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On the Seshadri constants of adjoint line bundles

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Abstract. In the present paper we study the possible values of Seshadri constants. While in general every positive rational number appears as the local Seshadri constant of some ample line bundle, we point out that for adjoint line bundles there are explicit lower bounds depending only on the dimension of the underlying variety. In the surface case, where the optimal lower bound is 1/2, we characterize all possible values in the range between 1/2 and 1—there are surprisingly few. As expected, one obtains even more restrictive results for the Seshadri constants of adjoints of very ample line bundles. Finally, we study Seshadri constants of adjoint line bundles in the multi-point setting.

1. Introduction

Seshadri constants are interesting invariants of ample line bundles on algebraic varieties. They were introduced by Demailly in [5] and may be thought of as capturing the local positivity of a given line bundle. A nice introduction to this circle of ideas is given in [11, Sect. 5], an overview of recent results can be found in [4]. Here we merely recall the basic definition:

Let X be a smooth projective variety, L an ample line bundle on X, and $x \in X$ a point on X. The number

$$\varepsilon(L, x) := \inf_{C \ni x} \frac{L \cdot C}{\operatorname{mult}_x C}$$

is the *Seshadri constant of* L *at* x, whereas

$$\varepsilon(L) := \inf_{x \in X} \varepsilon(L, x)$$

is the *Seshadri constant of* L.

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While $\varepsilon(L)$ is always a positive number, Miranda [11, Example 5.2.1] showed that there is no uniform positive lower bound for Seshadri constants of ample line bundles on varieties of fixed dimension. The purpose of the present note is to show that for adjoint line bundles, Seshadri constants exhibit surprisingly regular behavior.

Here is a more detailed description of the content of this paper:

(1) While every positive rational number occurs as a Seshadri constant of some (integral) ample line bundle (Proposition 2.1), we show that there exists a uniform lower bound in the adjoint setting, i.e., for ample bundles $K_X + L$, where L is nef (Theorem 3.2).
(2) On surfaces we show that the potential values that $\varepsilon(K_X + L, x)$ can assume in the interval (0, 1) form a monotone increasing sequence with limit 1 (Theorem 4.1).
(3) Still on surfaces, we prove that in the 'hyper-adjoint' setting no values below 1 occur for $\varepsilon(K_X + L, x)$, and we classify the borderline case (Theorem 4.6).
(4) We complete the picture by looking at the multi-point setting, where we provide a uniform lower bound for adjoint bundles (Proposition 5.6). On surfaces we answer the question corresponding to (2) by showing that there are only finitely many possible values (Theorem 5.7).

We work throughout over the field of complex numbers.

2. Possible values of Seshadri constants

The first observation is that in general every positive rational number occurs as a Seshadri constant:

Proposition 2.1. *For every rational number $q > 0$ there exists a smooth projective surface X, an (integral) ample line bundle L on X, and a point $x \in X$ such that*

$$\varepsilon(L, x) = q.$$

Proof. We write $q = \frac{d}{m}$. In the proof we follow closely Miranda's idea (cf. [10, Prop. 5.2]). We construct X as a blow-up of the projective plane, but in fact an analogous argument using [2, Lemma 3.5] would work on a suitable blow-up of an arbitrary smooth projective surface. For a suitably large integer k, the following holds true:

(i) There exists an irreducible plane curve C_1 of degree k with a point x of multiplicity m.
(ii) There exists another curve C_2 of the same degree k such that
   - C_1 and C_2 intersect transversally in k^2 distinct points, and
   - all curves in the pencil V generated by C_1 and C_2 are irreducible.

The existence of C_1 and C_2 is basically a dimension count on sections of $\mathcal{O}_{\mathbb{P}^2}(k)$ plus Bertini's theorem. Let now $f : X \longrightarrow \mathbb{P}^2$ be the blow-up of $\mathbb{P}^2$ at the intersection points of C_1 and C_2, with exceptional divisors $E_1, \ldots, E_{k^2}$. Thus, by (ii), the surface X is fibred over $\mathbb{P}^1$ by the irreducible curves from the pencil V. It is easy to verify that the line bundle $L = E + 2C$ is ample, where C denotes the class of the fiber on X and E is a fixed exceptional divisor. With a slight abuse of notation, we denote the preimage of x on X again by x. Then we have by (i)

$$\frac{L \cdot \widetilde{C_1}}{\mathrm{mult}_x \widetilde{C_1}} = \frac{1}{m}$$

for the proper transform $\widetilde{C_1}$ of C_1, so that in any case $\varepsilon(L, x) \leqslant \frac{1}{m}$. But for any irreducible curve D on X passing through x (hence different from E) and different from $\widetilde{C_1}$ we have

$$L \cdot D = (E + 2\widetilde{C_1}) \cdot D \geqslant 2m \cdot \mathrm{mult}_x D, \tag{1}$$

so that in fact $\varepsilon(L, x) = \frac{1}{m}$. Replacing L by dL we get $\varepsilon(dL, x) = \frac{d}{m}$, as claimed. □

Remark 2.2. One can easily generalize this construction to arbitrary dimension $n + 2 \geqslant 3$, following the idea of [11, Example 5.2.2]: to this end, let $Y = X \times \mathbb{P}^n$, where X is the surface constructed in the proof of Proposition 2.1, let $M := \mathrm{pr}_1^* L \times \mathrm{pr}_2^* H$, where L is the line bundle from the previous proof and H is the hyperplane bundle on $\mathbb{P}^n$. Furthermore, let $p \in \mathbb{P}^n$ be a fixed point and $Y_p = X \times \{p\}$. (We identify Y_p with X, in particular we view now the curve $\widetilde{C_1}$ as a subvariety of Y_p.) Then

$$\varepsilon(M, (x, p)) = \frac{1}{m}.$$

In fact, it follows from the projection formula that

$$M \cdot \widetilde{C_1} = L \cdot \widetilde{C_1} = 1,$$

so that in the point (x, p) we have in any case

$$\varepsilon(M, (x, p)) \leqslant \frac{1}{m}.$$

Let now D be another curve passing through the point (x, p). If D is not contained in Y_p, then

$$M \cdot D \geqslant \mathrm{pr}_2^* H \cdot D \geqslant \mathrm{mult}_{(x,p)} D,$$

which shows that D cannot give a lower Seshadri quotient than $\widetilde{C_1}$. If on the other hand D is contained in Y_p, then we conclude the same exactly as in (1).

Thus we saw that every positive rational number appears as the Seshadri constant of some ample line bundle on a variety of dimension $\geqslant 2$. On the other hand it is not known—and it would be extremely interesting to know—whether there exist irrational Seshadri constants [11, Remark 5.1.13].

3. Seshadri constants of adjoint line bundles

Now we show that there exists a uniform lower bound on Seshadri constants of adjoint line bundles. This is a direct consequence of the following result of Angehrn and Siu [1, Theorem 0.1], but it seems that it has not been explicitly noticed so far.

Theorem 3.1. (Angehrn-Siu) *Let X be a smooth projective variety of dimension n and let A be an ample divisor on X. Assume that*

$$(A^d \cdot Z) \geqslant \left(\binom{n+1}{2}+1\right)^d$$

for every irreducible subvariety $Z \subset X$ of positive dimension d. Then the adjoint line bundle $K_X + A$ is globally generated.

Theorem 3.2. *Let X be a smooth projective variety of dimension n. Let L be a nef line bundle on X and assume that the adjoint line bundle $K_X + L$ is ample. Then*

$$\varepsilon(K_X + L) \geqslant \frac{2}{n^2+n+4}.$$

Proof. We claim that

$$m(K_X + L) \text{ is globally generated for } m \geqslant \binom{n+1}{2} + 2.$$

In fact, take an integer $m \geqslant \binom{n+1}{2} + 2$ and let $A := (m-1)(K_X + L) + L$. This line bundle is ample and it satisfies the inequality

$$(A^d \cdot Z) \geqslant (m-1)^d$$

for any subvariety $Z \subset X$ of positive dimension d. Therefore the numerical condition in Theorem 3.1 is satisfied, and hence the adjoint bundle

$$K_X + A = m(K_X + L)$$

is globally generated.

Now, Seshadri constants of globally generated ample line bundles are at least 1 (see [11, Example 5.1.18]), and this implies the assertion after dividing by m. □

Remark 3.3. One can obtain an improved bound for $\varepsilon(K_X + L)$ by using Heier's result [7], which says that for any nef line bundle N and any integer $m \geqslant (e + \frac{1}{2}) n^{\frac{4}{3}} + \frac{1}{2} n^{\frac{2}{3}} + 1$ the bundle $K_X + mL + N$ is base-point free.[1] Writing $m(K_X + L) = K_X + (m-1)(K_X + L) + L$ and arguing as in the proof of Theorem 3.1, we get

$$\varepsilon(K_X + L) \geqslant \frac{1}{\left(e + \frac{1}{2}\right) n^{\frac{4}{3}} + \frac{1}{2} n^{\frac{2}{3}} + 2}.$$

[1] Actually, in the Main Theorem of [7] there is no mention of a nef bundle N, but as G. Heier informed us, his result remains true in the form needed here.

Remark 3.4. It is quite unlikely that the particular bounds on $\varepsilon(K_X + L)$ given by Theorem 3.2 and Remark 3.3 are sharp. The important observation is that Seshadri constants of adjoint line bundles are bounded from below by a universal number depending only on the dimension of the underlying variety.

There are two important classes of varieties where all ample line bundles can be written as adjoints of ample bundles. On these varieties we have universal lower bounds valid for all ample line bundles in all points. In particular we have in these cases a positive answer to the following problem raised by Demailly [5, Question 6.9].

Question 3.5. Let $\varepsilon(X)$ be the infimum of the numbers $\varepsilon(L)$ taken over all integral ample line bundles on X. Is the number $\varepsilon(X)$ positive, and if so, is there an effective lower bound on $\varepsilon(X)$?

Corollary 3.6. *Let X be a variety of dimension n with nef anti-canonical divisor. Then*

$$\varepsilon(X) \geqslant \frac{2}{n^2 + n + 4}.$$

So in particular there is a universal lower bound for Seshadri constants on (*weak*) *Fano varieties and varieties with numerically trivial canonical divisor.*

Remark 3.7. Note that the lower bound $\varepsilon(X) \geqslant \frac{1}{n-2}$ was proved before for Fano varieties of dimension $n \geqslant 3$ by Lee [12, Theorem 1.1] under the additional assumption that the anticanonical bundle $-K_X$ be globally generated. It seems that the existence of a lower bound valid without any restrictions is new.

4. Seshadri constants of adjoint line bundles on surfaces

For surfaces, i.e., $n = 2$, Theorem 3.2 gives $\frac{1}{5}$ as the lower bound. One could invoke Reider's theorem in this case to improve this number to $\frac{1}{4}$. However, we show here that the optimal lower bound for the Seshadri constants of an adjoint line bundle on a surface is $\frac{1}{2}$, and we give further restrictions for the possible values in the range below 1.

Theorem 4.1. *Let X be a smooth projective surface and L a nef line bundle such that $K_X + L$ is ample. If for some point $x \in X$ the Seshadri constant $\varepsilon(K_X + L, x)$ lies in the interval* $(0, 1)$*, then*

$$\varepsilon(K_X + L, x) = \frac{m-1}{m}$$

for some integer $m \geqslant 2$.

Proof. Let x be a point such that $\varepsilon(K_X + L, x) < 1$. Then there exists a curve $C \subset X$ such that

$$\varepsilon(K_X + L, x) = \frac{(K_X + L) \cdot C}{\operatorname{mult}_x(C)} = \frac{d}{m}.$$

By assumption, we have

$$d \leqslant m - 1. \tag{2}$$

By the Index Theorem, we have

$$d^2 = ((K_X + L) \cdot C)^2 \geqslant C^2 (K_X + L)^2, \tag{3}$$

so that in any case $C^2 \leqslant d^2$. The nefness of L and the adjunction formula imply that we have the following upper bound on the arithmetic genus of C:

$$p_a(C) = 1 + \frac{1}{2}C^2 + \frac{1}{2}C \cdot K_X \leqslant 1 + \frac{1}{2}d^2 + \frac{1}{2}C \cdot (K_X + L) = 1 + \frac{d(d+1)}{2}.$$

On the other hand, a curve having a point of multiplicity m is subject to the following inequality

$$p_a(C) \geqslant \binom{m}{2} = \frac{m(m-1)}{2}. \tag{4}$$

Combining these two inequalities, we see that for $m \geqslant 2$ we must have

$$d \geqslant m - 1.$$

Together with (2) this gives the claim. □

The following lower bound is a direct consequence of the above Proposition.

Corollary 4.2. *Let X be a smooth projective surface and L a nef line bundle such that $K_X + L$ is ample. Then*

$$\varepsilon(K_X + L, x) \geqslant \frac{1}{2}$$

for every point $x \in X$.

Now we show that the bound in Corollary 4.2 is sharp.

Example 4.3. Let X be a general surface of degree 10 in weighted projective space $\mathbb{P}(1, 1, 2, 5)$. Then X is smooth, K_X is ample with $K_X^2 = 1$, and there is a point $x_0 \in X$ such that there exists a canonical curve $D \in |K_X|$ with a double point in x_0. For details we refer to [3, Example 1.2]. Taking L to be the trivial line bundle, we see that

$$\varepsilon(K_X + L, x_0) = \frac{(K_X + L) \cdot D}{\operatorname{mult}_{x_0} D} = \frac{1}{2}.$$

This example was extreme in the sense that K_X was already ample and we took L to be trivial. In the next example we show that the Seshadri constant $\frac{1}{2}$ is possible also at the other extreme, i.e., when K_X trivial and L ample.

Example 4.4. Let X be a K3 surface with intersection matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix}.$$

Such a surface exists by [13, Corollary 2.9]. Moreover, by [9, Theorem 2] there exist effective curves Γ and E such that $\Gamma^2 = -2$, $E^2 = 0$ generating the Picard group of X. In particular, we have $\Gamma \cdot E = 1$. The line bundle $L = \mathcal{O}_X(\Gamma + 3E)$ is ample. It intersects every curve in the pencil $|E|$ with multiplicity 1, so that there are no reducible curves in the pencil. On the other hand, the elliptic fibration defined by $|E|$ must have singular fibers. If E_0 is such a singular fiber, then it has a double point x_0. We have again

$$\varepsilon(K_X + L, x_0) = \frac{L \cdot E_0}{\operatorname{mult}_{x_0} E_0} = \frac{1}{2}.$$

Remark 4.5. If both K_X and L are ample, then the Seshadri constant of $K_X + L$ is at least 1. To see this, it suffices to repeat the proof of Theorem 4.1, taking into account that the self-intersection of $K_X + L$ is in that case at least 4, so that the Index Theorem as in (3) implies now $C^2 \leqslant \frac{1}{4}$. Combining this again with the lower bound on $p_a(C)$ we get a contradiction to (2).

One might hope that there exist statements stronger than Corollary 4.2 for 'hyper-adjoint' bundles, i.e., for adjoints $K_X + L$ of *very* ample line bundles L. This is indeed the case:

Theorem 4.6. *Let X be a smooth projective surface and L a very ample line bundle on X such that $K_X + L$ is ample. Then*

(a) $\varepsilon(K_X + L) \geqslant 1$.
(b) If $\varepsilon(K_X + L, x) = 1$ for all $x \in X$, then either $(X, L) = (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(4))$ or X is a ruled surface. In the latter case, one has $L = -3C_0 + s \cdot f$, where C_0 is a section, f a fiber of the ruling, and s a positive integer.

Proof. (a) Let $x \in X$ and let $C \subset X$ be an irreducible curve passing through x with $m := \operatorname{mult}_x C$. We will show that $\frac{1}{m}(K_X + L) \cdot C \geqslant 1$. Suppose first that $L \cdot C \leqslant 2$. As L is very ample, the curve C is then a line or a smooth conic, and therefore $\frac{1}{m}(K_X + L) \cdot C$ is an integer $\geqslant 1$ in that case. Suppose then that $L \cdot C \geqslant 3$. This inequality implies, when arguing as in the proof of Theorem 4.1, that

$$p_a(C) + \frac{3}{2} \leqslant 1 + \frac{1}{2}C^2 + \frac{1}{2}C(K_X + L) \leqslant 1 + \frac{d(d+1)}{2}$$

Using the inequality (4) we get $d \geqslant m$, and this completes the proof of (a).

There is an alternative, adjunction-theoretic approach for the proof of assertion (a) as follows. The situation described in the proposition was studied by Sommese and Van de Ven: In [14, Theorem 0.1] they showed that the adjoint line bundle $K_X + L$ is globally generated unless

- $X = \mathbb{P}^2$ and $L = \mathcal{O}(d)$, with d equal to either 1 or 2, or
- X is a smooth quadric in $\mathbb{P}^3$ and L is the hyperplane bundle, or
- X is a $\mathbb{P}^1$ bundle over a smooth curve and L restricted to any fiber is $\mathcal{O}_{\mathbb{P}^1}(1)$.

It is easy to see that under our assumptions none of the exceptional cases is possible, so that the claim follows using the fact that the Seshadri constants of ample and globally generated line bundles are $\geqslant 1$ (see [11, Example 5.1.18]).

(b) We will make use of the adjunction mapping

$$\varphi_{K_X+L} : X \to \mathbb{P}^N,$$

which by the cited result of Sommese and Van de Ven is a morphism.

Suppose first that $(K_X + L)^2 = 1$. Then the image of φ_{K_X+L} is $\mathbb{P}^2$ and we are done.

So it remains to consider that case that $(K_X + L)^2 \geqslant 2$. By assumption there exists a family of curves $C \subset X$ and points $x \in X$ such that

$$\frac{(K_X + L) \cdot C}{m} = 1$$

where $m = \text{mult}_x C$ (cf. [6]). We claim first that

$$m = (K_X + L) \cdot C = 1. \tag{5}$$

Indeed, if we had $m \geqslant 2$, then by [15, Lemma 1] (or [8, Theorem A]) we would have the inequality

$$C^2 \geqslant m(m-1) + 1.$$

Upon using the Index theorem, this implies

$$4(m(m-1)+1) \leqslant 4C^2 \leqslant (K_X + L)^2 C^2 \leqslant ((K_X + L) \cdot C)^2 = m^2$$

and this is a contradiction, establishing (5).

Next we wish to show that $C^2 = 0$. In fact, applying the Index theorem again, we see that

$$4C^2 \leqslant C^2(K_X + L)^2 \leqslant ((K_X + L) \cdot C)^2 = 1$$

and hence $C^2 \leqslant 0$. The possibility that $C^2 < 0$ is excluded as the curves move in a family.

We next claim that the curves C are smooth and rational with $K_X \cdot C = -2$. Indeed, we have $K_X \cdot C < (K_X + L) \cdot C = 1$, hence $K_X \cdot C \leqslant 0$. Using this inequality, together with $C^2 = 0$ and the adjunction formula

$$0 \leqslant p_a(C) = 1 + \frac{1}{2}C^2 + \frac{1}{2}K_X \cdot C$$

implies the claim.

In order to prove now that X is a ruled surface, we show that for some integer $k \geqslant 1$ the linear series $|kC|$ is a basepoint-free pencil. To this end, consider for $k \geqslant 1$ the short exact sequence

$$0 \to \mathcal{O}_X((k-1)C) \to \mathcal{O}_X(kC) \to \mathcal{O}_C(kC) \to 0$$

Its cohomology sequence tells us that if $h^0(X, (k-1)C) = h^0(X, kC)$, then $h^1(X, kC) < h^1(X, (k-1)C)$. Therefore there exists a k such that

$$h^0(X, kC) > h^0(X, (k-1)C) \tag{6}$$

and hence $|kC|$ is a pencil. The curve C is the only possible base curve, but we see from (6) that it cannot be the base part of $|kC|$.

Finally, after taking the Stein factorization and normalizing, we may assume that the general element f of $|kC|$ is irreducible. We then see from $0 \leqslant p_a(f) = 1 + \frac{1}{2}k^2C^2 + \frac{1}{2}C \cdot K_X = 1 - k$ that $k = 1$, and therefore $L \cdot f = 3$. This implies that L is of the form that is asserted in the statement of the theorem. □

Remark 4.7. (a) The example of the projective plane $\mathbb{P}^2$ and $L = \mathcal{O}_{\mathbb{P}^2}(4)$ shows that the bound in part (a) of the previous proposition cannot be improved.

(b) One might hope that in part (b) of the theorem it could suffice to ask that $\varepsilon(K_X + L, x) = 1$ holds for *infinitely many* points x instead of requiring it on *all* points x. But the example of a smooth quartic surface $X \subset \mathbb{P}^3$ containing a line ℓ, with $L = \mathcal{O}_X(1)$ and $x \in \ell$ shows that this is not the case.

5. Multi-point Seshadri constants of adjoint line bundles

Some applications, notably in multivariate interpolation and in Nagata and Harbourne-Hirschowitz problems require knowledge of the multi-point version of the Seshadri constants defined in the introduction.

Definition 5.1. Let X be a smooth projective variety and L be an ample line bundle on X. Let r be a positive integer and $x_1, \ldots, x_r$ be arbitrary pairwise distinct points on X. The real number

$$\varepsilon(L; x_1, \ldots, x_r) = \inf_{C \cap \{x_1, \ldots, x_r\} \neq \varnothing} \frac{L \cdot C}{\sum_{i=1}^{r} \operatorname{mult}_{x_i} C}$$

is the *multi-point Seshadri constant* of L at $x_1, \ldots, x_r$.

It is easy to check that

$$\varepsilon(L; x_1, \ldots, x_r) \geqslant \frac{1}{\sum_{i=1}^{r} \frac{1}{\varepsilon(L, x_i)}}, \tag{7}$$

so that a lower bound on $\varepsilon(L)$ gives an immediate lower bound on $\varepsilon(L; x_1, \ldots, x_r)$.

Without any restrictions on L we can again produce examples of line bundles with arbitrary rational multi-point Seshadri constants quite along lines of Proposition 2.1:

Proposition 5.2. *For every rational number $q > 0$ and every positive integer r there exists a smooth projective surface X, an integral ample line bundle L on X, and points $x_1, \dots, x_r \in X$ such that*

$$\varepsilon(L; x_1, \dots, x_r) = q.$$

Proof. It suffices to produce examples with $\varepsilon(L; x_1, \dots, x_r) = \frac{1}{m+r-1}$, where m is a given positive integer. All other rational numbers can be obtained as multiples of these numbers.

We modify slightly the construction from the proof of Proposition 2.1. In fact, keeping the notation from this proposition, we simple put $x_1 = x$ and take $x_2, \dots, x_r$ as arbitrary pairwise distinct points on $\widetilde{C_1}$. Then we have certainly

$$\frac{L \cdot \widetilde{C_1}}{\operatorname{mult}_{x_1} \widetilde{C_1} + \cdots + \operatorname{mult}_{x_r} \widetilde{C_1}} = \frac{1}{m + 1 + \cdots + 1} = \frac{1}{m + r - 1}.$$

Now, if D is an irreducible curve different from $\widetilde{C_1}$, then we have

$$\begin{aligned} L \cdot D &= (E + 2\widetilde{C_1}) \cdot D \\ &\geqslant 2(m \cdot \operatorname{mult}_{x_1} D + \operatorname{mult}_{x_2} D + \cdots + \operatorname{mult}_{x_r} D) \\ &\geqslant 2 \cdot \sum_{i=1}^{r} \operatorname{mult}_{x_i} D, \end{aligned}$$

and this implies that $\varepsilon(L; x_1, \dots, x_r)$ is computed by $\widetilde{C_1}$.

Remark 5.3. Of course one can again modify the proof of Proposition 5.2 to obtain examples in arbitrary dimension, quite as in Remark 2.2.

On the other hand, in the adjoint case, for X, L and $K_X + L$ as in Theorem 3.2, we see from (7) and Theorem 3.2 that one has

$$\varepsilon(K_X + L; x_1, \dots, x_r) \geqslant \frac{1}{r} \cdot \frac{2}{n^2 + n + 4} \tag{8}$$

for all r-tuples $x_1, \dots, x_r \in X$.

Alternatively one can invoke the following generalization of Theorem 3.1 from [1, Theorem 0.3].

Theorem 5.4. (Angehrn-Siu) *Let r be a positive integer. If*

$$(L^d \cdot Z) \geqslant \left(\frac{1}{2} n(n + 2r - 1) + 1\right)^d$$

for all irreducible subvarieties $Z \subset X$ of positive dimension $d \geqslant 1$, then

$$K_X + L$$

separates any set of arbitrary r distinct points.

Combining this with the following Lemma leads to the improved lower bound expressed in Proposition 5.6.

Lemma 5.5. *Let r be a positive integer and let M be a line bundle such that the linear series $|M|$ separates any set of $r+1$ distinct points. Then*

$$\varepsilon(M; x_1, \ldots, x_r) \geqslant 1$$

for all r-tuples $x_1, \ldots, x_r$.

Proof. Let C be a curve passing through at least one of the points $x_1, \ldots, x_r$ and having multiplicities $m_1, \ldots, m_r$ at these points. Furthermore let y be a point on C distinct from $x_1, \ldots, x_r$. Then, by the assumption on point separation, there exists a divisor $D \in |M|$ which contains points $x_1, \ldots, x_r$ in its support and which avoids y. So it intersects C properly, from which we get

$$M \cdot C = D \cdot C \geqslant \sum_{i=1}^{r} m_i,$$

and the assertion follows. □

Proposition 5.6. *Let X, L and $K_X + L$ be as in Theorem 3.2. Then*

$$\varepsilon(K_X + L; x_1, \ldots, x_r) \geqslant \frac{2}{n^2 + (2r+1)n + 1}.$$

This bound is better than (8), but still it is quite unlikely that it is sharp. As before we turn now our attention to surfaces, where further restrictions are better accessible.

Corollary 4.2 together with (7) implies that $\varepsilon(K_X + L; x_1, \ldots, x_r) \geqslant \frac{1}{2r}$. On the other hand it is easy to construct examples of surfaces of arbitrary Kodaira dimension, adjoint ample line bundles on them and r-tuples $x_1, \ldots, x_r$ such that $\varepsilon(K_X + L; x_1, \ldots, x_r) = \frac{1}{r}$. A sample list of these is the following:

- $\kappa(X) = -\infty$: Take $X = \mathbb{P}^2$, $L = \mathcal{O}_{\mathbb{P}^2}(1)$ and r points on a line,
- $\kappa(X) = 0$: Take a product $X = E_1 \times E_2$ of two elliptic curves, $L = E_1 + E_2$ and r points on E_1,
- $\kappa(X) = 1$: Take a product $X = E \times C$ of an elliptic curve E and a smooth curve C of genus $\geqslant 2$, with $L = E + C$ and r points on E,
- $\kappa(X) = 2$: Take the surface X from Example 4.3, $L = K_X$ and r points on a canonical curve.

So the interesting question is what values are possible in the range from $\frac{1}{2r}$ to $\frac{1}{r}$. We show:

Theorem 5.7. *We fix an integer $r \geqslant 2$. Let X be a smooth projective surface and let L be a nef line bundle on X such that $K_X + L$ is ample. If for some distinct points $x_1, \ldots, x_r \in X$ the Seshadri constant $\varepsilon(K_X + L; x_1, \ldots, x_r)$ lies in the interval $(0, \frac{1}{r})$, then*

$$\varepsilon(K_X + L; x_1, \ldots, x_r) = \frac{1}{r+1} \quad \textit{or} \quad \frac{1}{r+2},$$

unless $r = 2$ and $\varepsilon(K_X + L; x_1, x_2) = \frac{2}{5}$.

Proof. The proof is quite similar to that of Theorem 4.1. Let C be a curve on X passing through $x_1, \dots, x_r$ with multiplicities $m_1, \dots, m_r$ and such that

$$\frac{(K_X + L) \cdot C}{m_1 + \cdots + m_r} = \frac{d}{m} < \frac{1}{r}.$$

Then as in the proof of Theorem 4.1 we have

$$p_a(C) \leqslant 1 + \frac{d(d+1)}{2}. \tag{9}$$

On the other hand, there is the lower bound

$$p_a(C) \geqslant \binom{m_1}{2} + \cdots + \binom{m_r}{2} = \frac{1}{2}\left(\sum_{i=1}^{r} m_i^2 - \sum_{i=1}^{r} m_i\right) \tag{10}$$

$$\geqslant \frac{1}{2}\left(\frac{1}{r}m^2 - m\right). \tag{11}$$

Using the assumption $rd \leqslant m - 1$ and combining (9) and (10) we get

$$r(m^2 - rm - 2r) \leqslant (m-1) \cdot (m + r - 1).$$

This implies that either

(i) $m \leqslant 2r$, or
(ii) $r = 2$ and $m = 5$.

In case (ii) we get the exceptional value $\frac{2}{5}$. In case (i) we must have $d = 1$, and using (9) we get $p_a(C) \leqslant 2$. Therefore there can be at most two double points among the x_i, and hence m is bounded by $r + 2$. This implies the assertion. □

We conclude by showing that both main cases in the preceding theorem actually occur. To obtain $\frac{1}{r+1}$ as a Seshadri constant is easy. Indeed, we can either start from Example 4.3 or Example 4.4 and take $r - 1$ additional smooth points on the curve D or E_0 respectively. To get $\frac{1}{r+2}$ requires a little bit more work. The idea is to construct a surface X as in Example 4.3 containing a canonical curve with two double points:

Example 5.8. In the weighted projective plane $H = \mathbb{P}(1, 2, 5)$ with variables y, z, w let C be the curve that is defined by the homogeneous equation of degree 10

$$f(y, z, w) = w^2 + z^2 \cdot (z + y^2)^2 \cdot (z - y^2).$$

Note that the curve C omits both singular points $P_1 = (0 : 1 : 0)$ and $P = (0 : 0 : 1)$ of H. It follows that C is irreducible. Indeed, it is elementary to check that all polynomials of degree $\leqslant 5$ vanish either at P_1 or P_2. The curve C has arithmetic genus 2 and two double points at $x_1 := (1 : 0 : 0)$ and $x_2 := (1 : -1 : 0)$. We want to realize the curve C as the hyperplane section $H \cap X$ of a surface $X \subset \mathbb{P}(1, 1, 2, 5)$

of degree 10. To this end let D be a curve in H defined by a homogeneous polynomial $g(y, z, w)$ of degree 9 intersecting C transversally. Then we let X be the surface defined by the equation

$$F(x, y, z, w) := f(y, z, w) + x \cdot g(y, z, w) = 0.$$

We claim that X is smooth. Taking the partial derivative of F with respect to x we see that the only singular points of X could be the intersection points of C and D. Since the intersection is transversal, we obtain a local coordinate system at each of the intersection points and this shows that X is smooth. For details cf. [2, Lemma 2.2], where an analogous construction in $\mathbb{P}^3$ is carried out.

Now, taking $x_3, \ldots, x_r$ on C pairwise different and different from x_1 and x_2, we get for the canonical bundle $K_X = \mathcal{O}_X(1)$ the equation

$$\frac{K_X \cdot C}{\sum_{i=1}^{r} \operatorname{mult}_{x_i} C} = \frac{1}{r+2},$$

as desired.

We do not know if the exotic value $\frac{2}{5}$ can be actually obtained as a two-point Seshadri constant.

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