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# Regularization-scheme dependence of QCD amplitudes in the massive case

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ABSTRACT: We investigate QCD amplitudes with massive quarks computed in the four-dimensional helicity scheme (FDH) and dimensional reduction at NNLO and describe how they are related to the corresponding amplitudes computed in conventional dimensional regularization. To this end, the scheme dependence of the heavy quark and the velocity-dependent cusp anomalous dimensions is determined using soft-collinear effective theory. The results are checked against explicit computations of massive form factors in FDH at NNLO. Our results complete the description of the scheme dependence of QCD amplitudes at NNLO.

## Contents

## 1 Introduction

The most common procedure to regularize ultraviolet (UV) and infrared (IR) singularities of scattering amplitudes is to apply conventional dimensional regularization (CDR), whereby all relevant quantities are treated as $D = 4 - 2\epsilon$ dimensional. In CDR, IR singularities of next-to-next-to leading order (NNLO) scattering amplitudes in massless QCD have a remarkably simple structure [1–4]. Key ingredients are the cusp anomalous dimension $\gamma_{\mathrm{cusp}}$ and the anomalous dimensions of quarks and gluons, $\gamma_q$ and $\gamma_g$, respectively.

For practical computations it is sometimes advantageous to apply certain variants of CDR, such as the 't Hooft-Veltman scheme (HV) [5], dimensional reduction (DRED) [6] or the four-dimensional helicity scheme (FDH) [7]. This leads to the question how virtual amplitudes computed in these schemes are related to the corresponding amplitudes computed in CDR. In the massless case at NNLO, this question has been answered in ref. [8], where, drawing on earlier results [9–18], it has been shown that the IR structure of CDR is only modified through changes in the anomalous dimensions. We indicate this regularization-scheme (RS) dependence by the shifts $\gamma_{\mathrm{cusp}} \to \gamma_{\mathrm{cusp}}^{\mathrm{RS}}$, $\gamma_q \to \gamma_q^{\mathrm{RS}}$ and $\gamma_g \to \gamma_g^{\mathrm{RS}}$. The explicit expressions of the anomalous dimensions as well as the $\beta$ functions of the various couplings in the different schemes have been determined at least up to NNLO.

In the presence of massive quarks there are additional structures in the IR singularities of QCD amplitudes [19]. Hence, the scheme dependence will also have to be generalized. At NLO the scheme-dependence has been discussed in ref. [20]. The generalization of the scheme dependence at NNLO to QCD amplitudes including massive quarks is the main result of this paper. As we will show, once the scheme-dependent UV renormalization has been carried out, this scheme dependence is contained entirely in two additional anomalous dimensions, the velocity-dependent cusp anomalous dimension $\gamma_{\text{cusp}}^{\text{RS}}(\beta)$ and the anomalous dimension of a heavy quark $\gamma_Q^{\text{RS}}$. In fact, the scheme dependence of $\gamma_{\text{cusp}}^{\text{RS}}(\beta)$ itself is induced solely through the scheme dependence of the cusp anomalous dimension $\gamma_{\text{cusp}}^{\text{RS}}$ from the massless case.

With the results presented here it is possible to convert any NNLO QCD amplitude between the four schemes CDR, HV, FDH, and DRED. This allows for using whatever scheme is most convenient in the computation of the virtual amplitude and then combine this with the real corrections, typically computed in CDR. In fact, for the generalization to the massive case it is sufficient to consider the difference between the FDH and the HV (or CDR) scheme. If there are no external gluons, FDH is equivalent to DRED. Hence, the IR anomalous dimensions are the same, e.g. $\gamma_{\text{cusp}}^{\text{FDH}}(\beta) = \gamma_{\text{cusp}}^{\text{DRED}}(\beta)$ and $\gamma_Q^{\text{FDH}} = \gamma_Q^{\text{DRED}}$. Furthermore, CDR and HV also have the same anomalous dimensions, $\gamma_{\text{cusp}}^{\text{CDR}}(\beta) = \gamma_{\text{cusp}}^{\text{HV}}(\beta)$ and $\gamma_Q^{\text{CDR}} = \gamma_Q^{\text{HV}}$. These schemes differ simply in the dimension of the polarization sum of external gluons.

Apart from the four schemes treated in this paper, other possibilities to regularize virtual amplitudes have been considered. The FDH scheme has been adapted to the so-called FDF scheme (four-dimensional formulation) for using unitary-based methods to compute NLO amplitudes [21, 22]. There are also proposals to abandon dimensional regularization altogether and perform computations completely in four dimensions in the context of implicit regularization [23–26], FDR (four-dimensional regularization/renormalization) [27–29], and using loop-tree duality to deal with IR singularities at the integrand level [30–32].

While this list is by no means exhaustive it shows that despite the impressive technical advances in computing higher-order corrections in CDR there is considerable interest in exploring alternative methods. The results presented here complete the description at NNLO of a first step away from a fully $D$ dimensional treatment of the problem. Apart from allowing to perform computations in FDH and DRED, we hope it also helps to understand better the relation between CDR and the different four-dimensional approaches mentioned above. The ultimate goal is, of course, to develop efficient methods to explicitly perform ever more accurate computations.

The paper is organized as follows: in section 2 we briefly review the various schemes, the IR structure of amplitudes and its extension to the massive case. We also discuss the UV renormalization, emphasizing the special features of FDH in the presence of massive quarks. Section 3 is devoted to the computation of $\gamma_Q^{\text{RS}}$ and $\gamma_{\text{cusp}}^{\text{RS}}(\beta)$ at NNLO in the FDH scheme. These results are obtained by direct computations using soft-collinear effective theory. In order to obtain an independent test of the scheme dependence of NNLO amplitudes, in section 4 we compare the heavy-quark and heavy-to-light form factors in the FDH and CDR schemes and verify that the results are in agreement with the expected scheme dependence

obtained from the anomalous dimensions. We also provide a guide on how to actually perform computations in the FDH scheme and show that the modifications compared to CDR are minimal. Finally we present our conclusion in section 5.

## 2 UV and IR structure of massive QCD

### 2.1 DRED and FDH

As has been shown in a series of papers [8, 12, 17, 18, 33], a consistent formulation of the dimensional reduction (DRED) and the four-dimensional helicity (FDH) scheme in the framework of massless QCD requires the introduction of three vector spaces. In this work we investigate how this can be extended to the case of massive partons. In doing so we do not consider processes including external vector fields. The names FDH and DRED are in the following therefore used synonymously, meaning that whenever a statement about the FDH schemes is made, the same argument also applies in DRED. For a detailed discussion and a precise definition of the schemes, of the related vector spaces and their algebraic relations we refer to ref. [14]. Here we only provide the most important characteristics.

In FDH, the underlying quasi 4-dimensional space $Q4S$ with metric $g^{\mu\nu}$ is split into a direct sum of the quasi $D$-dimensional space of CDR with metric $\hat{g}^{\mu\nu}$ and a disjoint space $Q2\epsilon S$ with metric $\tilde{g}^{\mu\nu}$:

$$g^{\mu\nu} = \hat{g}^{\mu\nu} + \tilde{g}^{\mu\nu}. \tag{2.1}$$

In order to have full control over the contributions originating from $Q2\epsilon S$, we *define* complete contractions of the corresponding metric tensors as

$$\tilde{g}^{\mu\nu}\tilde{g}_{\mu\nu} := N_\epsilon. \tag{2.2}$$

As a consequence, arbitrary FDH quantities in general depend on $N_\epsilon$. They are in the following denoted by a bar.

At the level of the Lagrangian, the structure of the different vector spaces is reflected in a split of the quasi 4-dimensional gluon field into a $D$-dimensional gluon field and an $\epsilon$-scalar field: $A^\mu = \hat{A}^\mu + \tilde{A}^\mu$. The 'particles' associated with these fields are in the following denoted by $g$ and $\tilde{g}$, respectively. In refs. [15, 34, 35], it has been shown that because of this split in principle five different couplings need to be distinguished in the bare theory: the gauge coupling $\alpha_s = g_s^2/(4\pi)$, the $\tilde{g}q\bar{q}$ coupling $\alpha_e = g_e^2/(4\pi)$, and three different quartic $\tilde{g}$-couplings. However, for the calculations presented in this work it is sufficient to consider only $\alpha_s$ and $\alpha_e$.

For later purposes it turns out to be useful to include repeatedly occurring universal factors in the definition of the bare couplings

$$a_i(m^2) := e^{-\epsilon\gamma_E}(4\pi)^\epsilon \left(\frac{1}{m^2}\right)^\epsilon \left(\frac{\alpha_i^0}{4\pi}\right) = \left(\frac{\mu^2}{m^2}\right)^\epsilon \bar{Z}_{\alpha_i}\left(\frac{\alpha_i}{4\pi}\right), \tag{2.3}$$

where $\gamma_E$ is the Euler-Mascheroni constant, $m$ is the mass of a heavy fermion, and $a_i \in \{a_s, a_e\}$. As renormalization prescription for the couplings we use the $\overline{\text{MS}}$ scheme

throughout this work. The constants $\bar{Z}_{\alpha_i}$ in FDH are given in e.g. ref. [17]. The perturbative expansion of FDH/DRED quantities in terms of the UV renormalized couplings is in the following written as

$$X^{\text{FDH/DRED}}(\{\alpha\}, N_\epsilon) = \bar{X}(\{\alpha\}, N_\epsilon) \equiv \sum_{m,n}^{\infty} \left(\frac{\alpha_s}{4\pi}\right)^m \left(\frac{\alpha_e}{4\pi}\right)^n \bar{X}_{mn}(N_\epsilon). \qquad (2.4)$$

## 2.2 IR factorization at NNLO in the FDH scheme

In CDR, the IR divergence structure of scattering amplitudes including massive external partons has been investigated up to the two-loop level in ref. [19]. Using a combination of soft-collinear effective theory (SCET) (for an introduction see e.g. ref. [36]) and heavy-quark effective theory (HQET) (for an introduction see e.g. ref. [37]) it has been shown that amplitudes with an arbitrary number of massive and massless legs factorize into a hard and a soft function, where the latter depends on both massive and massless Wilson lines. For amplitudes including massive partons, the corresponding IR anomalous dimension has less constraints compared to the massless case and additional color structures arise.

Starting from the CDR expression for the IR anomalous dimension, we write the two-parton correlation terms of the respective quantity in FDH as

$$\begin{aligned}
\bar{\mathbf{\Gamma}}\left(\{\underline{p}\}, \{\underline{m}\}, \mu\right)\Big|_{\text{2-parton}} &= \sum_{(i,j)} \frac{\mathbf{T}_i \cdot \mathbf{T}_j}{2} \bar{\gamma}_{\text{cusp}}(\{\alpha\}) \ln \frac{\mu^2}{-s_{ij}} + \sum_i \bar{\gamma}_i(\{\alpha\}) \\
&\quad - \sum_{(IJ)} \frac{\mathbf{T}_I \cdot \mathbf{T}_J}{2} \bar{\gamma}_{\text{cusp}}(\beta_{IJ}, \{\alpha\}) + \sum_I \bar{\gamma}_I(\{\alpha\}) \\
&\quad + \sum_{(Ij)} \frac{\mathbf{T}_I \cdot \mathbf{T}_j}{2} \bar{\gamma}_{\text{cusp}}(\{\alpha\}) \ln \frac{m_I \mu}{-s_{Ij}},
\end{aligned} \qquad (2.5)$$

where the capital indices $I, J$ correspond to the massive partons and the angle $\beta_{IJ}$ is defined as

$$\beta_{IJ} := \text{arcosh}\left(\frac{-s_{IJ}}{2\, m_I m_J}\right). \qquad (2.6)$$

For the definition of the color generators $\mathbf{T}_i$, of the kinematic variable $s_{ij}$, and of the sets $\{\underline{p}\}, \{\underline{m}\}$ we refer to [19].

In eq. (2.5), the first line corresponds to contributions from massless partons, already discussed in refs. [8, 17, 18]; the remainder is given by additional terms arising in the massive theory. Suppressing the dependence on the couplings, the complete set of IR anomalous dimensions in FDH/DRED is given by

$$\bar{\gamma}_{\text{cusp}}, \qquad \bar{\gamma}_i \in \{\bar{\gamma}_q, \bar{\gamma}_g, \bar{\gamma}_{\tilde{g}}\}, \qquad (2.7a)$$

$$\bar{\gamma}_{\text{cusp}}(\beta_{IJ}), \qquad \bar{\gamma}_I \in \{\bar{\gamma}_Q\}, \qquad (2.7b)$$

where $\bar{\gamma}_{\tilde{g}}$ only appears in DRED. The quantities in the first line have been computed up to the two-loop level in refs. [8, 17, 18]; the values of $\bar{\gamma}_{\text{cusp}}(\beta_{IJ})$ and $\bar{\gamma}_Q$ are so far unknown and will be given in section 3. Since there is no difference between the IR anomalous dimensions appearing both in FDH and DRED, relation (2.5) also holds in DRED.

In the massive theory, the IR anomalous dimension also contains three-parton correlation terms which we write in FDH as

$$\bar{\mathbf{\Gamma}}\left(\{\underline{p}\},\{\underline{m}\},\mu\right)\Big|_{\text{3-partons}} = if^{abc}\sum_{(I,J,K)}\mathbf{T}_I^a\mathbf{T}_J^b\mathbf{T}_K^c\, F_1\left(\beta_{IJ},\beta_{JK},\beta_{KI}\right)$$

$$+\, if^{abc}\sum_{(I,J)}\sum_k \mathbf{T}_I^a\mathbf{T}_J^b\mathbf{T}_k^c\, f_2\left(\beta_{IJ},\ln\frac{-\sigma_{Ik}\,v_I\cdot p_k}{-\sigma_{Jk}\,v_J\cdot p_k}\right),\quad (2.8)$$

including the four-velocities of the massive partons

$$v_I^\mu := \frac{p_I^\mu}{m_I}\,,\qquad v_I^2 \equiv 1\,.\qquad\qquad (2.9)$$

In refs. [38, 39], the functions $F_1$ and $f_2$ are given for the case of CDR. Since in FDH these functions do not receive evanescent contributions from the $\epsilon$-scalar up to NNLO, eq. (2.8) is a scheme-independent quantity at this order. Its value in FDH is therefore the same as in CDR.

In analogy to the massless case [8, 17], we subtract all IR divergences of QCD loop amplitudes by means of a factor $\bar{\mathbf{Z}}$ which is given by a path-ordered integral over $\bar{\mathbf{\Gamma}}$ (compare with eqs. (2.8) and (2.12) of ref. [8]). This renormalization factor is given in the effective theory where the heavy quarks have been integrated out. Hence, it is written in terms of $\alpha_i$, the couplings defined in the massless theory. In the massive case, however, we also need to take into account contributions from heavy-quark loops. To reproduce the correct IR behavior of the effective low-energy theory we therefore have to perform a matching of the couplings between the full and the effective theory. For an amplitude describing a process with $n$ external partons then the following relation holds:

$$\lim_{\epsilon\to 0}\ \bar{\mathbf{Z}}^{-1}(\{\alpha\})\left[\left|\mathcal{M}_n(\{\alpha\}^f)\right\rangle\right]_{\alpha_i^f\to\zeta_{\alpha_i}\alpha_i} = \text{ finite }.\qquad (2.10)$$

As mentioned above, $\alpha_i$ is a coupling in the effective theory, meaning that the heavy quark flavors have been integrated out. It is related to the corresponding coupling of the full theory via the decoupling relation $\alpha_i^f = \zeta_{\alpha_i}\alpha_i$. Explicit results for the decoupling constants in the FDH scheme will be given in section 2.4.

## 2.3 Mass renormalization of the $\epsilon$-scalar

In the case of massive fermions there is no symmetry that protects the propagator of the $\epsilon$-scalar from acquiring a mass term $\propto m^2\tilde{g}^{\mu\nu}$ where $m$ is a fermion mass. As a consequence, the $\epsilon$-scalar mass is effectively shifted away from zero, even if the $\epsilon$-scalar is massless at the tree-level. Therefore we have to introduce a mass counterterm $\delta m_\epsilon^2$ in the Lagrangian to restore the initial 'on-shell' condition of a vanishing $\epsilon$-scalar mass [40].

At the one-loop level there is only one diagram that effectively generates a mass term in the $\epsilon$-scalar propagator, see figure 1. To obtain the mass counterterm we need to compute the full one-particle irreducible (1PI) two-point function of the $\epsilon$-scalar, whose
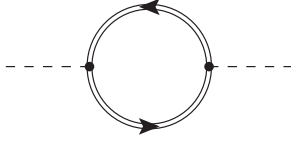
**Figure 1**. One-loop diagram that effectively generates an $\epsilon$-scalar mass at the one-loop level. Massive quarks are depicted by double lines.

tensor structure is given by

$$-i\tilde{\Pi}^{\mu\nu} = -i\tilde{\Pi}\,p^2\,\tilde{g}^{\mu\nu} = -i\left(A + \frac{m^2}{p^2}B\right)p^2\,\tilde{g}^{\mu\nu}\,, \tag{2.11}$$

including the dimensionless quantities $A$ and $B$. The mass counterterm can be extracted by writing the propagator of the $\epsilon$-scalar as

$$\frac{-i\tilde{g}_{\mu\nu}}{p^2\left(1+\tilde{\Pi}\right)+\delta m_\epsilon^2} = \frac{-i\tilde{g}_{\mu\nu}}{p^2\left(1+A\right)+m^2\,B+\delta m_\epsilon^2}\,. \tag{2.12}$$

In order to maintain the $\epsilon$-scalar massless we then require

$$\delta m_\epsilon^2 := -\,m^2\,B = -\,a_e(m^2)\,m^2\,N_H\left[\frac{2}{\epsilon} + 2 + \epsilon\left(2 + \frac{\pi^2}{6}\right) + \mathcal{O}(\epsilon^2)\right] + \mathcal{O}(a^2)\,, \tag{2.13}$$

where $N_H$ denotes the number of heavy quark flavors and the coupling is defined in eq. (2.3). As a consequence, any time we encounter a massive loop diagram insertion as in figure 1, we add the mass counterterm (2.13) in order to impose the on-shell condition of a massless $\epsilon$-scalar.

## 2.4 Decoupling transformations

The decoupling transformation needed in eq. (2.10) is well known for the gauge coupling. In order to extend it to $\alpha_e$ we apply the procedure described in ref. [41] and build an effective Lagrangian in which the heavy quark flavors have been integrated out. As a consequence, the parameters and fields of the effective theory are in general different from the ones of the full theory. To relate the two theories we introduce decoupling constants in the following way:

$$g^{0,f} = \zeta_g^0\,g^0, \qquad X^{0,f} = \sqrt{\zeta_X^0}\,X^0\,, \tag{2.14}$$

where $g$ and $X$ stand for parameters and fields of the theory, respectively. In this way we are able to relate the full and the effective bare QCD Lagrangian in terms of the re-scaled parameters and fields

$$\mathcal{L}^f\left(g_s^{0,f},\,g_e^{0,f},\,\hat{A}^{0,f},\,\tilde{A}^{0,f},\,\psi^{0,f},\dots\right) = \mathcal{L}\left(g_s^0,\,g_e^0,\,\hat{A}^0,\,\tilde{A}^0,\,\psi^0,\dots,\{\zeta_g^0\},\{\zeta_X^0\}\right). \tag{2.15}$$

The decoupling constants can be obtained from a matching calculation. For $\zeta_{\hat{A}}^0$ which is related to the gluon field decoupling, for example, we get

$$\frac{-\hat{g}_{\mu\nu}}{p^2 \left(1 + \hat{\Pi}^{0,f}\right)} = i \int \mathrm{d}^4 x \, e^{i\,px} \langle\, T \, \hat{A}_\mu^{0,f}(x) \, \hat{A}_\nu^{0,f}(0)\rangle \tag{2.16a}$$

$$= i \zeta_{\hat{A}}^0 \int \mathrm{d}^4 x \, e^{i\,px} \langle\, T \, \hat{A}_\mu^0(x) \, \hat{A}_\nu^0(0)\rangle = \zeta_{\hat{A}}^0 \frac{-\hat{g}_{\mu\nu}}{p^2 \left(1 + \hat{\Pi}^0\right)} \,, \tag{2.16b}$$

where $\hat{\Pi}^0$ only contains light degrees of freedom and $\hat{\Pi}^{0,f}$ receives virtual contributions from the heavy quarks. From eqs. (2.16) we then get

$$\zeta_{\hat{A}}^0 = \frac{1 + \hat{\Pi}^0}{1 + \hat{\Pi}^{0,f}} \,. \tag{2.17}$$

Since the l.h.s. does not depend on the kinematics of the process it is possible to consider the special case $p = 0$. The renormalization of the decoupling constant is done in the usual way by means of the gluon field renormalization constants in the effective and the full theory: $\zeta_{\hat{A}} = \bar{Z}_{\hat{A}}/\bar{Z}_{\hat{A}}^f \, \zeta_{\hat{A}}^0$.

The same method also applies to the decoupling of the $\epsilon$-scalar field where, however, according to the discussion in section 2.3 a mass counterterm has to be added in order to maintain the $\epsilon$-scalar massless. In fact, this counterterm is even required to ensure that

$$\zeta_{\tilde{A}}^0 = \left. \frac{1 + \tilde{\Pi}^0}{1 + \tilde{\Pi}^{0,f} + \delta m_\epsilon^2} \right|_{p \to 0} \tag{2.18}$$

is properly defined.

For the calculations in this work we need the decoupling transformations for $\alpha_s$ and $\alpha_e$ at the one-loop level which can be obtained from a matching of the $gq\bar{q}$ and $\tilde{g}q\bar{q}$ vertices, in analogy to eqs. (2.16)

$$\zeta_{g_s}^0 = \frac{1}{\zeta_\psi^0 \sqrt{\zeta_{\hat{A}}^0}} \frac{1 + \Gamma_{\hat{g}q\bar{q}}^{0,f}}{1 + \Gamma_{\hat{g}q\bar{q}}^0} \,, \qquad \zeta_{g_e}^0 = \frac{1}{\zeta_\psi^0 \sqrt{\zeta_{\tilde{A}}^0}} \frac{1 + \Gamma_{\tilde{g}q\bar{q}}^{0,f}}{1 + \Gamma_{\tilde{g}q\bar{q}}^0} \,. \tag{2.19}$$

Since $\zeta_\psi^0$, $(\Gamma_{\hat{g}q\bar{q}}^{0,f} - \Gamma_{\hat{g}q\bar{q}}^0)$, and $(\Gamma_{\tilde{g}q\bar{q}}^{0,f} - \Gamma_{\tilde{g}q\bar{q}}^0)$ are of $\mathcal{O}(\alpha^2)$, the (bare) one-loop decoupling constants for $g_s$ and $g_e$ are entirely given by $\zeta_{\hat{A}}^0$ and $\zeta_{\tilde{A}}^0$, respectively. Using $(\zeta_{g_s}^0)^2 = \zeta_{\alpha_s}^0$ and $\zeta_{\alpha_s} = \bar{Z}_{\alpha_s}/\bar{Z}_{\alpha_s}^f \, \zeta_{\alpha_s}^0$ and similar for the evanescent coupling we finally obtain

$$\zeta_{\alpha_s} = 1 + \left(\frac{\alpha_s}{4\pi}\right) N_H \frac{2}{3} \ln\left(\frac{\mu^2}{m^2}\right) + \mathcal{O}(\alpha^2) \,, \tag{2.20a}$$

$$\zeta_{\alpha_e} = 1 + \left(\frac{\alpha_e}{4\pi}\right) N_H \ln\left(\frac{\mu^2}{m^2}\right) + \mathcal{O}(\alpha^2) \tag{2.20b}$$

for the renormalized decoupling constants of $\alpha_s$ and $\alpha_e$.

## 2.5 Field and mass renormalization of the heavy quarks

To obtain UV-finite Green functions in the FDH scheme we need to perform a renormalization of the heavy quark field and mass, where the corresponding renormalization constants are defined by

$$\psi^0 = \sqrt{\bar{Z}_{2,h}}\,\psi, \qquad m^0 = \bar{Z}_m\,m\,. \tag{2.21}$$

Extending the standard CDR procedure for obtaining renormalization constants in the on-shell (OS) scheme, we write the 1PI self-energy of the heavy quark in FDH as

$$\bar{\Sigma}(p, m, N_\epsilon) = m\,\bar{\Sigma}_1(p^2, m, N_\epsilon) + (\not{p} - m)\,\bar{\Sigma}_2(p^2, m, N_\epsilon)\,. \tag{2.22}$$

The renormalization constants are then given by

$$\left(\bar{Z}_{2,h}\right)^{-1} = 1 + 2m^2 \frac{\partial}{\partial p^2}\bar{\Sigma}_1\big|_{p^2=m^2} + \bar{\Sigma}_2\big|_{p^2=m^2}\,, \tag{2.23a}$$

$$\bar{Z}_m = 1 + \bar{\Sigma}_1\big|_{p^2=m^2}\,. \tag{2.23b}$$

To obtain their values we calculated the quantities $\bar{\Sigma}_1$ and $\bar{\Sigma}_2$ up to the two-loop level, with sample diagrams shown in figure 2. One point of major importance is that apart from genuine two-loop diagrams we have to include contributions originating from UV (sub)renormalization. This in particular comprises the mass counterterm for the $\epsilon$-scalar given in eq. (2.13), see the r.h.s. of figure 2. In terms of the bare couplings we then get

$$
\begin{aligned}
\bar{Z}_{2,h} = {}& 1 + a_s(m^2)\,C_F\left[-\frac{3}{\epsilon} - 4 - \epsilon\left(8 + \frac{\pi^2}{4}\right)\right] + a_e(m^2)\,C_F N_\epsilon\left[-\frac{1}{2\epsilon} - \frac{1}{2} - \epsilon\left(\frac{1}{2} + \frac{\pi^2}{24}\right)\right] \\
&+ a_s^2(m^2)\bigg\{ C_F^2\left[\frac{9}{2\epsilon^2} + \frac{51}{4\epsilon} + \frac{433}{8} - \frac{49}{4}\pi^2 + 16\pi^2\ln(2) - 24\zeta(3)\right] \\
&\qquad\qquad + C_A C_F\bigg[-\frac{11}{2\epsilon^2} - \frac{101}{4\epsilon} - \frac{803}{8} + \frac{49}{12}\pi^2 - 8\pi^2\ln(2) + 12\zeta(3) \\
&\qquad\qquad\qquad\qquad + N_\epsilon\left(\frac{1}{4\epsilon^2} + \frac{11}{8\epsilon} + \frac{5}{24}\pi^2 + \frac{81}{16}\right)\bigg] \\
&\qquad\qquad + C_F N_F\left[\frac{1}{\epsilon^2} + \frac{9}{2\epsilon} + \frac{59}{4} + \frac{5}{6}\pi^2\right] + C_F N_H\left[\frac{2}{\epsilon^2} + \frac{19}{6\epsilon} + \frac{1139}{36} - \frac{7}{3}\pi^2\right]\bigg\} \\
&+ a_e^2(m^2)\,N_\epsilon\bigg\{ C_F^2\left[\frac{1}{\epsilon^2} + \frac{2}{\epsilon} + \frac{\pi^2}{2} - 3 + N_\epsilon\left(-\frac{1}{8\epsilon^2} - \frac{3}{16\epsilon} - \frac{13}{48}\pi^2 + \frac{91}{32}\right)\right] \\
&\qquad\qquad + C_A C_F\left[\left(-\frac{1}{2\epsilon^2} - \frac{1}{\epsilon} - \frac{\pi^2}{4} + \frac{3}{2}\right)\left(1 - \frac{N_\epsilon}{2}\right)\right] \\
&\qquad\qquad + C_F N_F\left[\frac{1}{4\epsilon^2} + \frac{7}{8\epsilon} + \frac{21}{16} + \frac{5}{24}\pi^2\right] + C_F N_H\left[\frac{1}{4\epsilon^2} + \frac{7}{8\epsilon} - \frac{3}{16} + \frac{\pi^2}{24}\right]\bigg\} \\
&+ a_s(m^2)\,a_e(m^2)\,N_\epsilon\bigg\{ C_F^2\left[\frac{3}{2\epsilon} + \frac{47}{4} - \pi^2\right] + C_A C_F\left[-\frac{9}{4\epsilon} - \frac{77}{8} + \frac{\pi^2}{6}\right]\bigg\} + \mathcal{O}(a^3)\,.
\end{aligned}
\tag{2.24}
$$

**Figure 2**. Sample two-loop contributions to the field renormalization of the heavy quark. The diagram on the r.h.s. shows the insertion of the mass counterterm $\delta m_\epsilon^2$.

For later purposes it is convenient to introduce a mass counterterm $\delta m = m - m^0 = m\left(1 - \bar{Z}_m\right)$ for the heavy quarks. Using eq. (2.23b), a direct calculation of $\bar{\Sigma}_1$ yields

$$
\frac{\delta m}{m} = a_s(m^2)\, C_F \left[\frac{3}{\epsilon} + 4 + \epsilon\left(8 + \frac{\pi^2}{4}\right)\right] + a_e(m^2)\, C_F N_\epsilon \left[\frac{1}{2\epsilon} + \frac{1}{2} + \epsilon\left(\frac{1}{2} + \frac{\pi^2}{24}\right)\right]
$$

$$
+ a_s^2(m^2) \Bigg\{ C_F^2 \left[ -\frac{9}{2\epsilon^2} - \frac{45}{4\epsilon} - \frac{199}{8} + \frac{17}{4}\pi^2 - 8\pi^2 \ln(2) + 12\zeta(3)\right]
$$

$$
+ C_A C_F \left[\frac{11}{2\epsilon^2} + \frac{91}{4\epsilon} + \frac{605}{8} - \frac{5}{12}\pi^2 + 4\pi^2 \ln(2) - 6\zeta(3)\right.
$$

$$
\left. + N_\epsilon\left(-\frac{1}{4\epsilon^2} - \frac{9}{8\epsilon} - \frac{5}{24}\pi^2 - \frac{63}{16}\right)\right]
$$

$$
+ C_F N_F \left[-\frac{1}{\epsilon^2} - \frac{7}{2\epsilon} - \frac{45}{4} - \frac{5}{6}\pi^2\right] + C_F N_H \left[-\frac{1}{\epsilon^2} - \frac{7}{2\epsilon} - \frac{69}{4} + \frac{7}{6}\pi^2\right]\Bigg\}
$$

$$
+ a_e^2(m^2) N_\epsilon \Bigg\{ C_F^2 \left[-\frac{1}{\epsilon^2} - \frac{3}{\epsilon} + \frac{\pi^2}{6} - 6 + N_\epsilon\left(\frac{1}{8\epsilon^2} + \frac{13}{16\epsilon} - \frac{11}{48}\pi^2 + \frac{75}{32}\right)\right]
$$

$$
+ C_A C_F \left[\left(\frac{1}{2\epsilon^2} + \frac{3}{2\epsilon} - \frac{\pi^2}{12} + 3\right)\left(1 - \frac{N_\epsilon}{2}\right)\right]
$$

$$
+ C_F N_F \left[-\frac{1}{4\epsilon^2} - \frac{5}{8\epsilon} - \frac{11}{16} - \frac{5}{24}\pi^2\right]
$$

$$
+ C_F N_H \left[-\frac{1}{4\epsilon^2} - \frac{5}{8\epsilon} - \frac{3}{16} - \frac{\pi^2}{24}\right]\Bigg\}
$$

$$
+ a_s(m^2)\, a_e(m^2)\, N_\epsilon \Bigg\{ C_F^2 \left[\frac{3}{2\epsilon} + \frac{23}{4} - \pi^2\right] + C_A C_F \left[\frac{3}{4\epsilon} + \frac{11}{8} + \frac{\pi^2}{2}\right]\Bigg\} + \mathcal{O}(a^3). \quad (2.25)
$$

up to the two-loop level. The pure $\alpha_s$ terms for $N_\epsilon = 0$ correspond to the CDR result.

## 2.6 Field renormalization of the light quarks

In analogy to the previous section we determine the field renormalization of the light quark fields where the corresponding renormalization constant is in the following denoted by $\bar{Z}_{2,l}$. As in the case of heavy quarks, $\bar{Z}_{2,l}$ receives contributions from heavy quark loops, see figure 3. However, there is no one-loop contribution since in dimensional regularization all corresponding loop integrals are scaleless. This also implies that up to the two-loop level
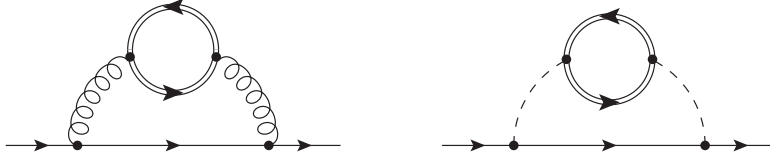
**Figure 3**. Two-loop contributions to the field renormalization of the light quark.

there is no contribution from the $\epsilon$-scalar mass counterterm. The explicit calculation then yields for the field renormalization of the light quark in the FDH scheme

$$\bar{Z}_{2,l} = 1 + C_F N_H \left[ a_s^2(m^2)\left( \frac{1}{2\epsilon} - \frac{5}{12} \right) + a_e^2(m^2) N_\epsilon \left( -\frac{1}{4\epsilon^2} + \frac{3}{8\epsilon} - \frac{13}{16} - \frac{\pi^2}{24} \right) \right] + \mathcal{O}(a^3). \quad (2.26)$$

As for the mass counterterm, the pure $\alpha_s$ terms are of course not new.

## 3 IR anomalous dimensions in the massive case

The aim of this section is to provide all so far unknown IR anomalous dimensions present in the general IR factorization formula (2.5), i.e. $\bar{\gamma}_Q$ and $\bar{\gamma}_{\text{cusp}}(\beta)$. As in the massless case [8], for this we use the SCET framework.

### 3.1 Scheme dependence of the heavy-to-light soft function and $\gamma_Q$

In ref. [42], it has been shown that the top quark decay factorizes into regions where only soft radiation and (or) radiation collinear to the massless partons are present. More precisely, the factorization consists of a hard function whose renormalization group equation (RGE) depends on the heavy-quark anomalous dimension, a quark jet function, and a soft function. In CDR, the jet and soft functions have been calculated up to the two-loop level in refs. [43] and [44], respectively. In FDH, so far only the jet function is known [8].

The general relation between the corresponding IR anomalous dimensions is given by

$$\gamma_Q^{\text{RS}} = \gamma_S^{\text{RS}} + \gamma_J^{\text{RS}} - \gamma_q^{\text{RS}}, \quad (3.1)$$

where $\gamma_S^{\text{RS}}$ and $\gamma_J^{\text{RS}}$ are the (RS-dependent) anomalous dimensions of the soft and jet function. Eq. (3.1) is a direct consequence of the fact that the RGE of the factorization formula does not depend on the factorization scale. The values of $\bar{\gamma}_J = \gamma_J^{\text{FDH/DRED}}$ and $\bar{\gamma}_q = \gamma_q^{\text{FDH/DRED}}$ have been calculated in ref. [8] up to the two-loop level. In order to obtain $\bar{\gamma}_Q = \gamma_Q^{\text{FDH/DRED}}$ we therefore have to compute $\bar{\gamma}_S = \gamma_S^{\text{FDH/DRED}}$.

Extending the approach of ref. [44], we define the scheme-dependent (bare) soft function as

$$S_{\text{bare}}^{\text{RS}}\left( \ln \frac{\Omega}{\mu}, \mu \right) := \int_0^\Omega d\omega \, \langle b_v | \, \bar{h}_v \, \delta(\omega + in \cdot D) \, h_v \, | b_v \rangle, \quad (3.2)$$

where $h_v$ are effective quark fields in HQET (see e.g. ref. [37]), $b_v$ are on-shell $b$-quark states with velocity $v$, and $n$ is a light-like 4-vector with $n \cdot v = 1$ and $n^2 = 0$. The normalization is fixed by $\langle b_v | \, \bar{h}_v \, h_v \, | b_v \rangle = 1$.
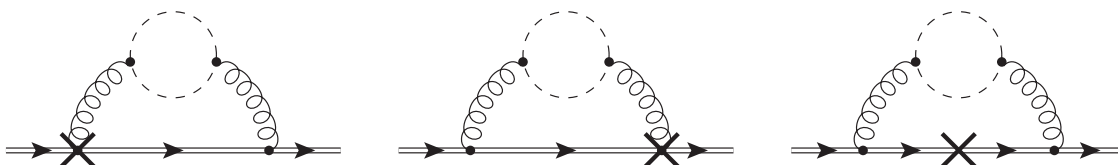
**Figure 4**. Evanescent two-loop contributions to the heavy-to-light soft anomalous dimension in the FDH scheme. The crosses denote the insertion of the operator $(\omega + in \cdot D + i0)^{-1}$.

For explicit calculations it is useful to express the soft function as a contour integral

$$S_{\text{bare}}^{\text{RS}}\left(\ln\frac{\Omega}{\mu}, \mu\right) = \frac{1}{2\pi i} \oint_{|\omega|=\Omega} d\omega \, \langle b_v | \, \bar{h}_v \, \frac{1}{\omega + in \cdot D + i0} \, h_v \, | b_v \rangle = \frac{1}{2\pi i} \oint_{|\omega|=\Omega} d\omega \, \mathcal{S}_{\text{bare}}^{\text{RS}}(\omega)$$
(3.3)

and to work in Laplace space

$$s_{\text{bare}}^{\text{RS}}(\Omega) := \int_0^\infty d\omega \, \exp\left(-\frac{\omega}{\Omega \, e^{\gamma_E}}\right) \frac{1}{\pi} \text{Im}\left[\mathcal{S}_{\text{bare}}^{\text{RS}}(\omega)\right].$$
(3.4)

Since $h_v$ and $b_v$ are Heisenberg fields, the usual perturbative expansion results in loop diagrams contributing to the heavy quark propagator. As in the massless case, the scheme dependence is related to the UV singularities of such diagrams.

At the one-loop level there are no evanescent contributions since the $\epsilon$-scalar does not couple to heavy quark lines, see also ref. [8]. There are exactly three diagrams that induce a scheme dependence of the soft function at the two-loop level. They are shown in figure 4. For the explicit computation we generated the diagrams with QGRAF [45] and applied a tensor reduction of the integrals with Reduze 2 [46], where the master integrals needed in FDH are identical to the ones of CDR given in ref. [44].

In FDH we then get up to the two-loop level

$$\bar{s}_{\text{bare}}(\Omega) = 1 + a_s(\Omega^2) C_F \left[ -\frac{2}{\epsilon^2} + \frac{2}{\epsilon} - \frac{5}{6}\pi^2 + \epsilon\left(\frac{5}{6}\pi^2 - \frac{14}{3}\zeta_3\right) - \epsilon^2\left(\frac{193}{720}\pi^4 - \frac{14}{3}\zeta_3\right) + \mathcal{O}(\epsilon^3) \right]$$

$$+ a_s^2(\Omega^2) C_F \left[ C_F \bar{K}_F(\epsilon) + C_A \bar{K}_A(\epsilon) + \frac{1}{2} N_F \bar{K}_f(\epsilon) \right] + \mathcal{O}(a^3),$$
(3.5)

with

$$\bar{K}_F(\epsilon) = \frac{2}{\epsilon^4} - \frac{4}{\epsilon^3} + \frac{2 + \frac{5\pi^2}{3}}{\epsilon^2} + \frac{-\frac{10}{3}\pi^2 + \frac{28}{3}\zeta(3)}{\epsilon} + \frac{5}{3}\pi^2 - \frac{56}{3}\zeta(3) + \frac{53}{60}\pi^4,$$
(3.6a)

$$\bar{K}_A(\epsilon) = -\frac{11}{6\epsilon^3} + \frac{-\frac{1}{18} + \frac{\pi^2}{6}}{\epsilon^2} + \frac{-\frac{55}{27} - \frac{37}{12}\pi^2 + 9\zeta(3)}{\epsilon} - \frac{326}{81} - \frac{41}{12}\pi^2 - \frac{437}{9}\zeta(3) + \frac{107}{180}\pi^4$$

$$+ N_\epsilon\left(\frac{1}{12\epsilon^3} + \frac{1}{18\epsilon^2} + \frac{\frac{1}{27} + \frac{\pi^2}{8}}{\epsilon} + \frac{2}{81} + \frac{\pi^2}{12} + \frac{25}{18}\zeta(3)\right),$$
(3.6b)

$$\bar{K}_f(\epsilon) = \frac{2}{3\epsilon^3} - \frac{2}{9\epsilon^2} + \frac{-\frac{4}{27} + \pi^2}{\epsilon} - \frac{8}{81} - \frac{\pi^2}{3} + \frac{100}{9}\zeta(3).$$
(3.6c)

Taking the limit $N_\epsilon \to 0$ in eq. (3.5) we obtain the CDR result which is in agreement with the one given in ref. [44].

As for the quark and gluon jet functions [8], all divergences of the soft function can be removed multiplicatively by means of a $Z$ factor

$$s_{\text{sub}}^{\text{RS}}(\Omega,\mu) = Z_S^{\text{RS}}(\Omega,\mu)\, s_{\text{bare}}^{\text{RS}}(\Omega)\,. \tag{3.7}$$

To relate $Z_S^{\text{RS}}(\Omega,\mu)$ with $\gamma_S^{\text{RS}}$ we compare the RGE of the soft function,

$$\frac{d}{d\ln\mu}\, s_{\text{sub}}^{\text{RS}}(\Omega,\mu) = \left[\left(\frac{d}{d\ln\mu}\, Z_S^{\text{RS}}(\Omega,\mu)\right)\left(Z_S^{\text{RS}}(\Omega,\mu)\right)^{-1}\right] s_{\text{sub}}^{\text{RS}}(\Omega,\mu)\,, \tag{3.8}$$

with the RGE written in terms of $\gamma_S^{\text{RS}}$,

$$\frac{d}{d\ln\mu}s_{\text{sub}}^{\text{RS}}(\Omega,\mu) = \left[C_F\,\gamma_{\text{cusp}}^{\text{RS}}\,L_\Omega - 2\gamma_S^{\text{RS}}\right] s_{\text{sub}}^{\text{RS}}(\Omega,\mu)\,, \tag{3.9}$$

where $L_\Omega = \ln(\Omega^2/\mu^2)$ and the cusp anomalous dimension is known from the massless case [8, 17, 18]. In FDH, the factor $\bar{Z}_S$ is given by

$$
\begin{aligned}
\ln\bar{Z}_S = {} & \left(\frac{\alpha_s}{4\pi}\right)\left[\frac{C_F\,\bar{\gamma}_{10}^{\text{cusp}}}{2\epsilon^2} - \frac{1}{\epsilon}\left(\frac{C_F\,\bar{\gamma}_{10}^{\text{cusp}}}{2}\,L_\Omega - \bar{\gamma}_{10}^S\right)\right] \\
& + \left(\frac{\alpha_s}{4\pi}\right)^2\left[-\frac{3\,C_F\,\bar{\gamma}_{10}^{\text{cusp}}\,\bar{\beta}_{20}^s}{8\epsilon^3} + \frac{\bar{\beta}_{20}^s}{2\,\epsilon^2}\left(\frac{C_F\,\bar{\gamma}_{10}^{\text{cusp}}}{2}\,L_\Omega - \bar{\gamma}_{10}^S\right) + \frac{C_F\,\bar{\gamma}_{20}^{\text{cusp}}}{8\,\epsilon^2}\right. \\
& \left. - \frac{1}{2\,\epsilon}\left(\frac{C_F\,\bar{\gamma}_{20}^{\text{cusp}}}{2}\,L_\Omega - \bar{\gamma}_{20}^S\right)\right] + \mathcal{O}(\alpha^3)
\end{aligned} \tag{3.10}
$$

and the coefficients of the $\beta$ function can be found e.g. in ref. [8]. Imposing minimal subtraction with $N_\epsilon$ as an independent quantity we can read off the soft anomalous dimension

$$
\begin{aligned}
\bar{\gamma}_S = {} & \left(\frac{\alpha_s}{4\pi}\right)\left(-2C_F\right) \\
& + \left(\frac{\alpha_s}{4\pi}\right)^2\left\{C_A C_F\left[\frac{110}{27} + \frac{\pi^2}{18} - 18\zeta(3) - N_\epsilon\left(\frac{2}{27} - \frac{\pi^2}{36}\right)\right] + C_F N_F\left[\frac{4}{27} + \frac{\pi^2}{9}\right]\right\} \\
& + \mathcal{O}(\alpha^3)\,,
\end{aligned} \tag{3.11}
$$

which is scheme independent at the one-loop level. Apart from $\bar{\gamma}_S$ it is also possible to extract the already known values of the cusp anomalous dimension as well as the $\beta$ functions in the FDH scheme, which provides a strong consistency check on the applied procedure. Using the obtained results together with eq. (3.1) we then find

$$
\begin{aligned}
\bar{\gamma}_Q = {} & \left(\frac{\alpha_s}{4\pi}\right)\left(-2C_F\right) \\
& + \left(\frac{\alpha_s}{4\pi}\right)^2\left\{C_A C_F\left[-\frac{98}{9} + \frac{2}{3}\pi^2 - 4\zeta(3) + \frac{8}{9}N_\epsilon\right] + C_F N_F\,\frac{20}{9}\right\} + \mathcal{O}(\alpha^3)
\end{aligned} \tag{3.12}
$$

for the IR anomalous dimension of the heavy quarks in the FDH scheme. Like $\bar{\gamma}_S$, at NLO it does not depend on $N_\epsilon$ and is therefore scheme independent, as already found in ref. [20]. However, at NNLO it receives RS-dependent contributions.
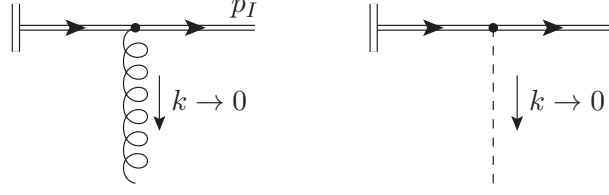
**Figure 5**. Coupling of a gluon (left) and an $\epsilon$-scalar (right) to a heavy quark propagator. In the eikonal approximation the latter vanishes.

Eq. (3.12) is the main result of this section. However, for the sake of completeness we give the result of the finite and scheme-independent soft function by setting $N_\epsilon = 2\epsilon$ and taking the subsequent limit $\epsilon \to 0$

$$s_{\text{fin}}(\Omega, \mu) = \lim_{N_\epsilon, \epsilon \to 0} s_{\text{sub}}^{\text{RS}}(\Omega, \mu) = 1 + \left(\frac{\alpha_s}{4\pi}\right)\left[ - C_F \gamma_{10}^{\text{cusp}} \frac{L_\Omega^2}{4} + \gamma_{10}^S L_\Omega + c_1^S \right]$$

$$+ \left(\frac{\alpha_s}{4\pi}\right)^2 \left[ C_F^2 (\gamma_{10}^{\text{cusp}})^2 \frac{L_\Omega^4}{32} + \left(2\gamma_{10}^S(\gamma_{10}^S - \beta_{20}^s) - C_F(\gamma_{20}^{\text{cusp}} + \gamma_{10}^{\text{cusp}} c_1^S)\right)\frac{L_\Omega^2}{4} \right.$$

$$\left. + \left(\beta_{20}^s - 3\gamma_{10}^S\right) C_F \gamma_{10}^{\text{cusp}} \frac{L_\Omega^3}{12} + \left(c_1^S(\gamma_{10}^S - \beta_{20}^s) + \gamma_{20}^S\right) L_\Omega + c_2^S \right], \tag{3.13}$$

with

$$c_1^S = C_F\left( - \frac{5\pi^2}{6} \right), \tag{3.14a}$$

$$c_2^S = C_F^2\left(\frac{25\pi^4}{72}\right) + C_F C_A\left( - \frac{326}{81} - \frac{233\pi^2}{36} - \frac{283\zeta(3)}{9} + \frac{107\pi^4}{180} \right)$$

$$+ C_F N_F\left( - \frac{4}{81} + \frac{7}{18}\pi^2 + \frac{22}{9}\zeta(3) \right). \tag{3.14b}$$

This result is in agreement with the one given in ref. [44].

## 3.2 Determination of $\bar{\gamma}_{\text{cusp}}(\beta)$

The velocity-dependent cusp anomalous dimensions can be extracted from the heavy-to-heavy soft anomalous dimension $\Gamma_{hh}$ for the pair production of massive quarks. Using CDR, $\Gamma_{hh}$ has been calculated in ref. [47] in the framework of the eikonal approximation. This method can also be used to derive the respective quantity in FDH.

In general, the eikonal approximation is suited for describing the emission of soft gluons from partons in a hard scattering process, see the l.h.s. of figure 5. For a vanishing gluon momentum, the Feynman rule for the coupling of a gluon to a massive quark propagator can be reduced to

$$\bar{u}(p_I)(-ig_s T^a)\hat{\gamma}^\mu\left[i\frac{\slashed{p}_I + \slashed{k} + m_I}{(p_I + k)^2 - m_I^2}\right] \to \bar{u}(p_I)\,g_s T^a\,\hat{\gamma}^\mu\left[\frac{\slashed{p}_I + m_I}{2\,p_I \cdot k}\right] \tag{3.15a}$$

$$= \bar{u}(p_I)\,g_s T^a\left[(p_I)_\nu \frac{\{\hat{\gamma}^\mu, \hat{\gamma}^\nu\}}{2\,p_I \cdot k}\right] \tag{3.15b}$$

$$= \bar{u}(p_I)\,g_s T^a\left[\frac{v_I^\mu}{v_I \cdot k}\right], \tag{3.15c}$$
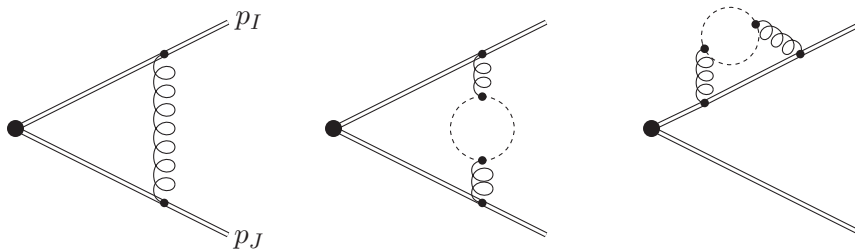
**Figure 6**. One- and two-loop contributions to the heavy-to-heavy soft anomalous dimension in the eikonal approximation. Since there is no direct coupling of $\epsilon$-scalars to massive quark propagators there is no evanescent contribution at the one-loop level.

where in the second line the Dirac equation $\bar{u}(p_I)(\slashed{p}_I - m_I) = 0$ has been used. Since the Feynman rule (3.15c) does not contain a Dirac matrix anymore, the evaluation of loop contributions is much simpler compared to ordinary QCD.

Extending this to the case of an $\epsilon$-scalar we get

$$\bar{u}(p_I)(-ig_e T^a)\tilde{\gamma}^\mu \left[ i\frac{\slashed{p}_I + \slashed{k} + m_I}{(p_I + k)^2 - m_I^2} \right] \to \bar{u}(p_I)\, g_e T^a \left[ (p_I)_\nu \frac{\{\tilde{\gamma}^\mu, \hat{\gamma}^\nu\}}{2\, p_I \cdot k} \right] = 0\,. \tag{3.16}$$

Due to the vanishing anticommutator, a direct coupling of $\epsilon$-scalars to massive quark propagators does not exist in the eikonal approximation.

Following the approach of ref. [47], the soft anomalous dimension for heavy-quark pair production can be obtained from the UV poles of corresponding eikonal diagrams with one- and two-loop examples shown in figure 6. Since there is no direct coupling of $\epsilon$-scalars to massive quarks, the soft anomalous dimension is scheme independent at the one-loop level. At the two-loop level, however, closed $\epsilon$-scalar loops yield evanescent contributions $\propto \alpha_s N_\epsilon$.

In the following, the scalar product of the two outgoing velocity vectors is fixed by $v_I \cdot v_J := -\cosh\beta_{IJ}$ with $\beta_{IJ}$ given in eq. (2.6), and the indices of $\beta$ are suppressed: $\beta_{IJ} =: \beta$. Generalizing eq. (14) of ref. [19] to the case of FDH, the result of the soft anomalous dimension can then be written as

$$\bar{\Gamma}_{hh}(v_I, v_J) = C_F\, \bar{\gamma}_{\text{cusp}}(\beta) + 2\, \bar{\gamma}_Q\,. \tag{3.17}$$

Using eq. (3.12), it is now possible to extract the velocity-dependent cusp anomalous dimension in FDH which in terms of the renormalized couplings reads

$$\bar{\gamma}_{\text{cusp}}(\beta) = \bar{\gamma}_{\text{cusp}}\, \beta \coth\beta + 8\, C_A \left(\frac{\alpha_s}{4\pi}\right)^2 \left\{ \beta^2 + \frac{\pi^2}{6} + \zeta_3 \right.$$

$$+ \coth\beta \left[ \text{Li}_2(e^{-2\beta}) - 2\,\beta \ln(1 - e^{-2\beta}) - \frac{\pi^2}{6}\,(1 + \beta) - \beta^2 - \frac{\beta^3}{3} \right]$$

$$\left. + \coth^2\beta \left[ \text{Li}_3(e^{-2\beta}) + \beta\,\text{Li}_2(e^{-2\beta}) - \zeta_3 + \frac{\pi^2}{6}\,\beta + \frac{\beta^3}{3} \right] \right\} + \mathcal{O}(\alpha^3)\,. \tag{3.18}$$

Since the terms in the curly brackets do not depend on $N_\epsilon$, the scheme dependence of $\bar{\gamma}_{\text{cusp}}(\beta)$ is entirely governed by the scheme dependence of the cusp anomalous dimension in the massless case, i.e. $\bar{\gamma}_{\text{cusp}}$.

## 4 Guideline for FDH calculations and checks of the results

In order to check the obtained results for the scheme dependence of IR divergences in massive QCD we compute the heavy and the heavy-to-light quark form factor in FDH up to the two-loop level. Apart from a pure check this section is also intended to provide a guideline how practical calculations in the FDH scheme can actually be done. For the two-loop calculations we therefore use the following approach:

- At the one-loop level we distinguish the $\epsilon$-scalar from the $D$-dimensional gluon since the related couplings $\alpha_s$ and $\alpha_e$ renormalize differently.

- At the two-loop level we use a (quasi) 4-dimensional Lorentz algebra for the evaluation of genuine two-loop diagrams and do not distinguish the $\epsilon$-scalar from the $D$-dimensional gluon.

- After having applied the UV renormalization we set equal the couplings $\alpha_s$ and $\alpha_e$ in contributions from one-loop counterterm diagrams.

- Throughout the calculations we identify $N_\epsilon = 2\epsilon$.

Using this setup it turns out that practical calculations in the FDH scheme are not significantly more complicated than the respective ones in CDR.

### 4.1 Heavy quark form factor

In CDR, the heavy quark form factor has been calculated up to NNLO in ref. [48]. In FDH, the Green function for the interaction of a virtual photon and two massive quarks can be written as

$$\bar{V}^\mu_{c_1 c_2}(p_1, p_2) = \bar{u}_{c_1}(p_1)\, \bar{\Gamma}^\mu_{c_1 c_2}(p_1, p_2)\, v_{c_2}(p_2)\,, \tag{4.1a}$$
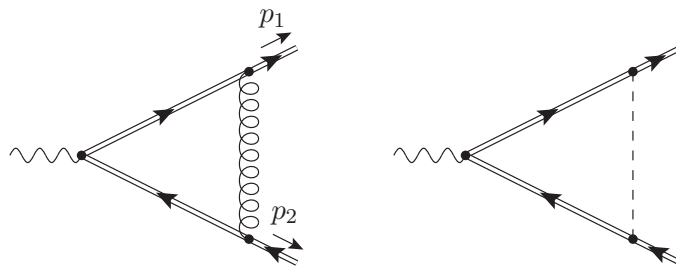
with

$$\bar{\Gamma}^\mu_{c_1 c_2}(p_1, p_2) = -i\, v_Q\, \delta_{c_1 c_2}\left[\bar{F}_1(x)\,\hat{\gamma}^\mu + \frac{1}{2m}\bar{F}_2(x)\, i\, \hat{\sigma}^{\mu\nu} q_\nu\right]. \tag{4.1b}$$

Here and in the following, $p_1$ and $p_2$ denote the (outgoing) momenta of the two external quarks with $p_1^2 = p_2^2 = m^2$ and $s = (p_1 + p_2)^2/m^2$. In general, the $\gamma$ matrices appearing in eq. (4.1b) are scheme-dependent. However, since we are only interested in the structure of $\bar{F}_1$ their dimensionality can be chosen arbitrarily. Here and in the following we therefore use $D$-dimensional $\gamma$ matrices in the Lorentz decomposition.

The IR anomalous dimensions can be obtained from the heavy-quark form factor, $\bar{F}_1$, which can be extracted from eq. (4.1b) by using an appropriate projection operator. For the proper definition of the projection and other details we refer to ref. [48]. In the FDH scheme, only two diagrams contribute to the form factor at the one-loop level, see figure 7. Using 1-dimensional harmonic polylogarithms [49, 50] of the variable

$$x = \frac{\sqrt{-s+4}-\sqrt{-s}}{\sqrt{-s+4}+\sqrt{-s}} \qquad (0 \le x \le 1) \tag{4.2}$$

**Figure 7**. One-loop diagrams contributing to the heavy-quark form factor in FDH.

and notation (2.3) for the couplings, we represent the bare one-loop coefficients of the form factor as[1]

$$\bar{F}_1(x) = 1 + a_s(m^2)\,\bar{\mathcal{F}}_{10}(x) + a_e(m^2)\,\bar{\mathcal{F}}_{01}(x) + \mathcal{O}(a^2)\,, \qquad (4.3)$$

with

$$
\begin{aligned}
\bar{\mathcal{F}}_{10}(x) = 2\,C_F \Bigg\{ & \frac{1}{\epsilon}\left[\frac{1}{2} + H(0;x)\,\frac{x^2+1}{x^2-1}\right] + \frac{1}{2}H(0;x)\,\frac{x+1}{x-1} \\
& - \left(\frac{\pi^2}{6} - H(0;x) - H(0,0;x) + 2H(-1,0;x)\right)\frac{x^2+1}{x^2-1} \\
& + \epsilon\left[\frac{\pi^2}{24} - \left(\frac{\pi^2}{12} - \frac{H(0,0;x)}{2} + H(-1,0;x)\right)\frac{x+1}{x-1} - \left(\frac{\pi^2}{6} - \left(4-\frac{\pi^2}{12}\right)H(0;x)\right.\right. \\
& + 2\,\zeta(3) - \frac{\pi^2}{3}\,H(-1;x) - H(0,0;x) + 2H(-1,0;x) - H(0,0,0;x) \\
& \left.\left. + 2H(-1,0,0;x) + 2H(0,-1,0;x) - 4H(-1,-1,0;x)\right)\frac{x^2+1}{x^2-1}\right] + \mathcal{O}(\epsilon^2) \Bigg\}, \quad (4.4\text{a}) \\
\bar{\mathcal{F}}_{01}(x) = C_F \Bigg\{ & 1 + \epsilon\left[1 + \frac{1-x}{1+x}H(0;x)\right] + \mathcal{O}(\epsilon^2) \Bigg\}. \qquad (4.4\text{b})
\end{aligned}
$$

To obtain the result at the two-loop level we evaluate the Feynman diagrams (see figure 8) using a quasi 4-dimensional Lorentz algebra. This in particular means that the absolute number of diagrams and master integrals [51, 52] we have to evaluate is exactly the same as in CDR. In line with that we do not have to introduce evanescent couplings like $\alpha_e$ in the computation of the genuine two-loop diagrams.

In the following we give the difference between the UV renormalized form factors in FDH and CDR at the two-loop level. For the renormalization of the couplings, the quark mass, and the fields we use eqs. (2.3), (2.25), and (2.24), respectively, and set $\alpha_s = \alpha_e$ after renormalization. Because of the appearing $\epsilon$-scalar propagator in the right diagram of figure 7 we also have to add the mass counterterm (2.13) of the $\epsilon$-scalar. Combining all

---

[1]Note that $\bar{F}_1$ denotes the (all-order) form factor in FDH whereas its perturbative coefficients are written using a calligraphic form, $\left[\bar{F}_1\right]_{mn} = \bar{\mathcal{F}}_{mn}$.
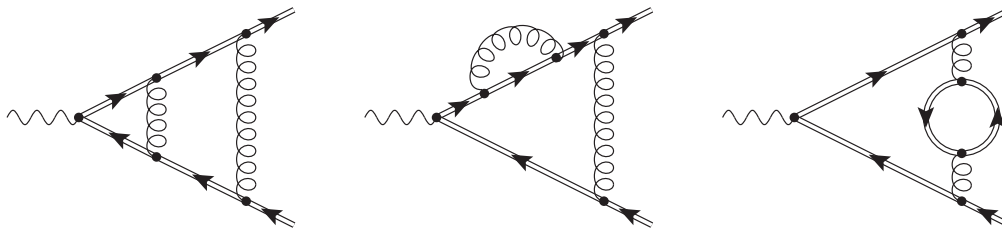
**Figure 8**. Sample two-loop diagrams contributing to the heavy-quark form factor in FDH. All gluons belong to the quasi 4-dimensional space $Q4S$.

results we finally get

$$\bar{F}_1(x) - F_1(x) = \left(\frac{\alpha_s}{4\pi}\right)^2 \left\{ C_A C_F \left[\frac{1}{3\epsilon} - \frac{8}{9}\right] \left(-1 + \frac{x^2+1}{x^2-1} H(0;x)\right) \right. $$
$$\left. + C_F^2 \left[\frac{2\left(x^2-1\right) H(0;x) - 4\left(x^2+1\right) H(0,0;x)}{(x+1)^2}\right] + \mathcal{O}(\epsilon^1) \right\} + \mathcal{O}(\alpha_s^3). \quad (4.5)$$

This difference can be expressed in terms of the IR anomalous dimensions and $\beta$ functions through eqs. (2.5) and (2.10), in a similar way as shown in ref. [8] for the case of massless partons:

$$\bar{F}_1(x) - F_1(x) = \left(\frac{\alpha_s}{4\pi}\right)^2 \left\{ -\frac{1}{\epsilon^2} C_F \left(\bar{\beta}_{20}^s - \beta_{20}^s\right) \left(-1 + \frac{x^2+1}{x^2-1} H(0;x)\right) \right. $$
$$+ \frac{1}{4\epsilon}\left[C_F\left(\bar{\gamma}_{20}^{\mathrm{cusp}}(\beta) - \gamma_{20}^{\mathrm{cusp}}(\beta) - 8\mathcal{F}_1^{\mathrm{diff}}\right) + 2\left(\bar{\gamma}_{20}^Q - \gamma_{20}^Q\right)\right.$$
$$\left.\left. + 8\,C_F\,\mathcal{F}_1^{\mathrm{diff}}\,\frac{x^2+1}{x^2-1}\,H(0;x)\right] + \mathcal{O}(\epsilon^1)\right\} + \mathcal{O}(\alpha_s^3), \quad (4.6)$$

where $\mathcal{F}_1^{\mathrm{diff}} = \bar{\mathcal{F}}_{10}^{\mathrm{ren}} + \bar{\mathcal{F}}_{01}^{\mathrm{ren}} - \mathcal{F}_1^{\mathrm{ren}}$ is the difference of the UV renormalized one-loop coefficients, i.e. including a field renormalization of the heavy quarks. The fact that the scheme dependence of the IR divergences related to the heavy-quark form factor can be predicted with the results from sections 2 and 3 constitutes a strong consistency check of the results obtained so far.
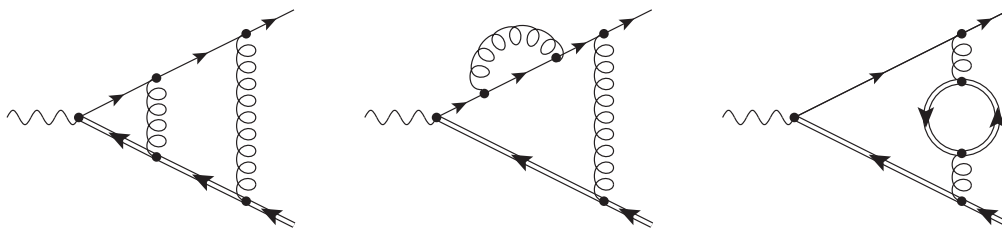
### 4.2 Heavy-to-light form factor

The CDR result for the decay process $b \to u\,W^* \to u\,l\,\bar{\nu}_l$ has been computed at NNLO in refs. [53–56]. Applying the procedure of the previous section we here extend the calculation to the case of FDH, with sample two-loop diagrams shown in figure 9.

In FDH, the tensor structure of the heavy-to-light form factor can be written as

$$\bar{\Gamma}^\mu(p_1, p_2) = \bar{F}_1(q^2)\,\hat{\gamma}^\mu + \frac{1}{2m}\bar{F}_2(q^2)\,\hat{\sigma}^{\mu\nu}\,q_\nu + \frac{i}{2m}\bar{F}_3(q^2)\,q^\mu + \bar{G}_1(q^2)\,\hat{\gamma}^\mu\,\gamma_5$$
$$+ \frac{i}{2m}\bar{G}_2(q^2)\,\gamma_5\,q^\mu + \frac{i}{2m}\bar{G}_3(q^2)\,\gamma_5\,(p_1^\mu - p_2^\mu), \quad (4.7)$$

with $q = p_1 + p_2$. Again, we are interested in the form factor $\bar{F}_1$ which can be extracted by means of a projection operator. Accordingly, the matrix $\hat{\gamma}^\mu$ is treated in $D$ dimensions.

**Figure 9**. Sample two-loop diagrams contributing to the heavy-to-light form factor in FDH.

We compute the bare diagrams up to NNLO and perform the UV renormalization exactly in the same way as described in the previous section, taking into account that here only one leg is massive. Again we have to add a counterterm to subtract the $\epsilon$-scalar mass shift. Using eq. (4.3) for the perturbative expansion of the form factor and expressing the result in terms of the dimensionless quantity

$$y := \frac{q^2}{m^2}\,, \tag{4.8}$$

we get for the bare one-loop coefficients

$$\bar{\mathcal{F}}_{10}(y) = -C_F\left[\frac{1}{\epsilon^2} + \frac{1 + 2\,H(1;y)}{\epsilon} + 4 + \frac{\pi^2}{12} + 3\,H(1;y) + 2\,H(0,1;y) + 4\,H(1,1;y)\right.$$

$$+ \epsilon\left(8 + \frac{\pi^2}{12} - \frac{\zeta(3)}{3} + \left(8 + \frac{\pi^2}{6}\right)H(1;y) + 3\,H(0,1;y) + 6\,H(1,1;y)\right.$$

$$+ 8\,H(1,1,1;y) + 4\,H(-1,0,-1;-y) + 4\,H(0,-1,-1;-y)$$

$$\left.\left. + 2\,H(0,0,1;y)\right)\right] + \mathcal{O}\left(\epsilon^2\right)\,, \tag{4.9a}$$

$$\bar{\mathcal{F}}_{01}(y) = C_F\left[1 + \epsilon\Big(1 + H(1;y)\Big)\right] + \mathcal{O}\left(\epsilon^2\right)\,. \tag{4.9b}$$

As in the previous section we give the difference between the UV renormalized form factors in FDH and CDR up to the two-loop level:

$$\bar{F}_1(y) - F_1(y) = \left(\frac{\alpha_s}{4\pi}\right)\frac{C_F}{2}$$

$$+ \left(\frac{\alpha_s}{4\pi}\right)^2\left\{C_A C_F\left[-\frac{1}{4\epsilon^2} + \frac{-\frac{25}{36} - \frac{1}{3}H(1;y) - \frac{L}{6}}{\epsilon} + \frac{965}{216} + \frac{\pi^2}{24} + \frac{8}{9}H(1;y) + \frac{4}{9}L\right]\right.$$

$$- C_F^2\left[\frac{1}{2\epsilon^2} + \frac{\frac{9}{4} + 2H(1;y) + L}{\epsilon} + \frac{49}{8} + \frac{\pi^2}{4} + \left(6 + 4L\right)H(1;y)\right.$$

$$\left. + 8H(1,1;y) + 2H(0,1;y) + \frac{7}{2}L + L^2\right]$$

$$\left. + C_F N_F\left[\frac{1}{4\epsilon} - \frac{3}{8}\right] - C_F N_H \frac{L}{2} + \mathcal{O}(\epsilon^1)\right\} + \mathcal{O}(\alpha_s^3)\,, \tag{4.10}$$

where $L$ is defined as $L = \ln\left(\frac{\mu^2}{m^2}\right)$. In terms of the IR anomalous dimensions, the $\beta$ functions, and the factor $\bar{\mathbf{Z}}$ defined in eq. (2.10) this difference is given by

$$
\begin{aligned}
\bar{F}_1(y) - F_1(y) = {} & \left(\frac{\alpha_s}{4\pi}\right)\frac{\bar{\gamma}_{01}^q}{2\epsilon} \\
& + \left(\frac{\alpha_s}{4\pi}\right)^2 \Bigg\{ \frac{3}{16\epsilon^3} C_F \gamma_{10}^{\text{cusp}} \left(\bar{\beta}_{20}^s - \beta_{20}^s\right) - \frac{1}{16\epsilon^2}\Bigg[ \bar{\gamma}_{01}^q \left(4(\bar{\beta}_{11}^e + \bar{\beta}_{02}^e) + 2\bar{\gamma}_{01}^q - 4\gamma_{10}^Q\right) \\
& \qquad + \left(\bar{\beta}_{20}^s - \beta_{20}^s\right)\left(4\left(\gamma_{10}^Q + \gamma_{10}^q\right) - 2\,C_F\gamma_{10}^{\text{cusp}}\left(2\,H(1;y) + L\right)\right) \\
& \qquad + C_F\left(\bar{\gamma}_{20}^{\text{cusp}} - \gamma_{20}^{\text{cusp}} - 8\bar{\gamma}_{01}^q\right) + 4\,C_F\gamma_{10}^{\text{cusp}}\mathcal{F}_1^{\text{diff}} \Bigg] \\
& \quad + \frac{1}{4\epsilon}\Bigg[ -\frac{1}{2}C_F\left(2\,H(1;y) + L\right)\left(\bar{\gamma}_{20}^{\text{cusp}} - \gamma_{20}^{\text{cusp}} + 2\,\gamma_{10}^{\text{cusp}}\mathcal{F}_1^{\text{diff}}\right) \\
& \qquad + (\bar{\gamma}_{20}^Q - \gamma_{20}^Q) + (\bar{\gamma}_{20}^q - \gamma_{20}^q) + \bar{\gamma}_{11}^q + \bar{\gamma}_{02}^q - 2\,N_H\,\bar{\gamma}_{01}^q\,L \\
& \qquad + 2\,\mathcal{F}_1^{\text{diff}}\left(\gamma_{10}^Q + \gamma_{10}^q + \bar{\gamma}_{01}^q\right) + 2\,\bar{\gamma}_{01}^q\,\mathcal{F}_1^{\text{fin}} \Bigg] + \mathcal{O}(\epsilon^1) \Bigg\} + \mathcal{O}(\alpha_s^3),
\end{aligned}
$$

(4.11)

with

$$
\mathcal{F}_1^{\text{diff}} = \bar{\mathcal{F}}_{10}^{\text{ren}} + \bar{\mathcal{F}}_{01}^{\text{ren}} - \mathcal{F}_1^{\text{ren}},
$$

(4.12a)

$$
\mathcal{F}_1^{\text{fin}} = \lim_{\epsilon \to 0}\left[\bar{\mathcal{F}}_{10}^{\text{ren}} + \delta\bar{\mathbf{Z}}_{10}\right] = \lim_{\epsilon \to 0}\left[\mathcal{F}_{10}^{\text{ren}} + \delta\mathbf{Z}_1\right].
$$

(4.12b)

The fact that eq. (4.10) matches with eq. (4.11) constitutes an additional and independent check of our results for the IR anomalous dimensions.

## 5 Conclusions

The scheme dependence of massless QCD amplitudes at NNLO had been discussed in ref. [8]. In this paper we complete this study by extending it to amplitudes containing massive quarks.

This requires modifications in the UV and IR sector. For the UV part, the presence of heavy quarks modifies the renormalization. In particular, the $\epsilon$-scalar field requires a mass counterterm. Also, the decoupling of $\alpha_e$ (the coupling of the $\epsilon$-scalars to the quarks) has to be determined. Furthermore, we have computed the additional contributions required in FDH in the quark mass and the quark wave-function renormalization.

Regarding the IR part, the important result is that the IR structure of massive QCD amplitudes in FDH (and DRED) is the same as in CDR (and HV). The only change is in the explicit scheme-dependent expressions of the various anomalous dimensions. In the massive case, there are two additional anomalous dimensions, the velocity-dependent cusp anomalous dimension and the heavy-quark anomalous dimension. We have computed them in the FDH scheme, using a SCET approach.

We have checked our results by computing the heavy-quark and heavy-to-light form factor in FDH at NNLO. These results differ from the corresponding expressions in CDR. After UV renormalization, the difference can be reproduced by the scheme dependence of the IR factorization formula. This provides us with a strong consistency check and establishes FDH as a consistent regularization scheme also in the massive case, at least to NNLO.

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