The Journal of Economic Inequality (2006) 4: 253–277
DOI: 10.1007/s10888-006-9020-1

The effect on inequality of changing one or two incomes

PETER J. LAMBERT[1,★] and GIUSEPPE LANZA[2]
[1]*Economics Department, 1285 University of Oregon, Eugene, OR 97404, USA, E-mail: plambert@uoregon.edu*
[2]*Dipartimento di Scienze Economiche, Università di Bari, Via Camillo Rosalba 53, 70124 Bari, Italy, E-mail: giuslanz@hotmail.com*

(Received: 3 February 2005; accepted: 31 January 2006; published online: 11 August 2006)

Abstract. We examine the effect on inequality of increasing one income, and show that for two wide classes of indices a benchmark income level or position exists, dividing upper from lower incomes, such that if a lower income is raised, inequality falls, and if an upper income is raised, inequality rises. We provide a condition on the inequality orderings implicit in two inequality indices under which the one has a lower benchmark than the other for all unequal income distributions. We go on to examine the effect on the same indices of simultaneously increasing one income and decreasing another higher up the distribution, deriving results which quantify the extent of the 'bucket leak' which can be tolerated without negating the beneficial inequality effect of the transfer. Our results have implications for the inequality and poverty impacts of different income growth patterns, and of redistributive programmes, leaky or not, which are briefly discussed.

Key words: inequality index, inequality ordering, leaky bucket.

1. Introduction

In an unequal two-person society, the effect on inequality of increasing one of the two incomes is clear: Inequality falls if we increase the lower income of the two, and rises if we increase the upper income. With more than two people, the effect on inequality of increasing one income is very much less clear. We obtain a range of definitive results here, showing that the insight from the two-person society carries over in essence to inequality indices, if not to the Lorenz configuration. Namely, if a low income is raised, inequality falls, and if a high income is raised, inequality rises; and there is a specific income level, or position in the distribution, determined by the particular inequality index one is using, which divides these effects. We call this the 'benchmark' income or position in what follows.

A condition between two inequality orderings, represented by indices, emerges which, if satisfied, ensures that the one index has an always lower benchmark than the other, whatever the income distribution to which both are applied. This condition evinces a Rawlsian-type measure which we call the 'lower tail concern' of an inequality ordering.

★ Corresponding author.

We go on to examine the so-called ‘leaky bucket paradox’ of Seidl [35]. We know that a pure rich-to-poor income transfer must reduce inequality for any Lorenz-consistent inequality index. Seidl demonstrates in respect of the Gini coefficient that the extent of the transaction cost or inefficiency ‘leak’ which can be tolerated, having taken \$1 from a person, and before giving the proceeds to another person further down the distribution, without negating the beneficial inequality effect of the transfer, can be surprising. Our analytics enable us to study this issue in considerable generality. The intuitively expected result is that the maximum permitted leak would be between 0% and 100%. However, as we shall show quite generally, not only can this case occur, but also – depending on the location of ‘donor’ and ‘recipient’ relative to the benchmark – the maximum permitted leak may exceed the amount taken away, so that the ‘recipient’ loses as well as the donor, or be negative, so that the recipient receives more than the donor gives up – somebody can be adding water to the bucket. This is the ‘leaky bucket paradox’ of Seidl [35], and it extends into a general proposition.

Our findings in this regard are quite distinct from the leaky bucket findings of authors such as Atkinson [3], Jenkins [23] and Duclos [16] in the welfare context, in which, following Okun [32, pp. 91–95], the maximum leak before a *welfare loss* is experienced is quantified; not least, for any monotonic social welfare function, such a leak cannot be negative, nor exceed 100%.

We emphasize that our focus is upon inequality *per se*, and not upon inequality as an ingredient of a social welfare function. The linkage between inequality and growth is, of course, much studied. Linkages between income inequality and aspects of health are also being investigated (Contoyannis and Forster [10]; Deaton and Paxson [12]) as well as between inequality, polarization and social exclusion (Wolfson [41]; Duclos [15]). Our results will be of interest in all of these scenarios. There are also implications for redistributive programmes.

The structure of the paper is as follows. In Section 2, we lay out the notation and preliminaries in terms of which the analysis will proceed. In Section 3, we comment briefly upon the implications for the Lorenz curve of increasing one income, and establish a central result: A benchmark income or position exists for any Lorenz-consistent inequality index. In Section 4, we examine the nature and properties of the benchmark for two wide classes of inequality indices, deriving explicit results for many familiar indices,[1] and a general insight that relates the benchmark to the lower tail concern of the underlying inequality ordering. In Section 5, we examine the leaky bucket issue in some depth. Section 6 concludes, with a discussion of some implications of our findings.

2. Notation and preliminaries

Let the population size be $N > 2$. Income distributions $\mathbf{x} = (x_1, x_2, \ldots, x_i, \ldots, x_N)$ will be assumed throughout to be unequal and non-decreasingly ordered, $\mathbf{x} \in \Omega_1 = \left\{\mathbf{x} \in \Re^N_{++} : x_1 \le x_2 \le \ldots \le x_i \le \ldots \le x_N \;\&\; x_1 < x_N\right\}$, with mean

$\mu(\mathbf{x}) = \frac{1}{N}\sum_i x_i$. For technical convenience we have disallowed zero incomes and will sometimes restrict attention to the subsets $\Omega_2 = \{\mathbf{x} \in \Re^N_{++} : x_1 < x_2 \leq \ldots \leq x_i \leq \ldots \leq x_N\}$ and $\Omega_3 = \{\mathbf{x} \in \Re^N_{++} : x_1 < x_2 < \ldots < x_i < \ldots < x_N\} \subset \Omega_2 \subset \Omega_1$. For $\mathbf{x} \in \Omega_1$, let $\delta(\mathbf{x}) = \min\{x_{i+1} - x_i : x_i \neq x_{i+1}\} > 0$ be the smallest gap between two adjacent, non-identical incomes, and for $1 \leq i \leq N$ and $0 < \delta < \delta(\mathbf{x})$ denote by $\mathbf{x}^i_\delta$ the vector obtained from $\mathbf{x}$ by adding δ to the income of person i. In general, $\mathbf{x}^i_\delta = (x_1, x_2, \ldots x_{i-1}, x_i + \delta, x_{i+1}, \ldots, x_N) \in \Omega_1$, but if $x_i = x_{i+1} = x$ then $\mathbf{x}^i_\delta \notin \Omega_1$, whereas its rearrangement $(x_1, x_2, \ldots, x, x + \delta, x_{i+2} \ldots, x_N)$, in which the ranks of persons i and $i + 1$ are reversed, does belong to Ω_1 (and has the same Lorenz curve as $\mathbf{x}^i_\delta$).[2]

For a Schur-convex inequality index $I : \Re^N_{++} \to \Re$ and distribution $\mathbf{x} \in \Omega_1$, and for $1 \leq i \leq N$ and $0 < \delta < \delta(\mathbf{x})$, we shall denote by $\Delta I(x_i, \delta)$ the change in inequality caused by increasing the income of individual i by the amount $\delta : \Delta I(x_i, \delta) = I(\mathbf{x}^i_\delta) - I(\mathbf{x}))$.

3. General results

The effect on the Lorenz curve for $\mathbf{x} \in \Omega_1$ of increasing one income, x_i, depends on which income this is. If the smallest income x_1 is unique, i.e. $x_1 < x_2$ (so that $\mathbf{x} \in \Omega_2$), and if x_1 is increased slightly, the Lorenz curve shifts upwards (just consider the effect on income shares), whilst if x_N is increased, the Lorenz curve shifts downwards (for all $\mathbf{x} \in \Omega_1$, and by similar reasoning). For $1 < i < N$, and also for $i = 1$ when $\mathbf{x} \in \Omega_1 \setminus \Omega_2$ (i.e when $x_1 = x_2$), the new Lorenz curve intersects the old one once, from below (again, just consider the income shares).[3]

What can we conclude about the effect on inequality indices of raising one income x_i by an amount δ, where $0 < \delta < \delta(\mathbf{x})$? Clearly, if $\mathbf{x} \in \Omega_2$ then $\Delta I(x_1, \delta) < 0$ for all Lorenz-consistent inequality indices I; and $\Delta I(x_N, \delta) > 0$ for all $\mathbf{x} \in \Omega_1$. These results for the lowest and highest incomes are in fact enough to establish the existence of a benchmark income, dividing positive from negative inequality effects for any Lorenz-consistent inequality index I:

THEOREM 1. *Given any Lorenz consistent inequality index I(.), income distribution* $\mathbf{x} \in \Omega_2$ *and a number* δ *such that* $0 < \delta < \delta(\mathbf{x})$, *there exists a benchmark income value* $x^* < x_N$ *such that* $x_i \leq x^* \Rightarrow \Delta I(x_i, \delta) \leq 0$ & $x_i > x^* \Rightarrow \Delta I(x_i, \delta) > 0$.

Proof. It is straightforward that for all $\mathbf{x}$, and for all i and j with $x_i < x_j$, $\mathbf{x}^i_\delta = \left(\left(\mathbf{x}^i_\delta\right)^j_\delta\right)^j_{-\delta} = \left(\left(\mathbf{x}^j_\delta\right)^i_\delta\right)^j_{-\delta}$, in other words that $\mathbf{x}^i_\delta$ is obtained from $\mathbf{x}^j_\delta$ by a progressive transfer of δ from j to i. Hence for any Lorenz-consistent inequality index I, we have $I\left(\mathbf{x}^i_\delta\right) < I\left(\mathbf{x}^j_\delta\right)$, whence $\Delta I(x_i, \delta) < \Delta I(x_j, \delta)$, $\forall\, i, j = (1, 2, \ldots, N)$ with $x_i < x_j$. Since we already know that, for $\mathbf{x} \in \Omega_2$, $\Delta I(x_1, \delta) < 0$ and $\Delta I(x_N, \delta) > 0$, necessarily $\exists\ k < N$ such that $\Delta I(x_i, \delta) \leq 0 \iff x_i \leq x_k$. By setting x^* equal to x_k (or, in fact, equal to any number between x_k and x_{k+1}), we establish the result. □

By this result, we establish the existence of a 'benchmark' income value x^* dividing positive from negative inequality effects for the inequality index $I(.)$ and income distribution $\mathbf{x} \in \Omega_2$. Plainly x^* need not be unique, for a given discrete income distribution, but if incomes are dense on (a subset of) the real line, a unique x^* must exist. In fact, for two large classes of inequality indices, the benchmark income level x^* can be uniquely determined, as a well-defined function of $\mathbf{x}$ and the index concerned, as we shall now see.

4. Further analysis for two general classes of indices

Some inequality indices depend on income shares alone, and others depend on income shares and ranks. We might call such indices rank-independent and rank-dependent, respectively, or non-positional and positional. Among the positional indices are the Gini coefficient, the extended Gini coefficients of Donaldson and Weymark [13], Weymark [39] and Yitzhaki [43], and the 'Lorenz family' of inequality indices introduced by Aaberge [1]. These are all members of the general class of 'linear measures' identified by Mehran [29]. Most of the familiar non-positional indices are related in one way or another to the generalized entropy family, shown by Bourguignon [5], Cowell [11] and Shorrocks [36] to be the unique additively decomposable indices. The mean logarithmic deviation and Theil index belong to the generalized entropy class, and the coefficient of variation and Atkinson index are monotonic transformations of indices in this class. We analyze indices of the two types separately here, using suitable general forms and then proceeding to specific indices afterwards. As we shall see, Theorem 1 extends from Ω_2 to Ω_1 for the non-positional indices, and provides a unique benchmark *relative* income $z^* = x^*/\mu(\mathbf{x})$, whilst for the positional indices, the benchmark can be expressed as a position (rank) rather than an income level when $\mathbf{x} \in \Omega_3$.

4.1. THE NON-POSITIONAL INDICES OF RELATIVE INEQUALITY FOR THE CLASS Ω_1

Many non-positional indices, including all the ones we have cited, can either be written in the form:

$$J(\mathbf{x}) = [1/N]\sum\nolimits_i u(x_i/\mu(\mathbf{x})) \tag{1}$$

where u: $\Re_{++} \rightarrow \Re$ is a twice-differentiable function such that u'' does not change sign, or are monotonic transformations of something in this form. Let $I(\mathbf{x})$ be such an inequality index; suppose that:

$$I(\mathbf{x}) = h(J(\mathbf{x})) \tag{2}$$

for all $\mathbf{x} \in \Omega_1$ where h: $\Re \to \Re$ is differentiable and such that h' does not change sign. For the transfer principle to hold, we require that $h'(J) > 0$ if u' is monotone increasing, and $h'(J) < 0$ if u' is monotone decreasing (recall that u'' does not change sign).[4]

This form encompasses most of the familiar non-positional inequality indices. For the mean logarithmic deviation D, set $u(z) = -\ln(z)$ and $h(J) = J$. The Theil index T is given by $u(z) = z \ln(z)$ and $h(J) = J$. (Both of these require normalized incomes z to be non-zero, which is true for $\mathbf{x} \in \Omega_1$). The generalized entropy class comprises indices $E(c)$, $c \in \Re$, of which $E(0) = D$, $E(1) = T$ and $E(c)$, $c \neq 0,1$ obtains when $u(z) = z^c - 1$ and $h(J) = J/[c(c-1)]$. For the coefficient of variation CV, set $u(z) = (z-1)^2$ and $h(J) = J^{1/2}$. For the Atkinson index $A(e)$, where $e > 0$ is the inequality aversion parameter, set $u(z) = z^{1-e}$ and $h(J) = 1 - J^{1/(1-e)}$ when $e \neq 1$ and set $u(z) = \ln(z)$ and $h(J) = 1 - e^J$ when $e = 1$. The coefficient of variation and Atkinson index for $0 < e \neq 1$ are monotonic transformations of generalized entropy indices: $\mathrm{CV} = \sqrt{[2E(2)]}$ and $A(e) = 1 - [1 - e(1-e)E(1-e)]^{1/(1-e)}$.

THEOREM 2. *Let I be a non-positional inequality index defined as in* (1) *and* (2), *let* $\mathbf{x} \in \Omega_1$. *and let $z_i = x_i/\mu(\mathbf{x})$ be income normalized by the mean*, $1 \leq i \leq N$. *Then* $\partial I/\partial x_k \gtrless 0 \iff z_k \gtrless z^*$ *where z^* is uniquely defined by* $u'(z^*) = [1/N].\sum z_i u'(z_i)$.

Proof. First, differentiate in (1) with respect to the income being increased, let this be x_k to distinguish it from the generic x_i:

$$\partial J/\partial x_k = \frac{1}{N}\left\{\left[\sum_{i \neq k} u'(x_i/\mu)\right]\left(-\frac{x_i}{N\mu^2}\right) + u'(x_k/\mu)\left[\frac{1}{\mu} - \frac{x_k}{N\mu^2}\right]\right\} \tag{3}$$

(in this, we have written μ for $\mu(\mathbf{x})$). Now differentiate in (2), substitute from (3) and rearrange:

$$\partial I/\partial x_k = \left[h'(J)\Big/N\mu\right]\left\{u'(x_k/\mu) - [1/N]\sum\nolimits_i (x_i/\mu)u'(x_i/\mu)\right\} \tag{4}$$

With z^* defined as in the statement of the theorem, (4) becomes:

$$\partial I/\partial x_k = [h'(J)/N\mu]\{u'(z_k) - u'(z^*)\} \tag{5}$$

from which the result follows (since $h' > 0$ if u' is increasing and $h' < 0$ if u' is decreasing). □

As this result demonstrates, the function u and income distribution $\mathbf{x}$ together uniquely determine the benchmark income level

$$x^* = \mu(\mathbf{x}).z^* \tag{6}$$

dividing positive from negative inequality effects, for all indices in our non-positional class (and for Ω_1 rather than the restricted Ω_2 of Theorem 1; ties, as in $\Omega_1 \setminus \Omega_2$, are immaterial for the non-positional indices). Notice that the function u alone defines the inequality ordering induced by I, and determines the benchmark, whereas the function h is also needed for the definition of I. It is now straightforward to obtain the benchmark income level for each of the familiar indices we have shown to be members of the non-positional class. For the mean logarithmic deviation D, for which $u(z) = -\ln(z)$ and $u'(z) = -1/z$, the critical z^* value is $z_D = 1$. Hence, if an above-average income is increased (slightly), D rises, and if a below-average income is raised, D falls. For the Theil index T, for which $u(z) = z\ln(z)$ and $u'(z) = 1 + \ln(z)$, we have $z_T = e^T$; for the generalized entropy index $E(c)$, we have $z_{E(c)} = [1 + c(c - 1)E(c)]^{1/(c-1)}$ $(c \neq 0,1)$; for the coefficient of variation, $z_{CV} = 1 + CV^2$; and for the Atkinson index $z_{A(e)} = [1 - A(e)]^{(e-1)/e}$ $(e \neq 1)$ and $z_{A(1)} = 1$.

There are some equivalences within this set of results. For example, using $E(2) = 1/2CV^2$, we see that $z_{E(2)} = [1 + 2E(2)] = z_{CV}$. This is as it ought to be, since the two indices are monotonically related. It can also be shown that $\lim_{c\to 0} z_{E(c)} = 1 = z_D = z_{A(1)}, \lim_{c\to 1} z_{E(c)} = e^T = z_T$ and $z_{A(e)} = z_{E(1-e)}$ for $e \neq 1$. These results clearly show that substantial change in the benchmark is possible – indeed almost inevitable – when changing the inequality index used for the measurement.

Let us now examine the benchmark $z_{E(c)}$ for the generalized entropy family more closely. Define $m_c = \frac{1}{N}\sum_{i=1}^{N} z_i^c$ and $M_c = \{m_c\}^{1/c}$ as the moment of order c and mean of order c, respectively, in the distribution of the z's. Then $z_{E(c)} = \{M_c\}^{c/(c-1)}$ for $c \neq 0,1$. The properties of M_c as a function of c, for a given distribution, are well-known in the statistical literature[5], and can be used to derive properties of the benchmark. In particular, for any given income distribution $\mathbf{x}$, $z_{E(c)}$ is continuous and increasing in c, and ranges in value from the minimum income relative to the mean, z_1, to the maximum, z_N: That is, $z_{E(c)} \to z_1$ as $c \to -\infty$ and $z_{E(c)} \to z_N$ as $c \to +\infty$. A particular consequence is that, for each person k in an income distribution $\mathbf{x} \in \Omega_1$ there exists a unique $c \in \Re$ such that $z_{E(c)} = x_k/\mu$: Each person can be considered to be at the benchmark position for exactly one generalized entropy index. Figure 1, obtained by simulation, shows graphs of M_c and $z_{E(c)}$ against c for the income distribution ($200, $500, $800, $1,100, $2,400).

We noted in Section 3 that for $1 < k < N$, and also for $k = 1$ when $\mathbf{x} \in \Omega_1 \setminus \Omega_2$ (i.e when $x_1 = x_2$), the new Lorenz curve, after x_k has been increased, intersects the old one once from below. Shorrocks and Foster [37] address such situations. They show that if the coefficient of variation is thereby increased, then inequality goes up for every transfer-sensitive inequality index I. Hence, if $x_k/\mu > z_{CV}$ then $\partial I/\partial x_k > 0$ for all transfer-sensitive inequality indices I. It follows that $z_{CV} = 1 + CV^2$ is an upper bound for the benchmarks z^* in the sub-class of non-positional indices which are also transfer-sensitive.[6]

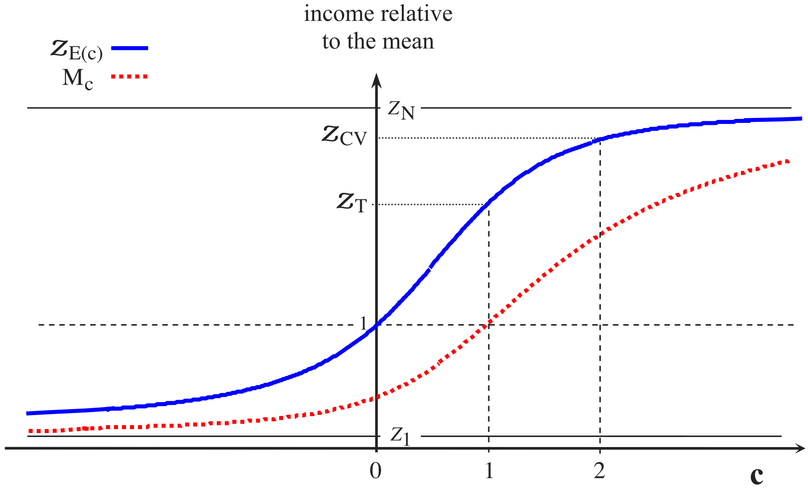

Figure 1. The generalized entropy benchmark as a function of the parameter c for the income distribution ($200, $500, $800, $1,100, $2,400).

Further insight into the relationship between the inequality ordering and benchmark income level can be gained with a simple transformation. Let $\pi_i = z_i/N$ be person i's income share, $1 \leq i \leq N$, so that $\sum \pi_i = 1$. Now set $U(z) = u'(z)$ where u is the function in (1) determining the inequality ordering. From Theorem 2, the benchmark income relative to the mean satisfies this equation:

$$U(z^*) = \sum \pi_i U(z_i) = E[U(\mathbf{Z})] \tag{7}$$

where $\mathbf{Z}$ is a risky prospect in which the return is z_i with probability π_i, $1 \leq i \leq N$. That is, $z^* = x^*/\mu$ is the certainty equivalent of $\mathbf{Z}$ for the 'utility function' U, in the sense of Pratt [34]. An extension of the Pratt theorem confirms the following result, linking the (relative) risk aversion of U, which takes the form:

$$P_u(Z) = -zu'''(z) \Big/ u''(z), \tag{8}$$

with the position of the benchmark:[7]

THEOREM 3. *Let I and $\hat{I}$ be inequality indices defined as in (1) and (2) by, respectively, h and u and $\hat{h}$ and $\hat{u}$, where $P_u(z) > P_{\hat{u}}(z)$ $\forall z$. Then for all unequal income distributions $\mathbf{x} \in \Omega_1$, the benchmark income for I is less than that for $\hat{I} : x^* < \hat{x}^*$.*

The higher is the measure $P_u(z)$ $\forall z$, the more confined is the lower-tail region $[0, x^*]$ in which an increase in a person's income is regarded as an inequality improvement, whatever the income distribution. In a clear sense, then, an inequality ordering with a higher P_u-measure is 'more Rawlsian.'

DEFINITION 1. *The function* $P_u(z) = -zu'''(z)/u''(z)$ *defined in* (8) *will be said to measure the* '*lower tail concern*' *of the non-positional inequality ordering defined by u in* (1), *of which the inequality index I defined in* (2) *is a cardinal representation.*[8]

All the specific indices we have been considering in fact have *constant* lower tail concern. This is because they all represent inequality orderings implicit in generalized entropy indices, for which $u(z) = z^c$ whence $P_{E(c)}(z) = 2 - c$, $\forall z$. It follows from Theorem 3 that the benchmark income for $E(c)$ is an increasing function of c whatever the income distribution **x**, as evidenced in Figure 1 for a specific income distribution. It can be checked directly, by inspecting the relevant u-functions, that for the mean logarithmic deviation, $P_{\mathrm{D}}(z) = 2$, $\forall z$; for the Theil index, $P_{\mathrm{T}}(z) = 1$, $\forall z$; for the coefficient of variation, $P_{\mathrm{CV}}(z) = 0$, $\forall z$; and for the Atkinson index, $P_{A(e)}(z) = e + 1$, $\forall z$.

The configuration of benchmarks for any two of the inequality indices we have catalogued can thus be ascertained, whatever the income distribution, by a simple comparison of scalar magnitudes. Notice that the inequality orderings with (constant) *negative* lower tail concern are precisely those represented by the generalized entropy indices $E(c)$ for $c > 2$. This ties in with a remark of Shorrocks [36, p. 623], that the indices $E(c)$, $c > 2$ "show little concern for equalization, except possibly among the very rich." In fact, within our class of non-positional indices, the sub-class having *positive* lower tail concern are precisely those which satisfy Kolm's [25] Principle of Diminishing Transfers.[9]

4.2. THE POSITIONAL INDICES OF RELATIVE INEQUALITY FOR THE CLASS Ω_3

Here we shall consider inequality indices in which people's incomes are weighted according to their positions in the distribution. Specifically, let $M(\mathbf{x})$ take the form

$$M(\mathbf{x}) = [1/N] \cdot \sum\nolimits_i w(i)x_i/\mu \tag{9a}$$

for $\mathbf{x} \in \Omega_3$, where w: $\Re \to \Re$ is such that $\sum_i w(i) = 0$ and $w(i + 1) > w(i)$ for $i = 1,2,\ldots N - 1$.

This specification covers the Gini coefficient G, for which $w_G(i) = (2i - N - 1)/N$, the extended Gini coefficient $G(\nu)$, $\nu > 1$, of Weymark [39], Donaldson and Weymark [13, 14] and Yitzhaki [43], for which $w_{G(\nu)}(i) = N.\{[(N - i)/N]^{\nu} - [(N - i + 1)/N]^{\nu}\}+ 1$ (the case $\nu = 2$ being that of the ordinary Gini coefficient),[10] and the illfare-ranked S-Gini coefficient $S(\beta)$, $0 \leq \beta < 1$, of Donaldson and Weymark [13], for which $w_{S(\beta)}(i) = 1 - \mathrm{N}.\{[i/N]^{\beta} - [(i - 1)/N]^{\beta}\}$.

Going slightly further, we shall assume that in (9a), the function w: $\Re \to \Re$ is strictly increasing and twice differentiable. Setting $\omega(p) = w(Np)$, so that ω : $[0, 1] \to \Re$ ascribes weights by *rank*, (9a) becomes:

$$M(\mathbf{x}) = [1/N] \cdot \sum_i \omega(p_i) x_i / \mu \tag{9b}$$

in which the rank of income x_i is written as $p_i = i/N$, so that $\omega(p_i) = w(i)$. This version of (9a) exactly describes the class of so-called 'linear inequality measures' identified by Mehran [29] and further studied by Weymark [39] and Yaari [42].[11]

THEOREM 4. *Let M be a positional inequality index defined for* $\mathbf{x} \in \Omega_3$ *as in* (9a), *with* w: $\Re \to \Re$ *continuous and strictly monotone increasing. Then* $\partial M / \partial x_k \gtrless 0 \Leftrightarrow k \gtrless k^*$ *where* $k^* = w^{-1}(M(\mathbf{x}))$.

Proof. For $\mathbf{x} \in \Omega_3$, M is differentiable in each x_i.[12] Differentiating in (9a), we have

$$\partial M / \partial x_k = [w(k) - M] / [N\mu] \gtrless 0 \Leftrightarrow w(k) \gtrless M \tag{10}$$

We know that $\partial M/\partial x_N > 0$ from Theorem 1. Hence $w(N) > M$; and since $\sum_i w(i) = 0$ by assumption, and w is increasing, we must have $w(1) < 0$. Then by continuity and monotonicity, there exists a unique real number k^* such that $w(k^*) = M$. This, with (10), proves the result. □

We have established the existence of a benchmark *position*, k^*, for indices in the positional class. Of course, k^* is unlikely to be an integer. It depends on the income distribution as well as upon the inequality index M itself. For the Gini coefficient, we have $k^*_G = [N(1 + G) + 1]/2 > N/2$, whence the benchmark is above the median (and by more, the more unequal is the distribution). For the extended Gini coefficient $G(\nu)$, the benchmark position $k^*_{G(\nu)}$ is the solution to the equation $w_\nu(k) = G(\nu)$, or $[(N - k + 1)/N]^\nu - [(N - k)/N]^\nu = [1 - G(\nu)]/N$, which is difficult to obtain explicitly. However, an approximation to $k^*_{G(\nu)}$ can be obtained quite easily. Define a function $g(s) = s^\nu$, so that $s^* = (N - k^*_{G(\nu)})/N$ is the solution of $[1 - G(\nu)]/N = g(s + 1/N) - g(s)$. For large N, $g(s + 1/N) - g(s) \approx \nu s^{\nu - 1}/N$, whence $s^* \approx \{[1 - G(\nu)]/\nu\}^{1/(\nu - 1)}$ i.e. $k^*_{G(\nu)} \approx N[1 - \{[1 - G(\nu)]/\nu\}^{1/(\nu - 1)}]$. In the case $\nu = 2$, this approximation becomes $k^*_{G(2)} \approx N[1 + G]/2$, whilst the true value, k^*_G, is $[N(1 + G) + 1]/2$ which is higher by 1/2. Hence the approximate benchmark is at most one position too high in this case. For the illfare-ranked S-Gini, by similar reasoning $k^*_{S(\beta)} \approx N\{[1 - S(\beta)]/\beta\}^{1/(\beta - 1)}]$.[13] For Aaberge's Lorenz family $B(\kappa)$, the benchmark position is given by $k^*_{B(\kappa)} = N[(\kappa B(\kappa) + 1)/(\kappa + 1)]^{1/\kappa}$.

We saw in Section 3 that for $\mathbf{x} \in \Omega_1$ and for any k for which $1 < k < N$, an increase in x_k causes a Lorenz shift involving a single intersection from below. Zoli [45] addresses such situations. He shows that if the Gini coefficient is thereby increased, then inequality goes up for all relative inequality indices I satisfying the positional transfer-sensitivity principle.[14] That is, if $k > k_G^* = [N(1 + G) + 1]/2$, then $\partial M / \partial x_k > 0$ for all such indices M. Therefore k_G^* is an upper bound for the benchmarks k^* in the sub-class of positional inequality indices which also satisfy the Positional Principle of Transfer Sensitivity (in particular, $k_G^* \geq k_{G(v)}^*$ for all $v > 2$).

A link between the lower tail concern of the inequality ordering represented by a positional inequality index M and the location of the benchmark k^* obtains, just as it did for the non-positional class in Theorem 3. Again setting $\pi_i = z_i / N$ as person i's income share, and treating it as a probability, and now using version (9b) of the definition of M, we have from (10) that the benchmark position k^* satisfies this equation:

$$\omega(p^*) = \sum \pi_i \omega(p_i) = E[\omega(\mathbf{K})] \tag{11}$$

where $p^* = k^*/N$ and $\mathbf{K}$ is a risky prospect in which the return is p_i with probability π_i, $1 \leq i \leq N$. That is, k^*/N is the certainty equivalent of $\mathbf{K}$ for ω, in the sense of Pratt [34]. Now define

$$Q_\omega(p) = -p\omega''(p)/\omega'(p) \tag{12}$$

as the relative risk aversion of the 'utility' function ω.

DEFINITION 2. *The function $Q_\omega(p) = -p\omega''(p)/\omega'(p)$ defined in (12) will be said to measure the lower tail concern of the positional inequality index M defined in (9b).*

THEOREM 5. *Let M and $\hat{M}$ be positional inequality indices defined for $\mathbf{x} \in \Omega_3$ as in (9b) by, respectively, ω and $\hat{\omega}$,* where *$Q_\omega(p) > Q_{\hat{\omega}}(p) \forall p$. Then for all unequal income distributions $\mathbf{x} \in \Omega_3$, the benchmark position is lower for M than for $\hat{M}$: $k^* < \hat{k}^*$.*

For the positional indices, lower tail concern $Q_\omega(p)$ is measured in terms of rank p (rather than relative income z), and is given by the concavity of the weighting function ω. The higher is the measure $Q_\omega(p)$ $\forall p$, the more confined is the set of lower tail *positions* $1 \leq k < k^*$ in which an increase in a person's income is regarded as an inequality improvement. If the population size N is large, the illfare-ranked S-Gini has constant (and positive) lower tail concern: $Q_{S(\beta)}(p) = 2 - \beta$ $\forall p$ (see footnote 11). Aaberge's Lorenz family also exhibits constant, though non-positive, lower tail concern: $Q_{B(\kappa)}(p) = 1 - \kappa$ (where κ is a positive integer). If we had defined $Q_\omega(p)$ slightly differently, as $Q_\omega^*(p) = -(1 -$

$p)\omega''(p)/\omega'(p)$, which would have no effect on the validity of the theorem, then it would be the extended Gini that had constant lower tail concern: $Q^*_{G(\nu)}(p) = \nu - 2\ \forall p$. This brings out a link between our tail concern measure and the Positional Principle of Transfer Sensitivity: Within the positional class, the sub-class having positive lower tail concern are precisely those which satisfy this Principle.[15]

5. The leaky bucket

We now address the leaky bucket issue. Suppose that, in an unequal distribution $\mathbf{x}$, a small amount δ is taken from individual ℓ and an amount $q\delta$ is given to individual j who is lower down the distribution ($j < \ell$). The effect on any differentiable inequality index I is readily obtained using the total differential:

$$dI = \left[q\partial I/\partial x_j - \partial I/\partial x_\ell\right]\cdot\delta \tag{13}$$

for an infinitesimally small δ. If $\mathbf{x} \in \Omega_1$ then $x_j \leq x_\ell$, whilst if $\mathbf{x} \in \Omega_3$ (or if $\ell = 2$ and $\mathbf{x} \in \Omega_2$) then $x_j < x_\ell$. As before, we can deal with the general case of $\mathbf{x} \in \Omega_1$ for the non-positional indices, but will restrict attention to $\mathbf{x} \in \Omega_3$ and $0 < \delta < \delta(\mathbf{x})$ for the positional ones. In both cases, the index is then differentiable. The value q_0 for which dI = 0 reveals the information we seek about the permitted leakiness of the bucket for a non-adverse inequality effect:

$$q_0 = \frac{\partial I(.)/\partial x_\ell}{\partial I(.)/\partial x_j} \tag{14}$$

The maximum permitted rate of leakage is $(1 - q_0)$. The intuitively agreeable scenario, that the size of the leak would not erase completely the amount of income to be received by the poor, corresponds to $0 < q_0 < 1$, whilst the other two cases, already identified by Seidl [35] in the case of the Gini coefficient and termed 'paradoxical,' that the leak could exceed 100% or even be negative, correspond to $q_0 < 0$ and $q_0 > 1$, respectively. As we shall see, it is possible to predict the circumstances in which each of these three cases occurs for all inequality indices in our two classes.

5.1. THE NON-POSITIONAL INDICES OF RELATIVE INEQUALITY

For an inequality index I defined as in (1) and (2), we obtain

$$q_0 = \frac{u'(z_\ell) - u'(z^*)}{u'(z_j) - u'(z^*)} \tag{15}$$

from (14), using (5). Since u' is monotonic, it follows[16] that the magnitude of the maximum permitted leak $(1 - q_0)$ depends crucially upon which side of the benchmark the donor and recipient lie:

THEOREM 6. *Let I be a non-positional inequality index defined as in* (1) *and* (2). *The fraction q_0 of a small amount δ taken from individual ℓ which must reach individual j (where $j < \ell$) for inequality neutrality depends upon the incomes of ℓ and j relative to the benchmark income x^* as follows:*

(i) $x^* > x_\ell > x_j \Rightarrow 0 < q_0 < 1$
(ii) $x_\ell > x^* > x_j \Rightarrow q_0 < 0$
(iii) $x_\ell > x_j > x^* \Rightarrow q_0 > 1$

The magnitude of the effect on inequality, of a leaky transfer from ℓ to j, depends on whether $q \gtrless q_0$, of course, as well as on the values $z_j = x_j/\mu$, $z_\ell = x_\ell/\mu$ and $z^* = x^*/\mu$: For any non-positional index, inequality increases or decreases according to the inefficiency level and the relative incomes of the individuals affected. Case (*i*), in which $0 < q_0 < 1$, is the one typically envisaged, and, our analytics reveal, *it can occur only when both the donor and recipient are below the benchmark.* In all other configurations of donor and recipient, the permitted leakage will either exceed the amount taken away ($q_0 < 0$ i.e., $(1 - q_0) > 1$), so that the 'recipient' may lose too, or be negative, so that the recipient may receive more than the donor gives up ($q_0 > 1$ i.e., $(1 - q_0) < 0$) with no adverse effect on inequality.

One can readily obtain the value of q_0 for any particular index using (15) and the appropriate function $u(.)$. For the mean logarithmic deviation D, $q_D = \frac{z_\ell^{-1}-1}{z_j^{-1}-1}$; for the Theil index T, $q_T = \frac{\ln z_\ell - T}{\ln z_j - T}$; for the generalized entropy index $E(c)$, $c \neq 0,1$, $q_{E(c)} = \frac{z_\ell^{c-1} - z_{E(c)}^{c-1}}{z_j^{c-1} - z_{E(c)}^{c-1}}$; for the coefficient of variation CV, $q_{CV} = \frac{z_\ell - z_{CV}}{z_j - z_{CV}}$; for the Atkinson index $A(e)$, $q_{A(e)} = \frac{z_\ell^{-e} - z_{A(e)}^{-e}}{z_j^{-e} - z_{A(e)}^{-e}} = q_{E(1-e)}$ for $0 < e \neq 1$ and $q_{A(1)} = q_D$.

In Table I, we illustrate how the benchmark income level x^* and maximum permitted rate of leakage $1 - q_0$ vary with inequality aversion e for the Atkinson index $A(e)$, using the income distribution (\$200, \$500, \$800, \$1,100, \$2,400) again and choosing $\ell = 4$ and $j = 2$. When \$1 is taken from the person with \$1,100 and an amount \$$q$ is given to the person with \$500, the leak \$$(1 - q)$ can be as big as the value $1 - q_0 = 1 - q_{A(e)}$ shown in the table before an inequality effect judged to be adverse would occur. As is clear, all three cases $0 < q_0 < 1$, $q_0 < 0$ and $q_0 > 1$ of Theorem 6 arise, for different ranges of inequality aversion e. In each such range the maximum permitted rate of leakage increases with e.

Figure 2 shows the maximum permitted rate of leakage $1 - q_{E(c)}$ for the class of generalized entropy indices $E(c)$ as a function of the parameter c, for this same income distribution, using the scenario $\ell = 4$ and $j = 2$ of Table I and three others each involving the richest and/or poorest person in the transfer. The results for the Atkinson index $A(e)$ for $0 < e \neq 1$ occur for $c < 1$ (recall that $q_{E(1-e)} = q_{A(e)}$). Panel 1 of Figure 2 thus replicates and extends the maximum leak values given in

Table I. The benchmark income level x^* and maximum permitted rate of leakage $1 - q_{A(e)}$ as a function of inequality aversion for the income distribution ($200, $500, $800, $1,100, $2,400) when $\ell = 4$ and $j = 2$

e	$A(e)$	$\mathbf{x}^*$	$1 - q_{\mathrm{A}(e)}$	Theorem 6, case:
0.1	0.0272	1,282.1811	0.8436	(*i*) $x^* > x_4 > x_2$ $\Rightarrow 0 < q_0 < 1$
0.2	0.0546	1,251.5924	0.8701	
0.3	0.0819	1,220.6203	0.8967	
0.4	0.1092	1,189.3367	0.9234	
0.5	0.1363	1,157.8210	0.9503	
0.6	0.1632	1,126.1599	0.9774	
0.8	0.2162	1,062.7796	1.0328	(*ii*) $x_4 > x^* > x_2$ $\Rightarrow q_0 < 0$
1	0.2673	1,000.0000	1.0909	
1.2	0.3160	938.6666	1.1535	
1.4	0.3617	879.6041	1.2230	
1.6	0.4041	823.5476	1.3033	
1.8	0.4428	771.0817	1.4001	
2	0.4778	722.6008	1.5222	
2.2	0.5092	678.2984	1.6849	
2.4	0.5370	638.1840	1.9160	
2.6	0.5615	602.1179	2.2737	
2.8	0.5831	569.8547	2.9028	
3	0.6020	541.0856	4.2955	
3.2	0.6186	515.4730	9.8986	
3.5	0.6398	482.2325	−6.9382	(*iii*) $x_4 > x_2 > x^*$ $\Rightarrow q_0 > 1$
4	0.6673	438.0625	−1.3731	
5	0.7032	378.4391	−0.3241	
6	0.7247	341.3486	−0.1117	
7	0.7387	316.5664	−0.0423	
10	0.7608	275.9386	−0.0026	
20	0.7823	234.9238	−0.0000	

Table I. It is clear from panels 3 and 4, however, that it is not always the case for the Atkinson index that the maximum permitted leak increases with inequality aversion.

When the richest person is the donor, in this example the maximum leak decreases with e in some or all ranges. *A fortiori*, there can be no clear *general* relationship between the lower tail concern of a non-positional inequality ordering, as measured by $P_u(z)$, and the maximum leak $1 - q_0$: An intuition that a more lower tail concerned inequality ordering would countenance bigger leaks, though tempting, must be wrong.

Our findings in Table I and Figure 2 may be set alongside those of Atkinson [3, p. 42] and Jenkins [23, pp. 28–9], which relate to the maximum tolerable leak for an Atkinson index *before a welfare loss is experienced* (rather than, as here, *before inequality is exacerbated*). Because the efficiency aspect gets taken into account in welfare, measured in these studies as $\mu[1 - A(e)]$, it is clear that very

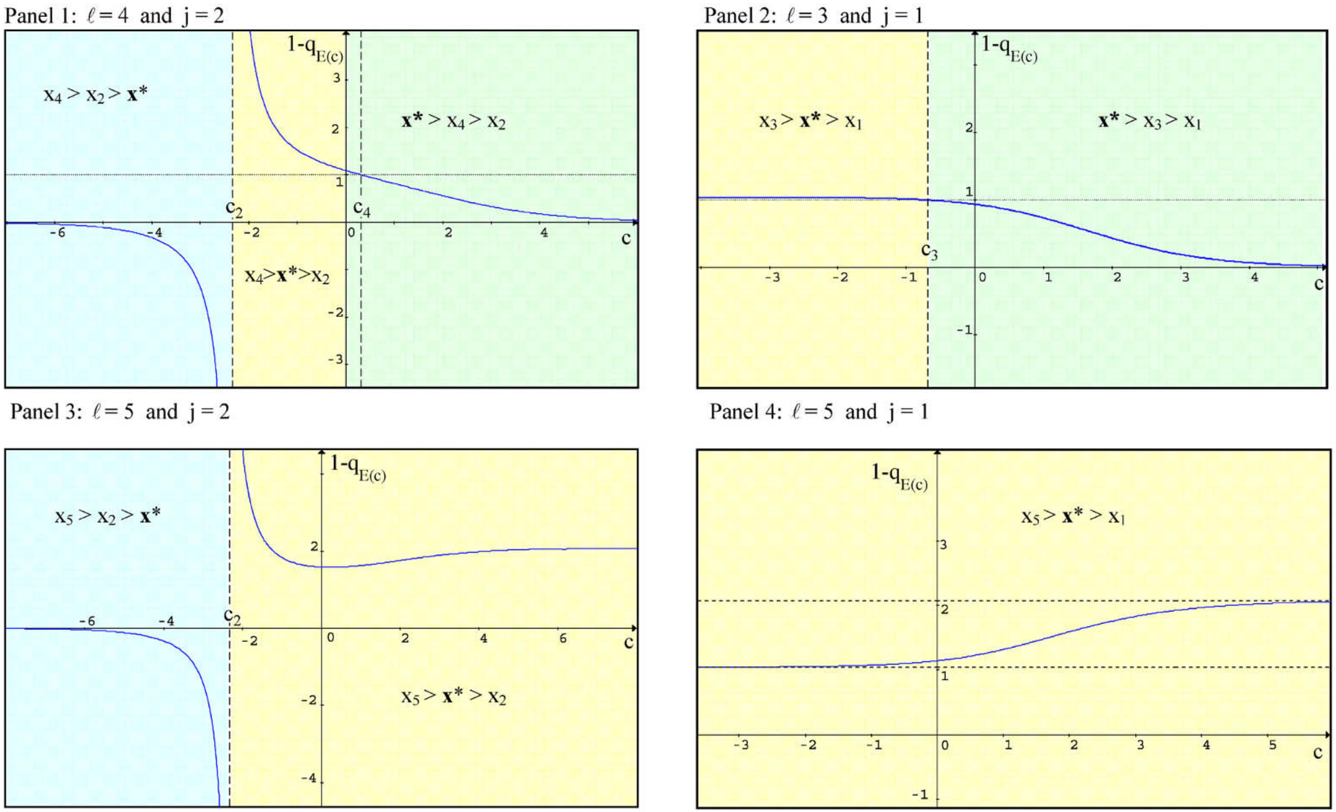

Figure 2. Maximum permitted leakage rate $1 - qE(c)$ for the generalized entropy index $E(c)$ as a function of c, for the scenario in Table I and three other scenarios involving the richest and/or the poorest person in the transfer.

big leaks could not be tolerated; Atkinson and Jenkins found maximum permitted leaks in the range 33%–75% for their particular numerical scenarios.

5.2. THE POSITIONAL INDICES OF RELATIVE INEQUALITY

If $\mathbf{x} \in \Omega_3$ and if $0 < \delta < \delta(\mathbf{x})$ then the resultant income distribution after the transfer, which is $\left(\mathbf{x}^{\ell}_{-\delta}\right)^{j}_{+q\delta}$, also belongs to Ω_3. Thus the form given in (9a) for a positional index $M(.)$ applies. Substituting from (10) into (14), the value of q_0 for the index M is:

$$q_0 = \frac{w(\ell) - M}{w(j) - M} \tag{16}$$

Now recall from Theorem 4 that the benchmark position for M is $k^* = w^{-1}(M)$.

Table II. The benchmark position k^* and maximum permitted rate of leakage $1 - q_{G(v)}$ as a function of inequality aversion for the same income distribution ($200, $500, $800, $1,100, $2,400) when $\ell = 4$ and $j = 2$

v	G(v)	k^*	$1 - q_{G(v)}$	Theorem 7, case:
1.2	0.1196	4.4054	0.7464	(i) $k^* > 4 > 2$ $\Rightarrow 0 < q_0 < 1$
1.4	0.2140	4.2976	0.8243	
1.6	0.2894	4.1941	0.8918	
1.8	0.3502	4.0949	0.9499	
2	0.4000	4.0000	1.0000	
3	0.5520	3.5895	1.1628	(ii) $4 > k^* > 2$ $\Rightarrow q_0 < 0$
4	0.6285	3.2724	1.2446	
5	0.6749	3.0244	1.2980	
6	0.7060	2.8249	1.3495	
7	0.7282	2.6607	1.4141	
8	0.7444	2.5225	1.5053	
9	0.7566	2.4046	1.6415	
10	0.7659	2.3026	1.8568	
11	0.7731	2.2135	2.2286	
12	0.7787	2.1351	2.9848	
13	0.7831	2.0655	5.2139	
14	0.7866	2.0034	84.5591	
15	0.7893	1.9477	−4.6751	(iii) $4 > 2 > k^*$ $\Rightarrow q_0 > 1$
16	0.7915	1.8975	−2.0133	
17	0.7932	1.8521	−1.1755	
18	0.7946	1.8108	−0.7730	
20	0.7965	1.7386	−0.3936	
25	0.7989	1.6028	−0.1036	
30	0.7996	1.5083	−0.0319	
40	0.8000	1.3866	−0.0033	

Hence

$$q_0 = \frac{w(\ell) - w(k^*)}{w(j) - w(k^*)} \tag{17}$$

(compare this with (15), which expresses q_0 in a similar form for the non-positional indices). The following results are immediate, given that $w(.)$ is strictly increasing:

THEOREM 7. *Let M be a positional inequality index defined for* $\mathbf{x} \in \Omega_3$ *as in* (9a)*, with w:* $\Re \rightarrow \Re$ *continuous and strictly monotone increasing. The fraction* q_0 *of a small amount* $0 < \delta < \delta(\mathbf{x})$ *taken from individual* ℓ *which must reach individual j* (*where* $j < \ell$) *for inequality neutrality depends upon the positions of* ℓ *and j relative to the benchmark position k* as follows:*

(i) $k^* > \ell > j \Rightarrow 0 < q_0 < 1$
(ii) $\ell > k^* > j \Rightarrow q_0 < 0$
(iii) $\ell > j > k^* \Rightarrow q_0 > 1$

The case $0 < q_0 < 1$ occurs only when both the donor and recipient are positioned below the benchmark k^*. In all other configurations, the permitted leakage will either exceed the amount taken away ($q_0 < 0$), so that the 'recipient' may lose too, or be negative, so that the recipient may receive more than the donor gives up ($q_0 > 1$) with no adverse effect on inequality. These results are analogous to the ones in Theorem 6 for the non-positional indices, in which the benchmark *income level* forms the divide; for the positional indices, it is the benchmark *position* which takes this role.

In the case of the Gini coefficient, for which $w(i) = (2i - N - 1)/N$, we have $q_G = (\ell - k_G^*)/(j - k_G^*)$ where $k_G^* = [N(1 + G) + 1]/2$. Seidl [35] obtained essentially this result by other means. The expression for q_0 for the extended Gini coefficient $G(v)$, $v > 1$, which is more complex, obtains by substituting $w_{G(v)}(i) = N\{[(N - i)/N]^v - [(N - i + 1)/N]^v\} + 1$ and $M = G(v)$ in (16).

Noting that for large N, $w_{G(v)}(i) \approx [1 - v.\{(N - i)/N\}^{v-1}]/N$, so that q_0 can be approximated from (17) as $q_0 \approx \left[\left(N - k^*_{G(v)}\right)^{v-1} - (N - \ell)^{v-1}\right] / \left[(N - k^*_{G(v)})^{v-1} - (N - j)^{v-1}\right]$, it follows from the further approximation $k^*_{G(v)} \approx N \left[1 - \{[1 - G(v)]/v\}^{1/(v-1)}\right]$ already noted that $q_{G(v)} \approx \frac{1 - G(v) - v(1 - p_l)^{v-1}}{1 - G(v) - v\left(1 - p_j\right)^{v-1}}$ where p_j and p_l are the ranks of j and l, respectively. Analogously, for the illfare-ranked S-Gini, $q_{S(\beta)} \approx \frac{1 - S(\beta) - \beta p_l^{\beta-1}}{1 - S(\beta) - \beta p_j^{\beta-1}}$ for large N. For the Lorenz family of Aaberge [1], we have $q_{B(\kappa)} = \frac{(\kappa+1)p_\ell^\kappa - \kappa B(\kappa) - 1}{(\kappa+1)p_j^\kappa - \kappa B(\kappa) - 1}$.

In Table II, we illustrate for the extended Gini coefficient how the benchmark position $k^*_{G(v)}$ and maximum permitted rate of leakage $1 - q_{G(v)}$ vary with the distributional judgment parameter v, using the same income distribution as in

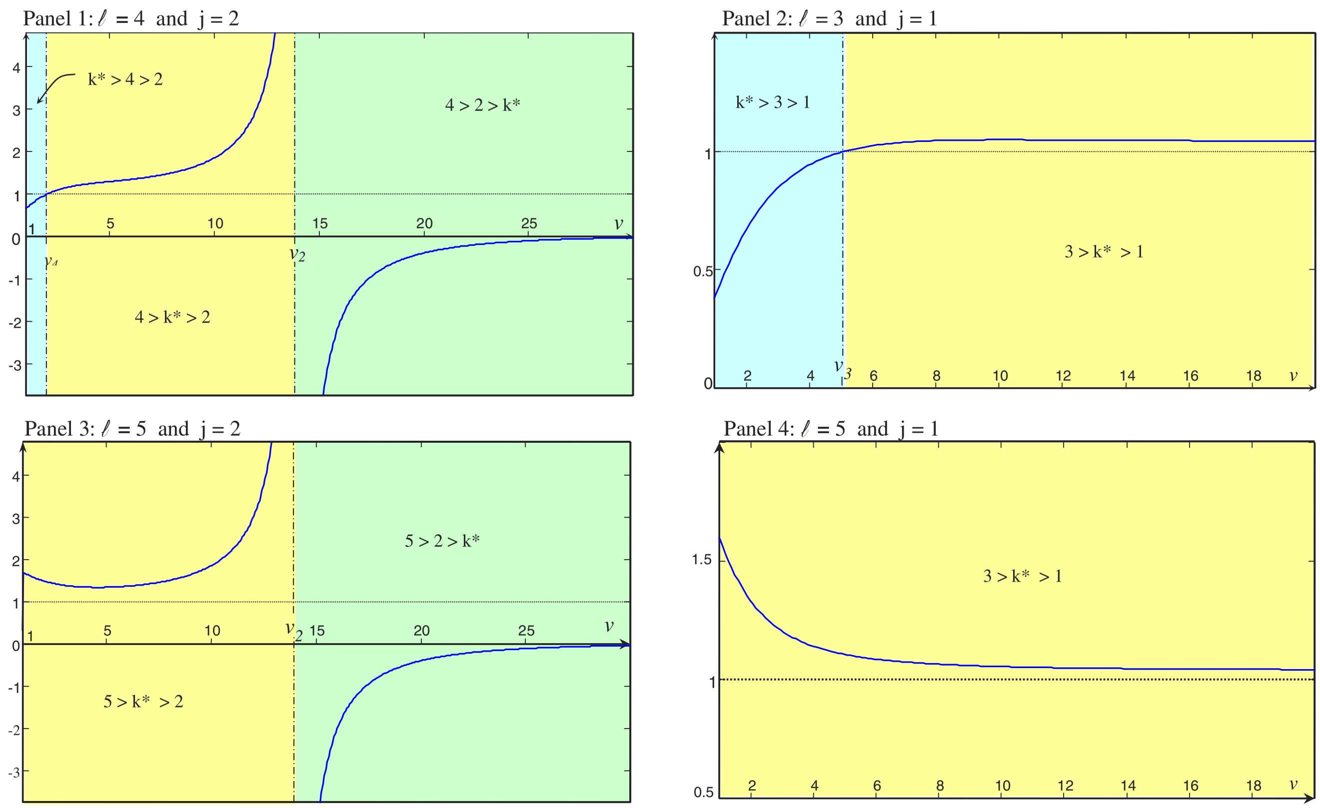

Figure 3. Maximum permitted leakage rate $1 - q_{G(v)}$ for the extended Gini coefficient $G(v)$ as a function of v, for the scenario in Table II and three other scenarios involving the richest and/or poorest person in the transfer.

Table I and choosing $\ell = 4$ and $j = 2$ as before. The cases $0 < q_0 < 1$, $q_0 < 0$ and $q_0 > 1$ of Theorem 7 all arise.

Figure 3 shows the dependence of $1 - q_{G(v)}$ on v graphically, for the same four scenarios as used in Figure 2 for $1 - q_{E(c)}$. As before, we see non-monotonicity in some scenarios between v and $1 - q_{G(v)}$. For the positional indices too, then, there can be no general link between the degree of lower tail concern of the inequality ordering and the maximum permitted leak.[17] The leakage rates shown in Table II and Figure 3 may be compared with those of Duclos [16, p.149–150], who calculates the maximum tolerable leaks for *no welfare loss*, where welfare is measured as $\mu[1 - G(v)]$. Duclos's maximum leaks are shown for various scenarios to be increasing in v and lying between 6.7% and 99.6%.

There is an analytical connection between our maximum leakage rate $(1 - q_0)$ for inequality and those of Atkinson, Jenkins and Duclos for welfare. Letting welfare be evaluated as $W = \mu[1 - I]$, where I is an inequality index in one of our two classes whose range is contained in the interval [0,1] (such as the Atkinson and extended Gini indices), the welfare effect of the leaky transfer is $dW = \left[q\partial W/\partial x_j - \partial W/\partial x_\ell\right].\delta$ (compare with (13)). The maximum permitted leak for a non-adverse welfare effect, call it $1 - q_W$, occurs at the value of q for which $\mathrm{d}W = 0$. It can easily be shown, in fact for any monotonic social welfare function, that $1 - q_W$ lies between 0 and 1. The welfare and inequality leakage rates in our case are linked by an equation of the form:

$$(1 - q_W) = (1 - q_0).\lambda \tag{18}$$

in which $\lambda \in (-\infty, 1)$ is a term that depends on the position of the recipient j relative to the benchmark.[18]

6. Summary and conclusions

It is important for economists to be able to compare inequality in income distributions with different means. Incomes can change due to growth, and also due to disincentive effects arising from the implementation of redistributive programmes. It is perhaps surprising, then, that one can find little in the inequality measurement literature about the inequality consequences of a single income growing, or of a single leaky transfer. The effects on welfare of such changes have, of course, been much discussed; our results in this paper have thrown light on the corresponding questions for inequality.

First, we looked at the effect on inequality of increasing one income. We confirmed the casual intuition that increasing a low income should reduce inequality and increasing a high one should surely raise it. In fact we proved that, for large classes of inequality indices, there is a benchmark income level or position dividing the two responses, which is different for each inequality index and income distribution. This benchmark can be both quantified and systematically related to a property of the underlying inequality ordering, its lower tail concern. The intuition for the aggregate, offered up by our analysis, that income growth in the lower part of a distribution will be equalizing, and income growth in the upper part disequalizing, seems unexceptionable, but it surely has not been appreciated before now that the divide between 'lower' and 'upper' that supports this intuition could differ so markedly for different inequality indices, and its determinants be understood.[19] In the pro-poor growth literature, which has lately departed from that on the growth-inequality relationship, a significant strand now focuses on the growth elasticity of poverty according to various measures. See Foster and Székely [20] for a discussion of this trend, and for a proposal that essentially reduces to computing pro-poorness as the growth elasticity of the Atkinson inequality index $A(e)$, whose benchmark income level, call it $x^*(e)$, equals $\mu[1 - A(e)]^{(e-1)/e}$ when $e \neq 1$ and μ when $e = 1$. An implication is that all growth taking place entirely below $x^*(e)$ counts as pro-poor, whilst growth taking place entirely above $x^*(e)$ may or may not do so, depending on its effect on μ; our analysis exposes this property, which holds without regard to any assumed poverty line.

The analytics we have pursued here in respect of 'changing one income' can surely be taken further. The inequality index form $I = [1/N]. \sum_i w(i)u(x_i/\mu)$ could be a starting point. This form embeds both our non-positional and positional classes, and would cover, for example, Berrebi and Silber's [4] construction.[20] Ebert [18] specifies a class of inequality indices which cuts across our two, containing some of the generalized entropy indices (those for which $c < 1$) and all of the Gini, extended Gini and S-Ginis, along with other indices which have not gained currency. Mosler and Muliere [31] specify a class of indices obeying the 'star-shaped principle of transfers', according to which only those rich-to-poor transfers which take place across a specific income value or position θ need reduce inequality. The extension of our results to these and other classes is left for future research.

In the second part of the paper, we turned to the leaky bucket scenario. We took for granted a rate of leakage $(1 - q)$ from the bucket and asked the question, how leaky would the bucket have to be before the intended inequality-ameliorating effect of a single rich-to-poor transfer would be negated? The answer was $(1 - q_0)$, with q_0 depending on the relative incomes or ranks of the donor and recipient, and, crucially, on which side of the benchmark they are located. We showed that a negative rate of leakage or even one exceeding 100% could be countenanced for some configurations. Only in case the donor and recipient are both in the lower part of the distribution is there a bound $0 < (1 - q_0) < 1$. So here too, we obtain an insight for the aggregate: The inefficiencies of redistributive programmes had better not be focussed entirely within the lower part of an income distribution.[21]

A further insight arises in the context of tax-transfer policy in a socially heterogeneous population of households, even in the absence of efficiency losses.

Let ℓ and j be two households, selected as the donor and recipient for a money transfer, respectively. If the equivalence scale deflators for $\ell's$ and j's money incomes are m_ℓ and m_j, each unit reduction in the living standard of ℓ is accompanied by an increase of $q = m_\ell / m_j$ units in the living standard of j. We can apply Theorems 6 and 7, to examine the effect of the (non-leaky) money transfer on inequality in the distribution of living standards for any non-positional or positional index. If j is below the benchmark in the living standards distribution, inequality reduction requires $q > q_0$ (where $0 < q_0 < 1$ if ℓ is also below the benchmark, and $q_0 < 0$ if ℓ is above it); and if j is above the benchmark, inequality reduction requires $q < q_0$ (in this case $q_0 > 1$).[22] These results pick up on, and extend, an insight of Glewwe [21], that some money transfers from the better-off to the worse-off can exacerbate inequality. Transfers taking place entirely below the benchmark may do this if from a less needy to a *very* needy type of household ($m_j > m_\ell / q_0$, where $0 < q_0 < 1$): We regard this as a strongly counter-intuitive result. Transfers taking place entirely above the benchmark may also exacerbate inequality, but only if directed to a very much less needy household type ($m_j < m_\ell / q_0$, where $q_0 > 1$); this seems less unreasonable. Transfers which are made across the benchmark are unambiguously inequality-reducing regardless of relative needs (because $q = m_\ell / m_j > q_0$ is always satisfied if $q_0 < 0$).

Although negative rates of 'leakage' and rates exceeding 100% have not been encountered in leaky bucket analytics addressing the *welfare* effect of transfers before now,[23] and may seem surprising in the inequality context (indeed were termed 'paradoxical' by Seidl [35]), the intuition is, after all, quite straightforward. Tolerance of a leakage exceeding 100% ($q_0 < 0$) occurs when donor and 'recipient' are either side of the benchmark. Taking from a rich person (above the benchmark) unambiguously reduces inequality. This effect is necessarily reinforced by giving to a poor person (below the benchmark). Hence, having taken from the rich, one can also take from the poor (up to a certain limit, that limit being $-q_0$) without eliminating the inequality gain. Similarly, a negative leak ($q_0 > 1$) is tolerated when the donor and recipient are both above the benchmark. Taking \$1 from a rich person and giving it to another, less rich but still above the benchmark, reduces inequality (by the Principle of Transfers); to restore inequality to the previous level, one may give extra to the recipient (namely, an additional amount of $q_0 - 1$). Our analytics have enabled these effects to be quantified, understood and compared for wide classes of inequality indices.

Acknowledgements

We wish to thank two anonymous referees of this journal, and Rolf Aaberge, Branko Milanovic, Henry Chiu, Valentino Dardanoni, Jean-Yves Duclos, Udo Ebert, Joan Esteban, Mike Hoy, Stephen Jenkins, Karl Mosler, Krishna Pendakur, Christian Seidl, Jacques Silber, Shlomo Yitzhaki, Buhong Zheng and

Claudio Zoli for valued comments, insights and suggestions regarding the content of this paper.

After this article went to press, the authors became aware of a paper "Effect of the rise of a person's income on inequality" by Rodolfo Hoffman, published on pp. 237–262 of volume 21 of the Brazilian Review of Econometrics in 2001. Inter alia, Hoffman demonstrates the existence of a benchmark income level for inequality measures, characterizes the benchmark as a "relative poverty line", computes its value for the mean logarithmic deviation, Theil index, generalized entropy family and Gini coefficient, and applies these constructions to Brazilian data.

Notes

[1] One class includes rank-independent indices such as the coefficient of variation, mean logarithmic deviation, generalized entropy index and Atkinson index; the other, rank-dependent (or positional) indices such as the Gini and extended Gini coefficients.

[2] In this notation, $\left(\mathbf{x}_{\alpha}^{j}\right)_{\beta}^{j} = \mathbf{x}_{\alpha+\beta}^{j}$ for all j such that $x_j \neq x_{j+1}$ and for α and β suitably restricted, whilst if $j > i$, $\left(\mathbf{x}_{-\delta}^{j}\right)_{\delta}^{i} = \left(\mathbf{x}_{\delta}^{i}\right)_{-\delta}^{j}$ is the distribution obtained from $\mathbf{x}$ by making a progressive transfer of δ from individual j to individual i.

[3] If zero incomes were admitted, then the effect of increasing x_1 when $x_1 = x_2 = 0$ would be to shift the Lorenz curve upwards.

[4] For the transfer principle to hold, $x_\ell > x_j \Rightarrow \partial I/\partial x_\ell > \partial I/\partial x_j \Rightarrow [h'(J)]\{u'(x_\ell/\mu) - u'(x_j/\mu)\} > 0$.

[5] For a proof of the properties of the mean of order c, see for example Hardy et al. [22, chapter 1].

[6] The transfer sensitive inequality indices are those which adhere to the Principle of Diminishing Transfers of Kolm [25]. For an index I in the non-positional class, if $h'(J) > 0$ then I satisfies Kolm's principle if and only if $u'' > 0$ and $u''' < 0$, and if $h'(J) < 0$ then I satisfies Kolm's principle if and only if $u'' < 0$ and $u''' > 0$. Thus $A(e)$ is transfer-sensitive for all e, and $E(c)$ is transfer sensitive for $c < 2$. The benchmarks for these indices are all below z_{CV} : $c < 2 \Rightarrow z_{\mathrm{CV}} > z_{E(c)} = z_{A(1-c)}$ (as Figure 1 shows).

[7] For a direct proof, just follow similar steps to those in Lambert's [27, theorem 4.1] proof of the Pratt theorem. These steps are spelt out explicitly in Lambert and Lanza [28], where additional material relevant to this paper may also be found.

[8] There is a formal link with Kimball's [24] concept of 'prudence' in the uncertainty context. We refrain from calling $P_u(z)$ 'downside inequality aversion,' as this would be inconsistent with Modica and Scarsini's [30] measure in the uncertainty context of downside risk aversion, which, in absolute form, is $-u'''(z)/u'(z)$. We also refrained from calling $P_u(z)$ 'downside-mindedness,' however apt, as this concept belongs to Wilthien [40]. Chiu [9] introduces a measure which he calls "the strength of an index's downside inequality aversion against its inequality aversion" that is ordinally equivalent to our $P_u(z)$. Chiu shows that the magnitude of his measure determines the ranking by the index of two distributions whose Lorenz curves cross once. Chiu interprets the raising of one income, low enough in the distribution, as "a special combination of a downside inequality increase and an inequality decrease" (ibid, pp. 16–17).

[9] Footnote 6 demonstrates this.

[10] For more on the extended Gini coefficient, see Lambert [27, chapter 5].

[11] In the case of a continuous distribution function $F(\mathrm{x})$, the Mehran index becomes $M_F = \int_0^\infty$

$x\omega(F(x))f(x)dx/\mu$ where $\int_0^1 w(p)dp = 0$ (see Lambert [27] for more on this). In this setting, the rank-weighting functions for the Gini, extended Gini and S-Gini are $\omega_G(p) = 2p - 1$, $\omega_{G(v)}(p) = 1 - v(1 - p)^{v-1}$ and $\Omega_{S(\beta)}(p) = 1 - \beta p^{\beta-1}$, respectively. These correspond to the discrete weighting functions $w_G(i)$, $w_{G(v)}(i)$ and $w_{S(\beta)}(i)$ cited above, making the identification $p = i/N$ and regarding $1/N$ as an infinitesimal. The rank-weighting function for Aaberge's [1] Lorenz family of inequality indices, $B(\kappa)$ where κ is a positive integer, is $\omega_{B(\kappa)}(p) = [(\kappa+1)p^\kappa - 1]/\kappa$. Notice that if we extend the functional forms defining $G(v)$ and $S(\beta)$ to all non-zero parameter values, then $-G(v)$ belongs to our positional class for $v < 1$ and $-S(\beta)$ belongs to it for $\beta > 1$. An inequality index outlined in Wang and Tsui [38] takes the form $J(c) = \text{sign}\,(c - 1)[G(c) - S(c)]$, $0 < c \neq 1$, and hence belongs to our class too. Another class of 'generalized Gini' indices, due to Aaberge [2], in which the weights depend on Lorenz curve values $L(p)$ rather than positions p, does not fall within the scope of our general form in (9a)–(9b). See also Chakravarty [7].

12 The form in (9a) can be extended to Ω_1, with the loss of differentiability, if the weights when $x_i = x_{i+1}$ are made the same for persons i and $i + 1$, and equal to $[w(i) + w(i + 1)]/2$. Without this change, a small amount taken from person i and given to person $i + 1$ would increase inequality, whereas the same amount taken from person $i + 1$ and given to person i would reduce it – yet the final income distribution would be the same in both cases.

13 Pendakur [33], addressing a slightly different question, identifies a unique threshold position (percentile) for the S-Gini, such that a lump-sum transfer from all agents but one, to that one, either raises or lowers inequality depending on whether the recipient is above or below the threshold position. See footnote 12, ibid.

14 The positional index M of (9a)–(9b) satisfies the strong version of the Positional Principle of Transfer Sensitivity when $w(i + 1) - w(i)$ is positive and strictly decreasing in i, or $\omega''(p) < 0\ \forall p \in (0,1)$. See Mehran [29, p. 808], Zoli [44] and Chateauneuf et al. [8, theorem 9] for more on this. Note that $w_{G(v)}(i + 1) - w_{G(v)}(i) = N\{[(N - i + 1)/N]^v + [(N - i - 1)/N]^v - 2[(N - i)/N]^v\} = 2N[E(Y^v) - (E(Y))^v]$ where Y is a random variable with realizations $(N - i + 1)/N$ and $(N - i - 1)/N$ each with probability one half. This is strictly positive because Y^v is a convex function of Y for $v > 1$. Similarly, by a slight abuse of notation, $\partial[w_{G(v)}(i + 1) - w_{G(v)}(i)]/\partial i = -2v[E(Y^{v-1}) - (E(Y))^{v-1}]$, which is negative for $v > 2$, zero for $v = 2$ and positive for $v < 2$. $G(v)$ thus satisfies the strong version of the Positional Principle only for $v > 2$.

15 In particular, the Gini coefficient is excluded. In Aaberge [1, pp. 648–9], criteria for the positional principle to apply to restricted classes of distributions are explored, which allow for negative lower tail concern, and in particular a role is found for the Gini coefficient. Yaari's [42] 'equality-mindedness' measure for the positional indices, which in our notation is $-\omega'(p)/[1 - \omega(p)]$, is based upon a leaky bucket experiment: see footnote 17 ahead for more on this.

16 It is a general property that if a function $g(.)$ is strictly monotonic, either increasing or decreasing, and if $d = [g(a) - g(b)]/[g(c) - g(b)]$, where $a > c$, then $d < 0$ if $a > b > c$, $d > 1$ if $a > c > b$, and $0 < d < 1$ if $b > a > c$.

17 Yaari's [42] equality–mindedness measure concerns a leaky bucket.Yaari suggests a thought experiment whereby the incomes of a given fractile of the poor are raised, at the expense of lowering the incomes of a certain fractile of the rich. A more equality–minded index M, he argues, would tolerate a bigger fractile of donors than a less equality-minded one, before regarding the 'leak' entailed as detrimental. Thus his leaks involve a loss of mass, whereas ours involve a loss of income.

18 A demonstration that (18) holds may be found in Lambert and Lanza [28]. For the Atkinson index with $e = 1$, $\lambda = 1 - x_j/\mu$, whilst for the Gini coefficient, $\lambda = [G - w(j)]/[1 - w(j)]$ where $w(j) = (2j - N - 1)/N$, which can also be written $\lambda = [k^*_G - j]/[N + 1/2 - j]$. Camacho-Cuena et al. [6] point out that the corresponding welfare function based on the generalized entropy inequality index, which would be $\mu[1 - E(c)]$, is in general non-monotonic: see their Theorem 14.

[19] Our analytics can in fact be extended to other types of index, for example to the variance of logarithms which, though not Lorenz consistent (Foster and Ok [19]), is popular among applied economists. The variance of logarithms has geometric mean income $\tilde{\mu}$ as its benchmark, and the value of q_0 for the leaky bucket analytics is $q_0 = \left(x_j/x_\ell\right)\frac{\ln x_\ell - \ln\tilde{\mu}}{\ln x_j - \ln\tilde{\mu}}$: see Lambert and Lanza [28, page 23].

[20] See Lambert [27, p. 131] and Duclos et al. [17] for an inequality index in this form which merges the Gini coefficient and Atkinson index.

[21] In Lambert [26], a labour supply model was investigated, in which wage rates were lognormally distributed and a piecewise linear negative income tax scheme was applied. It was shown that, for a wide range of tax and benefit parameter values, the efficiency loss of the tax-transfer system exceeded the size of the bucket.

[22] These requirements stem from (13), which shows that the inequality effect dI of the transfer is a negative or positive function of q, respectively.

[23] But see the very recent article of Camacho-Cuena et al. [6], in which leaky bucket analytics have been extended to the social welfare function $\mu[1 - E(c)]$ (cf. footnote 18), for which a benchmark income level is shown to exist with analogous properties for leaky transfers to those of our Theorem 6. Experiments are also conducted in this paper, in which student subjects coached in the transfer principle and basic welfare considerations were shown a hypothetical 7-person income distribution, and asked to adjust a named recipient's income each time another recipient's income was raised or lowered by a small amount, and to make the adjustment such that "the degree of income inequality within this society should be maintained" (p. 12). The authors' main finding is that their subjects' behaviour patterns did not accord with the leaky bucket analytics developed here, but instead followed a 'compensating justice' hypothesis, for which "income inequality measurement needs to be restructured along special axioms if it should comply." Here is another area for possible theoretical development and refinement.

References

1. Aaberge, R: Characterizations of Lorenz curves and income distributions, *Social Choice and Welfare* **17** (2000), 639–653.
2. Aaberge, R.: Axiomatic characterization of the Gini coefficient and Lorenz curve orderings, *Journal of Economic Theory* **101** (2001), 115–132.
3. Atkinson, A.B.: *Wealth, Income and Inequality* (Second Edition), Oxford University Press, 1980.
4. Berrebi, Z.M. and Silber, J.: Weighting income ranks and levels: A multi-parameter generalisation for absolute and relative inequality indices, *Economics Letters* **7** (1981), 391–397.
5. Bourguignon, F.: Decomposable inequality measures, *Econometrica* **47** (1979), 901–920.
6. Camacho-Cuena, E., Neugebauer, T. and Seidl, C.: Compensating justice beats leaky buckets: An experimental investigation. Economics Working Paper No. 2005–06, University of Kiel, 2005.
7. Chakravarty, S.R.: Extended Gini indices of inequality, *International Economic Review* **29** (1988), 147–156.
8. Chateauneuf, A., Gajdos, T. and Wilthien, P.-H.: The principle of strong diminishing transfer, *Journal of Economic Theory* **103** (2002), 311–332.
9. Chiu, W.H.: Intersecting Lorenz curves and the degree of downside inequality aversion, Mimeo, University of Manchester, 2004.
10. Contoyannis, P. and Forster, M.: The distribution of health and income: A theoretical framework, *Journal of Health Economics* **18** (1999), 605–622.

11. Cowell, F.A.: On the structure of additive inequality measures, *Review of Economic Studies* **47** (1980), 521–531.
12. Deaton, A. and Paxson, C.: Mortality, income, and income inequality over time in Britain and the United States, In: D. Wise (ed.), *Perspectives on the Economics of Aging*, University of Chicago, 2004, pp. 247–280.
13. Donaldson, D. and Weymark, J.A.: A single parameter generalization of the Gini indices of inequality, *Journal of Economic Theory* **22** (1980), 67–86.
14. Donaldson, D. and Weymark, J.A.: Ethically flexible indices for income distributions in the continuum, *Journal of Economic Theory* **29** (1983), 353–358.
15. Duclos, J.-Y.: Social evaluation functions, economic isolation and the Suits index of progressivity, *Journal of Public Economics* **69** (1998), 103–121.
16. Duclos, J.-Y.: Gini indices and the redistribution of income, *International Tax and Public Finance* **7** (2000), 141–162.
17. Duclos, J.-Y., Jalbert, V. and Araar, A.: Classical horizontal inequality and reranking: An integrating approach, *Research on Economic Inequality* **10** (2003), 65–100.
18. Ebert, U.: Measurement of inequality: An attempt at unification and generalization, *Social Choice and Welfare* **5** (1988), 147–169.
19. Foster, J.E. and Ok, E.A.: Lorenz dominance and the variance of logarithms, *Econometrica* **67** (1999), 901–907.
20. Foster, J.E. and Székely, M.: How good is growth?, *Asian Development Review* **18** (2000), 59–73.
21. Glewwe, P.: Household equivalence scales and the measurement of inequality: Transfers from the poor to the rich could decrease inequality, *Journal of Public Economics* **44** (1991), 211–216.
22. Hardy, G., Littlewood, J. and Polya, G.: *Inequalities,* Cambridge University Press, London, 1934.
23. Jenkins, S.P.: The measurement of income inequality, In: L. Osberg (ed.) *Economic Inequality and Poverty: International Perspectives*, Sharpe, Armonk NY, 1991, pp. 3–38.
24. Kimball, M.S.: Precautionary saving in the small and in the large, *Econometrica* **58** (1990), 53–73.
25. Kolm S.-C.: Unequal inequalities, I and II, *Journal of Economic Theory* **12** (1976), 416–442 and **13** (1976), 82–111.
26. Lambert, P.J.: Okun's bucket: A leak and two splashes?, *Journal of Economic Studies* **15** (1988), 71–78.
27. Lambert, P.J.: *The Distribution and Redistribution of Income*, University Press, Manchester, UK, 2001.
28. Lambert, P.J. and Lanza, G.: The effect on inequality of changing one or two incomes, Economics Discussion Paper No. 2003-15, University of Oregon, 2003.
29. Mehran, F.: Linear measures of income inequality, *Econometrica* **44** (1976), 805–809.
30. Modica, S. and Scarsini, M.: A note of comparative downside risk aversion, *Journal of Economic Theory* 122 (2005), 267–271.
31. Mosler, K. and Muliere, P.: Inequality indices and the starshaped principle of transfers, *Statistical Papers* **37** (1996), 343–364.
32. Okun, A.M.: *Equality and Efficiency: The Big Trade-Off,* Brookings Institution, Washington, District of Columbia, 1975.
33. Pendakur, K.: Changes in Canadian family income and family consumption inequality between 1978 and 1992, *Review of Income and Wealth* **44** (1998), 259–283.
34. Pratt, J.W.: Risk aversion in the small and in the large, *Econometrica* **32** (1964), 122–136.
35. Seidl, C.: Inequality measurement and the leaky-bucket paradox, *Economics Bulletin* **4**(6) (2001), 1–7.

36. Shorrocks, A.F.: The class of additively decomposable inequality measures, *Econometrica* **48** (1980), 613–625.
37. Shorrocks A.F. and Foster, J.: Transfer sensitive inequality measures, *Review of Economic Studies* **54** (1987), 485–497.
38. Wang, Y.-Q. and Tsui, K.-Y.: A new class of deprivation-based generalized Gini indices, *Economic Theory* **16** (2000), 363–377.
39. Weymark, J.A.: Generalized Gini inequality indices, *Mathematical Social Sciences* **1** (1981), 409–430.
40. Wilthien, P.-H.: Downside-mindedness. *Cahiers de la MSE n° 1999.97*, CERMSEM, Université Paris I Panthéon-Sorbonne, Paris, 1999.
41. Wolfson, M.C.: When inequalities diverge, *American Economic Review (AEA Papers and Proceedings)* **84** (1994), 353–358.
42. Yaari, M.: A controversial proposal concerning inequality measurement, *Journal of Economic Theory* **44** (1988), 381–397.
43. Yitzhaki, S.: On an extension of the Gini index, *International Economic Review* **24** (1983), 617–628.
44. Zoli, C.: Intersecting generalized Lorenz curves and the Gini index, *Social Choice and Welfare* **16** (1999), 183–196.
45. Zoli, C.: Inverse stochastic dominance, inequality measurement and Gini indices, *Journal of Economics* Supplement **9** (2002), 119–161.