Wang and Yin *Boundary Value Problems* 2012, **2012**:96
http://www.boundaryvalueproblems.com/content/2012/1/96

Boundary Value Problems
a SpringerOpen Journal

RESEARCH **Open Access**

Grow-up rate of solutions for the heat equation with a sublinear source

Liangwei Wang[1*] and Jingxue Yin[2]

*Correspondence:
wanglw08@163.com
[1] School of Mathematics and Statistics, Chongqing Three Gorges University, Chongqing, China
Full list of author information is available at the end of the article

Abstract
In this paper, we investigate the grow-up rate of solutions for the heat equation with a sublinear source. We find that if the initial value grows fast enough, then it plays a major role in the growing up of solutions, while if the initial value grows slowly, then the sublinear source prevails. As a direct application of these results, we show that the effect of the sublinear source is negligible in the asymptotic behavior of solutions as $t \to \infty$ if the initial value grows fast enough.
MSC: 35K55; 35B40

Keywords: grow-up; asymptotic behavior; heat equation; sublinear source

1 Introduction

We consider the Cauchy problem of the heat equation with the source

$$\frac{\partial u}{\partial t} - \Delta u = u^p, \quad (x,t) \in \mathbb{R}^N \times (0,\infty), \tag{1.1}$$

$$u(x,0) = u_0(x), \quad x \in \mathbb{R}^N. \tag{1.2}$$

Here $p > 0$, $N \geq 1$, and $u_0 \in L^\infty(\rho_\sigma) \equiv \{\varphi : \rho_\sigma \varphi \in L^\infty(\mathbb{R}^N)\}$ with $\rho_\sigma(x) = (1 + |x|^2)^{-\frac{\sigma}{2}}$.

After the famous work [1], this problem has been widely studied by several authors. It is well known that any positive solutions blow up in finite time if $1 < p \leq p_F \equiv 1 + \frac{2}{N}$ [1–3], while positive global solutions exist if $p > p_F$ [1, 4]. Let

$$p_c = \begin{cases} \frac{(N-2)^2 - 4N + 8\sqrt{N-1}}{(N-2)(N-10)}, & \text{if } N > 10, \\ \infty, & \text{if } 1 \leq N \leq 10. \end{cases}$$

If $p \geq p_c$, the existence of growing up global solutions, the solutions $u(x,t)$ exist for any $(x,t) \in \mathbb{R}^N \times (0,\infty)$ and $u(x,t) \to \infty$ as $t \to \infty$ in some senses, has been established by Poláčik and Yanagida [5, 6]. If $p > p_c$ and the initial data u_0 satisfy some conditions, Fila, Winkler and Yanagida [7] in 2004 precisely evaluated the grow-up rate of solutions of (1.1)-(1.2) and they found that for large t and some $\ell > 0$, the solution $u(x,t)$ satisfies

$$C_1 t^\ell \leq \|u(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} \leq C_2 t^\ell,$$

see also [8]. For the Cauchy-Dirichlet problem of (1.1), the existence of growing up global solutions and the grow-up rate of solutions has been investigated by Dold, Galaktionov, Lacey and Vázquez in [9], Galaktionov and King in [10]. If $p > 1 + \frac{2}{N}$, there are also a lot of papers which intensely investigate the solutions of (1.1)-(1.2) converging to zero at different algebraic rates [11–16].

For the sublinear case ($0 < p < 1$ in (1.1)), it was Aguirre and Escobedo [17] who first proved that if $0 < \sigma < \infty$, and the initial value u_0 satisfies

$$0 \le u_0(x) \in L^\infty(\rho_\sigma),$$

then the solutions $u(x,t)$ of (1.1)-(1.2) are global.

Our interest in this paper is to investigate the grow-up rate of solutions for the problem (1.1)-(1.2) with a sublinear source. We first show that if the initial value u_0 satisfies

$$0 \le u_0 \in L^\infty(\rho_\sigma) \tag{1.3}$$

and

$$\lim_{|x|\to\infty} |x|^{-\sigma} u_0(x) = A \quad \text{for some } A > 0, \tag{1.4}$$

then the solutions of (1.1)-(1.2) ($0 < p < 1$) are growing up solutions such that

$$C_1 t^{\frac{\ell_1}{2}} \le \left\| u(t) \right\|_{L^\infty(\rho_\sigma)} \le C_2 t^{\frac{\ell_2}{2}} \tag{1.5}$$

for large t. Here $\ell_1 = \ell_2 = \sigma$ if $\sigma > \frac{2}{1-p}$, and $\ell_1 = \frac{2}{1-p} < \ell_2 \le \frac{2}{1-p} + \epsilon$ for any $\epsilon > 0$ if $0 < \sigma \le \frac{2}{1-p}$. Moreover, as an application of these results, we get that if $\frac{2}{1-p} < \sigma < \infty$ and the initial value u_0 satisfies (1.3), (1.4), then the effect of the sublinear source is negligible in the asymptotic behavior of solutions as $t \to \infty$. While for $\sigma = \frac{2}{1-p}$, Aguirre and Escobedo [17] revealed that the effect of the sublinear source cannot be negligible in the asymptotic behavior of the solutions as $t \to \infty$. For the absorption case (u^p is replaced by $-u^p$ in (1.1)) and the supercritical case ($p > 1 + \frac{2}{N}$ in (1.1)), some similar results about the asymptotic behavior of solutions for these problems were established by a lot of papers, see [18–20].

The paper is organized as follows. The next section is devoted to giving the grow-up rate for the solutions of the problem (1.1)-(1.2) with $0 < p < 1$. In Section 3, we investigate the asymptotic behavior of solutions for the problem (1.1)-(1.2).

2 Growth-up rate of solutions

We take $0 < p < 1$ in the rest of this paper. For any $0 < \sigma < \infty$, we define a weighted L^∞ space as

$$L^\infty(\rho_\sigma) \equiv \left\{ \varphi(x); \rho_\sigma \varphi \in L^\infty\left(\mathbb{R}^N\right) \right\}$$

with the norm $\|\varphi\|_{L^\infty(\rho_\sigma)} = \|\rho_\sigma \varphi\|_{L^\infty(\mathbb{R}^N)}$, where $\rho_\sigma(x) = (1 + |x|^2)^{-\frac{\sigma}{2}}$. If $(1 + |x|^2)^{\frac{\sigma}{2}} \le u_0 \le C(1 + |x|^2)^{\frac{\sigma}{2}}$, then there exist two subsolutions of the problem (1.1)-(1.2):

$$t \to S(t)u_0(x) = (4\pi t)^{-\frac{N}{2}} \int_{\mathbb{R}^N} \exp^{(-\frac{|x-y|^2}{4t})} u_0(y)\, \mathrm{d}y \tag{2.1}$$

and

$$t \to \big((1-p)t\big)^{1/(p-1)}. \tag{2.2}$$

Using a similar method as in [21] (see the Appendix), we can get that there exist constants $C_1, C_2 > 0$ such that

$$C_1\big(1+t+|x|^2\big)^{\frac{\sigma}{2}} \le S(t)u_0(x) \le C_2\big(1+t+|x|^2\big)^{\frac{\sigma}{2}}. \tag{2.3}$$

So, for any $x \in \mathbb{R}^N$, those two growing up effects given by (2.1) and (2.2) can be compared as $t \to \infty$. When $0 < \sigma < \frac{2}{1-p}$, the one given by (2.2) prevails; when $\frac{2}{1-p} < \sigma < \infty$, the one given by (2.1) prevails; and they coincide in the critical case $\sigma = \frac{2}{1-p}$.

Inspired by the above discussions, in this paper we first study the grow-up rate of solutions for the problem (1.1)-(1.2). The mild solution $u(x,t)$ of the problem (1.1)-(1.2) is defined as follows:

$$u(x,t) = S(t)u_0(x) + \int_0^t S(t-s)u^p(x,s)\,\mathrm{d}s. \tag{2.4}$$

If the initial value $0 \not\equiv u_0 \in L^\infty(\rho_\sigma)$, the existence and uniqueness of a mild solution for the problem (1.1)-(1.2) has been given in [17].

Lemma 2.1 ([17]) *Suppose* $0 \le u_0 \in L^\infty(\rho_\sigma)$ *and* $u_0 \not\equiv 0$, *then there exists a unique mild global solution u for the problem (1.1)-(1.2) with* $0 < p < 1$ *such that*

I. $u \in C^\infty((0,\infty) \times \mathbb{R}^N) \cap L^\infty_{\mathrm{loc}}((0,\infty); L^\infty(\rho_\sigma))$;

II. $\lim_{t\to 0} u(x,t) = u_0(x)$ *for a.e.* $x \in \mathbb{R}^N$.

Moreover, if $u_0 \in C(\mathbb{R}^N)$, *the convergence is uniform on compact subsets of* $\mathbb{R}^N$.

Our results about the grow-up rate of solutions are the following two theorems.

Theorem 2.1 *Let* $0 < p < 1$, $A > 0$ *and* $\frac{2}{1-p} < \sigma < \infty$. *Suppose*

$$0 \le u_0 \in L^\infty(\rho_\sigma) \tag{2.5}$$

and

$$\lim_{|x|\to\infty} |x|^{-\sigma} u_0(x) = A. \tag{2.6}$$

Then there exist constants $T, C_1, C_2 > 0$, *such that*

$$C_1 t^{\frac{\sigma}{2}} \le \big\|u(t)\big\|_{L^\infty(\rho_\sigma)} \le C_2 t^{\frac{\sigma}{2}} \quad \textit{for } t > T. \tag{2.7}$$

Here $u(x,t)$ *is the solution of (1.1)-(1.2).*

Proof The hypothesis (2.6) clearly implies that there exists a constant $R > 0$ such that if $|x| \ge R$, then

$$\frac{A}{2}|x|^\sigma \le u_0(x) \le 2A|x|^\sigma.$$

So,

$$u_0(x) \ge \frac{A}{2}|x|^\sigma - \frac{A}{2}R^\sigma.$$

From the property of the heat semigroup, we have

$$S(t)u_0(x) \ge S(t)\varphi(x) - \frac{A}{2}R^\sigma,$$

where $\varphi(x) = \frac{A}{2}|x|^\sigma$. Using a similar method as [21] (see (A.5)), we obtain that there exists a constant $C_0 > 0$ such that

$$S(\tau)\varphi(x) \ge C_0\big(\tau + |x|^2\big)^{\frac{\sigma}{2}}.$$

So, for $\tau = 1 + 2(\frac{A}{2C_0})^{\frac{2}{\sigma}}R^2$, there exists a constant $C > 0$ (depending on A and σ) such that

$$S(\tau)u_0(x) \ge C\big(1 + |x|^2\big)^{\frac{\sigma}{2}}.$$

It follows from the comparison principle that

$$u(x,\tau) \ge S(\tau)u_0(x) \ge C\big(1 + |x|^2\big)^{\frac{\sigma}{2}}.$$

From $0 \le u_0 \in L^\infty(\rho_\sigma)$ and I of Lemma 2.1, we obtain that there exists a constant $C > 0$ (depending on τ) such that

$$u(x,t)\big(1 + |x|^2\big)^{-\frac{\sigma}{2}} \le \sup_{0\le s\le\tau}\big\|u(s)\big\|_{L^\infty(\rho_\sigma)} \le C \quad \text{for } 0 \le t \le \tau.$$

Therefore,

$$u(x,t) \le C\big(1 + |x|^2\big)^{\frac{\sigma}{2}} \quad \text{for } 0 \le t \le \tau.$$

So, from (2.3), we have

$$C_1\big(1 + t + |x|^2\big)^{\frac{\sigma}{2}} \le S(t)\big[u(\tau)\big](x) \le C_2\big(1 + t + |x|^2\big)^{\frac{\sigma}{2}}, \tag{2.8}$$

where C_1 and C_2 are positive constants depending on A, σ and τ. The hypothesis $\frac{2}{1-p} < \sigma < \infty$ indicates that

$$\sigma(1-p) - 2 > 0.$$

Let

$$a(t) = \left[\left(1 + C_1^{p-1}(1-p)\int_0^t (1+s)^{\frac{\sigma(p-1)}{2}}\,\mathrm{d}s\right)\right]^{\frac{1}{1-p}}.$$

So,

$$\eta \equiv C_1^{p-1}(1-p)\int_0^\infty (1+t)^{\frac{\sigma(p-1)}{2}}\,\mathrm{d}t = \frac{2C_1^{p-1}(1-p)}{\sigma(1-p)-2} > 0.$$

Therefore, $a(t)$ is an increasing function satisfying

$$\begin{cases} a(0) = 1, \\ a(t) \leq (1+\eta)^{\frac{1}{1-p}} \quad \text{for all } t \geq 0. \end{cases} \tag{2.9}$$

From (2.8), we have

$$\begin{aligned} a'(t) &= C_1^{p-1} a(t)^p (1+t)^{\frac{\sigma(p-1)}{2}} \\ &= a(t)^p \big[C_1(1+t)^{\frac{\sigma}{2}}\big]^{p-1} \geq a(t)^p \big[S(t)u(\tau)(x)\big]^{p-1}. \end{aligned} \tag{2.10}$$

Let $w(x,t) = S(t)u(\tau)(x)$, and assume that

$$\overline{w}(x,t) = a(t)w(x,t).$$

So, from (2.10), one can verify that $\overline{w}(x,t)$ is a supersolution of the following problem:

$$\begin{aligned} &\frac{\partial v}{\partial t} - \Delta v = v^p, \quad (x,t) \in \mathbb{R}^N \times (0,\infty), \\ &v(x,0) = \overline{w}(x,0) = u(x,\tau), \quad x \in \mathbb{R}^N. \end{aligned}$$

By (2.8), (2.9) and the comparison principle, we get that

$$\begin{aligned} C_1\big(1+t+|x|^2\big)^{\frac{\sigma}{2}} &\leq w(x,t) \leq v(x,t) \leq \overline{w}(x,t) \\ &= a(t)w(x,t) \leq (1+\eta)^{\frac{1}{1-p}} w(x,t) \leq C_2\big(1+t+|x|^2\big)^{\frac{\sigma}{2}}. \end{aligned}$$

This means that

$$C_1\big(1+t+|x|^2\big)^{\frac{\sigma}{2}} \leq u(x,t+\tau) \leq C_2\big(1+t+|x|^2\big)^{\frac{\sigma}{2}} \quad \text{for } t > \tau.$$

Let $T = 2\tau + 1$. So, there exist two constants, which we still write as C_1 and C_2, such that

$$C_1\big(1+t+|x|^2\big)^{\frac{\sigma}{2}} \leq u(x,t) \leq C_2\big(1+t+|x|^2\big)^{\frac{\sigma}{2}} \quad \text{for } t > T. \tag{2.11}$$

From this, we get (2.7) easily. So we complete the proof of this theorem. □

In the following theorem, we consider the grow-up rate for the solutions of the problem (1.1)-(1.2) when the nonnegative initial value $u_0 \in L^\infty(\rho_\sigma)$ with $0 < \sigma \leq \frac{2}{1-p}$.

Theorem 2.2 *Let $0 < p < 1$, and assume that $0 < \sigma \leq \frac{2}{1-p}$. If the initial value u_0 satisfies (2.5) and (2.6), then for any $\epsilon > 0$, there exist constants $C_1, C_2, T > 0$ such that*

$$C_1 t^{\frac{1}{1-p}} \leq \big\|u(t)\big\|_{L^\infty(\rho_\sigma)} \leq C_2 t^{\frac{1}{1-p}+\epsilon} \quad \textit{for } t > T.$$

Here $u(x,t)$ is also the solution of (1.1)-(1.2).

Proof Using the same method as the proof of (2.8), we can get if $0 < \sigma \le \frac{2}{1-p}$, then there exist $C_1, C_2, T_1 > 0$ such that

$$C_1\big(t + |x|^2\big)^{\frac{\sigma}{2}} \le S(t)u_0(x) \le C_2\big(t + |x|^2\big)^{\frac{\sigma}{2}} \quad \text{for } t > T_1 \text{ and } x \in \mathbb{R}^N. \tag{2.12}$$

So, by the comparison principle, we can get that there exists a constant $\tau > T_1$ satisfying

$$u(x,\tau) \ge S(\tau)u_0(x) \ge C_1\big(1 + |x|^2\big)^{\frac{\sigma}{2}}.$$

We first consider the case of $N > 1$. Let

$$B = (1-p)^{-1}C_1^{1-p} \quad \text{and} \quad \underline{w}(x,t) = (1-p)^{\frac{1}{1-p}}\Big(B\big(1+|x|^2\big)^{\frac{\sigma(1-p)}{2}} + t\Big)^{\frac{1}{1-p}}.$$

So, $\underline{w}$ is a subsolution of the following problem:

$$\begin{aligned} &\frac{\partial v}{\partial t} - \Delta v = v^p, \quad (x,t) \in \mathbb{R}^N \times (0,\infty), \\ &v(x,0) = u(x,\tau), \quad x \in \mathbb{R}^N. \end{aligned} \tag{2.13}$$

Here we have used the facts that $\underline{w}(x,0) = C_1(1+|x|^2)^{\frac{\sigma}{2}} \le u(x,\tau)$ and $N > 1$. Therefore, by the comparison principle, for $t > 0$, there exists a constant C satisfying

$$C\Big(\big(1+|x|^2\big)^{\frac{\sigma(1-p)}{2}} + t\Big)^{\frac{1}{1-p}} \le \underline{w}(x,t) \le u(x,t+\tau).$$

From this, we can get that there exist $C, T > 0$ such that

$$C(1+t)^{\frac{1}{1-p}} \le \big\|u(t)\big\|_{L^\infty(\rho_\sigma)} \quad \text{for } t > T. \tag{2.14}$$

Now, we consider the case of $N = 1$. Let

$$C_3 = \min\big(C_1, (2\sigma)^{-\frac{1}{1-p}}\big) \quad \text{and} \quad B_1 = (1-p)^{-1}C_3^{1-p}.$$

Then, we define the function

$$\underline{w_1}(x,t) = (1-p)^{\frac{1}{1-p}}\left(B_1\big(1+|x|^2\big)^{\frac{\sigma(1-p)}{2}} + \frac{1}{2}t\right)^{\frac{1}{1-p}}.$$

Therefore, $\underline{w_1}$ is also a subsolution of the problem (2.13). In fact,

$$\frac{\partial \underline{w_1}}{\partial t} - \underline{w_1}^p = -\frac{1}{2}(1-p)^{\frac{p}{1-p}}\left(B_1\big(1+|x|^2\big)^{\frac{\sigma(1-p)}{2}} + \frac{1}{2}t\right)^{\frac{p}{1-p}}$$

and

$$\frac{\partial^2 \underline{w_1}}{\partial x^2} \ge -(1-p)^{\frac{1}{1-p}}B_1\sigma\left(B_1\big(1+x^2\big)^{\frac{\sigma(1-p)}{2}} + \frac{1}{2}t\right)^{\frac{p}{1-p}},$$

so,

$$\frac{\partial \underline{w_1}}{\partial t} - \underline{w_1}^p - \frac{\partial^2 \underline{w_1}}{\partial x^2} \le 0.$$

Using the same method as above, we can get that (2.14) holds for $N = 1$. Without loss of generality, we can assume that $t > T$ in the rest of this proof. From the definition of the mild solutions with (2.12), we have

$$\begin{aligned} u(x,t) &= S(t)u_0(x) + \int_0^t S(t-s)u^p(x,s)\,\mathrm{d}s \\ &\le C\big(1+t+|x|^2\big)^{\frac{\sigma}{2}} + \int_0^t S(t-s)\big[\big(1+|x|^2\big)^{\frac{\sigma}{2}} u(x,s)\big(1+|x|^2\big)^{-\frac{\sigma}{2}}\big]^p\,\mathrm{d}s \\ &\le C\big(1+t+|x|^2\big)^{\frac{\sigma}{2}} + C\sup_{0\le s\le t}\big\|u(s)\big\|^p_{L^\infty(\rho_\sigma)}\int_0^t \big(1+|x|^2+(t-s)\big)^{\frac{\sigma p}{2}}\,\mathrm{d}s \\ &\le C\big(1+t+|x|^2\big)^{\frac{\sigma}{2}} + C\sup_{0\le s\le t}\big\|u(s)\big\|^p_{L^\infty(\rho_\sigma)}\big(1+|x|^2+t\big)^{\frac{\sigma p}{2}} t \\ &\le C\big(1+|x|^2\big)^{\frac{\sigma}{2}}(1+t)^{\frac{\sigma}{2}} + C\sup_{0\le s\le t}\big\|u(s)\big\|^p_{L^\infty(\rho_\sigma)}\big(1+|x|^2\big)^{\frac{\sigma p}{2}}(1+t)^{\frac{\sigma p}{2}+1} \\ &\le C\big(1+|x|^2\big)^{\frac{\sigma}{2}}(1+t)^{\frac{\sigma}{2}} + C\sup_{0\le s\le t}\big\|u(s)\big\|^p_{L^\infty(\rho_\sigma)}\big(1+|x|^2\big)^{\frac{\sigma}{2}}(1+t)^{\frac{\sigma p}{2}+1}. \end{aligned}$$

Here we have used $0 < p < 1$ and Lemma A.1, see the Appendix. The assumption $0 < \sigma \le \frac{2}{1-p}$ implies that

$$\frac{\sigma}{2} \le \frac{\sigma p}{2} + 1.$$

Therefore,

$$(1+t)^{\frac{\sigma}{2}} \le (1+t)^{\frac{\sigma p}{2}+1}.$$

By (2.14), we deduce that there exists a constant C such that

$$u(x,t) \le C\sup_{0\le s\le t}\big\|u(s)\big\|^p_{L^\infty(\rho_\sigma)}\big(1+|x|^2\big)^{\frac{\sigma}{2}}(1+t)^{\frac{\sigma p}{2}+1}.$$

This implies that

$$\sup_{0\le s\le t}\big\|u(s)\big\|_{L^\infty(\rho_\sigma)} \le C(1+t)^{\frac{\sigma p}{2(1-p)}+\frac{1}{1-p}}.$$

Using the integral expression (2.4) again, we have

$$\begin{aligned} u(x,t) &\le C\big(1+t+|x|^2\big)^{\frac{\sigma}{2}} + C\sup_{0\le s\le t}\big\|u(s)\big\|^p_{L^\infty(\rho_\sigma)}\int_0^t \big(1+t-s+|x|^2\big)^{\frac{\sigma p}{2}}\,\mathrm{d}s \\ &\le C\big(1+t+|x|^2\big)^{\frac{\sigma}{2}} + C(1+t)^{\frac{\sigma p^2}{2(1-p)}+\frac{p}{1-p}}\big(1+t+|x|^2\big)^{\frac{\sigma p}{2}} t. \end{aligned}$$

Here we have used the fact that

$$\left(1+|x|^2+t\right)^\alpha \le (1+t)^\alpha\left(1+|x|^2\right)^\alpha \quad \text{for } \alpha>0.$$

Notice that for $t>s$ and $m>0$,

$$S(t-s)\phi(x) \le C\left(1+t+|x|^2\right)^{\frac{m}{2}},$$

where $\phi(x)=(1+s+|x|^2)^{\frac{m}{2}}$. So,

$$\begin{aligned}
u(x,t) &= S(t)u_0(x)+\int_0^t S(t-s)u^p(x,s)\,\mathrm{d}s \le C\left(1+t+|x|^2\right)^{\frac{\sigma}{2}} \\
&\quad + C\int_0^t \left[\left(1+t+|x|^2\right)^{\frac{p\sigma}{2}} + (1+s)^{\frac{\sigma p^3}{2(1-p)}+\frac{p^2}{1-p}}\left(1+t+|x|^2\right)^{\frac{\sigma p^2}{2}} s^p\right]\mathrm{d}s \\
&\le C\left[\left(1+t+|x|^2\right)^{\frac{\sigma}{2}} + \left(1+t+|x|^2\right)^{\frac{p\sigma}{2}} t \right.\\
&\quad \left. + (1+t)^{\frac{\sigma p^3}{2(1-p)}+\frac{p^2}{1-p}}\left(1+t+|x|^2\right)^{\frac{\sigma p^2}{2}} t^{1+p}\right]. \qquad (2.15)
\end{aligned}$$

Iterating (2.15) $n-1$ times, we get that

$$\begin{aligned}
u(x,t) &\le C\left[\left(1+t+|x|^2\right)^{\frac{\sigma}{2}} + \left(1+t+|x|^2\right)^{\frac{p\sigma}{2}} t + \left(1+t+|x|^2\right)^{\frac{p^2\sigma}{2}} t^{1+p} + \cdots \right.\\
&\quad + \left(1+t+|x|^2\right)^{\frac{p^{n-1}\sigma}{2}} t^{\frac{1-p^{n-1}}{1-p}} \\
&\quad \left. + (1+t)^{[\frac{\sigma p}{2(1-p)}+\frac{1}{1-p}]p^n}\left(1+t+|x|^2\right)^{\frac{\sigma p^n}{2}} t^{\frac{1}{1-p}-\frac{p^n}{1-p}}\right] \\
&\le C\left[\left(1+|x|^2\right)^{\frac{\sigma}{2}}(1+t)^{\frac{\sigma}{2}} + \left(1+|x|^2\right)^{\frac{p\sigma}{2}}(1+t)^{1+\frac{p\sigma}{2}} \right.\\
&\quad + \left(1+|x|^2\right)^{\frac{p^2\sigma}{2}}(1+t)^{\frac{1-p^2}{1-p}+\frac{p^2\sigma}{2}} + \cdots + \left(1+|x|^2\right)^{\frac{p^{n-1}\sigma}{2}}(1+t)^{\frac{1-p^{n-1}}{1-p}+\frac{p^{n-1}\sigma}{2}} \\
&\quad \left. + (1+t)^{\frac{\sigma p^n}{2(1-p)}+\frac{1}{1-p}}\left(1+|x|^2\right)^{\frac{\sigma p^n}{2}}\right] \\
&\le C(n)\left(1+|x|^2\right)^{\frac{\sigma}{2}}\left[(1+t)^{\frac{1}{1-p}+\frac{\sigma p^n}{2(1-p)}} + (1+t)^{\frac{1}{1-p}+\frac{\sigma p^{n-1}}{2}}\right]. \qquad (2.16)
\end{aligned}$$

Here we have used the facts that

$$\frac{\sigma}{2} \le \frac{p\sigma}{2}+1 \le \frac{p^2\sigma}{2}+\frac{1-p^2}{1-p} \le \cdots \le \frac{p^{n-1}\sigma}{2}+\frac{1-p^{n-1}}{1-p} \le \frac{p^{n-1}\sigma}{2}+\frac{1}{1-p}$$

and

$$\left(1+|x|^2\right)^{\frac{\sigma p^m}{2}} \le \left(1+|x|^2\right)^{\frac{\sigma}{2}} \quad \text{for } m>0.$$

So, for any $\epsilon>0$, we can select n large enough to satisfy

$$\epsilon > \max\left(\frac{p^n\sigma}{2(1-p)}, \frac{p^{n-1}\sigma}{2}\right).$$

From (2.16), we thus have

$$\left\|u(t)\right\|_{L^\infty(\rho_\sigma)} \le C(1+t)^{\frac{1}{1-p}+\epsilon}.$$

Combining this with (2.14), we can get the desired results. So we complete the proof of this theorem. □

Remark 2.1 From Theorem 2.1 and Theorem 2.2, we find that if $\sigma > \frac{2}{1-p}$, then the main effect on the growing up of solutions comes from the initial value; while if $0 < \sigma \le \frac{2}{1-p}$, then the sublinear source has a major effect.

3 Asymptotic behavior

In this section, we will use the fact that the mild solutions of the problem (1.1)-(1.2) given by Lemma 2.1 also satisfy the following integral identity:

$$\int_0^T \int_{\mathbb{R}^N} \left[\left(\frac{\partial \xi}{\partial t} + \Delta \xi\right) u + \xi u^p\right] \mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}^N} \xi(x,0)u_0(x)\,\mathrm{d}x = 0, \tag{3.1}$$

for any $\xi \in C^{1,2}([0,T]\times\mathbb{R}^N)$ which vanishes for large $|x|$ and at $t = T$.

The following result gives the fact that if $\sigma > \frac{2}{1-p}$, then the sublinear source is negligible in the asymptotic behavior of the rescaled solution $t^{-\frac{\sigma}{2}}u(x,t)$ as $t \to \infty$. Similar to [19, 22, 23], we follow the framework by Kamin and Peletier [19] to give the proof of our result.

Theorem 3.1 *Let $0 < p < 1$ and $\frac{2}{1-p} < \sigma < \infty$. If the initial value u_0 satisfies (2.5) and (2.6), then*

$$\lim_{t\to\infty} t^{-\frac{\sigma}{2}}\left|u(x,t) - S(t)\varphi(x)\right| = 0 \tag{3.2}$$

uniformly on sets $\{(y,s); |y| \le \gamma s^{\frac{1}{2}}\}$, $\gamma > 0$. Here $u(x,t)$ is the mild solution of (1.1)-(1.2) and $\varphi(x) = A|x|^\sigma$.

Proof We first define the functions

$$w(x,t) = S(t)u_0(x),$$
$$u_\lambda(x,t) = \lambda^{-\sigma}u(\lambda x, \lambda^2 t)$$

and

$$w_\lambda(x,t) = \lambda^{-\sigma}\left[S(\lambda^2 t)u_0\right](\lambda x).$$

Using the comparison principle, we know that

$$w(x,t) \le u(x,t),$$

and

$$w_\lambda(x,t) \le u_\lambda(x,t) \quad \text{for all } \lambda \ge 1.$$

For $t > 0$, without loss of generality, we can assume that λ is large enough to satisfy $\lambda t > T$, where T is the constant given by Theorem 2.1. So, from (2.11), we have

$$\begin{aligned} u_\lambda(x,t) &\le C\lambda^{-\sigma}\left[1+\lambda^2 t+\lambda^2|x|^2\right]^{\frac{\sigma}{2}} \le C\left(\lambda^{-2}+t+|x|^2\right)^{\frac{\sigma}{2}} \\ &\le C\left(\lambda^{-2}+t\right)^{\frac{\sigma}{2}}\left(1+\left(\lambda^{-2}+t\right)^{-1}|x|^2\right)^{\frac{\sigma}{2}} \\ &\le C\left(\lambda^{-2}+t\right)^{\frac{\sigma}{2}}\left(1+\left(\lambda^{-2}+t\right)^{-\frac{1}{2}}|x|\right)^{\sigma}. \end{aligned} \tag{3.3}$$

So, if $\lambda > 2$ and $0 < \tau < \frac{1}{4}$, then

$$\begin{aligned} \int_0^\tau \int_{B_1} u_\lambda(x,t)\,\mathrm{d}x\,\mathrm{d}t &\le C\int_{\lambda^{-2}}^{\tau+\lambda^{-2}} s^{\frac{N+\sigma}{2}} \int_0^{s^{-\frac{1}{2}}} (1+r)^{N+\sigma-1}\,\mathrm{d}r\,\mathrm{d}s \\ &\le C\int_{\lambda^{-2}}^{\tau+\lambda^{-2}} s^{\frac{N+\sigma}{2}}\left[\left(1+s^{-\frac{1}{2}}\right)^{N+\sigma}-1\right]\mathrm{d}s \\ &\le C\int_{\lambda^{-2}}^{\tau+\lambda^{-2}} s^{\frac{N+\sigma}{2}}\left(2s^{-\frac{1}{2}}\right)^{N+\sigma}\mathrm{d}s \le C\tau. \end{aligned}$$

Similarly, for any $q > 0$, from (3.3), we can obtain the following integral estimates:

$$\begin{aligned} \int_0^\tau \int_{B_1} u_\lambda(x,t)^q\,\mathrm{d}x\,\mathrm{d}t &\le C\int_{\lambda^{-2}}^{\tau+\lambda^{-2}} s^{\frac{N+q\sigma}{2}}\left[\left(1+s^{-\frac{1}{2}}\right)^{N+q\sigma}-1\right]\mathrm{d}s \\ &\le C\int_{\lambda^{-2}}^{\tau+\lambda^{-2}} \left(1+s^{\frac{1}{2}}\right)^{\frac{N+\sigma}{2}}\mathrm{d}s \le C\left(1+\tau+\lambda^{-2}\right)^{N+q\sigma}\int_{\lambda^{-2}}^{\tau+\lambda^{-2}}\mathrm{d}s \\ &\le C\left(1+\tau+\lambda^{-2}\right)^{N+q\sigma}\tau. \end{aligned} \tag{3.4}$$

Using the same method as above and the comparison principle, we can get the similar integral estimates for $w_\lambda(x,t)$. For any $T_1 > t > 0$, from (3.1), we have

$$\iint_{S_\tau+S_{T_1}^\tau}\left[\xi_t(u_\lambda-w_\lambda)+\Delta\xi(u_\lambda-w_\lambda)\right]\mathrm{d}t\,\mathrm{d}x = \iint_{\overline{S_{T_1}}}\lambda^{-\kappa}\xi u_\lambda^p\,\mathrm{d}x\,\mathrm{d}t, \tag{3.5}$$

where $S_\tau \equiv (0,\tau]\times\mathbb{R}^N$, $S_{T_1}^\tau \equiv (\tau,T_1]\times\mathbb{R}^N$, $\kappa = \sigma(1-p)-2 > 0$ and $\xi \in C^{1,2}(\overline{S_{T_1}})$ which vanishes for large $|x|$ and at $t = T_1$. For any $\epsilon > 0$, by the integral estimates of $u_\lambda(x,t)$ and $w_\lambda(x,t)$, there exists $\tau > 0$ such that

$$\iint_{S_\tau}\left[\xi_t(w_\lambda-u_\lambda)+\Delta\xi(w_\lambda-u_\lambda)\right]\mathrm{d}x\,\mathrm{d}t < \frac{\epsilon}{3}. \tag{3.6}$$

From the fact that $\kappa > 0$ and (3.4), we can get that there exists λ_1 such that if $\lambda > \lambda_1$, then

$$\iint_{S_{T_1}}\lambda^{-\kappa}\xi u_\lambda^p\,\mathrm{d}x\,\mathrm{d}t < \frac{\epsilon}{3}. \tag{3.7}$$

Let $z_\lambda(x,t) = u_\lambda(x,t) - w_\lambda(x,t)$. From (3.1), we can get that for any compact subset K of S_{T_1}, $u_\lambda(x,t)$ and $w_\lambda(x,t)$ have a uniform upper bound, which means that the sequence $z_\lambda(x,t)$

is equicontinuous on K (see [17, 24, 25]). So, we can get that there exist a subsequence $z_{\lambda'}(x,t)$ and a function $z(x,t) \in C(S_{T_1})$ such that

$$z_{\lambda'}(x,t) \to z(x,t)$$

as $\lambda' \to \infty$ uniformly on K. Therefore, we have as $\lambda' \to \infty$, omitting the primes,

$$\iint_{S_{T_1}^{\tau}} \left[\xi_t(u_\lambda - w_\lambda) + \Delta\xi(u_\lambda - w_\lambda)\right] \mathrm{d}t\,\mathrm{d}x \to \iint_{S_{T_1}^{\tau}} [\xi_t + \Delta\xi] z\,\mathrm{d}t\,\mathrm{d}x.$$

Combining this with (3.5)-(3.7), we obtain that

$$\iint_{S_{T_1}} [\xi_t + \Delta\xi] z\,\mathrm{d}t\,\mathrm{d}x = 0.$$

Therefore, it follows from the uniqueness of the solutions of the heat equation that

$$z(x,t) = 0 \quad \text{for all } (x,t) \in S_{T_1}.$$

Thus the entire sequence z_λ converges to $z = 0$. Therefore, we have proved that for any $0 < t < T < \infty$,

$$u_\lambda(x,t) - w_\lambda(x,t) \to 0 \quad \text{as } \lambda \to \infty$$

uniformly on any compact subset of $\mathbb{R}^N$. Thus, taking $t = 1$ and $\lambda = s^{\frac{1}{2}}$, we obtain

$$\lim_{s\to\infty} s^{-\frac{\sigma}{2}} \left\| u\left(s^{\frac{1}{2}}\cdot, s\right) - w\left(s^{\frac{1}{2}}\cdot, s\right)\right\|_{L^\infty_{\mathrm{loc}}(\mathbb{R}^N)} = 0. \tag{3.8}$$

From (2.8) and $0 \le u_0 \in L^\infty(\rho_\sigma)$, we have

$$w_{t^{\frac{1}{2}}}(x,1) = t^{-\frac{\sigma}{2}} S(t) u_0\left(t^{\frac{1}{2}}x\right) \le C\left(t^{-\frac{1}{2}} + 1 + |x|^2\right)^{\frac{\sigma}{2}}.$$

So, for any $x \in \mathbb{R}^N$, by Lebesgue's dominated convergence theorem, we have

$$\begin{aligned} w_{t^{\frac{1}{2}}}(x,1) &= t^{-\frac{\sigma}{2}} S(t) u_0\left(t^{\frac{1}{2}}x\right) \\ &= (4\pi)^{-\frac{N}{2}} \int_{\mathbb{R}^N} \exp\left(-\frac{|x-y|^2}{4}\right) t^{-\frac{\sigma}{2}} u_0\left(t^{\frac{1}{2}}y\right) \mathrm{d}y \to S(1)\varphi(x) \end{aligned} \tag{3.9}$$

as $t \to \infty$. The uniform upper bound of $w_{t^{\frac{1}{2}}}(x,1)$ on any compact subset M of $\mathbb{R}^N$ implies that the sequence $w_{t^{\frac{1}{2}}}(x,1)$ is equicontinuous on M, see [17, 24, 25]. Therefore, from (3.9), we have

$$w_{t^{1/2}}(x,1) = t^{-\frac{\sigma}{2}} S(t) u_0\left(t^{\frac{1}{2}}x\right) \to S(1)\varphi(x)$$

uniformly on any compact sets of $\mathbb{R}^N$ as $t \to \infty$. By (3.8), we thus have (3.2). So we complete the proof of this theorem. □

Appendix

Lemma A.1 *Let $M_1, M_2 > 0$ and $0 < \sigma < \infty$. If*

$$M_1\left(1+|x|^2\right)^{\frac{\sigma}{2}} \le u_0 \le M_2\left(1+|x|^2\right)^{\frac{\sigma}{2}}, \tag{A.1}$$

then there exist two constants $C(M_1,\sigma), C(M_2,\sigma) > 0$ such that

$$C(M_1,\sigma)\left(1+t+|x|^2\right)^{\frac{\sigma}{2}} \le S(t)u_0(x) \le C(M_2,\sigma)\left(1+t+|x|^2\right)^{\frac{\sigma}{2}}. \tag{A.2}$$

Proof Consider the following problem:

$$\begin{aligned}
&\frac{\partial v}{\partial t} - \Delta v = 0, \quad \text{in } \mathbb{R}^N \times (0,\infty),\\
&v(x,0) = v_0(x) = M|x|^\sigma, \quad \text{in } \mathbb{R}^N,
\end{aligned}$$

where $M > 0$ is a constant. For $\lambda > 1$, from (2.1), we can get

$$\lambda^{-\frac{\sigma}{2}}\left[S(\lambda t)v_0\right]\left(\lambda^{\frac{1}{2}}x\right) = S(t)\left[\lambda^{-\frac{\sigma}{2}}v_0\left(\lambda^{\frac{1}{2}}\cdot\right)\right](x) = S(t)v_0(x). \tag{A.3}$$

By existence and regularity theories for solutions, we can obtain that for $t > 0$,

$$0 < S(t)v_0 \in C^\infty\left((0,\infty)\times\mathbb{R}^N\right),$$

see [24, 25]. Now taking $t = 1$, $\lambda = s$ and $g(x) = S(1)v_0(x)$ in the expression (A.3), we have

$$S(s)v_0(x) = s^{\frac{\sigma}{2}}g\left(s^{-\frac{1}{2}}x\right). \tag{A.4}$$

The fact that $S(s)v_0(x) \in C([0,\infty)\times\mathbb{R}^N)$ clearly implies that for $|x| = 1$,

$$s^{\frac{\sigma}{2}}g\left(s^{-\frac{1}{2}}x\right) = S(s)v_0(x) \to v_0(x) = M|x|^\sigma = M$$

as $s \to 0$. Let

$$y = s^{-\frac{1}{2}}x.$$

So

$$|y| \to \infty \quad \text{as } s \to 0.$$

Therefore,

$$|y|^{-\sigma}g(y) - M \to 0$$

as $|y| \to \infty$. So, there exist constants $0 < C_1(M) \le C_2(M) < \infty$ satisfying

$$C_1(M)\left(1+|x|^2\right)^{\frac{\sigma}{2}} \le g(x) \le C_2(M)\left(1+|x|^2\right)^{\frac{\sigma}{2}}.$$

By (A.4), we thus have

$$C_1(M)\big(s+|x|^2\big)^{\frac{\sigma}{2}} \le S(s)v_0(x) \le C_2(M)\big(s+|x|^2\big)^{\frac{\sigma}{2}}. \tag{A.5}$$

Let $\varphi(x) = M(1+|x|^2)^{\frac{\sigma}{2}}$. So there exist two constants $C_1(M,\sigma), C_2(M,\sigma) > 0$ such that

$$C_1(M,\sigma)\big(1+v_0(x)\big) \le \varphi(x) \le C_2(M,\sigma)\big(1+v_0(x)\big).$$

Therefore, by the comparison principle and (A.5), we can get that for all $t \ge 0$, there exist constants $C_1(M,\sigma), C_2(M,\sigma) > 0$ such that

$$C_1(M,\sigma)\big(1+s+|x|^2\big)^{\frac{\sigma}{2}} \le S(t)\varphi(x) \le C_2(M,\sigma)\big(1+s+|x|^2\big)^{\frac{\sigma}{2}}.$$

By (A.1) and the comparison principle, we have (A.2). So we complete the proof of this lemma. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The paper is the result of joint work of all authors who contributed equally to the final version of the paper. All authors read and approved the final manuscript.

Author details
[1]School of Mathematics and Statistics, Chongqing Three Gorges University, Chongqing, China. [2]School of Mathematics Science, South China Normal University, Guangzhou, China.

Acknowledgements
The authors are grateful to the referees for their useful comments. This work are supported by NSFC, the Research Fund for the Doctoral Program of Higher Education of China, the Natural Science Foundation Project of "CQ CSTC", the Scientific and Technological Projects of Chongqing Municipal Commission of Education.

Received: 13 June 2012 Accepted: 20 August 2012 Published: 31 August 2012

References

1. Fujita, H: On the blowing up of solutions of the Cauchy problem for $u_t = \Delta u + u^{1+\alpha}$. J. Fac. Sci., Univ. Tokyo, Sect. 1A, Math. **13**, 109-124 (1966)
2. Hayakawa, K: On the nonexistence of global solutions of some semilinear parabolic equations. Proc. Jpn. Acad. **49**, 503-525 (1974)
3. Galaktionov, VA: Blow-up for quasilinear heat equations with critical Fujita's exponents. Proc. R. Soc. Edinb. A **124**(3), 517-525 (1994) (English summary)
4. Lee, T-Y, Ni, W-M: Global existence, large time behavior and life span of solutions of a semilinear parabolic Cauchy problem. Trans. Am. Math. Soc. **333**(1), 365-378 (1992)
5. Poláčik, P, Yanagida, E: On bounded and unbounded global solutions of a supercritical semilinear heat equation. Math. Ann. **327**(4), 745-771 (2003) (English summary)
6. Poláčik, P, Yanagida, E: Nonstabilizing solutions and grow-up set for a supercritical semilinear diffusion equation. Differ. Integral Equ. **17**(5-6), 535-548 (2004) (English summary)
7. Fila, M, Winkler, M, Yanagida, E: Grow-up rate of solutions for a supercritical semilinear diffusion equation. J. Differ. Equ. **205**(2), 365-389 (2004) (English summary)
8. Mizoguchi, N: Grow-up of solutions for a semilinear heat equation with supercritical nonlinearity. J. Differ. Equ. **227**(2), 652-669 (2006) (English summary)
9. Dold, JW, Galaktionov, VA, Lacey, AA, Vázquez, JL: Rate of approach to a singular steady state in quasilinear reaction-diffusion equations. Ann. Sc. Norm. Super. Pisa, Cl. Sci. (4) **26**(4), 663-687 (1998)
10. Galaktionov, VA, King, JR: Composite structure of global unbounded solutions of nonlinear heat equations with critical Sobolev exponents. J. Differ. Equ. **189**(1), 199-233 (2003) (English summary)
11. Galaktionov, V, Vázquez, JL: Continuation of blow-up solutions of nonlinear heat equations in several space dimensions. Commun. Pure Appl. Math. **50**, 1-67 (1997)
12. Gui, C, Ni, W-M, Wang, X: Further study on a nonlinear heat equation. J. Differ. Equ. **169**(2), 588-613 (2001)
13. Fila, M, King, JR, Winkler, M, Yanagida, E: Linear behaviour of solutions of a superlinear heat equation. J. Math. Anal. Appl. **340**(1), 401-409 (2008) (English summary)
14. Stinner, C: Very slow convergence to zero for a supercritical semilinear parabolic equation. Adv. Differ. Equ. **14**(11-12), 1085-1106 (2009) (English summary)

15. Fila, M, Winkler, M, Yanagida, E: Slow convergence to zero for a parabolic equation with a supercritical nonlinearity. Math. Ann. **340**(3), 477-496 (2008) (English summary)
16. Stinner, C: Very slow convergence rates in a semilinear parabolic equation. Nonlinear Differ. Equ. Appl. **17**(2), 213-227 (2010) (English summary)
17. Aguirre, J, Escobedo, MA: Cauchy problem for $u_t - \Delta u = u^p$ with $0 < p < 1$. Asymptotic behaviour of solutions. Ann. Fac. Sci. Toulouse Math. (5) **8**(2), 175-203 (1986/87)
18. Galaktionov, VA, Vázquez, JL: Asymptotic behaviour of nonlinear parabolic equations with critical exponents. A dynamical systems approach. J. Funct. Anal. **100**(2), 435-462 (1991)
19. Kamin, S, Peletier, LA: Large time behaviour of solutions of the heat equation with absorption. Ann. Sc. Norm. Super. Pisa, Cl. Sci. (4) **12**(3), 393-408 (1985)
20. Ishige, K, Ishiwata, M, Kawakami, T: The decay of the solutions for the heat equation with a potential. Indiana Univ. Math. J. **58**(6), 2673-2707 (2009) (English summary)
21. Cazenave, T, Dickstein, F, Escobedo, M, Weissler, FB: Self-similar solutions of a nonlinear heat equation. J. Math. Sci. Univ. Tokyo **8**(3), 501-540 (2001) (English summary)
22. Kamin, S, Peletier, LA: Large time behaviour of solutions of the porous media equation with absorption. Isr. J. Math. **55**(2), 129-146 (1986)
23. Kwak, M: A porous media equation with absorption. I. Long time behaviour. J. Math. Anal. Appl. **223**(1), 96-110 (1998)
24. Widder, DV: The Heat Equation. Pure and Applied Mathematics, vol. 67. Academic Press, New York (1975)
25. Wu, Z, Yin, J, Wang, C: Elliptic and Parabolic Equations. World Scientific, Hackensack (2006)

doi:10.1186/1687-2770-2012-96
Cite this article as: Wang and Yin: Grow-up rate of solutions for the heat equation with a sublinear source. *Boundary Value Problems* 2012 **2012**:96.