$

Since $q_i^* = 0$, for all $i \geq 2$, our goal is to show that there exists a pair (λ, β) such that $\bar{q}_i(\lambda, \beta) = 0$, for all $i \geq 2$ and $\bar{q}_1(\lambda, \beta) = Q^*$. From the derivatives $\partial \Pi_i(q_i, \bar{q}_{-i}(\lambda, \beta))/\partial q_i$, it suffices to check the feasibility of

$$\frac{1 - (\lambda\beta - 1)\frac{b}{u_1 - b}}{2\lambda} = 1, \quad \theta \leq (\lambda\beta - 1)\frac{b}{u_1 - b}, \tag{3}$$

and that $\lambda \geq 0$ and $\beta \geq 1$. Note that the second condition in (3) guarantees that $\bar{q}_i(\lambda, \beta) = 0$, for all $i \geq 2$, and the first condition results in $\bar{q}_1(\lambda, \beta) = Q^*$. If we choose

$$\lambda \in \left(0, \frac{1-\theta}{2}\right] \quad \text{and} \quad \beta = \frac{1 + \frac{b}{u_1 - b} - 2\lambda - \lambda \frac{b}{u_1 - b}}{\lambda b/(u_1 - b)} + 1$$

accordingly, this combination is feasible to (3) and $\lambda \geq 0$ and $\beta \geq 1$.

Now consider the case when the first k users have the same utilities, i.e., $u_1 = \cdots = u_k > u_{k+1} \geq \cdots \geq u_n$. In this case, to induce the system optimal solution in the Nash equilibrium, we must ensure that $\bar{q}_i = 0$, for all $i \geq k + 1$, and $\sum_{i=1}^{k} \bar{q}_i(\lambda, \beta) = Q^*$. In this case, we can follow the above argument to check the feasibility of

$$\frac{1 - (\lambda\beta - 1)\frac{b}{u_1 - b}}{(k+1)\lambda} = 1, \quad \theta \leq (\lambda\beta - 1)\frac{b}{u_1 - b}.$$

Any solution for which

$$\lambda \in \left(0, \frac{1-\theta}{k+1}\right], \quad \beta = \frac{1 + \frac{b}{u_1 - b} - (k+1)\lambda - \lambda \frac{b}{u_1 - b}}{\lambda b/(u_1 - b)} + 1$$

is feasible. Consequently, the efficiency can be achieved in a decentralized equilibrium in this case as well. □

In the case with linear utilities, system optimality can be achieved by a uniform price that depends on the cost functions and the ratio between the (distinct) marginal utilities of the highest two users. Note that as opposed to VCG mechanism, our price mechanism does not require the system designer to identify the users' identities, and splitting or combining orders is not profitable for any user. Due to the multiplicity of Nash equilibria, we cannot follow the approach in Sect. 4 to extend to the network case.

5.2 Concave utility functions

Now we switch to the case with nonlinear utilities. Given that the efficiency can be achieved when the users' utilities are linear, one may conjecture that this continues to hold even with nonlinear utilities. However, the following example ends the hope. Suppose that $n = 2$, the cost functions $C(Q) = Q + Q^2$, and the utilities are respectively

$$U_1(q_1) = \begin{cases} 3q_1 - \frac{1}{2}q_1^2, & q_1 \leq 3, \\ \frac{9}{2}, & q_1 > 3, \end{cases} \quad U_2(q_2) = \begin{cases} 2q_2 - \frac{1}{2}q_2^2, & q_2 \leq 2, \\ 2, & q_2 > 2. \end{cases}$$

It can be verified that the centralized solution is $(q_1^*, q_2^*) = (2/3, 0)$. Suppose, on the contrary, that there exist a pair of nonnegative constants (A, B) such that $(2/3, 0)$ is a Nash equilibrium with $P(Q) = B + AQ$. Then the first-order conditions yield

$$3 - \frac{2}{3} - B - \frac{2}{3}A - \frac{2}{3}A = 0 \quad \text{and} \quad 2 - B - \frac{2}{3}A \leq 0,$$

which has a unique solution $(A, B) = (0, 7/3)$. But this would imply $\lambda = 0$ and $\lambda\beta = 7/3$, a contradiction.

The above example shows that the efficiency cannot be achieved in all the single-link problems. The question that remains is whether knowing the users' preference profile allows the system designer to achieve a better performance guarantee. We will show that bounds of coordination ratios can be further improved from 0.686 if utility functions satisfy the following regularity conditions:

Assumption 2 The set of utility functions $\{U_i(\cdot)\}$'s satisfy:

1. $U_i^{'}$ exists, and $\max_{i \in I} U_i^{'}(0) < \infty$.
2. For all positive flow q, $U_i^{'}(q) \geq U_j^{'}(q)$ whenever $i \leq j$.

Condition 1 in Assumption 2 guarantees the existence of utilities' derivatives and ensures that the marginal utility for any user is bounded. The second condition is usually referred to as the single-crossing (sorting) condition, or Spence-Mirrlees condition. This condition is commonly adopted in a variety of literature, including nonlinear pricing [16], Nash implementation [17], auctions [15], and operations management [5]. Examples that conform the single-crossing condition include the popular utility functions $U_i(q) = \log(q + C_i)$ for empirical work, where constants C_i's satisfy $C_i \leq C_j$ whenever $i \leq j$ and $C_i > 1, \forall i$ to ensure that the utilities are nonnegative. Another class is $U_i(q) = u_i q + G(q)$, with $G(q)$ being concave in q and nonnegative. Note that the linear utility $U_i(q) = u_i q$ is a special case of this class.

The single-crossing condition is adopted in mechanism design literature to ensure the possibility of truth-telling mechanism [8], in games with incomplete information to establish the existence of pure-strategy Nash equilibrium [1], and in Nash implementation to warrant game forms that Nash implements certain social choice correspondences [17]. As we demonstrate later, here the single-crossing condition helps us to show that the same user has the highest marginal utility in any Nash equilibrium, which enables us to design a corresponding price parameter. Without this assumption, the highest marginal utility in equilibrium can switch among users while varying the price schemes, and the continuity, a required property for us to establish the upper bound of price of anarchy, becomes problematic. It is also worth mentioning that our analysis is significantly different from Nash implementation approach, albeit complete information among agents is required in both settings. Our proposed mechanism does not implement the centralized solution. On the contrary, we use a simple common price and guarantee a bound of price of anarchy. Following [17], the canonical mechanism for Nash implementation requires each agent to report preference profiles of all agents, and includes a numbering system to achieve incentive compatibility.

Note that $U_i(q)$ should also be nonnegative, increasing, and concave, as discussed in Sect. 2. From Assumption 1, we obtain that $U_1^{'}(0) > b$. when the single-crossing condition holds. With Assumption 2, we are able to provide structural properties of users' behavior in equilibrium. Note that the users' utilities $\{U_i\}$'s depend on only through the flow profiles they select (i.e., $\{\bar{q}_i(\lambda, \beta)\}$'s). We first show that equilibrium flows can be ordered.

Lemma 6 *Consider the single-link problem with concave utilities and quadratic cost and suppose that Assumption 2 holds. Then for every pair* $\lambda > 0$ *and* $\beta \geq 1$, *we have* $\bar{q}_i(\lambda, \beta) \geq \bar{q}_j(\lambda, \beta)$, *for all* $i \geq j$.

Proof From the first-order condition for user i, we have

$$\bar{q}_i(\lambda, \beta)[U_i^{'}(\bar{q}_i(\lambda, \beta)) - \lambda\beta - \lambda a\bar{Q}(\lambda, \beta) - \lambda a\bar{q}_i(\lambda, \beta)] = 0.$$

Suppose that $\bar{q}_j(\lambda, \beta) > \bar{q}_i(\lambda, \beta)$, $i \leq j$. From the concavity and single-crossing condition, we obtain that $U_i^{'}(\bar{q}_i(\lambda, \beta)) \geq U_i^{'}(\bar{q}_j(\lambda, \beta)) > U_j^{'}(\bar{q}_j(\lambda, \beta))$. Hence,

$$\begin{aligned} U_i^{'}(\bar{q}_i(\lambda, \beta)) - \lambda\beta b - \lambda a\bar{Q}(\lambda, \beta) - \lambda a\bar{q}_i(\lambda, \beta) > U_j^{'}(\bar{q}_j(\lambda, \beta)) - \lambda\beta b - \lambda a\bar{Q}(\lambda, \beta) \\ -\lambda a\bar{q}_j(\lambda, \beta) = 0, \end{aligned}$$

because $\bar{q}_j(\lambda, \beta) > \bar{q}_i(\lambda, \beta) \geq 0$. But then $\bar{q}_i(\lambda, \beta)$ cannot be a best response. □

We now establish the monotonicity of $U_i^{'}(\cdot)$'s in equilibrium.

Lemma 7 *Consider the single-link problem with concave utilities and quadratic cost and suppose that Assumption 2 holds. Then for every pair* $\lambda > 0$ *and* $\beta \geq 1$, $U_i^{'}(\bar{q}_i(\lambda, \beta)) \geq U_j^{'}(\bar{q}_j(\lambda, \beta))$, *for all* $i \leq j$.

Proof Suppose that $\bar{q}_i(\lambda, \beta) > 0$. From the first-order conditions, in equilibrium we have

$$\begin{aligned} U_j^{'}(\bar{q}_j(\lambda, \beta)) \leq \lambda\beta b + \lambda a\bar{Q}(\lambda, \beta) + \lambda a\bar{q}_j(\lambda, \beta) \leq \lambda\beta b + \lambda a\bar{Q}(\lambda, \beta) + \lambda a\bar{q}_i(\lambda, \beta) \\ = U_i^{'}(\bar{q}_i(\lambda, \beta)). \end{aligned}$$

If $\bar{q}_i(\lambda, \beta) = 0$, then $\bar{q}_j(\lambda, \beta) = 0$ by Lemma 6, and Assumption 2 leads to $U_j^{'}(0) \leq U_i^{'}(0)$. □

Now we state our main result in this section. We propose a price scheme with coordination ratio $8n(n+1)/(3n+1)^2$, which converges to $8/9 > 4(3 - 2\sqrt{2})$.

Theorem 4 *Consider the single-link problem with concave utilities and quadratic cost. Suppose that Assumptions 1 and 2 hold and the information of the users' preferences is available. We can choose* $\lambda = (n-1)/[4(n+1)]$ *and* $\beta \geq 1$ *(which depends on utilities) such that the coordination ratio is at least* $8n(n+1)/(3n+1)^2$.

Proof We first assume that the following equation

$$\lambda\beta - 1 = \frac{2(n+1)}{3n+1} \frac{\max_{i\in I} U_i^{'}(\bar{q}_i(\lambda, \beta)) - b}{b} \quad (4)$$

has a solution $\beta \geq 1$ when $\lambda = (n-1)/[4(n+1)]$. If we use linear approximation for utility functions, i.e., $u_i = U_i^{'}(\bar{q}_i(\lambda, \beta))$, from the proof of Lemma 3 we obtain that

$$\bar{\Pi}(\lambda, \beta) \geq (\lambda\beta - 1)b\left(\frac{1}{\lambda}Q^* - \frac{b(\lambda\beta - 1)}{a\lambda}\right) + a\left(\lambda - \frac{1}{2} + \frac{\lambda}{n-1}\right)$$
$$\times \left(\frac{1}{\lambda}Q^* - \frac{b(\lambda\beta - 1)}{a\lambda}\right)^2 + \left(\frac{2n+2}{n-1}\lambda - \frac{1}{2}\right)a\bar{q}_1^2$$
$$+ \left[-\frac{(2n+2)\lambda - n + 1}{\lambda(n-1)}aQ^* + (\lambda\beta - 1)b\left(-1 + \frac{1}{\lambda}\frac{(2n+2)\lambda - n + 1}{n-1}\right)\right]\bar{q}_1.$$

As we choose $\lambda = (n-1)/[4(n+1)]$ and correspondingly

$$\beta = \frac{2(n+1)}{3n+1}\frac{aQ^*}{\lambda b} + \frac{1}{\lambda},$$

both the coefficients of $\bar{q}_1$ and $\bar{q}_1^2$ become zero. Moreover,

$$\lambda\beta - 1 = \frac{2(n+1)}{3n+1}\frac{aQ^*}{b},$$

and therefore the left-over terms on the right-hand side can be combined as a single term

$$\frac{8n(n+1)}{(3n+1)^2}\frac{1}{2}aQ^*,$$

where $aQ^*/2 = \Pi^*$. Thus, we obtain a lower bound of the aggregate payoff that is independent of $\bar{q}_1$:

$$\bar{\Pi}(\lambda, \beta) \geq \frac{8n(n+1)}{(3n+1)^2}\Pi^*.$$

□

We now prove that (4) indeed has a fixed point when $\lambda = (n-1)/[4(n+1)]$. We in the following provide more general results that work for any $\lambda \in (0, 1)$, and decompose the proof into a sequence of lemmas. Note that Lemma 7 allows us to consider merely $U_1^{'}(\bar{q}_1(\lambda, \beta)) = \max_{i \in I} U_i^{'}(\bar{q}_i(\lambda, \beta))$ in (4) for any Nash equilibrium. We first establish the continuity of $\bar{q}_i(\lambda, \beta)$ while varying β.

Lemma 8 *Fixed any $\lambda > 0$, $\bar{q}_i(\lambda, \beta)$ is continuous in β, and $\bar{Q}(\lambda, \beta)$ is also continuous in β.*

Proof We first show that $\bar{Q}(\lambda, \beta)$ is decreasing in β. Suppose the claim is not true, then there must exist $\hat{\beta}$ and β such that $\hat{\beta} > \beta$ but $\bar{Q}(\lambda, \hat{\beta}) > \bar{Q}(\lambda, \beta)$. Consider a user i with $\bar{q}_i(\lambda, \hat{\beta}) > 0$. The KKT condition for such user i becomes $U_i^{'}(\bar{q}_i(\lambda, \hat{\beta})) - \lambda\beta b - \lambda a \bar{Q}(\lambda, \hat{\beta}) - \lambda a \bar{q}_i(\lambda, \hat{\beta}) = 0$. We obtain that as long as $\bar{q}_i(\lambda, \hat{\beta}) > 0$,

$$
\begin{aligned}
U_i^{'}(0) - [\lambda\beta b + \lambda a \bar{Q}(\lambda, \beta)] &> U_i^{'}(0) - [\lambda\beta b + \lambda a \bar{Q}(\lambda, \hat{\beta})] \\
&> U_i^{'}(0) - U_i^{'}(\bar{q}_i(\lambda, \hat{\beta})) + \lambda a \bar{q}_i(\lambda, \hat{\beta}),
\end{aligned}
$$

and therefore $U_i^{'}(0) - [\lambda\beta b + \lambda a \bar{Q}(\lambda, \beta)] > 0$ from concavity of $U_i(\cdot)$. Hence 0 is not user i's best response, i.e., $\bar{q}_i(\lambda, \beta) > 0$.

Since $\bar{q}_i(\lambda, \beta) > 0$, the associated first-order condition must be binding at β. Hence as long as $\bar{q}_i(\lambda, \hat{\beta}) > 0$,

$$
\begin{aligned}
U_i^{'}(\bar{q}_i(\lambda, \beta)) - \lambda a \bar{q}_i(\lambda, \beta) &= \lambda\beta b + \lambda a \bar{Q}(\lambda, \beta) < \lambda\beta b + \lambda a \bar{Q}(\lambda, \hat{\beta}) \\
&= U_i^{'}(\bar{q}_i(\lambda, \hat{\beta}) - \lambda a \bar{q}_i(\lambda, \hat{\beta}).
\end{aligned}
$$

Thus, by concavity of $U_i(\cdot)$, we have $\bar{q}_i(\lambda, \beta) > \bar{q}_i(\lambda, \hat{\beta})$ as long as $\bar{q}_i(\lambda, \hat{\beta}) > 0$. Summing over i for which $\bar{q}_i(\lambda, \hat{\beta}) > 0$, we have

$$
\begin{aligned}
\bar{Q}(\lambda, \hat{\beta}) = \sum_{j=1}^{n} \bar{q}_j(\lambda, \hat{\beta}) &= \sum_{i:\bar{q}_i(\lambda,\hat{\beta})>0} \bar{q}_i(\lambda, \hat{\beta}) < \sum_{i:\bar{q}_i(\lambda,\hat{\beta})>0} \bar{q}_i(\lambda, \beta) \\
&\leq \sum_{j=1}^{n} \bar{q}_j(\lambda, \beta) = \bar{Q}(\lambda, \beta),
\end{aligned}
$$

which contradicts $\bar{Q}(\lambda, \hat{\beta}) > \bar{Q}(\lambda, \beta)$.

Now we prove by contradiction that for fixed λ, $\bar{q}_i(\lambda, \beta)$ is continuous in $\beta \geq 1$. Suppose that the flow of user i is not continuous at some $\beta \geq 1$, i.e., there exists $\epsilon > 0$ such that for every $\delta > 0$, we can find $\hat{\beta} \in [\beta, \beta + \delta)$ and $|\bar{q}_i(\lambda, \hat{\beta}) - \bar{q}_i(\lambda, \beta)| > \epsilon$, where we have applied the same notation as above. Choose $\delta < a\epsilon/(nb)$. Since $\bar{Q}$ is decreasing, there must exist a user j, such that $\bar{q}_j(\lambda, \hat{\beta}) < \bar{q}_j(\lambda, \beta) - \epsilon/n$ (If $\bar{q}_i(\lambda, \hat{\beta}) < \bar{q}_i(\lambda, \beta) - \epsilon$, then choose $j = i$; otherwise, there exists another user j that satisfies it). Now consider the incentive of user j:

$$
\begin{aligned}
&U_j^{'}(\bar{q}_j(\lambda, \hat{\beta})) - \lambda a \bar{q}_j(\lambda, \hat{\beta}) - \lambda\beta b - \lambda a \bar{Q}(\lambda, \hat{\beta}) \\
&\quad \geq U_j^{'}(\bar{q}_j(\lambda, \beta)) - \lambda a \bar{q}_j(\lambda, \beta) - \lambda\beta b - \lambda a \bar{Q}(\lambda, \hat{\beta}) + \lambda a \frac{\epsilon}{n} \\
&\quad \geq U_j^{'}(\bar{q}_j(\lambda, \beta)) - \lambda a \bar{q}_j(\lambda, \beta) - \lambda\beta b - \lambda a \bar{Q}(\lambda, \beta) + \lambda \left(a\frac{\epsilon}{n} - b\delta\right),
\end{aligned}
$$

where the first inequality is by concavity of $U_j(\cdot)$, and the second inequality follows from monotonicity of $\bar{Q}(\lambda, \beta)$. Thus,

$$
U_j^{'}(\bar{q}_j(\lambda, \hat{\beta})) - \lambda a \bar{q}_j(\lambda, \hat{\beta}) - \lambda\hat{\beta} b - \lambda a \bar{Q}(\lambda, \hat{\beta}) \geq \lambda \left(a\frac{\epsilon}{n} - b\delta\right) > 0,
$$

because $\bar{q}_j(\lambda, \beta) > \bar{q}_j(\lambda, \hat{\beta}) + \epsilon/n > 0$. This contradicts the fact that $\bar{q}_i(\lambda, \hat{\beta})$ is user i's best response. The continuity of $\bar{Q}(\lambda, \beta)$ follows from $\bar{Q}(\lambda, \beta) = \sum_{i \in I} \bar{q}_i(\lambda, \beta)$. □

Recall that our goal is to show the existence of fixed point for (4). Since $\max_{i \in I} U_i^{'}(\bar{q}_i(\lambda, \beta)) = U_1^{'}(\bar{q}_1(\lambda, \beta))$, for every pair $\lambda > 0$ and $\beta \geq 1$, we can rearrange (4) as follows:

$$\beta = \frac{2(n+1)}{3n+1} \frac{U_1^{'}(\bar{q}_1(\lambda, \beta))}{\lambda b} + \frac{1}{\lambda}\left[1 - \frac{2(n+1)}{3n+1}\right]. \tag{5}$$

Define

$$f(\beta) = \frac{2(n+1)}{3n+1} \frac{U_1^{'}(\bar{q}_1(\lambda, \beta))}{\lambda b} + \frac{1}{\lambda}\left[1 - \frac{2(n+1)}{3n+1}\right].$$

Then it suffices to show that $f(\beta) = \beta$ has a solution. Recall that $U_1^{'}(0) > b$, and denote $\Delta = U_1^{'}(0) - b > 0$. It is easy to verify that

$$\frac{U_1^{'}(0)}{\lambda b} - 1 - \frac{\Delta}{\lambda b}\left(1 - \frac{2(n+1)}{3n+1}\right) \geq 0.$$

Now we are ready to prove the following lemma.

Lemma 9 *If* $\lambda \in (0, 1)$, *then for any*

$$\beta \in \left[1, \frac{U_1^{'}(0)}{\lambda b} - \frac{\Delta}{\lambda b}\left(1 - \frac{2(n+1)}{3n+1}\right)\right],$$

we have

$$f(\beta) = \frac{2(n+1)}{3n+1} \frac{\lambda \beta b + \lambda a \bar{Q}(\lambda, \beta) + \lambda a \bar{q}_1(\lambda, \beta)}{\lambda b} + \frac{1}{\lambda}\left[1 - \frac{2(n+1)}{3n+1}\right], \tag{6}$$

and $f(\beta)$ *is continuous in* β.

Proof We first show that if

$$\beta \in \left[1, \frac{U_1^{'}(0)}{\lambda b} - \frac{\Delta}{\lambda b}\left(1 - \frac{2(n+1)}{3n+1}\right)\right],$$

then $\bar{q}_1(\lambda, \beta)$ must be positive. Otherwise, assume $\bar{q}_1(\lambda, \beta) = 0$, then by Lemma 6, $\bar{q}_i(\lambda, \beta) = 0$ for all $i \in I$ and thus $\bar{Q}(\lambda, \beta) = 0$. However, on the other hand, we know that

$$U_1^{'}(\bar{q}_1(\lambda, \beta)) - \lambda \beta b - \lambda a \bar{Q}(\lambda, \beta) - \lambda a \bar{q}_1(\lambda, \beta) \leq 0.$$

It follows that $U_1'(0) - \lambda\beta b \le 0$, which contradicts to the assumption that

$$\beta \le \frac{U_1'(0)}{\lambda b} - \frac{\Delta}{\lambda b}\left(1 - \frac{2(n+1)}{3n+1}\right).$$

Therefore, it must be the case that $\bar{q}_1(\lambda, \beta) > 0$. From the first order condition we have $U_1'(\bar{q}_1(\lambda, \beta)) - \lambda\beta b - \lambda a\bar{Q}(\lambda, \beta) - \lambda a\bar{q}_1(\lambda, \beta) = 0$. Rearranging the terms, we get $U_1'(\bar{q}_1(\lambda, \beta)) = \lambda\beta b + \lambda a\bar{Q}(\lambda, \beta) + \lambda a\bar{q}_1(\lambda, \beta)$. Therefore, (6) holds. By Lemma 8, both $\bar{q}_1(\lambda, \beta)$ and $\bar{Q}(\lambda, \beta)$ are continuous in β. □

Finally, we prove that (5) has a solution.

Lemma 10 *If* $\lambda \in (0, 1)$, $f(\beta)$ *has a fixed point* $\beta \ge 1$.

Proof For any

$$\beta \in \left[1, \frac{U_1'(0)}{\lambda b} - \frac{\Delta}{\lambda b}\left(1 - \frac{2(n+1)}{3n+1}\right)\right],$$

from Lemma 9,

$$\begin{aligned} f(\beta) &= \frac{2(n+1)}{3n+1}\frac{\lambda\beta b + \lambda a\bar{Q}(\lambda, \beta) + \lambda a\bar{q}_1(\lambda, \beta)}{\lambda b} + \frac{1}{\lambda}\left[1 - \frac{2(n+1)}{3n+1}\right] \\ &\ge \frac{2(n+1)}{3n+1}\frac{\lambda\beta b}{\lambda b} + \frac{1}{\lambda}\left[1 - \frac{2(n+1)}{3n+1}\right] \ge \frac{2(n+1)}{3n+1}\frac{\lambda b}{\lambda b} + \frac{1}{\lambda}\left[1 - \frac{2(n+1)}{3n+1}\right], \end{aligned}$$

and hence

$$f(\beta) = \left(1 - \frac{1}{\lambda}\right)\left[1 - \frac{2(n+1)}{3n+1}\right] + 1 \ge 1.$$

On the other hand,

$$\begin{aligned} f(\beta) &= \frac{2(n+1)}{3n+1}\frac{U_1'(\bar{q}_1(\lambda, \beta))}{\lambda b} + \frac{1}{\lambda}\left[1 - \frac{2(n+1)}{3n+1}\right] \le \frac{2(n+1)}{3n+1}\frac{U_1'(0)}{\lambda b} \\ &\quad + \frac{1}{\lambda}\left[1 - \frac{2(n+1)}{3n+1}\right] \le \frac{2(n+1)}{3n+1}\frac{U_1'(0)}{\lambda b} + \frac{U_1'(0) - \Delta}{\lambda b} \\ &\quad \times \left[1 - \frac{2(n+1)}{3n+1}\right] \le \frac{U_1'(0)}{\lambda b} - \frac{\Delta}{\lambda b}\left(1 - \frac{2(n+1)}{3n+1}\right). \end{aligned}$$

Therefore, we have shown that for any

$$\beta \in \left[1, \frac{U_1'(0)}{\lambda b} - \frac{\Delta}{\lambda b}\left(1 - \frac{2(n+1)}{3n+1}\right)\right],$$

$f(\beta)$ is continuous and

$$f(\beta) \in \left[1, \frac{U_1'(0)}{\lambda b} - \frac{\Delta}{\lambda b}\left(1 - \frac{2(n+1)}{3n+1}\right)\right].$$

Thus, from fixed-point theorem, there exists β such that $f(\beta) = \beta$. □

Note that $\lambda = 2(n+1)/(3n+1)$ satisfies the conditions in all of the above lemmas. Since $f(\beta) = \beta$ has a solution when $\lambda = 2(n+1)/(3n+1)$, for any instance $U_i(\cdot)$'s, we can find a price mechanism that provides a bound of coordination ratio that is at least $8n(n+1)/(3n+1)^2$. The proof of Theorem 4 is now complete. □

Note that albeit λ is a constant, β depends on the utilities. Hence, this price mechanism does not fall into the category studied in Sect. 4. The optimal price mechanism is at least as good as the proposed scheme, and therefore the price of anarchy under the optimal mechanism shall be between 0 and $(n^2 - 2n - 1)/(3n+1)^2$, where the latter converges to $1/9 \approx 11.1\%$ in the limit. Thus, the price of anarchy is further suppressed from 31.4% to 11.1% when we allow the prices to be contingent on the users' preferences.

The case with linear utilities achieves the lower bound when price mechanisms are restricted to depend only on the cost structures. However, it becomes the best scenario if users' preferences are available to the system designer. This strict contrast demonstrates the value of information in this special case.

6 Conclusion

In this paper, we study the design of price mechanisms for communication network problems in which the congestion cost exhibits quadratic structure. We investigate the price mechanisms that are characterized by a set of axioms, and we obtain the price mechanisms that provide the minimum price of anarchy. We show that, given the non-decreasing and concave utilities of users and the convex quadratic congestion costs in each link, if the price mechanism cannot depend on utility functions, the best achievable price of anarchy is $4(3 - 2\sqrt{2}) \approx 31.4\%$. Thus, the marginal cost pricing is nearly optimal, whereas the average cost pricing performs extremely poorly. We also investigate the scenario in which the price mechanisms can be made contingent on the users' preference profile while such information is available.

As a possible extension, when the cost functions are not quadratic, one may need to look for other potentially more complicated price mechanisms. Whether there exists a price mechanism that provides a constant performance guarantee is still an open question, and it remains a research priority.

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