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Research Article

On Jordan Type Inequalities for Hyperbolic Functions

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This paper deals with some inequalities for trigonometric and hyperbolic functions such as the Jordan inequality and its generalizations. In particular, lower and upper bounds for functions such as $(\sin x)/x$ and $x/\sinh x$ are proved.

1. Introduction

During the past several years there has been a great deal of interest in trigonometric inequalities [1–7]. The classical Jordan inequality [8, page 31]

$$\frac{2}{\pi}x \le \sin x \le x, \quad 0 < x < \frac{\pi}{2} \tag{1.1}$$

has been in the focus of these studies and many refinements have been proved for it by Wu and Srivastava [9, 10], Zhang et al. [11], J.-L. Li and Y.-L. Li [5, 12], Wu and Debnath [13–15], Özban [16], Qi et al. [17], Zhu [18–29], Sándor [30, 31], Baricz and Wu [32, 33], Neuman and Sándor [34], Agarwal et al. [35], Niu et al. [36], Pan and Zhu [37], and Qi and Guo [38]. For a long list of recent papers on this topic see [7] and for an extensive survey see [17]. The proofs are based on familiar methods of calculus. In particular, a method based on a l'Hospital type criterion for monotonicity of the quotient of two functions from Anderson et al. [39] is a key tool in these studies. Some other applications of this criterion are reviewed in [40, 41]. Pinelis has found several applications of this criterion in [42] and in several other papers.

The inequality

$$\frac{1+\cos x}{2} \le \frac{\sin x}{x} \le \frac{2+\cos x}{3}, \tag{1.2}$$

where $x \in (-\pi/2, \pi/2)$ is well-known and it was studied recently by Baricz in [43, page 111].

The second inequality of (1.2) is given in [8, page 354, 3.9.32] for $0 \le x \le \pi$. For a refinement of the first inequality in (1.2) see Remark 1.3(1) and of the second inequality see Theorem 2.4.

This paper is motivated by these studies and it is based on the Master Thesis of Visuri [44]. Some of our main results are the following theorems.

Theorem 1.1. *For $x \in (0, \pi/2)$*

$$\frac{x^2}{\sinh^2 x} < \frac{\sin x}{x} < \frac{x}{\sinh x}. \tag{1.3}$$

Theorem 1.2. *For $x \in (0,1)$*

$$\left(\frac{1}{\cosh x}\right)^{1/2} < \frac{x}{\sinh x} < \left(\frac{1}{\cosh x}\right)^{1/4}. \tag{1.4}$$

We will consider quotients $\sin x/x$ and $x/\sinh x$ at origin as limiting values $\lim_{x\to 0} \sin x/x = 1$ and $\lim_{x\to 0} x/\sinh x = 1$.

Remark 1.3. (1) Let

$$g_1(x) = \frac{1+2\cos x}{3} + \frac{x\sin x}{6}, \qquad g_2(x) = \frac{1+\cos x}{2}. \tag{1.5}$$

Then $g_1(x) - g_2(x) > 0$ on $(0, \pi/2)$ because

$$\frac{d}{dx}\left(g_1(x) - g_2(x)\right) = \frac{x\cos x}{6} > 0. \tag{1.6}$$

In [45, (27)] it is proved that for $x \in (0, \pi/2)$

$$\frac{\sin x}{x} \ge g_1(x). \tag{1.7}$$

Hence $(1 + 2\cos x)/3 + (x\sin x)/6$ is a better lower bound for $(\sin x)/x$ than (1.2) for $x \in (0, \pi/2)$.

(2) Observe that

$$\frac{2+\cos x}{3} = \frac{2+2\cos^2(x/2)-1}{3} \le \cos\frac{x}{2}, \tag{1.8}$$

which holds true as equality if and only if $\cos(x/2) = (3\pm 1)/4$. In conclusion, (1.8) holds for all $x \in (-2\pi/3, 2\pi/3)$. Together with (1.2) we now have

$$\cos^2\frac{x}{2} = \frac{1+\cos x}{2} > \cos x, \tag{1.9}$$

and by (1.8)

$$\cos^2\frac{x}{2} < \frac{\sin x}{x} < \cos\frac{x}{2}. \tag{1.10}$$

2. Jordan's Inequality

In this section we will find upper and lower bounds for $(\sin x)/x$ by using hyperbolic trigonometric functions.

Theorem 2.1. *For* $x \in (0, \pi/2)$

$$\frac{1}{\cosh x} < \frac{\sin x}{x} < \frac{x}{\sinh x}. \tag{2.1}$$

Proof. The lower bound of $\sin x/x$ holds true if the function $f(x) = \sin x \cosh x - x$ is positive on $(0, \pi/2)$. Since

$$f''(x) = 2\cos x \sinh x, \tag{2.2}$$

we have $f''(x) > 0$ for $x \in (0, \pi/2)$ and $f'(x)$ is increasing on $(0, \pi/2)$. Therefore

$$f'(x) = \cos x \cosh x + \sin x \sinh x - 1 > f'(0) = 0, \tag{2.3}$$

and the function $f(x)$ is increasing on $(0, \pi/2)$. Now $f(x) > f(0) = 0$ for $x \in (0, \pi/2)$.

The upper bound of $\sin x/x$ holds true if the function $g(x) = x^2 - \sin x \sinh x$ is positive on $(0, \pi/2)$. Let us denote $h(x) = \tan x - \tanh x$. Since $\cos x < 1 < \cosh x$ for $x \in (0, \pi/2)$ we have $h'(x) = \cosh^{-2}x - \cos^{-2}x > 0$ and $h(x) > h(0) = 0$ for $x \in (0, \pi/2)$. Now

$$g'''(x) = 2(\cos x \cosh x)h(x), \tag{2.4}$$

which is positive on $(0, \pi/2)$, because $\cos x \cosh x > 0$ and $h(x) > 0$ for $x \in (0, \pi/2)$. Therefore

$$\begin{aligned} g''(x) &= 2(1 - \cos x \cosh x) > g(0) = 0, \\ g'(x) &= 2x - \cos x \sinh x - \sin x \cosh x > g'(0) = 0 \end{aligned} \tag{2.5}$$

for $x \in (0, \pi/2)$. Now $g(x) > g(0) = 0$ for $x \in (0, \pi/2)$. □

Proof of Theorem 1.1. The upper bound of $\sin x/x$ is clear by Theorem 2.1. The lower bound of $\sin x/x$ holds true if the function $f(x) = \sin x \sinh^2 x - x^3$ is positive on $(0, \pi/2)$.

Let us assume $x \in (0, \pi/2)$. Since $\sin x > x - x^3/6 = (6x - x^3)/6$ we have $f(x) > ((6x - x^3)\sinh^2 x)/6 - x^3$. We will show that

$$g(x) = \frac{6 - x^2}{6}\sinh^2 x - x^2 \tag{2.6}$$

is positive which implies the assertion.

Now $g(x) > 0$ is equivalent to

$$\frac{\sinh x}{x} > \frac{\sqrt{6}}{\sqrt{6-x^2}}. \tag{2.7}$$

Since $x^{-1}\sinh x > 1 + x^2/6$ it is sufficient to show that $1 + x^2/6 > \sqrt{6}/\sqrt{6-x^2}$, which is equivalent to

$$x^2\left(-x^4 - 6x^2 + 36\right) > 0. \tag{2.8}$$

Let us denote $h(x) = -x^4 - 6x^2 + 36$. Now $h'(x) = -4x(x^2+3)$ and therefore $h'(x) \neq 0$ and $h(x) > \min\{h(0), h(\pi/2)\} > 0$. Therefore inequality (2.8) holds for $x \in (0, \pi/2)$ and the assertion follows. ☐

We next show that for $x \in (0,1)$ the upper and lower bounds of (1.2) are better than the upper and lower bounds in Theorem 2.1.

Theorem 2.2. (i) *For* $x \in (-1,1)$

$$\frac{2+\cos x}{3} \leq \frac{x}{\sinh x}. \tag{2.9}$$

(ii) *For* $x \in (-\pi/2, \pi/2)$

$$\frac{1}{\cosh x} \leq \frac{1+\cos x}{2} = \cos^2\frac{x}{2}. \tag{2.10}$$

(iii) *For* $x \in (-\pi/2, \pi/2)$

$$\frac{1}{1+\sin^2 x} \leq \frac{1+\cos^2 x}{2} \leq \frac{1+\cos x}{2}. \tag{2.11}$$

Proof. (i) The claim holds true if the function $f(x) = 3x - 2\sinh x - \sinh x \cos x$ is nonnegative on $[0,1)$. By a simple computation we obtain $f''(x) = 2(\cosh x \sin x - \sinh x)$. Inequality $f''(x) \geq 0$ is equivalent to $\sin x \geq \tanh x$. By the series expansions of $\sin x$ and $\tanh x$ we obtain

$$\begin{aligned} \sin x - \tanh x &= \sum_{n\geq 3,\, n\equiv 1(\bmod 2)} \frac{(-1)^{(n-1)/2}(n+1) - 2^{n+1}\left(2^{n+1}-1\right)B_{n+1}}{(n+1)!} x^n \\ &= \sum_{n\geq 3,\, n\equiv 1(\bmod 2)} c_n x^n, \end{aligned} \tag{2.12}$$

where B_j is the jth Bernoulli number. By the properties of the Bernoullin numbers $c_1 = 1/6$, $c_3 = -1/8$, coefficients c_n, for $n \equiv 1(2)$, form an alternating sequence, $|c_n x^n| \to 0$ as $n \to \infty$ and $|c_{2m+1}| > |c_{2m+3}|$ for $m \geq 1$. Therefore by Leibniz Criterion

$$\sin x - \tanh x \geq \frac{x^3}{6} - \frac{x^5}{8} = \frac{x^3}{24}\left(4 - 3x^2\right) \tag{2.13}$$

and $\sin x \geq \tanh x$ for all $x \in [0,1)$. Now $f(x)$ is a convex function on $[0,1)$ and $f'(x)$ is nondecreasing on $[0,1)$ with $f'(0) = 0$. Therefore $f(x)$ is nondecreasing and $f(x) \geq f(0) = 0$.

(ii) The claim holds true if the function $g(x) = \cosh x(1 + \cos x) - 2$ is nonnegative on $[0, \pi/2)$. By the series expansion of $\cos x$ we have $\cos x - 1 + x^2/2 \geq 0$ and therefore by the series expansion of $\cosh x$

$$\begin{aligned} g(x) &\geq \left(1 + \frac{x^2}{2}\right)(1 + \cos x) - 2 \\ &= \cos x - 1 + \frac{x^2}{2} + \frac{x^2 \cos x}{2} \\ &\geq \cos x - 1 + \frac{x^2}{2} \geq 0, \end{aligned} \tag{2.14}$$

and the assertion follows.

(iii) Clearly we have

$$\left(1 + \cos^2 x\right)\left(1 + \sin^2 x\right) = 2 + \sin^2 x \cos^2 x \geq 2, \tag{2.15}$$

which implies the first inequality of the claim. The second inequality is trivial since $\cos x \in (0,1)$. □

Theorem 2.3. *Let* $x \in (0, \pi/2)$. *Then*

(i) *the function*

$$f(t) = \cos^t \frac{x}{t} \tag{2.16}$$

is increasing on $(1, \infty)$,

(ii) *the function*

$$g(t) = \sin^t \frac{x}{t} \tag{2.17}$$

is decreasing on $(1, \infty)$,

(iii) *the functions* $\overline{f}(t) = \cosh^t(x/t)$ *and* $\overline{g}(t) = \sinh^t(x/t)$ *are decreasing on* $(0, \infty)$.

Proof. (i) Let us consider instead of $f(x)$ the function

$$f_1(y) = \frac{x}{y} \log \cos y \tag{2.18}$$

for $y \in (0, x)$. Note that $f(t) = \exp(f_1(x/t))$ and therefore the claim is equivalent to the function $f_1(y)$ being decreasing on $(0, x)$. We have

$$f_1'(y) = -\frac{x}{y^2}(\log \cos y + y \tan y), \tag{2.19}$$

and $f_1'(y) \le 0$ is equivalent to $f_2(y) = \log \cos y + y \tan y \ge 0$. Since $f_2'(y) = y/\cos^2 y \ge 0$ we have $f_2(y) \ge f_2(0) = 0$. Therefore $f(t)$ is increasing on $(1, \infty)$.

(ii) We will consider instead of $g(x)$ the function

$$g_1(y) = \frac{x}{y} \log \sin y \tag{2.20}$$

for $y \in (0, x)$. Note that $g(t) = \exp(g_1(x/t))$ and therefore the claim is equivalent to the function $g_1(y)$ being increasing on $(0, x)$. We have

$$g_1'(y) = \frac{x}{y^2}\left(\frac{y}{\tan y} - \log \cos y\right), \tag{2.21}$$

and $g_1'(y) \ge 0$ is equivalent to $g_2(y) = y/\tan y - \log \cos y \ge 0$. Since $g_2'(y) = ((1/\cos y) - (y/\sin y))/\sin y \ge 0$ we have $g_2(y) \ge f_2(0) = 1$. Therefore $g_1'(y) \ge 0$ and the assertion follows.

(iii) We will show that $h_1(y) = (x/y) \log \cosh y$ is increasing on $(0, \infty)$. Now $h_1'(y) = (x(y \tanh y - \log \cosh y))/y^2$,

$$\frac{d(y \tanh y - \log \cosh y)}{dy} = \frac{y}{\cosh^2 y} > 0, \tag{2.22}$$

and $y \tanh y - \log \cosh y \ge 0$. Therefore the function $h_1(y)$ is increasing on $(0, \infty)$ and $\overline{f}(t)$ is decreasing on $(0, \infty)$.

We will show that $h_2(y) = (x/y) \log \sinh y$ is increasing on $(0, \infty)$. Now $h_2'(y) = (x(y \coth y - \log \sinh y))/y^2$,

$$\frac{d(y \coth y - \log \sinh y)}{dy} = -\frac{y}{\sinh^2 y} < 0, \tag{2.23}$$

and $\coth y - (\log \sinh y)/y \ge \lim_{y\to\infty} \coth y - (\log \sinh y)/y = 0$. Therefore the function $h_2(y)$ is increasing on $(0, \infty)$ and $\overline{g}(t)$ is decreasing on $(0, \infty)$. □

We next will improve the upper bound of (1.2).

Theorem 2.4. *For* $x \in (-\sqrt{27/5}, \sqrt{27/5})$

$$\cos^2\frac{x}{2} \le \frac{\sin x}{x} \le \cos^3\left(\frac{x}{3}\right) \le \frac{2+\cos x}{3}. \tag{2.24}$$

Proof. The first inequality of (2.24) follows from (1.2).

By the series expansions of $\sin x$ and $\cos x$

$$\frac{\sin x}{x} \le 1 - \frac{x^2}{6} + \frac{x^4}{120} \le \left(1 - \frac{x^2}{18}\right)^3 \le \cos^3\left(\frac{x}{3}\right), \tag{2.25}$$

where the second inequality is equivalent to $x^4(27-5x^2)/29160 \ge 0$ and the second inequality of (2.24) follows.

By the identity $\cos^3 x = (\cos 3x + 3\cos x)/4$ the upper bound of (2.24) is equivalent to $0 \le 8 + \cos x - 9\cos(x/3)$. By the series expansion of $\cos x$

$$8 + \cos x - 9\cos\left(\frac{x}{3}\right) = \sum_{n=2}^{\infty}(-1)^n \frac{3^{2n}-9}{3^{2n}(2n)!}x^{2n}, \tag{2.26}$$

and by the Leibniz Criterion the assertion follows. □

3. Hyperbolic Jordan's Inequality

In this section we will find upper and lower bounds for the functions x/sinhx and $\cosh x$.

Theorem 3.1. *For* $x \in (-\pi/2, \pi/2)$

$$1 - \frac{x^2}{6} \le \frac{\sin x}{x} \le 1 - \frac{2x^2}{3\pi^2}. \tag{3.1}$$

Proof. We obtain from the series expansion of $\sin x$

$$\frac{\sin x}{x} \ge 1 - \frac{x^2}{6}, \tag{3.2}$$

which proves the lower bound.

By using the identity $1 - \cos x = 2\sin^2(x/2)$ the chain of inequalities (1.2) gives

$$\frac{\sin x}{x} \le 1 - \frac{2\sin^2(x/2)}{3} \tag{3.3}$$

and the assertion follows from inequality $\sin^2(x/2) \ge (x/\pi)^2$. □

Remark 3.2. J.-L. Li and Y.-L. Li have proved [12, (4.9)] that

$$\frac{\sin x}{x} < p(x)\left(1 - \frac{x^2}{\pi^2}\right) < 1 - \frac{x^2}{\pi^2}, \quad 0 < x < \pi, \tag{3.4}$$

where $p(x) = 1/\sqrt{1 + 3(x/\pi)^4} < 1$. This result improves Theorem 3.1.

Lemma 3.3. *For* $x \in (0, 1)$

(i) $\sinh x < x + x^3/5$,

(ii) $\cosh x < 1 + 5x^2/9$,

(iii) $1/\cosh x < 1 - x^2/3$.

Proof. (i) For $x \in (0, 1)$ we have $x^2(1 - x^2) > 0$ which is equivalent to

$$\frac{1}{1 - x^2/6} < 1 + \frac{x^2}{5}. \tag{3.5}$$

By Theorems 2.1, 3.1, and (3.5)

$$\sinh x \leq \frac{x^2}{\sin x} \leq \frac{x}{1 - x^2/6} < x + \frac{x^3}{5}. \tag{3.6}$$

(ii) Since $(2n)! > 6^n$ for $n \geq 3$ we have

$$\begin{aligned} 1 + \frac{5x^2}{9} - \cosh x &= \frac{x^2}{18} - \sum_{n=2}^{\infty} \frac{x^{2n}}{(2n)!} \\ &\geq \frac{x^2}{18} - \sum_{n=2}^{\infty} \frac{x^2}{(2n)!} \\ &= x^2\left(\frac{1}{72} - \sum_{n=3}^{\infty} \frac{1}{(2n)!}\right) \\ &\geq x^2\left(\frac{1}{72} - \sum_{n=3}^{\infty} \frac{1}{6^n}\right) \\ &> 0. \end{aligned} \tag{3.7}$$

(iii) By the series expansion of $\cosh x$ we have

$$\cosh x\left(1 - \frac{x^2}{3}\right) \geq \left(1 + \frac{x^2}{2}\right)\left(1 - \frac{x^2}{3}\right) = 1 + \frac{x^2}{6} - \frac{x^4}{6} > 1. \tag{3.8}$$

□

Proof of Theorem 1.2. The lower bound of $x/\sinh x$ follows from Lemma 3.3 and Theorem 3.1 since

$$\frac{1}{\cosh x} < 1 - \frac{x^2}{3} < \left(1 - \frac{x^2}{6}\right)^2 \leq \left(\frac{x}{\sinh x}\right)^2. \tag{3.9}$$

The upper bound of $x/\sinh x$ holds true if the function $g(x) = \sinh^4 x - x^4 \cosh x$ is positive on $(0,1)$. By the series expansion it is clear that

$$\sinh x > x + \frac{x^3}{6}. \tag{3.10}$$

By Lemma 3.3 and (3.10)

$$g(x) > \left(x + \frac{x^3}{6}\right)^4 - x^4\left(1 + \frac{2x^2}{3}\right) = \frac{x^6}{1296}\left(x^6 + 24x^4 + 216x^2 + 144\right) > 0, \tag{3.11}$$

and the assertion follows. □

Theorem 3.4. *For $x \in (0,\pi/4)$*

$$\cosh x < \frac{\cos x}{\sqrt{(\cos x)^2 - (\sin x)^2}}. \tag{3.12}$$

Proof. The upper bound of $\cosh x$ holds true if the function $f(x) = \cos^2 x - \cosh^2 x(\cos^2 x - \sin^2 x)$ is positive on $(0,\pi/4)$. Since

$$f''(x) = 4\sin(2x)\sinh(2x) > 0, \tag{3.13}$$

we have

$$f'(x) = \sin(2x)\sinh(2x) - \cos(2x)\cosh(2x) > f'(0) = 0. \tag{3.14}$$

Therefore $f(x) > f(0) = 0$ and the assertion follows. □

Theorem 3.5. *For $x \in (0,\pi/4)$*

$$\frac{1}{(\cos x)^{2/3}} < \cosh x < \frac{1}{\cos x}. \tag{3.15}$$

Proof. The upper bound of $\cosh x$ holds true if the function $f(x) = 1 - \cos x \cosh x$ is positive on $(0,\pi/4)$. Since $f''(x) = 2\sin x \sinh x > 0$ the function $f'(x) = \cosh x \sin x - \cos x \sinh x$ is increasing. Therefore $f'(x) > f'(0) = 0$ and $f(x) > f(0) = 0$.

The lower bound of $\cosh x$ holds true if the function $g(x) = \cos^2 x \cosh^3 x - 1$ is positive on $(0, \pi/4)$. By the series expansions we have

$$g(x) > \left(1 - \frac{x^2}{2}\right)^2 \left(1 + \frac{x^2}{2}\right)^3 - 1 = \frac{x^2}{32}\left(x^8 + 2x^6 - 8x^4 - 16x^2 + 16\right). \tag{3.16}$$

By a straightforward computation we see that the polynomial $h(x) = x^8 + 2x^6 - 8x^4 - 16x^2 + 16$ is strictly decreasing on $(0, \pi/4)$. Therefore

$$\begin{aligned} h(x) &> h(\pi/4) \\ &= 16 - \pi^2 - \frac{\pi^4}{32} + \frac{\pi^6}{2048} + \frac{\pi^8}{65536} \\ &> 16 - \frac{16}{5}^2 - 32^{-1}\frac{16}{5}^4 + \frac{3^6}{2048} + \frac{3^8}{65536} \\ &= \frac{120392497}{40960000} \\ &> 0, \end{aligned} \tag{3.17}$$

and the assertion follows. □

Remark 3.6. Baricz and Wu have shown in [33, page 276-277] that the right hand side of Theorem 2.1 is true for $x \in (0, \pi)$ and the right hand side of Theorem 3.5 is true for $x \in (0, \pi/2)$. Their proof is based on the infinite product representations.

Note that for $x \in (0, \pi/4)$

$$\frac{1}{\cos x} \le \frac{\cos x}{\sqrt{(\cos x)^2 - (\sin x)^2}}. \tag{3.18}$$

Hence, the upper bound in Theorem 3.5 is better that in Theorem 3.4.

4. Trigonometric Inequalities

Theorem 4.1. *For* $x \in (0, 1)$ *the following inequalities hold*

(i) $x/\arcsin x \le \sin x/x$,

(ii) $x/\operatorname{arcsinh} x \le \sinh x/x$,

(iii) $x/\arctan x \le \tan x/x$,

(iv) $x/\mathit{arctanh}\ x \le \tanh x/x$.

Proof. (i) By setting $x = \sin t$ the assertion is equivalent to

$$\operatorname{sinc} t \le \operatorname{sinc}(\sin t), \tag{4.1}$$

which is true because $\operatorname{sinc} t = (\sin t)/t$ is decreasing on $(0, \pi/2)$ and $\sin t \le t$.

(ii) By the series expansions of $\sinh x$ and $\operatorname{arcsinh}\, x$ we have by Leibniz Criterion

$$
\begin{aligned}
(\sinh x)\operatorname{arcsinh}\, x - x^2 &\geq \left(x + \frac{x^3}{6}\right)\left(x - \frac{x^3}{6} + \frac{3x^5}{40} - \frac{5x^7}{112}\right) - x^2 \\
&= \frac{x^6}{10080}\left(-75x^4 - 324x^2 + 476\right),
\end{aligned}
\tag{4.2}
$$

and since $-75x^4 - 324x^2 + 476 > 77$ on $(0,1)$ the assertion follows.

(iii) By the series expansions of $\tan x$ and $\arctan x$ we have by Leibniz Criterion

$$
\begin{aligned}
(\tan x) \arctan x - x^2 &\geq \left(x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315}\right)\left(x - \frac{x^3}{3}\right) - x^2 \\
&= \frac{x^6}{945}\left(21 + 9x^2 - 17x^4\right),
\end{aligned}
\tag{4.3}
$$

and since $21 + 9x^2 - 17x^4 > 4$ on $(0,1)$ the assertion follows.

(iv) By the series expansions of $\tanh x$ and $\operatorname{arctanh} x$ we have by Leibniz Criterion

$$
\begin{aligned}
(\tanh x)\mathit{arctanh}\ x - x^2 &\geq \left(x - \frac{x^3}{3}\right)\left(x + \frac{x^3}{3} + \frac{x^5}{5}\right) - x^2 \\
&= \frac{x^6}{45}\left(4 - 3x^2\right),
\end{aligned}
\tag{4.4}
$$

and since $4 - 3x^2 > 1$ on $(0,1)$ the assertion follows. □

Remark 4.2. Similar inequalities to Theorem 4.1 have been considered by Neuman in [46, page 34-35].

Theorem 4.3. *Let $k \in (0,1)$. Then*

(i) *for $x \in (0,\pi)$*

$$\frac{\sin x}{x} \leq \frac{\sin(kx)}{kx}, \tag{4.5}$$

(ii) *for $x > 0$*

$$\frac{\sinh x}{x} \geq \frac{\sinh(kx)}{kx}, \tag{4.6}$$

(iii) *for $x \in (0,1)$*

$$\frac{\tanh x}{x} \leq \frac{\tanh(kx)}{kx}. \tag{4.7}$$

Proof. (i) The claim follows from the fact that sinc is decreasing on $(0, \pi)$.

(ii) The claim is equivalent to saying that the function $f(x) = (\sinh x)/x$ is increasing for $x > 0$. Since $f'(x) = (\cosh x)/x - (\sinh x)/x^2 \geq 0$ and $f'(x) \geq 0$ is equivalent to $\tanh x \leq x$ the assertion follows.

(iii) The claim is equivalent to $\tanh(kx) - k \tanh x \geq 0$. By the series expansion of $\tanh x$ we have

$$\tanh(kx) - k \tanh x = k \sum_{n=1}^{\infty} \frac{4^{n+1}(4^{n+1}-1) B_{2(n+1)} x^{2n+1}}{(2n+2)!} \left(k^{2n} - 1\right), \tag{4.8}$$

where B_j is the jth Bernoulli number ($B_0 = 1, B_1 = -1/2, B_2 = 1/6, \ldots$). The assertion follows from the Leibniz Criterion, if

$$\frac{k - k^3}{3} x^3 - \frac{2(k - k^5)}{15} x^5 > 0 \tag{4.9}$$

for all $x \in (0,1)$. Since (4.9) is equivalent to

$$x^2 < \frac{5}{2(1+k^2)}, \tag{4.10}$$

the assertion follows from the assumptions $k \in (0,1)$ and $x \in (0,1)$. □

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