An improved forward modeling method for two-dimensional electromagnetic induction problems with bathymetry

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Recently, electromagnetic observations have become common not only on land but also on the seafloor. In particular, thanks to the development of instruments for use in shallow seas, one can conduct observations along land-sea arrays. However, since there is a large contrast in conductivity between sea water and rocks in the crust and the mantle, we have to pay more attention to the accuracy of forward solvers used for electromagnetic induction problems, including bathymetry. In this paper, we develop a two-dimensional forward code using triangular finite elements, and confirm the accuracy of the new code by TM mode responses. The accuracy of the improved solver was tested by comparison with an analytical solution in a hemi-cylindrical geometry. We also show that triangular elements are more reliable than rectangular elements in determining conductivity structures beneath land-sea arrays. Our results indicate the importance of precisely discretized bathymetry and the accuracy of spatial derivatives of electromagnetic field components, especially in the vicinity of coastlines.

Key words: Magnetotellurics, electrical conductivity structures, land-sea arrays, the finite element method, bathymetry.

1. Introduction

Recently, it has become quite common to carry out electromagnetic (EM) observations on the seafloor. For instance, the development of EM instruments applicable to shallow seas enables us to investigate subsurface conductivity structures near coastlines in more detail. However, magnetotelluric (MT) responses obtained in the vicinity of coastlines are known to be influenced strongly by large differences in conductivities between land and sea. Careful consideration, therefore, is necessary as to how electric currents flow within conductive sea water, which greatly depends on bathymetry, as well as whether this is reproduced accurately in forward modeling or not.

Many numerical approaches developed so far may be used in regions including coastlines theoretically. The finite element method (FEM) is one of very popular approaches because it is capable of including arbitrary bathymetry/topography by adopting various forms of elements. For example, Utada (1987) developed a two-dimensional (2-D) FEM code using triangular elements. On the other hand, Uchida (1993)/Ogawa and Uchida (1996) adopted rectangular elements for their FEM forward solver. There exist respective advantages for triangular and rectangular elements. Rectangular elements are conceptually simple to use, say, in generating numerical meshes and coding many desired mathematical/physical formulations. On the other hand, triangular elements are very useful in expressing complicated topography/bathymetry accurately, especially in regions around coastlines where at least one triangular element is indispensable at the very edge of the land-sea boundary in 2-D problems. In this study, we have adopted triangular elements in our 2-D FEM modeling and improved Utada's (1987) forward solver, which we henceforth call UT, in order to develop a 2-D forward code that enables precise modeling of bathymetry. Expressions of bathymetry by triangular elements, however, may not be sufficient to obtain reliable MT responses near coastlines. To improve UT's calculation algorithm itself, we also applied Li *et al.*'s (2008) differentiation method. In addition, we have extended the code so that one can use electric and magnetic fields at different observation sites to calculate the desired EM responses. This improvement helps us to calculate MT responses at sites where only electric variations were observed. In the following section, we will explain the improvements we have made in detail. However, we work with only 2-D problems because it is rather simple to show how MT responses near coastlines are affected by bathymetry, and the accuracy of the forward code, in two dimensions.

2. Improvement of the 2-D FEM Forward Code

In general, 2-D FEM calculations, i.e., solutions of a matrix equation are obtained in terms of either electric or magnetic fields, which correspond to TE and TM mode solutions, respectively. In order to calculate 2-D MT responses consisting of complex ratios between electric and magnetic components, auxiliary fields have to be calculated by the spatial differentiation of primarily obtained electric or magnetic fields in both modes. Li *et al.* (2008) proposed a precise auxiliary field calculation method on conductivity boundaries by taking finite differences, which extrapolates

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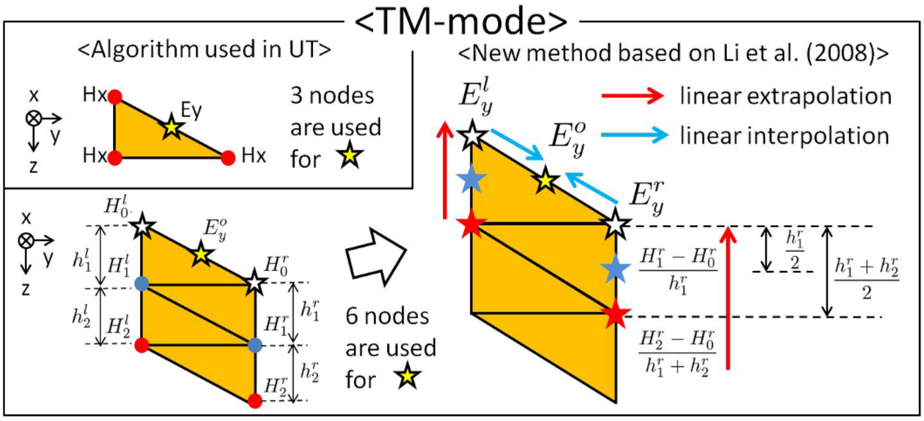

Fig. 1. The algorithm to obtain MT responses using Li *et al.*'s (2008) method in TM mode. Prior to the present improvement, the auxiliary electric field was calculated using only three nodes within the same element. If we apply Li *et al.*'s (2008) method, the along-strike magnetic fields at six nodes (i.e., two sets of H_0, H_1 and H_2) are utilized. Here, H_0^s, H_1^s and H_2^s $(s = l, r)$, denote the along-strike magnetic fields at the open stars, blue and red circles on the left or right side, respectively. E_y^l, E_y^r, and E_y^o indicate the auxiliary electric fields at the left and right open stars, and the yellow stars, respectively.

spatial derivatives of the along-strike fields from the resistive side. There are two essential points in their formulation. One is to use extrapolation. In their extrapolation method, the spatial derivatives of the along-strike fields should be obtained by not only using the nodes defining the element considered but also using more nodes of the neighbouring elements. In contrast, the shape function is often simply linear and spatial derivatives of the along-strike fields are evaluated using the nodes defining the desired element in the case of ordinary 2-D FEM forward codes. We applied Li *et al.*'s (2008) method to our code so that auxiliary fields are always calculated using six nodes in total even in the case of triangular elements. Figure 1 illustrates the improved algorithm for spatial derivatives. Specifically, the auxiliary field, $E_y(= 1/\sigma \cdot \partial H_x/\partial z)$, in TM mode is given by;

$$\sigma E_y^s = \frac{(H_1^s - H_0^s)\cdot(h_2^s + h_1^s)}{h_1^s \cdot h_2^s} - \frac{(H_2^s - H_0^s)\cdot h_1^s}{(h_1^s + h_2^s)\cdot h_2^s}, \qquad (1)$$

where $s = l, r$, indicating whether value is on the left or right side of the observation sites denoted by the yellow stars in Fig. 1. σ is the conductivity on the resistive side. As for the other notations, refer to Fig. 1. Equation (1) indicates that derivatives on the interface where EM responses were observed are calculated by a linear extrapolation using three nodes in a row (six nodes in total); one is on the interface and two are in the resistive side. Finally, the auxiliary electric fields at observation sites (denoted by E_y^o in Fig. 1) are linearly interpolated using the two electric fields obtained at both side nodes. In the previous algorithm, spatial derivatives, evaluated using only three nodes within one triangular element, were considered to compute the auxiliary field on the interface. The new method, therefore, has an advantage over the previous one because it can appreciate spatial variations of the derivatives more precisely.

Another essential point is that the extrapolation should be started from the resistive side. This is because the amplitude and phase of the along-strike fields vary more severely when the magnetic/electric fields propagate through conductive bodies. For instance, the auxiliary fields on the seafloor can be calculated more accurately by extrapolation from the sub-seafloor side than from the sea water side. From this point of view, we extrapolated the spatial derivatives from the sub-seafloor side in obtaining the auxiliary fields on the seafloor for both modes.

Furthermore, we modified the code so as to calculate MT responses using electric and magnetic fields observed at different sites. In marine and/or land EM observations, only electric fields are often observed at a significant number of sites and MT responses are calculated using magnetic fields obtained at other sites assuming a spatial uniformity of the inducing horizontal geomagnetic field. This modification enabled us to increase the number of observed MT responses for use in further forward modeling and/or inversion.

3. The Appraisal Method of the Improved Code

In this section, we test the accuracy of the improved 2-D FEM forward code using an analytical solution. Wannamaker *et al.* (1986) considered a hemi-cylindrical geometry, the analytical responses of which are identical at the dc limit to that of a cylinder excited by a time-varying horizontal electric field in the lower half-space. To test the accuracy of our new 2-D code for regions including coastlines, we considered the case where a cylinder full of a conductive (4 S/m) medium corresponding to seawater is embedded in a resistive (100 ohm.m) whole-space. Under this circumstance, a geometry whose upper half-space is replaced by an

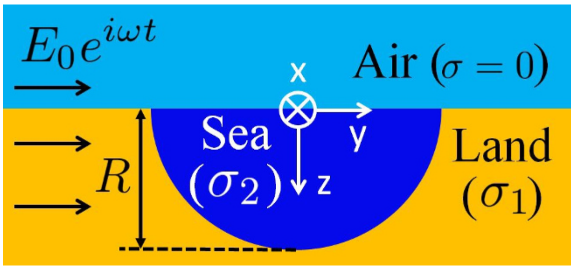

Fig. 2. A hemi-cylindrical configuration for the analytical solution that corresponds to a coastal area.

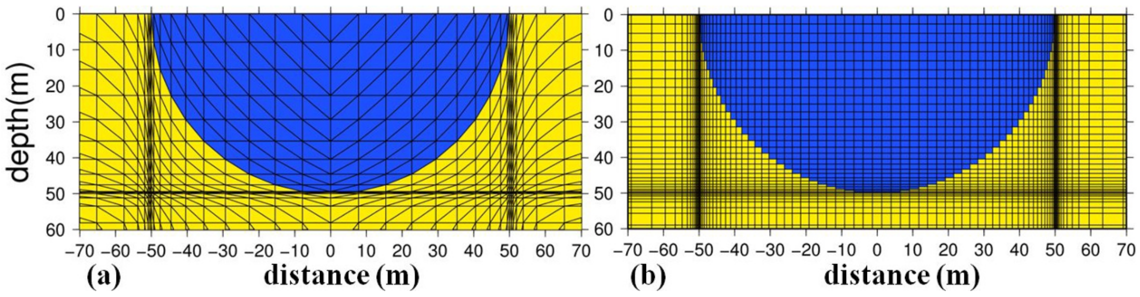

Fig. 3. (a) The numerical grid using triangular elements. (b) The numerical grid using rectangular elements.

insulator can be regarded as a land-sea configuration where the hemi-cylinder, the lower half-space excluding the cylinder, and the upper half-space insulator correspond to the sea, land and the air, respectively. In addition, we did not apply dc electric currents but horizontal electric fields oscillating with an angular frequency, ω, as the inducing field for the TM mode.

The configuration we considered here is summarized in Fig. 2. In this configuration, analytical responses for the TM mode external to the hemi-cylinder can be given by the following formulae (Ward and Hohmann, 1988, chap. 5);

$$\begin{aligned}
&H_x = -\sqrt{\frac{\sigma_1}{i\omega\mu}}E_0 - \sigma_1 E_0 R^2 \beta \frac{z}{\rho^2} \\
&E_y = E_0 + E_0 R^2 \beta \frac{y^2 - z^2}{\rho^4} \qquad (\rho \geq R) \qquad\qquad (2) \\
&\left(\beta = \frac{\sigma_2 J_1(k_2 R) - \sigma_1[k_2 R J_0(k_2 R) - J_1(k_2 R)]}{\sigma_2 J_1(k_2 R) + \sigma_1[k_2 R J_0(k_2 R) - J_1(k_2 R)]}\right),
\end{aligned}$$

where R is the radius of the cylinder, μ is the magnetic permeability for the whole space, E_0 is the amplitude of the inducing electric field, $\rho = \sqrt{y^2 + z^2}$, $k_2 = \sqrt{-i\omega\mu\sigma_2}$, J_0 and J_1 are the Bessel functions of the first kind, respectively. It should be noted here that the analytical solution assumes that the inducing electric field is very slowly declining with z (viz., depth). This assumption leads to the restriction that the radius of the cylinder must be small so that damping of the inducing field strength is small enough. Hence, we used $R = 50$ m here. The validity of this decision will be considered later.

We prepared two numerical grids which were able to represent the land-sea configuration described above. One is based on triangular elements as shown in Fig. 3(a). The other used rectangular elements as shown in Fig. 3(b). In the triangular grid, the half circle was represented by 20 nodes, and the curve was expressed by the sides of the triangles. On the other hand, in the rectangular grid, the half circle was specified by ~60 nodes, and the curve was approximated by rectangular steps. As a result, the ratio of the total number of elements of the rectangular grid to that of the triangular grid is approximately 4.5. Using these grids, we compared three numerical solutions with the analytical solution in terms of MT responses in the TM mode calculated along the seafloor and the air-land interface. Three sets of calculation were given by the original UT's forward code, our new 2-D FEM code using Li *et al.*'s (2008) differentiation/extrapolation method, and Ogawa and Uchida's (1996) forward code using the rectangular grid. The former two used the triangular grid shown in Fig. 3(a). We adopted Ogawa and Uchida's 2-D FEM forward code as a representative of 2-D FEM forward codes using rectangular elements. Since Ogawa and Uchida's code allows useful parameters such as static shifts, it is widely used in the EM induction community (e.g., Ichihara *et al.*, 2008).

4. Results and Discussion

Figure 4 shows the results of the appraisal in terms of the apparent resistivity and phase in the TM mode. We henceforth call the calculated result by Utada's (1987) original 2-D triangular FEM code, that with Li *et al.*'s (2008) method, and that by Ogawa and Uchida's (1996) 2-D rectangular FEM code, as T0, T1 and R0, respectively. The numerical responses were calculated for the periods 8, 16 and 32 s. The skin depths in seawater are approximately 0.7, 1.0 and 1.4 km, respectively, which are much larger than the radius of the hemi-cylinder, 50 m. This means that the condition of the non-decaying horizontal electric field was fulfilled.

It is evident from the figure that the MT responses of R0 are very different from those of the analytical solution. In particular, the biggest discrepancies between them, both in apparent resistivity and phase, occur at the edge of the hemi-cylinder. This means that one should be very careful in applying the rectangular FEM code, especially in the vicinity of coastlines. On the other hand, both the T0 and T1 responses fit the analytical solution at the coastline very well. The reason why only R0 failed to reproduce the analytical solution is because the simulation of bathymetric slopes using rectangular elements are much inferior to that using triangular elements. This can be attributed to the presence of rectangular steps along the seafloor and at the coastline. In the rectangular grid, vertical walls arising from of the steps, even if they are small, cause zigzag electric currents at each small step. For plane wave sources, electric currents tend to flow in the horizontal direction basically. If they encounter a resistive wall in seawater, they will be deflected to flow vertically. As a result, the deflected electric currents finally concentrate at the wedge of seawater near the coastline. This implies that discretization of bathymetry, especially in the vicinity of coastlines, is very important for the accurate evaluation of MT responses on the seafloor and at the coast. The fact that the largest discrepancy in the calculated responses is present at the coast supports this conjecture.

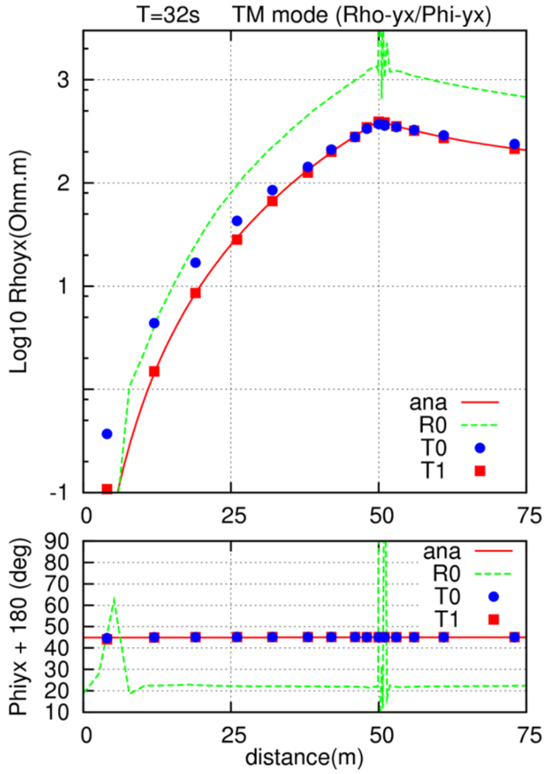

Fig. 4. Comparison of the MT responses in TM mode among the three numerical solutions and the analytical solution for the period of 32 s. The top and bottom panels show the apparent resistivity and phase, respectively. In each panel, the red line (ana) is the analytical solution, the green broken line (R0) is the calculated result using the rectangular grid shown in Fig. 3(b). T0 and T1 indicate the calculated results by the original Utada's (1987) forward code and the new 2-D FEM code developed in this study. T0 and T1 used the triangular grid shown in Fig. 3(a).

Furthermore, in our numerical experiments, the rectangular grid has more than four times as many elements as the triangular grid. Therefore, Fig. 4 also illustrates that, regarding rectangular elements, a large number of elements are not sufficient to achieve the same accuracy as in the case of triangular elements, and a much finer discretization of the hemi-cylinder (i.e., bathymetry) is needed especially in the vicinity of the coastline. This implies that it is very critical in 2-D EM FEM modeling near coastlines that appropriate numerical grids are employed that allow smooth and continuous tangential components of the electric field with respect to bathymetry.

As for the two numerical solutions using triangular elements, T1 becomes superior to T0 with regard to the apparent resistivity close to the bottom of the hemi-cylinder, while there are almost no differences between the two from the coastline to landward. This suggests that the accuracy of the spatial derivatives may greatly affect MT responses on the deep seafloor. It can be stressed that the improvements we achieved on the 2-D FEM forward code for EM induction in the Earth is necessary in regions including bathymetry and coastlines.

5. Summary

We have developed a new 2-D FEM forward code, which is useful especially for EM induction problems with bathymetry and coastlines. The FEM code adopts triangular elements which has an obvious advantage over rectangular elements. The improvements achieved in this study are two-fold: First, we applied Li *et al.*'s (2008) differentiation/extrapolation method in order to evaluate more accurate MT responses on the seafloor and in the vicinity of coastlines. Second, we enabled the code to calculate EM responses allowing any combination of observation sites and EM components.

We tested the accuracy of the new code by a comparison with the analytical solution in the hemi-cylindrical geometry. It was clearly shown that the new code yielded most reliable MT responses especially on the seafloor and at the coastline.

In conclusion, careful considerations are needed for 2-D EM FEM modeling on the seafloor and in the vicinity of coastlines. The determination of bathymetry near coastlines by numerical meshes is particularly critical because deflected electric currents can concentrate at the shore. We recommend avoiding rectangular grids to determine bathymetry, in which zigzag electric currents occur along bathymetry, especially in the vicinity of coastlines, in order to determine conductivity structures beneath coastal regions. It was found difficult, or very expensive, to achieve the same accuracy as by triangular grids by rectangular grids near coastlines. Without any tests of accuracy, the indiscriminate use of rectangular elements may cause critical errors in estimating theoretical EM response functions near coastlines. In addition, a differentiation method with a more precise extrapolation should be applied as well when seafloor EM observations are to be modeled accurately.

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