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BPS state counting in $N=8$ supersymmetric string theory for pure D-brane configurations

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ABSTRACT: Exact results for the BPS index are known for a class of BPS dyons in type II string theory compactified on a six dimensional torus. In this paper we set up the problem of counting the same BPS states in a duality frame in which the states carry only Ramond-Ramond charges. We explicitly count the number of states carrying the lowest possible charges and find agreement with the result obtained in other duality frames. Furthermore, we find that after factoring out the supermultiplet structure, each of these states carry zero angular momentum. This is in agreement with the prediction obtained from a representation of these states as supersymmetric black holes.

KEYWORDS: Black Holes in String Theory, D-branes

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Contents

1	Introduction	1
2	The system	3
3	The low energy dynamics of the D-brane system	5
4	Supersymmetric solutions	7
5	Non-abelian generalization	10
6	Conclusion	12
A	Effect of metric and 2-form background	14
B	Duality transformation	16
C	Explicit solutions to (4.10)	17

1 Introduction

Understanding the microscopic origin of Bekenstein-Hawking entropy is one of the important problems in any theory of quantum gravity, and in particular in string theory. In recent years there has been considerable progress towards this direction, including precision counting of microscopic states in certain string theories with 16 or more unbroken supersymmetries [1–7]. One of these theories is type IIA or IIB compactified on a six dimensional torus. In this theory, for certain configurations carrying a combination of Kaluza-Klein (KK) monopole charge, momentum along one of the circles of the torus and D-brane wrapping charges along some of the cycles of the torus, one can carry out the exact counting for the number of microscopic BPS states [7]. On the other hand, for large charges this system can be described by a supersymmetric black hole with a finite area event horizon. Thus, by comparing the logarithm of the number of microstates with the Bekenstein-Hawking entropy of the corresponding black hole, one can verify the equality of the macroscopic and microscopic entropy of the black hole.

Although the counting of microscopic states was carried out for a specific system of KK monopoles and D-branes carrying momentum along a compact circle, using duality symmetry we can map it to other systems. In particular it is possible to map this configuration to a system that contains only D-brane charges. Duality symmetry predicts that the BPS index of this system computed from microscopic counting should give the same result as the original system to which it is dual. Nevertheless, it is of some interest to count

the number of microscopic states of the new system directly. At the least, this will provide us with another non-trivial test of duality symmetry which, although has been tested in many ways, has not been proven. Another motivation for this is that by learning how to count states of pure D-brane systems in type II string theory on T^6 we may eventually gain some insight into similar counting for D-branes wrapped on various cycles of Calabi-Yau manifolds. Indeed, for type II compactification on Calabi-Yau manifolds, all charges are associated with D-branes wrapped on various cycles of Calabi-Yau manifolds as there are no non-contractible circles and hence no momentum, KK monopole charges or winding numbers of fundamental strings and NS 5-branes. Earlier attempts to count states of pure D-brane systems describing a black hole can be found in [8–10].

Another reason for studying representations of black holes as pure D-brane systems is as follows. One knows on general grounds that supersymmetric black holes in 3+1 dimensions describe an ensemble of states each of which carries strictly zero angular momentum [11, 12] after factoring out the fermion zero mode contribution whose quantization generates the supermultiplet structure. This leads to many non-trivial conjectures about the sign of the index of supersymmetric black holes which have been verified explicitly [13, 14]. However, in microscopic counting of the same system, one often finds BPS states carrying non-zero angular momentum. This does not represent a contradiction between microscopic and macroscopic results, since only the index, and not the detailed information about angular momentum, is protected as we go from the weak coupling regime where microscopic calculation is valid, to the strong coupling regime where the black hole computation is valid. Nevertheless, one could ask if there is a duality frame in which the detailed information about the angular momentum in the microscopic and macroscopic descriptions matches. Since in the macroscopic description all black holes carry zero angular momentum, in the microscopic description this will demand that all states are singlets under the $SU(2)$ rotation group. Recent analysis of some microstates of $\mathcal{N} = 2$ supersymmetric black holes revealed that when we describe them as D-branes wrapped on certain internal cycles of Calabi-Yau manifolds we indeed get exactly zero angular momentum for the microstates of single centered black holes [15–17]. Assuming this to be a general phenomenon led to the conjectured Coulomb branch formula for computing the spectrum of quiver quantum mechanics and of general systems of multicentered black holes [18–20].

Now in $\mathcal{N} = 2$ supersymmetric string theories, the above analysis is made complicated due to the fact that the index receives a contribution from both single and multi-centered black holes. Since the latter do not necessarily carry zero angular momentum, we need to carefully subtract the contribution from multi-centered black holes before we can verify that D-brane microstates representing single centered black holes carry zero angular momentum. This can be done [18, 21], and was used in the analysis of [18–20]. However, in type II string theory on T^6 , which has $\mathcal{N} = 8$ supersymmetry, the multi-centered black holes do not contribute to the index, and hence we expect that only single centered black holes will survive at a generic point in the moduli space of the theory [22]. Generalization of the observations in $\mathcal{N} = 2$ supersymmetric string theories made above would then suggest that representing a supersymmetric black hole in type II on T^6 as a system carrying only RR charges associated with various D-brane sources, we may get a system whose microstates

would have strictly zero angular momentum after factoring out the goldstino fermion modes whose quantization generates the supermultiplet structure. Now, after factoring out these fermionic zero modes and the bosonic zero modes associated with various translational symmetries, the BPS states of the D-brane system correspond to the cohomology of the moduli space of classical solutions of the world-line theory of the system, with the space-time rotation group acting as the Lefschetz $SU(2)$ action on the cohomology [15, 23]. This shows that in order to get only zero angular momentum states, all states must come from the middle cohomology. Since any compact manifold has a non-trivial 0-form and a top form, the only way that a manifold can have only middle cohomology is if it becomes zero dimensional, i.e. a collection of points.¹ Verification of this conjecture is another motivation for our analysis.

In this paper we shall analyze a pure D-brane system in type II theory compactified on T^6 that is dual to the system for which the microscopic result is known, and test the result by direct computation of the microscopic index of the D-brane system. We introduce the system in section 2, and derive its world-line theory for the lowest possible values of the charges in section 3. In section 4 we explicitly count the index of supersymmetric states of this system. This is shown to reduce to counting the number of independent solutions of a set of polynomial equations — a problem that can be easily solved. We find that the solution contains a set of isolated points provided we work at a generic point in the moduli space of the theory parametrized by constant background values of the metric and 2-form fields along the internal torus. Hence, at least in this example, the microstates carry strictly zero angular momentum in agreement with the macroscopic results. In section 5 we briefly discuss possible generalization of our analysis to cases where we replace each D-brane of the system described in section 2 by a stack of parallel D-branes. We conclude with a discussion of our results in section 6. In appendix A we derive the relation between some of the parameters of the D-brane world-volume theory and the background values of the metric and 2-form field along T^6 . In appendix B we describe the chain of dualities that relate the system under consideration to the system analyzed in [7]. Finally, in appendix C we give explicit solutions to the polynomial equations which appear in the analysis of section 4.

2 The system

Let us consider for definiteness a type IIA string theory on T^6 labelled by the coordinates x^4, \dots, x^9 and in this theory we take a system containing N_1 D2-branes wrapped along 4-5 directions, N_2 D2-branes wrapped along 6-7 directions, N_3 D2-branes wrapped along 8-9 directions, N_4 D6-branes wrapped along 4-5-6-7-8-9 directions and N_5 D4-branes along 6-7-8-9 directions. By a series of duality transformations described in appendix B, this configuration is related to a system of N_1 KK monopoles associated with the 4-direction, $-N_2$ units of momentum along the 5-direction, N_3 D1-branes along the 5-direction, N_4 D5-branes along 5-6-7-8-9 directions and $-N_5$ units of momentum along the 4-direction. The microscopic index of this system was computed explicitly in [7] for $N_1 = 1$. By a further series of U-duality transformations reviewed e.g. in [24], this system can be mapped to a system in type IIA string theory on T^6 with only NS-NS sector charges, containing

¹We thank Boris Pioline for discussion on this point.

$-N_2$ units of momentum along the 5-direction, N_1 fundamental strings wound along the 5-direction, N_4 KK monopoles associated with the 4-direction, $-N_3$ NS 5-branes wrapped along 5-6-7-8-9 directions and N_5 NS 5-branes along 4-6-7-8-9 directions. In the notation of [24], the electric charge vector Q and magnetic charge vector P for this state in the latter description are represented as

$$Q = \begin{pmatrix} 0 \\ -N_2 \\ 0 \\ -N_1 \end{pmatrix}, \quad P = \begin{pmatrix} N_3 \\ N_5 \\ N_4 \\ 0 \end{pmatrix}. \quad (2.1)$$

The T-duality invariant inner product matrix between charges was given by $\begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}$. With this we get

$$Q^2 = 2 N_1 N_2, \quad P^2 = 2 N_3 N_4, \quad Q \cdot P = -N_1 N_5. \quad (2.2)$$

We also define

$$\begin{aligned} \ell_1 &= \gcd\{Q_i P_j - Q_j P_i\} = \gcd\{N_1 N_3, N_1 N_4, N_2 N_3, N_2 N_4, N_5 N_1\}, \\ \ell_2 &= \gcd\{Q^2/2, P^2/2, Q \cdot P\} = \gcd\{N_1 N_2, N_3 N_4, N_1 N_5\}. \end{aligned} \quad (2.3)$$

We shall consider configurations for which

$$\gcd\{\ell_1, \ell_2\} = 1, \quad i.e. \quad \gcd\{N_1 N_3, N_1 N_4, N_2 N_3, N_2 N_4, N_1 N_2, N_3 N_4, N_1 N_5\} = 1. \quad (2.4)$$

In this case, following [25, 26] one can show that there is a further series of duality transformations that map this system to one with $N_1 = 1$ [27] for which the microscopic index is known from the analysis of [7]. Expressed in terms of the more general set of variables (N_1, \dots, N_5) , the result for the BPS index for this system, which in this case corresponds to the 14-th helicity supertrace B_{14} , takes the form [28]

$$B_{14} = (-1)^{Q \cdot P + 1} \sum_{s|\ell_1 \ell_2} s \widehat{c}(\Delta/s^2), \quad \Delta \equiv Q^2 P^2 - (Q \cdot P)^2 = 4 N_1 N_2 N_3 N_4 - (N_1 N_5)^2, \quad (2.5)$$

where $\widehat{c}(u)$ is defined through the relation [6, 7]

$$-\vartheta_1(z|\tau)^2 \eta(\tau)^{-6} \equiv \sum_{k,l} \widehat{c}(4k - l^2) e^{2\pi i(k\tau + lz)}. \quad (2.6)$$

$\vartheta_1(z|\tau)$ and $\eta(\tau)$ are respectively the odd Jacobi theta function and the Dedekind eta function.

In this paper we shall analyze the simplest of these configurations with

$$N_1 = N_2 = N_3 = N_4 = 1, \quad N_5 = 0. \quad (2.7)$$

For this, (2.6) predicts

$$B_{14} = 12. \quad (2.8)$$

We shall verify this by direct counting of microstates of the D-brane system.

3 The low energy dynamics of the D-brane system

The combined system of four D-branes that we have introduced in section 2 with the choice of N_i 's given in (2.7) preserves 4 out of the 32 supersymmetries. This is equivalent to $\mathcal{N} = 1$ supersymmetry in 3+1 dimensions. Since we are dealing with a quantum mechanical system, we can effectively regard this as an $\mathcal{N} = 1$ supersymmetric theory in 3+1 dimensions, dimensionally reduced to 0+1 dimensions. Thus we can use the $\mathcal{N} = 1$ superfield formalism, but ignore all spatial derivatives and integration over spatial directions while writing the action. We shall follow the normalization conventions of [29] in constructing this action.

Since the four D-branes we have are related to each other by T-duality, each of them individually has the same low energy theory given by the dimensional reduction of $\mathcal{N} = 4$ supersymmetric U(1) gauge theory from 3+1 to 0+1 dimensions. We begin with one of the four different D-branes. In the language of $\mathcal{N} = 1$ supersymmetry in 3+1 dimensions, each D-brane has one U(1) vector superfield V and three chiral superfields Φ_1, Φ_2, Φ_3 . A vector multiplet, after dimensional reduction to 0+1 dimensions, has three scalars corresponding to three spatial components of the gauge field and a gauge field A_0 . We can use the gauge $A_0 = 0$ and interpret the three scalars as the coordinates giving the location of the D-brane along the three non-compact directions. We shall denote these three scalars by X_1, X_2, X_3 . The three chiral multiplets $\{\Phi_i\}$ contains three complex scalars $\{\Phi_i\}$.² These complex scalars give the coordinates or Wilson lines along $x^4 + ix^5, x^6 + ix^7$ and $x^8 + ix^9$ directions respectively. For example, for the D6-brane all three complex scalars correspond to Wilson lines, while for the D2-brane wrapped along the 4-5 directions, Φ_1 corresponds to a Wilson line along $x^4 + ix^5$ but Φ_2 and Φ_3 correspond to positions of the brane along $x^6 + ix^7$ and $x^8 + ix^9$ respectively. Finally, we shall use a superscript (k) to label the four different D-branes, with $k = 1, 2, 3$ corresponding to D2-branes wrapped along the 4-5, 6-7 and 8-9 directions and $k = 4$ corresponding to the D6-brane along 4-5-6-7-8-9 directions. Besides these fields, for every pair of D-branes labelled by (k, ℓ) we have two chiral superfields $Z^{(k\ell)}$ and $Z^{(\ell k)}$ arising from open strings stretched between the two D-branes. They carry respectively 1 and -1 units of charge under the vector superfield $V^{(k)}$ and -1 and 1 units of charge under the vector superfield $V^{(\ell)}$.

We shall now write down the action involving these fields. To begin with we shall assume that the six circles of T^6 are orthonormal to each other, with each circle having radius $\sqrt{\alpha'}$ and that there is no background 2-form field along T^6 . From now on, we shall set $\alpha' = 1$. In this case the action takes the form

$$S_{\text{kinetic}} + \int dx^0 \left[\int d^4\theta \sum_{k=1}^4 \sum_{\substack{\ell=1 \\ \ell \neq k}}^4 \left\{ \bar{Z}^{(k\ell)} e^{2V^{(\ell)} - 2V^{(k)}} Z^{(k\ell)} \right\} + \int d^2\theta \mathcal{W} + \int d^2\bar{\theta} \bar{\mathcal{W}} \right], \quad (3.1)$$

where S_{kinetic} denotes the kinetic terms for the vector superfields $V^{(k)}$ and the gauge neutral chiral superfields $\Phi_i^{(k)}$. These have the standard form and will not be written down explicitly. The superpotential \mathcal{W} has two different components. The first component describes

²Following usual notation, we shall use the same symbol to denote a superfield and its scalar component.

the coupling of the superfields $\Phi^{(k)}$ to $Z^{(k\ell)}$ and takes the form

$$\mathcal{W}_1 = \sqrt{2} \left[\sum_{k,\ell,m=1}^3 \varepsilon^{k\ell m} \Phi_m^{(k)} Z^{(k\ell)} Z^{(\ell k)} + \sum_{k=1}^3 \left(\Phi_k^{(k)} - \Phi_k^{(4)} \right) Z^{(4k)} Z^{(k4)} \right], \quad (3.2)$$

where $\varepsilon^{k\ell m}$ is the totally antisymmetric symbol with $\varepsilon^{123} = 1$. The second component describes the cubic self-coupling between the $Z^{(k\ell)}$'s and takes the form

$$\mathcal{W}_2 = \sqrt{2} C \sum_{\substack{k,\ell,m=1 \\ k < \ell, m; \ell \neq m}}^4 Z^{(k\ell)} Z^{(\ell m)} Z^{(mk)}, \quad (3.3)$$

where C is a constant whose value can be computed in principle by analyzing the coupling between open strings stretched between different branes, but we shall not need it for our analysis. The sum over k, ℓ, m runs over all distinct values of k, ℓ and m which are not related by cyclic permutations of (k, ℓ, m) . There could also be gauge invariant quartic and higher order terms involving the $Z^{(k\ell)}$'s, but as we shall see, these can be ignored in our analysis.

So far we have assumed that background metric along T^6 is diagonal and that there are no background 2-form fields. We shall now study the effect to switching on small background values of the off-diagonal components of the metric and 2-form fields. As reviewed in appendix A, this has two effects. First it introduces Fayet-Iliopoulos (FI) term with coefficient $c^{(k)}$ for each of the four vector superfields, satisfying

$$\sum_{k=1}^4 c^{(k)} = 0. \quad (3.4)$$

Second, it generates a linear term in the superpotential of the form

$$\mathcal{W}_3 = \sqrt{2} \left[\sum_{k,\ell,m=1}^3 c^{(k\ell)} \varepsilon^{k\ell m} \Phi_m^{(k)} + \sum_{k=1}^3 c^{(k4)} \left(\Phi_k^{(k)} - \Phi_k^{(4)} \right) \right],$$

$$c^{(\ell k)} = c^{(k\ell)} \quad \text{for } 1 \leq k < \ell \leq 4. \quad (3.5)$$

Explicit expressions for $c^{(k)}$ and $c^{(k\ell)}$ for $1 \leq k < \ell \leq 4$ in terms of the off-diagonal components of the metric and 2-form fields have also been given in appendix A.

Let us now write down the potential involving the scalar fields derived from the above action. This consists of three pieces. The first comes from the usual quartic coupling between the gauge field components $X_i^{(k)}$ and the charged scalars $Z^{(k\ell)}$ and takes the form

$$V_{\text{gauge}} = \sum_{i=1}^3 \sum_{k=1}^4 \sum_{\substack{\ell=1 \\ \ell \neq k}}^4 (X_i^{(k)} - X_i^{(\ell)}) (X_i^{(k)} - X_i^{(\ell)}) \left(\bar{Z}^{(k\ell)} Z^{(k\ell)} + \bar{Z}^{(\ell k)} Z^{(\ell k)} \right), \quad (3.6)$$

where ‘bar’ denotes complex conjugation. The second component of the potential is the D-term contribution. This takes the form

$$V_D = \frac{1}{2} \sum_{k=1}^4 \left\{ \sum_{\substack{\ell=1 \\ \ell \neq k}}^4 \left(\bar{Z}^{(k\ell)} Z^{(k\ell)} - \bar{Z}^{(\ell k)} Z^{(\ell k)} \right) - c^{(k)} \right\}^2. \quad (3.7)$$

The third component is the F-term potential which takes the form

$$V_F = \sum_{k=1}^4 \sum_{i=1}^3 \left| \frac{\partial W}{\partial \Phi_i^{(k)}} \right|^2 + \sum_{k=1}^4 \sum_{\substack{\ell=1 \\ \ell \neq k}}^4 \left| \frac{\partial W}{\partial Z^{(k\ell)}} \right|^2. \quad (3.8)$$

For finding a supersymmetric configuration we have to look for configurations with vanishing potential. Since the potential is a sum of squares, this requires setting each of these terms to zero. In section 4 we shall look for solutions to these conditions.

Note that the potential has the following flat directions

$$\begin{aligned} \Phi_m^{(k)} &\rightarrow \Phi_m^{(k)} + \xi_m, & \text{for } 1 \leq k \leq 3, \quad k \neq m; \quad 1 \leq m \leq 3, \\ \Phi_k^{(k)} &\rightarrow \Phi_k^{(k)} + \zeta_k, & \Phi_k^{(4)} \rightarrow \Phi_k^{(4)} + \zeta_k, & \text{for } 1 \leq k \leq 3, \\ X_i^{(k)} &\rightarrow X_i^{(k)} + a_i, & \text{for } 1 \leq k \leq 4, \quad 1 \leq i \leq 3, \end{aligned} \quad (3.9)$$

where ξ_m and ζ_k are arbitrary complex numbers and a_i are arbitrary real numbers. The a_i 's represent overall translation of the system along the non-compact directions. The symmetries generated by ξ_m and ζ_k imply that the potential has six complex flat directions.³ This corresponds to six exactly massless chiral multiplets. Since each chiral multiplet contains a Weyl fermion in 3+1 dimensions which has four real components, we have altogether $6 \times 4 = 24$ real fermion zero modes after dimensional reduction to 0+1 dimensions. The vector superfield $\sum_{k=1}^4 V^{(k)}$ also decouples from the action, reflecting the symmetry parametrized by the a_i 's. The Majorana fermion belonging to this multiplet gives 4 more fermion zero modes. Thus altogether we have $24 + 4 = 28$ fermion zero modes. These are the Goldstino modes associated with supersymmetry breaking; since a 1/8 BPS black hole in $\mathcal{N} = 8$ supersymmetric string theory preserves 4 out of 32 supersymmetries, we expect $32 - 4 = 28$ broken supersymmetries. Quantization of these 28 fermion zero modes gives the 2^{14} fold degenerate supermultiplet which is the right degeneracy for a 1/8 BPS state in a theory with 32 supersymmetries.

4 Supersymmetric solutions

We shall now look for configurations preserving supersymmetry, i.e. configurations which make the potential vanish. As noted below (3.8), this requires setting each term in V_{gauge} , V_D and V_F to zero. Furthermore, due to the $U(1)^4$ gauge symmetry of the original theory, we need to classify solutions up to equivalence relations under these $U(1)$ gauge symmetries:

$$Z^{(k\ell)} \rightarrow \exp \left[i \left(\theta^{(k)} - \theta^{(\ell)} \right) \right] Z^{(k\ell)}, \quad (4.1)$$

where $\theta^{(k)}$ for $1 \leq k \leq 4$ are the gauge transformation parameters. Note that the overall $U(1)$ — obtained by setting all the $\theta^{(k)}$'s equal — acts trivially on the $Z^{(k\ell)}$'s. Furthermore,

³These directions are all compact since they are associated with translations along T^6 and the dual torus \tilde{T}^6 . Thus the quantization of the zero modes associated with these bosonic flat directions does not cause any problem and gives a unique zero energy ground state.

since we have fixed $A_0^{(k)} = 0$ gauge, we need to demand equivalence only under the subgroup of the gauge group that preserves this gauge condition. This leaves us with the global part of the gauge group, labelled by time independent $\theta^{(k)}$'s.

We begin by examining the equations $\partial\mathcal{W}/\partial\Phi_i^{(k)} = 0$ for $1 \leq k \leq 4$ and $1 \leq i \leq 3$. Using (3.2), (3.5) we see that this gives

$$Z^{(k\ell)} Z^{(\ell k)} = -c^{(k\ell)} \quad \text{for } 1 \leq k < \ell \leq 4. \quad (4.2)$$

It follows from this that as long as the $c^{(k\ell)}$ are non-zero for every k, ℓ in the range $1 \leq k < \ell \leq 4$, none of the $Z^{(k\ell)}$'s can vanish. Eq. (3.6) now gives

$$X_i^{(k)} = 0 \quad \text{for } 1 \leq k \leq 4, \quad 1 \leq i \leq 3, \quad (4.3)$$

up to the translation symmetry parametrized by the constants a_i in eq. (3.9). Next we consider the $\partial\mathcal{W}/\partial Z^{(k\ell)} = 0$ equations. This gives

$$\begin{aligned} \sum_{m=1}^3 \varepsilon^{k\ell m} \left(\Phi_m^{(k)} - \Phi_m^{(\ell)} \right) Z^{(\ell k)} + C \sum_{\substack{m=1 \\ m \neq k, \ell}}^4 Z^{(\ell m)} Z^{(mk)} &= 0 \quad \text{for } 1 \leq k, \ell \leq 3, \\ \left(\Phi_k^{(k)} - \Phi_k^{(4)} \right) Z^{(k4)} + C \sum_{\substack{\ell=1 \\ \ell \neq k}}^3 Z^{(k\ell)} Z^{(\ell 4)} &= 0 \quad \text{for } 1 \leq k \leq 3, \\ \left(\Phi_k^{(k)} - \Phi_k^{(4)} \right) Z^{(4k)} + C \sum_{\substack{m=1 \\ m \neq k}}^3 Z^{(4m)} Z^{(mk)} &= 0 \quad \text{for } 1 \leq k \leq 3. \end{aligned} \quad (4.4)$$

These equations serve two purposes. First they determine the combinations

$$\Phi_m^{(k)} - \Phi_m^{(\ell)} \quad \text{for } 1 \leq k, \ell, m \leq 3, \quad k, \ell, m \text{ distinct,} \quad \text{and} \quad \Phi_k^{(k)} - \Phi_k^{(4)} \quad \text{for } 1 \leq k \leq 3, \quad (4.5)$$

in terms of the $Z^{(k\ell)}$'s. This gives 6 linear combinations of the 12 complex scalars $\Phi_i^{(k)}$. The rest of the $\Phi_i^{(k)}$'s are associated with flat directions and hence remain undetermined. Second they give the following relations among the $Z^{(k\ell)}$'s:

$$\begin{aligned} Z^{(k\ell)} \sum_{\substack{m=1 \\ m \neq k, \ell}}^4 Z^{(\ell m)} Z^{(mk)} &= Z^{(\ell k)} \sum_{\substack{m=1 \\ m \neq k, \ell}}^4 Z^{(km)} Z^{(m\ell)} \quad \text{for } 1 \leq k, \ell \leq 3, \\ Z^{(4k)} \sum_{\substack{\ell=1 \\ \ell \neq k}}^3 Z^{(k\ell)} Z^{(\ell 4)} &= Z^{(k4)} \sum_{\substack{m=1 \\ m \neq k}}^3 Z^{(4m)} Z^{(mk)} \quad \text{for } 1 \leq k \leq 3. \end{aligned} \quad (4.6)$$

Finally let us turn to the D-term constraints. It is well known that the effect of the D-term constraints together with quotienting by the U(1) gauge groups is to convert the space spanned by the coordinates $Z^{(k\ell)}$ to a toric variety. This is parametrized by the coordinates $Z^{(k\ell)}$ modded out by the complexified U(1) gauge groups *after removing appropriate submanifolds of complex codimension ≥ 1 from the space spanned by the $Z^{(k\ell)}$'s.*

These submanifolds are obtained by setting one or more $Z^{(k\ell)}$'s to zero, and depend on the FI parameters $c^{(k)}$. However, since we have seen that the F-term constraints force all the $Z^{(k\ell)}$'s to be non-zero, removal of these complex submanifolds has no effect on the final solutions.⁴ Thus, we can proceed by parametrizing the variety by an appropriate set of gauge invariant polynomials and forget about the D-term constraints. Since to start with there are $4 \times 3 = 12$ independent $Z^{(k\ell)}$'s, and we quotient by a $U(1)^3$ gauge group — the overall $U(1)$ having trivial action on all the $Z^{(k\ell)}$'s — we need $12 - 3 = 9$ independent gauge invariant coordinates. We take them to be

$$\begin{aligned} u_1 &\equiv Z^{(12)} Z^{(21)}, & u_2 &\equiv Z^{(23)} Z^{(32)}, & u_3 &\equiv Z^{(31)} Z^{(13)}, \\ u_4 &\equiv Z^{(14)} Z^{(41)}, & u_5 &\equiv Z^{(24)} Z^{(42)}, & u_6 &\equiv Z^{(34)} Z^{(43)}, \\ u_7 &\equiv Z^{(12)} Z^{(24)} Z^{(41)}, & u_8 &\equiv Z^{(13)} Z^{(34)} Z^{(41)}, & u_9 &\equiv Z^{(23)} Z^{(34)} Z^{(42)}. \end{aligned} \tag{4.7}$$

We now note that (4.2) fixes u_1, \dots, u_6 completely. Thus, the only remaining variables are u_7, u_8, u_9 and the equations to be solved are given in (4.6). These actually give three independent equations

$$\begin{aligned} Z^{(23)} Z^{(31)} Z^{(12)} + Z^{(23)} Z^{(34)} Z^{(42)} &= Z^{(32)} Z^{(21)} Z^{(13)} + Z^{(32)} Z^{(24)} Z^{(43)}, \\ Z^{(24)} Z^{(41)} Z^{(12)} + Z^{(24)} Z^{(43)} Z^{(32)} &= Z^{(42)} Z^{(21)} Z^{(14)} + Z^{(42)} Z^{(23)} Z^{(34)}, \\ Z^{(34)} Z^{(41)} Z^{(13)} + Z^{(34)} Z^{(42)} Z^{(23)} &= Z^{(43)} Z^{(31)} Z^{(14)} + Z^{(43)} Z^{(32)} Z^{(24)}. \end{aligned} \tag{4.8}$$

Defining

$$m_{k\ell} = m_{\ell k} = -c^{(k\ell)} \quad \text{for } 1 \leq k < \ell \leq 4 \tag{4.9}$$

and using the solutions for u_1, \dots, u_6 given in (4.2), eqs.(4.8) can be expressed as

$$\begin{aligned} u_7 u_8^{-1} &= m_{24} \left(\frac{m_{24} m_{23} m_{12} u_9^{-1} - u_7 u_8^{-1} u_9}{m_{31} u_7 u_8^{-1} u_9 - m_{23} m_{24}^2 m_{34} u_9^{-1}} \right), \\ u_7 u_9 &= m_{24} \left(\frac{m_{12} m_{14} u_9 - m_{34} m_{23} u_7}{u_7 - u_9} \right), \\ u_8 u_9^{-1} &= \left(\frac{m_{34} m_{31} m_{14} - u_8 u_9}{u_8 u_9 - m_{34} m_{23} m_{24}} \right), \end{aligned} \tag{4.10}$$

respectively. The solutions to the system (4.10) are given in table 1 of appendix C. The important point to note from table 1 is that there are 12 distinct solutions. This shows that there are 12 supersymmetric ground states, in perfect agreement with the prediction (2.8)

⁴Put another way, for a generic toric variety, if some equation is given in terms of homogeneous coordinates, it may have solutions in more than one patch. Thus, when we translate the equations in terms of coordinates of any single patch (which does not cover the whole variety) and look for the solutions, we always have the risk of not having all the solutions. Fortunately this is not the case here. If we closely look into what are the regions that are not covered by an arbitrary single patch, we see that these are the regions where some of the coordinates vanish. But our $Z^{(k\ell)}$'s cannot vanish due to the constraint $Z^{(k\ell)} Z^{(\ell k)} = m_{k\ell}$. Thus, although such regions exist in the toric variety, they are not part of the solution of our equations. Hence it is enough to work in a single patch only, which is what we do.

from the dual description. Furthermore since the moduli space of solutions is zero dimensional, all the solutions carry zero angular momentum after factoring out the contribution of the goldstino fermion zero modes. This is in agreement with the prediction from the black hole side.

It is clear from the form of the potential as well as the solutions given in table 1 that under a uniform scaling of all the $c^{(k)}$'s and $c^{(k\ell)}$'s by a real parameter λ , the $Z^{(k\ell)}$'s and $\Phi_m^{(k)}$'s at the solution (except the ones associated with flat directions) scale as $\lambda^{1/2}$. Thus by taking λ to be small we can ensure that each $Z^{(k\ell)}$ and $\Phi_m^{(k)}$ at the solution is small. In this case the contributions from the quartic and higher order terms in the superpotential are small compared to the cubic terms that we have included. This justifies our ignoring such terms for studying these solutions. This also justifies our ignoring the fact that $\Phi_i^{(k)} - \Phi_i^{(\ell)}$ for $1 \leq k, \ell \leq 3$ and $\Phi_k^{(k)} - \Phi_k^{(4)}$ for $1 \leq k \leq 3$ are periodic variables while solving the eqs. (4.4).

Note however that we have not ruled out existence of solutions where $\Phi_i^{(k)} - \Phi_j^{(k)}$ and $Z^{(k\ell)}$'s are of order unity measured in the string scale. In such cases we must take into account possible higher order terms in the superpotential, and must also include the effect of $\Phi_i^{(k)}$'s being periodic variables so that we have to include in our analysis also open string states which wind around the various circles on their way from one D-brane to another. In other words, full stringy dynamics is needed for examining the existence of these states. Our experience with BPS state counting tells us however that the BPS states arise only from low energy fluctuations on the branes and hence it seems unlikely that there will be new BPS states from the stringy configurations of the type described above.

5 Non-abelian generalization

In this section we shall generalize the analysis of section 3 to the case where some of the stacks have more than one brane, i.e. the N_i 's introduced in section 2 are not all equal to 1. We shall focus on the scalar fields and their potential since this is what is needed for the counting of supersymmetric solutions.

We begin with a discussion of how the scalar degrees of freedom change in this case. First of all, the complex scalars $\Phi_i^{(k)}$ and the real scalars $X_i^{(k)}$ become $N_k \times N_k$ hermitian matrices transforming in the adjoint representation of $U(N_k)$. On the other hand, the complex scalar $Z^{(k\ell)}$ becomes $N_k \times N_\ell$ complex matrix transforming in the (N_k, \bar{N}_ℓ) representation of $U(N_k) \times U(N_\ell)$.

Let us now describe the modification of the potential. The superpotential \mathcal{W}_1 given in (3.2) is generalized to

$$\mathcal{W}_1 = \sqrt{2} \left[\sum_{k,\ell,m=1}^3 \varepsilon^{k\ell m} \text{Tr} \left(\Phi_m^{(k)} Z^{(k\ell)} Z^{(\ell k)} \right) + \sum_{k=1}^3 \text{Tr} \left(Z^{(4k)} \Phi_k^{(k)} Z^{(k4)} \right) - \sum_{k=1}^3 \text{Tr} \left(\Phi_k^{(4)} Z^{(4k)} Z^{(k4)} \right) \right]. \tag{5.1}$$

The generalization of (3.3) takes the form

$$\mathcal{W}_2 = \sqrt{2} C \left[\sum_{\substack{k,\ell,m=1 \\ k<\ell,m;\ell\neq m}}^4 \text{Tr} \left(Z^{(k\ell)} Z^{(\ell m)} Z^{(mk)} \right) \right]. \quad (5.2)$$

The generalization of (3.5) is

$$\mathcal{W}_3 = \sqrt{2} \left[\sum_{k,\ell,m=1}^3 c^{(k\ell)} \varepsilon^{k\ell m} N_\ell \text{Tr} \left(\Phi_m^{(k)} \right) + \sum_{k=1}^3 c^{(k4)} \left[N_4 \text{Tr} \left(\Phi_k^{(k)} \right) - N_k \text{Tr} \left(\Phi_k^{(4)} \right) \right] \right]. \quad (5.3)$$

There is also an additional superpotential

$$\mathcal{W}_4 = -\sqrt{2} \sum_{k=1}^4 \text{Tr} \left(\Phi_1^{(k)} \left[\Phi_2^{(k)}, \Phi_3^{(k)} \right] \right). \quad (5.4)$$

(3.6) generalizes to

$$\begin{aligned} V_{\text{gauge}} = & \sum_{k=1}^4 \sum_{\substack{\ell=1 \\ \ell \neq k}}^4 \sum_{i=1}^3 \text{Tr} \left[\left(X_i^{(k)} Z^{(k\ell)} - Z^{(k\ell)} X_i^{(\ell)} \right)^\dagger \left(X_i^{(k)} Z^{(k\ell)} - Z^{(k\ell)} X_i^{(\ell)} \right) \right] \\ & + \sum_{k=1}^4 \sum_{j=1}^3 \text{Tr} \left([X_i^{(k)}, \Phi_j^{(k)}]^\dagger [X_i^{(k)}, \Phi_j^{(k)}] \right) + \frac{1}{4} \sum_{k=1}^4 \sum_{i,j=1}^3 \text{Tr} \left([X_i^{(k)}, X_j^{(k)}]^\dagger [X_i^{(k)}, X_j^{(k)}] \right). \end{aligned} \quad (5.5)$$

Finally, the D-term potential (3.7) is generalized to

$$V_D = \frac{1}{2} \sum_{k=1}^4 \text{Tr} \left[\left(\sum_{\substack{\ell=1 \\ \ell \neq k}}^4 Z^{(k\ell)} Z^{(k\ell)\dagger} - \sum_{\substack{\ell=1 \\ \ell \neq k}}^4 Z^{(\ell k)\dagger} Z^{(\ell k)} + \sum_{i=1}^3 [\Phi_i^{(k)}, \Phi_i^{(k)\dagger}] - c^{(k)} I_{N_k} \right)^2 \right], \quad (5.6)$$

where I_{N_k} denotes $N_k \times N_k$ identity matrix. The FI parameters $c^{(k)}$ now satisfy

$$\sum_{k=1}^4 c^{(k)} N_k = 0. \quad (5.7)$$

The coefficients $c^{(k\ell)}$ and $c^{(k)}$ are to be chosen so that they reproduce the masses of the $Z^{(k\ell)}$'s correctly. The equations take the form of (A.4) and (A.5) with identical right hand sides, but the left hand sides are different since the masses of $Z^{(k\ell)}$'s expressed in terms of $c^{(k)}$'s and $c^{(k\ell)}$'s have additional dependence on the N_k 's.

The potential given above has a shift symmetry generalizing (3.9)

$$\begin{aligned} \Phi_m^{(k)} & \rightarrow \Phi_m^{(k)} + \xi_m I_{N_k}, & \text{for } 1 \leq k \leq 3, \quad k \neq m; \quad 1 \leq m \leq 3, \\ \Phi_k^{(k)} & \rightarrow \Phi_k^{(k)} + \zeta_k I_{N_k}, & \Phi_k^{(4)} & \rightarrow \Phi_k^{(4)} + \zeta_k I_{N_4}, \quad \text{for } 1 \leq k \leq 3, \\ X_i^{(k)} & \rightarrow X_i^{(k)} + a_i I_{N_k}, & \text{for } 1 \leq i \leq 3. \end{aligned} \quad (5.8)$$

This generates six complex translations along compact directions and three real translations along the non-compact directions.

The $\partial\mathcal{W}/\partial\Phi_m^{(k)} = 0$ equations give

$$\begin{aligned} Z^{(k\ell)} Z^{(\ell k)} &= -c^{(k\ell)} N_\ell I_{N_k} + [\Phi_k^{(k)}, \Phi_\ell^{(k)}] \quad \text{for } 1 \leq k, \ell \leq 3, \\ Z^{(k4)} Z^{(4k)} &= -c^{(k4)} N_4 I_{N_k} + \sum_{\ell, m=1}^3 \varepsilon^{k\ell m} \Phi_\ell^{(k)} \Phi_m^{(k)}, \quad 1 \leq k \leq 3, \\ Z^{(4k)} Z^{(k4)} &= -c^{(k4)} N_k I_{N_4} - \sum_{\ell, m=1}^3 \varepsilon^{k\ell m} \Phi_\ell^{(4)} \Phi_m^{(4)}, \quad 1 \leq k \leq 3, \end{aligned} \tag{5.9}$$

generalizing (4.2). The $\partial\mathcal{W}/\partial Z^{(k\ell)}$ equations give

$$\begin{aligned} \sum_{m=1}^3 \varepsilon^{k\ell m} \left(Z^{(\ell k)} \Phi_m^{(k)} - \Phi_m^{(\ell)} Z^{(\ell k)} \right) + C \sum_{\substack{m=1 \\ m \neq k, \ell}}^4 Z^{(\ell m)} Z^{(mk)} &= 0 \quad \text{for } 1 \leq k, \ell \leq 3, \\ \left(\Phi_k^{(k)} Z^{(k4)} - Z^{(k4)} \Phi_k^{(4)} \right) + C \sum_{\substack{\ell=1 \\ \ell \neq k}}^3 Z^{(k\ell)} Z^{(\ell 4)} &= 0 \quad \text{for } 1 \leq k \leq 3, \\ \left(Z^{(4k)} \Phi_k^{(k)} - \Phi_k^{(4)} Z^{(4k)} \right) + C \sum_{\substack{m=1 \\ m \neq k}}^3 Z^{(4m)} Z^{(mk)} &= 0 \quad \text{for } 1 \leq k \leq 3, \end{aligned} \tag{5.10}$$

generalizing (4.4).

It seems reasonable to assume that up to the translation symmetry described in the last line of (5.8), all the $X_i^{(k)}$'s vanish at the zeroes of the potential since this makes all the terms in V_{gauge} vanish. This will also make the classical bound state have zero size in the non-compact directions. Furthermore, the effect of D-term constraints is to take the quotient of the space of solutions to (5.9), (5.10) by complexified $\prod_{k=1}^4 \text{U}(N_k)$ gauge transformations. Let \mathcal{M} be the space of gauge inequivalent solutions to (5.9), (5.10) after factoring out the zero mode directions associated with the shift symmetry (5.8). The number of supersymmetric states (or more precisely the index B_{14}) will be given by the Euler number of \mathcal{M} . Thus, duality symmetry of string theory predicts that

$$\chi(\mathcal{M}) = -\hat{c}(4N_1 N_2 N_3 N_4), \tag{5.11}$$

where $\hat{c}(u)$ has been defined in (2.6). If \mathcal{M} is zero dimensional, then $\chi(\mathcal{M})$ just counts the number of solutions as in the abelian case. In that case all the microstates would carry strictly zero angular momentum after factoring out the contribution due to the goldstino fermion modes.

6 Conclusion

In this paper we have set up the general equations whose solutions describe the BPS states of type II string theory compactified on T^6 carrying only RR charges. We have been able

to solve the equations explicitly when the charges take the lowest possible values. The result is in perfect agreement with the counting of the same states in a U-dual description.

Admittedly this is only a small beginning of the much more ambitious project. Nevertheless even at this level our analysis provides a non-trivial test of duality symmetry, since the counting leading to the magic number 12 is very different from the one that was used to arrive at the formula (2.6). As far as test of black hole entropy is concerned, a black hole carrying charges given in (2.7) has large curvature at the horizon and hence the Bekenstein-Hawking entropy is not expected to agree with $\ln 12$. Nevertheless explicit computation of Bekenstein-Hawking entropy, together with one loop logarithmic corrections [30, 31], give a macroscopic entropy

$$S_{\text{macro}} = \pi\sqrt{\Delta} - 2 \ln \Delta + \dots \simeq 2\pi - 2 \ln 4 \simeq 3.51, \quad (6.1)$$

which is not very different from the microscopic entropy

$$S_{\text{micro}} = \ln 12 = 2.48. \quad (6.2)$$

Thus it is not unreasonable to regard our analysis as the counting of microstates of a black hole made solely of D-branes although the curvature at the horizon of the black hole is large. Just for comparison we note that for $\Delta = 100$, $\ell_1 \ell_2 = 1$ we shall have

$$S_{\text{macro}} = \pi\sqrt{100} - 2 \ln 100 + \dots \simeq 22.2056, \quad S_{\text{micro}} = \ln 3627000060 = 22.012. \quad (6.3)$$

In recent years there has also been progress in computing the macroscopic entropy of these black holes by evaluating the supergravity path integral in the near horizon geometry of the black hole using localization techniques [32–36]. In this approach one regards the $\mathcal{N} = 8$ supersymmetric theory in 3+1 dimensions as an $\mathcal{N} = 2$ supersymmetric theory with vector, hyper, gravitino and Weyl multiplets and evaluates the path integral. Although the arguments are not complete due to the inability to extend the analysis to include hypermultiplets and gravitino multiplets in the language of $\mathcal{N} = 2$ supergravity, if we ignore this problem then the result of localization gives the following result for S_{macro} from the leading saddle point [34]

$$S_{\text{macro}} \simeq \ln \left[\sqrt{2\pi} \Delta^{-7/4} I_{7/2}(\pi\sqrt{\Delta}) \right], \quad (6.4)$$

where $I_n(x)$ is the standard Bessel function. For $\Delta = 4$ this gives

$$S_{\text{macro}} = 2.50, \quad (6.5)$$

which is quite close to the microscopic result (6.2).

Finally we must mention that there is one important aspect of our result which could have significant impact on our understanding of black hole microstates in the future. All the microstates of the D-brane system we have constructed have zero angular momentum after factoring out the contribution due to fermion zero modes, in agreement with the prediction from the black hole side. Although the D-brane and black hole descriptions hold in different regions of the moduli space of the theory, and hence the detailed results

on the angular momentum need not match, the results mentioned above indicate that the D-brane description may be closer to the actual microstates of the black hole than what one might naively expect. This could eventually help us identify the microstates of the black hole in the region of the moduli space where the black hole description is actually valid.

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A Effect of metric and 2-form background

The undeformed system that we started with in section 3 had the six circles of T^6 orthonormal to each other and each having period 2π . The system of D6-D2-D2-D2 branes that we have considered will be described by a set of massless degrees of freedom with action governed by the supersymmetric action with superpotentials \mathcal{W}_1 and \mathcal{W}_2 and all the FI parameters set to zero. In this appendix we shall show that the effect of small off-diagonal components of the metric and the 2-form field is to generate the superpotential \mathcal{W}_3 given in (3.5) and the FI terms labelled by the $c^{(k)}$'s.

Our strategy will be to compute the mass of the open strings stretched between different D-branes in the presence of the deformation and compare it with the mass computed from the deformed action given in section 3. Consider for example the open string stretched between the D2-brane along the 4-5 directions and the D6-brane along the 4-5-6-7-8-9 directions, labelled by the 0+1 dimensional fields $Z^{(14)}$ and $Z^{(41)}$. The quadratic terms involving these fields, computed from the action in section 3, is given by

$$(c^{(4)} - c^{(1)})(|Z^{(14)}|^2 - |Z^{(41)}|^2) + 4c^{(14)*}Z^{(14)}Z^{(41)} + 4c^{(14)}Z^{(14)*}Z^{(41)*}. \tag{A.1}$$

After diagonalization we find the renormalized (mass)² of the fields to be

$$\pm \sqrt{16|c^{(14)}|^2 + (c^{(1)} - c^{(4)})^2}. \tag{A.2}$$

On the other hand, we can calculate the mass of the open string stretched between the D2-brane along the 4-5 directions and the D6-brane along the 4-5-6-7-8-9 directions as follows. First we make a T-duality transformation along the 4-5-directions to convert this into a D0-D4 system. This leaves unchanged the components of the metric and 2-form field along the 6-7-8-9 direction. Now for small values of the background 2-form field, the (mass)² of the open string stretched between the D0-brane and the D4-brane along the 6-7-8-9 directions takes the form [37, 38]

$$\pm \sqrt{\frac{1}{2} \sum_{m,n} b_{mn} \left(b^{mn} + \frac{1}{2} \sum_{p,q} \epsilon^{mnpq} b_{pq} \right)} = \pm \sqrt{(b_{67} + b_{89})^2 + (b_{68} - b_{79})^2 + (b_{69} + b_{78})^2}, \tag{A.3}$$

up to an overall proportionality constant. Here ϵ^{mnpq} denotes the component of the invariant totally anti-symmetric rank 4 tensor along the D4-brane world-volume. Comparing (A.2) and (A.3) we get

$$16 |c^{(14)}|^2 + (c^{(1)} - c^{(4)})^2 = (b_{67} + b_{89})^2 + (b_{68} - b_{79})^2 + (b_{69} + b_{78})^2. \quad (\text{A.4})$$

A similar analysis of open strings stretched between other brane pairs, and comparison with the result derived from the deformed action yields the results

$$\begin{aligned} 16 |c^{(12)}|^2 + (c^{(1)} - c^{(2)})^2 &= (g_{47} + g_{56})^2 + (b_{45} - b_{67})^2 + (g_{46} - g_{57})^2, \\ 16 |c^{(13)}|^2 + (c^{(1)} - c^{(3)})^2 &= (g_{49} + g_{58})^2 + (b_{45} - b_{89})^2 + (g_{48} - g_{59})^2, \\ 16 |c^{(23)}|^2 + (c^{(2)} - c^{(3)})^2 &= (g_{69} + g_{78})^2 + (b_{67} - b_{89})^2 + (g_{68} - g_{79})^2, \\ 16 |c^{(24)}|^2 + (c^{(2)} - c^{(4)})^2 &= (b_{45} + b_{89})^2 + (b_{48} - b_{59})^2 + (b_{49} + b_{58})^2, \\ 16 |c^{(34)}|^2 + (c^{(3)} - c^{(4)})^2 &= (b_{45} + b_{67})^2 + (b_{46} - b_{57})^2 + (b_{47} + b_{56})^2. \end{aligned} \quad (\text{A.5})$$

Note that in the mass formula, only 6 independent combinations of metric components and 9 independent combinations of 2-form field components appear. This gives a total of 15 independent real quantities. On the other hand, in our Lagrangian we have 3 independent FI parameters and six complex parameters $c^{(k\ell)}$. This also gives a total of 15 real parameters. Nevertheless the solutions are not unique since, for example, the left hand sides of eqs. (A.4), (A.5) are insensitive to the phases of the $c^{(k\ell)}$'s. A similar symmetry exists on the right hand side. A possible choice of $c^{(k)}$'s and $c^{(k\ell)}$'s is

$$\begin{aligned} c^{(1)} &= \frac{1}{2} (b_{45} - b_{67} - b_{89}), & c^{(2)} &= \frac{1}{2} (b_{67} - b_{45} - b_{89}), \\ c^{(3)} &= \frac{1}{2} (b_{89} - b_{45} - b_{67}), & c^{(4)} &= \frac{1}{2} (b_{45} + b_{67} + b_{89}), \end{aligned} \quad (\text{A.6})$$

$$\begin{aligned} 16 |c^{(12)}|^2 &= (g_{47} + g_{56})^2 + (g_{46} - g_{57})^2, & 16 |c^{(13)}|^2 &= (g_{49} + g_{58})^2 + (g_{48} - g_{59})^2, \\ 16 |c^{(14)}|^2 &= (b_{68} - b_{79})^2 + (b_{69} + b_{78})^2, & 16 |c^{(23)}|^2 &= (g_{69} + g_{78})^2 + (g_{68} - g_{79})^2, \\ 16 |c^{(24)}|^2 &= (b_{48} - b_{59})^2 + (b_{49} + b_{58})^2, & 16 |c^{(34)}|^2 &= (b_{46} - b_{57})^2 + (b_{47} + b_{56})^2. \end{aligned} \quad (\text{A.7})$$

It is also clear that given a set of $c^{(k)}$'s and $c^{(k\ell)}$'s satisfying $\sum_{k=1}^4 c^{(k)} = 0$ it is always possible to find b_{ij} 's and g_{ij} 's satisfying (A.6), (A.7). For example, if we denote by the subscripts R and I the real and imaginary parts of $c^{(k\ell)}$, then we can invert (A.6), (A.7) as

$$\begin{aligned} b_{45} &= \frac{1}{2} (c^{(1)} - c^{(2)} - c^{(3)} + c^{(4)}), & b_{67} &= \frac{1}{2} (-c^{(1)} + c^{(2)} - c^{(3)} + c^{(4)}), \\ b_{89} &= \frac{1}{2} (-c^{(1)} - c^{(2)} + c^{(3)} + c^{(4)}), \end{aligned} \quad (\text{A.8})$$

$$\begin{aligned} g_{47} + g_{56} &= 4 c_R^{(12)}, & g_{46} - g_{57} &= 4 c_I^{(12)}, & g_{49} + g_{58} &= 4 c_R^{(13)}, & g_{48} - g_{59} &= 4 c_I^{(13)}, \\ b_{68} - b_{79} &= 4 c_R^{(14)}, & b_{69} + b_{78} &= 4 c_I^{(14)}, & g_{69} + g_{78} &= 4 c_R^{(23)}, & g_{68} - g_{79} &= 4 c_I^{(23)}, \\ b_{48} - b_{59} &= 4 c_R^{(24)}, & b_{49} + b_{58} &= 4 c_I^{(24)}, & b_{46} - b_{57} &= 4 c_R^{(34)}, & b_{47} + b_{56} &= 4 c_I^{(34)}. \end{aligned} \quad (\text{A.9})$$

This establishes that the effect of an arbitrary constant background metric and 2-form field along T^6 can be encoded in an appropriate choice of the constants $c^{(k)}$ and $c^{(k\ell)}$. On the other hand, a generic deformation of the D-brane world-volume theory characterized by the constants $c^{(k)}$ and $c^{(k\ell)}$ can be produced by suitably choosing the background values of the metric and 2-form fields.

B Duality transformation

In [7], the counting of BPS states was done for a system consisting of N_1 KK monopoles associated with the 4-direction, $-N_2$ units of momentum along the 5-direction, N_3 D1-branes along the 5-direction, N_4 D5-branes along 5-6-7-8-9 directions and $-N_5$ units of momentum along the 4-direction.⁵ Our goal will be to show that via a series of duality transformations this can be related to the system introduced in section 2. During this analysis we shall ignore all the signs (which can in principle be determined by following some specific sign convention, e.g. the one given in appendix A of [24]). This way we shall at most miss the signs of the charges carried by the final D-brane configurations. However, our analysis of the world-line theory of the D-brane system is independent of the signs of these charges as long as the signs are chosen to give a configuration that preserves 4 out of 32 supersymmetries.

Consider the following series of duality transformations:

- T-duality transformations along the 4 and 5 directions: this gives a configuration of N_1 NS-5-branes along 5-6-7-8-9 directions, N_2 fundamental strings along the 5-direction, N_3 D1-branes along the 4-direction, N_4 D5-branes along 4-6-7-8-9 directions and N_5 fundamental strings along the 4-direction.
- T-duality transformation along 8 and 9 directions: this gives a configuration of N_1 NS-5-branes along 5-6-7-8-9 directions, N_2 fundamental strings along the 5-direction, N_3 D3-branes along 4-8-9 directions, N_4 D3-branes along 4-6-7 directions and N_5 fundamental strings along the 4-direction.
- S-duality: this gives a configuration of N_1 D5-branes along 5-6-7-8-9 directions, N_2 D1-branes along the 5-direction, N_3 D3-branes along 4-8-9 directions, N_4 D3-branes along 4-6-7 directions and N_5 D1-branes along the 4-direction.
- T-duality along 5, 8 and 9 directions: this gives a configuration of N_1 D2-branes along 6-7 directions, N_2 D2-branes along 8-9 directions, N_3 D2-branes along 4-5 directions, N_4 D6-branes along 4-5-6-7-8-9 directions and N_5 D4-branes along the 4-5-8-9 directions.
- Cyclic permutation of 6-7→4-5→8-9→6-7: this gives a configuration of N_1 D2-branes along 4-5 directions, N_2 D2-branes along 6-7 directions, N_3 D2-branes along 8-9 directions, N_4 D6-branes along 4-5-6-7-8-9 directions and N_5 D4-branes along the 6-7-8-9 directions.

For $N_5 = 0$, this reduces to the configuration described in section 3.

⁵The actual computation was done for $N_1 = 1$ but we shall consider a more general situation.

	u_7	u_8	u_9
1	$-\sqrt{m_{12} m_{14} m_{24}}$	$-\sqrt{m_{31} m_{14} m_{34}}$	$-\sqrt{m_{23} m_{24} m_{34}}$
2	$\sqrt{m_{12} m_{14} m_{24}}$	$-\sqrt{m_{31} m_{14} m_{34}}$	$-\sqrt{m_{23} m_{24} m_{34}}$
3	$-\sqrt{m_{12} m_{14} m_{24}}$	$\sqrt{m_{31} m_{14} m_{34}}$	$-\sqrt{m_{23} m_{24} m_{34}}$
4	$\sqrt{m_{12} m_{14} m_{24}}$	$\sqrt{m_{31} m_{14} m_{34}}$	$-\sqrt{m_{23} m_{24} m_{34}}$
5	$-\sqrt{m_{12} m_{14} m_{24}}$	$-\sqrt{m_{31} m_{14} m_{34}}$	$\sqrt{m_{23} m_{24} m_{34}}$
6	$\sqrt{m_{12} m_{14} m_{24}}$	$-\sqrt{m_{31} m_{14} m_{34}}$	$\sqrt{m_{23} m_{24} m_{34}}$
7	$-\sqrt{m_{12} m_{14} m_{24}}$	$\sqrt{m_{31} m_{14} m_{34}}$	$\sqrt{m_{23} m_{24} m_{34}}$
8	$\sqrt{m_{12} m_{14} m_{24}}$	$\sqrt{m_{31} m_{14} m_{34}}$	$\sqrt{m_{23} m_{24} m_{34}}$
9	$-(d/g)\sqrt{b+c}$	$(e/h)\sqrt{b+c}$	$-\sqrt{(b+c)/2}$
10	$(d/g)\sqrt{b+c}$	$-(e/h)\sqrt{b+c}$	$\sqrt{(b+c)/2}$
11	$-(f/g)\sqrt{b-c}$	$-(e'/h)\sqrt{b-c}$	$-\sqrt{(b-c)/2}$
12	$(f/g)\sqrt{b-c}$	$(e'/h)\sqrt{b-c}$	$\sqrt{(b-c)/2}$

Table 1. Solutions to (4.10).

C Explicit solutions to (4.10)

In this appendix we shall describe the solutions to (4.10) in terms of the quantities

$$\begin{aligned}
 a &\equiv m_{12}^2 (m_{31} m_{14} - m_{23} m_{24})^2 \\
 &\quad - 2 m_{12} m_{34} [m_{31}^2 m_{14} m_{23} + m_{31} m_{24} (m_{14}^2 - 4 m_{14} m_{23} + m_{23}^2) + m_{14} m_{23} m_{24}^2] \\
 &\quad + m_{34}^2 (m_{31} m_{23} - m_{14} m_{24})^2, \\
 b &\equiv -\frac{m_{23}^2 (m_{31} - m_{24}) (m_{12} - m_{34})}{m_{14} - m_{23}} - \frac{m_{14} (m_{12} m_{31} - m_{24} m_{34})^2}{(m_{31} - m_{24}) (m_{12} - m_{34})} \\
 &\quad + \frac{m_{23} [-m_{12}^2 m_{31} (m_{31} - 2 m_{24}) + 2 m_{12} m_{31} m_{34} (m_{31} - 2 m_{24}) + m_{24}^2 m_{34}^2]}{(m_{31} - m_{24}) (m_{12} - m_{34})}, \\
 c &\equiv \sqrt{a} \left[-\frac{m_{23}}{m_{14} - m_{23}} - \frac{m_{12} m_{31}}{(m_{31} - m_{24}) (m_{12} - m_{34})} + \frac{m_{24} m_{34}}{(m_{31} - m_{24}) (m_{12} - m_{34})} \right],
 \end{aligned}$$

$$\begin{aligned}
 d &\equiv -m_{12} \left[m_{14} \sqrt{a} + 2 m_{31} m_{14} m_{23} m_{34} + m_{24} m_{34} (m_{14}^2 - 4 m_{14} m_{23} + m_{23}^2) \right] \\
 &\quad + m_{23} m_{34} (\sqrt{a} + m_{31} m_{23} m_{34} - m_{14} m_{24} m_{34}) + m_{12}^2 m_{14} (m_{31} m_{14} - m_{23} m_{24}), \\
 e &\equiv -\sqrt{a} + m_{12} (m_{31} m_{14} + m_{23} m_{24}) - m_{34} (m_{31} m_{23} + m_{14} m_{24}), \\
 e' &\equiv -\sqrt{a} - m_{12} (m_{31} m_{14} + m_{23} m_{24}) + m_{34} (m_{31} m_{23} + m_{14} m_{24}), \\
 f &\equiv \sqrt{a} m_{12} m_{14} + m_{23} m_{34} (-\sqrt{a} + m_{31} m_{23} m_{34} - m_{14} m_{24} m_{34}) \\
 &\quad + m_{12}^2 m_{14} (m_{31} m_{14} - m_{23} m_{24}) \\
 &\quad - m_{12} m_{34} \left[2 m_{31} m_{14} m_{23} + m_{24} (m_{14}^2 - 4 m_{14} m_{23} + m_{23}^2) \right], \\
 g &\equiv 2\sqrt{2} m_{23} m_{24} m_{34} (m_{12} - m_{34}) (m_{14} - m_{23}), \\
 h &\equiv 2\sqrt{2} m_{23} m_{24} (m_{12} - m_{34}). \tag{C.1}
 \end{aligned}$$

The solutions are given in table 1. The important point to note is that there are 12 solutions, in agreement with the microscopic results.

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References

- [1] R. Dijkgraaf, E.P. Verlinde and H.L. Verlinde, *Counting dyons in $N = 4$ string theory*, *Nucl. Phys. B* **484** (1997) 543 [[hep-th/9607026](#)] [[INSPIRE](#)].
- [2] D.P. Jatkar and A. Sen, *Dyon spectrum in CHL models*, *JHEP* **04** (2006) 018 [[hep-th/0510147](#)] [[INSPIRE](#)].
- [3] J.R. David, D.P. Jatkar and A. Sen, *Dyon spectrum in generic $N = 4$ supersymmetric Z_N orbifolds*, *JHEP* **01** (2007) 016 [[hep-th/0609109](#)] [[INSPIRE](#)].
- [4] S. Banerjee, A. Sen and Y.K. Srivastava, *Partition functions of torsion > 1 dyons in heterotic string theory on T^6* , *JHEP* **05** (2008) 098 [[arXiv:0802.1556](#)] [[INSPIRE](#)].
- [5] A. Dabholkar, J. Gomes and S. Murthy, *Counting all dyons in $N = 4$ string theory*, *JHEP* **05** (2011) 059 [[arXiv:0803.2692](#)] [[INSPIRE](#)].
- [6] J.M. Maldacena, G.W. Moore and A. Strominger, *Counting BPS black holes in toroidal type II string theory*, [hep-th/9903163](#) [[INSPIRE](#)].
- [7] D. Shih, A. Strominger and X. Yin, *Counting dyons in $N = 8$ string theory*, *JHEP* **06** (2006) 037 [[hep-th/0506151](#)] [[INSPIRE](#)].
- [8] D. Gaiotto et al., *D4-D0 branes on the quintic*, *JHEP* **03** (2006) 019 [[hep-th/0509168](#)] [[INSPIRE](#)].
- [9] D. Gaiotto, A. Strominger and X. Yin, *The M5-brane elliptic genus: modularity and BPS states*, *JHEP* **08** (2007) 070 [[hep-th/0607010](#)] [[INSPIRE](#)].
- [10] F. Denef and G.W. Moore, *Split states, entropy enigmas, holes and halos*, *JHEP* **11** (2011) 129 [[hep-th/0702146](#)] [[INSPIRE](#)].
- [11] A. Sen, *Arithmetic of quantum entropy function*, *JHEP* **08** (2009) 068 [[arXiv:0903.1477](#)] [[INSPIRE](#)].

- [12] A. Dabholkar, J. Gomes, S. Murthy and A. Sen, *Supersymmetric index from black hole entropy*, *JHEP* **04** (2011) 034 [[arXiv:1009.3226](#)] [[INSPIRE](#)].
- [13] A. Sen, *How do black holes predict the sign of the Fourier coefficients of Siegel modular forms?*, *Gen. Rel. Grav.* **43** (2011) 2171 [[arXiv:1008.4209](#)] [[INSPIRE](#)].
- [14] K. Bringmann and S. Murthy, *On the positivity of black hole degeneracies in string theory*, *Commun. Num. Theor Phys.* **07** (2013) 15 [[arXiv:1208.3476](#)] [[INSPIRE](#)].
- [15] I. Bena, M. Berkooz, J. de Boer, S. El-Showk and D. Van den Bleeken, *Scaling BPS solutions and pure-Higgs states*, *JHEP* **11** (2012) 171 [[arXiv:1205.5023](#)] [[INSPIRE](#)].
- [16] S.-J. Lee, Z.-L. Wang and P. Yi, *Quiver invariants from intrinsic Higgs states*, *JHEP* **07** (2012) 169 [[arXiv:1205.6511](#)] [[INSPIRE](#)].
- [17] S.-J. Lee, Z.-L. Wang and P. Yi, *BPS states, refined indices and quiver invariants*, *JHEP* **10** (2012) 094 [[arXiv:1207.0821](#)] [[INSPIRE](#)].
- [18] J. Manschot, B. Pioline and A. Sen, *A fixed point formula for the index of multi-centered $N = 2$ black holes*, *JHEP* **05** (2011) 057 [[arXiv:1103.1887](#)] [[INSPIRE](#)].
- [19] J. Manschot, B. Pioline and A. Sen, *From black holes to quivers*, *JHEP* **11** (2012) 023 [[arXiv:1207.2230](#)] [[INSPIRE](#)].
- [20] J. Manschot, B. Pioline and A. Sen, *On the Coulomb and Higgs branch formulae for multi-centered black holes and quiver invariants*, *JHEP* **05** (2013) 166 [[arXiv:1302.5498](#)] [[INSPIRE](#)].
- [21] J. de Boer, S. El-Showk, I. Messamah and D. Van den Bleeken, *Quantizing $N = 2$ multicenter solutions*, *JHEP* **05** (2009) 002 [[arXiv:0807.4556](#)] [[INSPIRE](#)].
- [22] A. Sen, *$N = 8$ dyon partition function and walls of marginal stability*, *JHEP* **07** (2008) 118 [[arXiv:0803.1014](#)] [[INSPIRE](#)].
- [23] M. Berkooz and H.L. Verlinde, *Matrix theory, AdS/CFT and Higgs-Coulomb equivalence*, *JHEP* **11** (1999) 037 [[hep-th/9907100](#)] [[INSPIRE](#)].
- [24] A. Sen, *Black hole entropy function, attractors and precision counting of microstates*, *Gen. Rel. Grav.* **40** (2008) 2249 [[arXiv:0708.1270](#)] [[INSPIRE](#)].
- [25] A. Dabholkar, D. Gaiotto and S. Nampuri, *Comments on the spectrum of CHL dyons*, *JHEP* **01** (2008) 023 [[hep-th/0702150](#)] [[INSPIRE](#)].
- [26] S. Banerjee and A. Sen, *Duality orbits, dyon spectrum and gauge theory limit of heterotic string theory on T^6* , *JHEP* **03** (2008) 022 [[arXiv:0712.0043](#)] [[INSPIRE](#)].
- [27] A. Sen, *U -duality invariant dyon spectrum in type-II on T^6* , *JHEP* **08** (2008) 037 [[arXiv:0804.0651](#)] [[INSPIRE](#)].
- [28] A. Sen, *Arithmetic of $N = 8$ black holes*, *JHEP* **02** (2010) 090 [[arXiv:0908.0039](#)] [[INSPIRE](#)].
- [29] L. Álvarez-Gaumé and S.F. Hassan, *Introduction to S duality in $N = 2$ supersymmetric gauge theories: a pedagogical review of the work of Seiberg and Witten*, *Fortsch. Phys.* **45** (1997) 159 [[hep-th/9701069](#)] [[INSPIRE](#)].
- [30] S. Banerjee, R.K. Gupta and A. Sen, *Logarithmic corrections to extremal black hole entropy from quantum entropy function*, *JHEP* **03** (2011) 147 [[arXiv:1005.3044](#)] [[INSPIRE](#)].

- [31] S. Banerjee, R.K. Gupta, I. Mandal and A. Sen, *Logarithmic corrections to $N = 4$ and $N = 8$ black hole entropy: a one loop test of quantum gravity*, *JHEP* **11** (2011) 143 [[arXiv:1106.0080](#)] [[INSPIRE](#)].
- [32] N. Banerjee, S. Banerjee, R.K. Gupta, I. Mandal and A. Sen, *Supersymmetry, localization and quantum entropy function*, *JHEP* **02** (2010) 091 [[arXiv:0905.2686](#)] [[INSPIRE](#)].
- [33] A. Dabholkar, J. Gomes and S. Murthy, *Quantum black holes, localization and the topological string*, *JHEP* **06** (2011) 019 [[arXiv:1012.0265](#)] [[INSPIRE](#)].
- [34] A. Dabholkar, J. Gomes and S. Murthy, *Localization & exact holography*, *JHEP* **04** (2013) 062 [[arXiv:1111.1161](#)] [[INSPIRE](#)].
- [35] R.K. Gupta and S. Murthy, *All solutions of the localization equations for $N = 2$ quantum black hole entropy*, *JHEP* **02** (2013) 141 [[arXiv:1208.6221](#)] [[INSPIRE](#)].
- [36] A. Dabholkar, J. Gomes and S. Murthy, *Nonperturbative black hole entropy and Kloosterman sums*, [arXiv:1404.0033](#) [[INSPIRE](#)].
- [37] N. Seiberg and E. Witten, *String theory and noncommutative geometry*, *JHEP* **09** (1999) 032 [[hep-th/9908142](#)] [[INSPIRE](#)].
- [38] J.R. David, *Tachyon condensation in the $D0/D4$ system*, *JHEP* **10** (2000) 004 [[hep-th/0007235](#)] [[INSPIRE](#)].