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Fractional type Marcinkiewicz integral operators associated to surfaces

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Abstract
In this paper, we discuss the boundedness of the fractional type Marcinkiewicz integral operators associated to surfaces, and we extend a result given by Chen *et al.* (J. Math. Anal. Appl. 276:691-708, 2002). They showed that under certain conditions the fractional type Marcinkiewicz integral operators are bounded from the Triebel-Lizorkin spaces $\dot{F}^{\alpha}_{pq}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$. Recently the second author, together with Xue and Yan, greatly weakened their assumptions. In this paper, we extend their results to the case where the operators are associated to the surfaces of the form $\{x = \phi(|y|)y/|y|\} \subset \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$. To prove our result, we discuss a characterization of the homogeneous Triebel-Lizorkin spaces in terms of lacunary sequences.
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Keywords: L^p boundedness; Marcinkiewicz integral; fractional integral operator; Triebel-Lizorkin spaces; Sobolev spaces

1 Introduction

The fractional type Marcinkiewicz operator is defined by

$$\mu_{\Omega,\rho,\alpha} f(x) = \left(\int_0^\infty \left| \frac{1}{t^{\rho+\alpha}} \int_{B(t)} f(x-y) \frac{\Omega(y/|y|)}{|y|^{n-\rho}} \, dy \right|^2 \frac{dt}{t} \right)^{\frac{1}{2}}, \tag{1.1}$$

where we write $B(r) = \{|x| < r\} \subset \mathbb{R}^n$ for $r > 0$ here and below. The operator $\mu_{\Omega,\rho,\alpha} f$ is the so called singular integral operator. In this paper, we shall prove that this operator is bounded under a certain highly weak integrability assumption. To this end, we plan to employ a modified Littlewood-Paley decomposition adapted to our situation. It turns out that we can relax the integrability assumption on Ω and that the integral operator itself can be generalized to a large extent.

Let S^{n-1} be the unit sphere in the n-dimensional Euclidean space $\mathbb{R}^n$ ($n \geq 2$), with the induced Lebesgue measure $d\sigma = d\sigma(x')$ and $\Omega \in L^1(S^{n-1})$. In the sequel, we often suppose that Ω satisfies the cancellation condition

$$\int_{S^{n-1}} \Omega(y')\, d\sigma(y') = 0. \tag{1.2}$$

Here, for the symbols x' and y', we adopt the following convention: Sometimes they stand for points in S^{n-1}. But for $x \in \mathbb{R}^n \setminus \{0\}$, we abbreviate $x/|x|$ to x' in the present paper. We make this slight abuse of notation since no confusion is likely to occur.

In the present paper we deal with operators of Marcinkiewicz type. Define

$$\mu_{\Omega,\rho,\alpha,q}f(x) := \left(\int_0^\infty \left|\frac{1}{t^{\rho+\alpha}}\int_{B(t)} f(x-y)\frac{\Omega(y')}{|y|^{n-\rho}}\,dy\right|^q \frac{dt}{t}\right)^{\frac{1}{q}} \quad \left(x \in \mathbb{R}^n\right). \tag{1.3}$$

As a special case, by letting $\rho = 1$, $\alpha = 0$, $q = 2$, we recapture the Marcinkiewicz integral operator that Stein introduced in 1958 [1]. In 1960, Hörmander considered the parametric Marcinkiewicz integral operator $\mu_{\Omega,\rho,\alpha,2}$ [2]. Since then, about Marcinkiewicz type integral operators, many works appeared. A nice survey is given by Lu [3].

We formulate our results in the framework of Triebel-Lizorkin spaces of homogeneous type. For $\alpha \in \mathbb{R}$ and $p, q \in (1,\infty)$, we let $\dot{F}^\alpha_{pq}(\mathbb{R}^n)$ be the Triebel-Lizorkin space defined in [4, 5]. Note that the space $\mathcal{S}_\infty(\mathbb{R}^n)$ given by

$$\mathcal{S}_\infty\left(\mathbb{R}^n\right) := \bigcap_{\alpha \in (\mathbb{N}\cup\{0\})^n} \left\{f \in \mathcal{S}\left(\mathbb{R}^n\right) : \int_{\mathbb{R}^n} x^\alpha f(x)\,dx = 0\right\}$$

is dense in $\dot{F}^\alpha_{pq}(\mathbb{R}^n)$ as long as $\alpha \in \mathbb{R}$ and $p, q \in (1,\infty)$. If $u \in (1,\infty)$, then define $u' = \frac{u}{u-1}$ and $\tilde{u} = \max(u, u')$. Here and below a tacit understanding in the present paper is that the letter C is used for constants that may change from one occurrence to another, that is, the letter C will denote a positive constant which may vary from line to line but will remain independent of the relevant quantities. Our main theorem in the simplest form reads as follows:

Theorem 1 *Let* $\rho > 0$, $1 < p, q < \infty$ *and* $\Omega \in L^1(S^{n-1})$.

(i) *If* $\alpha \in (0, 4/(\tilde{p}\tilde{q}))$ *and* Ω *satisfies the cancellation condition* (1.2), *then*

$$\|\mu_{\Omega,\rho,\alpha,q}f\|_{L^p(\mathbb{R}^n)} \le C\|\Omega\|_{L^1(S^{n-1})}\|f\|_{\dot{F}^\alpha_{pq}(\mathbb{R}^n)} \tag{1.4}$$

for all $f \in \mathcal{S}_\infty(\mathbb{R}^n)$.

(ii) *If* $\alpha \in (-\min\{\frac{4\beta}{\tilde{p}\tilde{q}}, \rho\}, 0)$ *and*

$$Z_\Omega := \sup_{\xi' \in S^{n-1}} \int_{S^{n-1}} \frac{|\Omega(y')|}{|y'\cdot\xi'|^\beta}\,d\sigma\left(y'\right) < +\infty, \tag{1.5}$$

for some $0 < \beta \le 1$, *then*

$$\|\mu_{\Omega,\rho,\alpha,q}f\|_{L^p(\mathbb{R}^n)} \le CZ_\Omega\|f\|_{\dot{F}^\alpha_{pq}(\mathbb{R}^n)} \tag{1.6}$$

for all $f \in \mathcal{S}_\infty(\mathbb{R}^n)$.

(iii) *If* $\alpha = 0$ *and* $\Omega \in L\log L(S^{n-1})$ *satisfies the cancellation condition* (1.2), *then*

$$\|\mu_{\Omega,\rho,\alpha,q}f\|_{L^p(\mathbb{R}^n)} \le C\|\Omega\|_{L\log L(S^{n-1})}\|f\|_{\dot{F}^\alpha_{pq}(\mathbb{R}^n)} \tag{1.7}$$

for all $f \in \mathcal{S}_\infty(\mathbb{R}^n)$.

In any case, by density we can extend (1.4), (1.6) *and* (1.7) *and have them for all* $f \in \dot{F}^\alpha_{pq}(\mathbb{R}^n)$.

In 2002, Chen *et al.* obtained a result about the fractional type Marcinkiewicz integral operator [6], which we recall now.

Theorem A *Let* $1 < p, q < \infty$ *and* $1 < r \le \infty$. *Suppose* $\Omega \in L^r(S^{n-1})$ *satisfies the cancellation condition* (1.2). *If* $|\alpha| < 2/(r'\tilde{p}\tilde{q})$ *and* $\rho = 1$, *then*

$$\|\mu_{\Omega,\rho,\alpha,q}f\|_{L^p(\mathbb{R}^n)} \le C\|\Omega\|_{L^r(S^{n-1})}\|f\|_{\dot{F}^\alpha_{pq}(\mathbb{R}^n)}$$

for all $f \in \mathcal{S}_\infty(\mathbb{R}^n)$.

Si, Wang and Jiang discussed ones of somewhat different type [7]. About Theorems 1 and A, a couple of remarks may be in order.

Remark 1 If $0 < \beta < 1$, $1/(1-\beta) < r \le \infty$ and $\Omega \in L^r(S^{n-1})$, it is easily seen that the condition (1.5) is satisfied. In this case

$$Z_\Omega \le C\|\Omega\|_{L^r(S^{n-1})}.$$

So, our result includes completely Theorem A, where they assumed that $\Omega \in L^r(S^{n-1})$. Let $r > 1$ and define

$$\Omega_0(y') = \operatorname{sgn}(y' \cdot (1,0,\dots,0))\bigl|(y' \cdot (1,0,\dots,0))\bigr|^{-1/r}. \tag{1.8}$$

Then it is also easily checked that Ω is in $L^1(S^{n-1}) \setminus L^r(S^{n-1})$ and satisfies (1.5) for any $\beta \in (0, 1/r')$.

In the case $\alpha = 0$, $\rho = 1$ and $q = 2$, the conclusion in Theorem 1(iii) is shown to hold even when $\Omega \in L\log L^{1/2}(S^{n-1})$ in [8].

Remark 2 We can relax the condition on α: $|\alpha| < 4/(r'\tilde{p}\tilde{q})$ suffices. Indeed, one can get $|(\Omega(\cdot)|\cdot|^{-n+1}\chi_{B(1)})^\wedge(\xi)| \le C|\xi|^{-1/r'}$ by direct computation.

By reexamining their proof, we can parametrize Theorem A: we can prove

$$\left(\int_{\mathbb{R}^n}\left(\int_0^\infty\left|\frac{1}{t^{\rho+\alpha}}\int_{B(t)} f(x-y)\frac{\Omega(y)}{|y|^{n-\rho}}\,dy\right|^q \frac{dt}{t}\right)^{p/q} dx\right)^{1/p} \le C\|f\|_{\dot{F}^\alpha_{pq}(\mathbb{R}^n)}, \tag{1.9}$$

provided $|\alpha| < 4\min\{\frac{1}{r'}, \min(\rho,1)\}\frac{1}{\tilde{p}\tilde{q}}$. Comparing (1.9) with Theorem 1, one concludes that our theorem outranges Theorem A in view of the case when $\min(\rho,1) < 1/r'$. In our earlier paper [9], we improved Theorem A by relaxing the conditions postulated on Ω.

Our method is also applicable even in more generalized settings. For $\rho > 0$, $\alpha \in \mathbb{R}$ and $\Omega \in L^1(S^{n-1})$, we define the fractional type Marcinkiewicz integral operator by (1.1) and the fractional type Marcinkiewicz integral operator associated to surfaces $\{(x,y) : x = \phi(|y|)y'\} \in \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$ by

$$\mu_{\Omega,\rho,\phi,\alpha}f(x) = \left(\int_0^\infty\left|\frac{1}{t^\rho\phi(t)^\alpha}\int_{B(t)} f\bigl(x-\phi(|y|)y'\bigr)\frac{\Omega(y)}{|y|^{n-\rho}}\,dy\right|^2\frac{dt}{t}\right)^{\frac{1}{2}}. \tag{1.10}$$

Theorem 1 extends further to the case when the operator is equipped with a function space Δ_γ with $\gamma \ge 1$. Regarding to Calderón-Zygmund singular integral and Marcinkiewicz integral operators, many authors discussed those operators with modified kernel $b(|\cdot|)\Omega(\cdot)$ in place of $\Omega(\cdot)$, where b belongs to the class of all measurable functions $h:[0,\infty)\to\mathbb{C}$ satisfying $\|h\|_{\Delta_\gamma}=\sup_{R>0}(R^{-1}\int_0^R|h(t)|^\gamma\,dt)^{1/\gamma}<\infty$ $(1\le\gamma\le\infty)$, see [10–14], etc. We note that

$$L^\infty(\mathbb{R}_+)\subset\Delta_\beta(\mathbb{R}_+)\subset\Delta_\alpha(\mathbb{R}_+)\quad\text{for } 1\le\alpha<\beta,$$

and that all these inclusions are proper. We refer to [15–17] for extension and generalization of the space Δ_γ.

We define the modified fractional type Marcinkiewicz operator $\mu^{(b)}_{\Omega,\rho,\alpha,q}$ by

$$\mu^{(b)}_{\Omega,\rho,\alpha,q}f(x)=\left(\int_0^\infty\left|\frac{1}{t^{\rho+\alpha}}\int_{B(t)}f(x-y)\frac{b(|y|)\Omega(y')}{|y|^{n-\rho}}\,dy\right|^q\frac{dt}{t}\right)^{\frac{1}{q}}. \tag{1.11}$$

We can recover Theorem 1 by letting $b\equiv 1$ in the next theorem.

Theorem 2 *Suppose that we are given $\Omega\in L^1(S^{n-1})$ and parameters p,q,α,γ,ρ satisfying*

$$1<p,q<\infty,\qquad\gamma>\frac{1}{2}\max\{\tilde{p},\tilde{q}\},\qquad\rho>0.$$

(i) *Let $\alpha\in(0,\frac{4(1/\tilde{p}-1/(2\gamma))(1/\tilde{q}-1/(2\gamma))}{(1-1/\gamma)^2})$. If $b\in\Delta_\gamma(\mathbb{R}_+)$ and Ω satisfies the cancellation condition* (1.2), *then*

$$\left\|\mu^{(b)}_{\Omega,\rho,\alpha,q}f\right\|_{L^p(\mathbb{R}^n)}\le C\|\Omega\|_{L^1(S^{n-1})}\|b\|_{\Delta_\gamma}\|f\|_{\dot{F}^\alpha_{pq}(\mathbb{R}^n)} \tag{1.12}$$

for all $f\in\mathcal{S}_\infty(\mathbb{R}^n)$.

(ii) *Assume $\alpha\in(-\min\{2\beta\frac{1/\tilde{p}-1/(2\gamma)}{1-1/\gamma}\cdot\frac{1/\tilde{q}-1/(2\gamma)}{1-1/\gamma},\rho\},0)$ with $\beta\in(0,1]$. If $b\in\Delta_{\max(\gamma,2)}$ and*

$$W_\Omega:=\sqrt{\sup_{\xi'\in S^{n-1}}\int_{S^{n-1}\times S^{n-1}}\frac{|\Omega(y')\Omega(z')|}{|(y'-z')\cdot\xi'|^\beta}\,d\sigma(y')\,d\sigma(z')}<+\infty, \tag{1.13}$$

then

$$\left\|\mu^{(b)}_{\Omega,\rho,\alpha,q}f\right\|_{L^p(\mathbb{R}^n)}\le CW_\Omega\|b\|_{\max(\gamma,2)}\|f\|_{\dot{F}^\alpha_{pq}(\mathbb{R}^n)} \tag{1.14}$$

for all $f\in\mathcal{S}_\infty(\mathbb{R}^n)$.

(iii) *Assume $\alpha=0$. If $b\in\Delta_{\max(\gamma,2)}$, $\Omega\in L\log L(S^{n-1})$ and Ω satisfies the cancellation condition* (1.2), *then*

$$\left\|\mu^{(b)}_{\Omega,\rho,\alpha,q}f\right\|_{L^p(\mathbb{R}^n)}\le C\|\Omega\|_{L\log L(S^{n-1})}\|b\|_{\max(\gamma,2)}\|f\|_{\dot{F}^\alpha_{pq}(\mathbb{R}^n)} \tag{1.15}$$

for all $f\in\mathcal{S}_\infty(\mathbb{R}^n)$.

In any case, by density we can extend (1.12), (1.14) *and* (1.15) *and have them for all* $f \in \dot{F}^{\alpha}_{pq}(\mathbb{R}^n)$.

Remark 3 In Theorem 1(ii) a modification of the proof changes 4β into 2β. We cannot estimate directly the Fourier transform of the measure σ_t in Section 3, and we use the idea given by Duoandikoetxea and Rubio de Francia [11, p.551] as in Chen *et al.* [6].

If $0 < \beta < 1$, $1/(1-\beta) < r \leq \infty$ and $\Omega \in L^r(S^{n-1})$, it is easily seen that the condition (1.13) is satisfied. In this case

$$W_\Omega \leq C\|\Omega\|_{L^r(S^{n-1})}.$$

In the case $\alpha = 0$, $\rho = 1$ and $q = 2$, it is again known in [18] that the conclusion in Theorem 2(iii) holds even when $\Omega \in L\log L^{1/2}(S^{n-1})$.

In the earlier paper [9], in Theorem 1(ii) (respectively, in Theorem 2(ii)), we needed to postulate the additional conditions $\rho > \beta$ (respectively, $2\rho > \beta$) and the cancellation condition on Ω. However, these are no longer necessary in the new theorems.

Remark In [19], instead of W_Ω, the following quantity is proposed:

$$\sup_{\xi' \in S^{n-1}} \left| \int_{S^{n-1}} \int_{S^{n-1}} \Omega(y')\overline{\Omega(z')} \log\left(\frac{2}{|\xi' \cdot y'|^2 + |\xi' \cdot z'|^2}\right)^{1/2} d\sigma(y')\, d\sigma(z') \right| < \infty.$$

In addition to the factor of b, we can even distort the convolution. For $\alpha > 0$, $1 \leq q < \infty$, a kernel Ω and a positive function ϕ on $\mathbb{R}_+$, we define the operator $\mu_{\Omega,\rho,\phi,\alpha,q}$ and the modified one $\mu^{(b)}_{\Omega,\rho,\phi,\alpha,q}$ by

$$\mu_{\Omega,\rho,\phi,\alpha,q}f(x) = \left(\int_0^\infty \left| \frac{1}{t^\rho \phi(t)^\alpha} \int_{B(t)} f\bigl(x - \phi(|y|)y'\bigr) \frac{\Omega(y)}{|y|^{n-\rho}}\, dy \right|^q \frac{dt}{t}\right)^{\frac{1}{q}}, \tag{1.16}$$

and

$$\mu^{(b)}_{\Omega,\rho,\phi,\alpha,q}f(x) = \left(\int_0^\infty \left| \frac{1}{t^\rho \phi(t)^\alpha} \int_{B(t)} f\bigl(x - \phi(|y|)y'\bigr) \frac{b(|y|)\Omega(y)}{|y|^{n-\rho}}\, dy \right|^q \frac{dt}{t}\right)^{\frac{1}{q}}. \tag{1.17}$$

Now we formulate our main theorem. Here and below we write $\mathbb{R}_+ := (0,\infty)$.

Theorem 3 *Let* $\rho > 0$, $1 < p, q < \infty$ *and* $\Omega \in L^1(S^{n-1})$. *Let* $c_0 > 1$ *and* $c_1 > 0$. *Suppose that* $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ *is a nonnegative increasing* C^1*-function such that*

$$\phi(2t) \leq c_0\phi(t) \quad \textit{for all } t \in \mathbb{R}_+ \tag{1.18}$$

and that

$$\phi(t) \leq c_1 t\phi'(t) \quad \textit{for all } t \in \mathbb{R}_+. \tag{1.19}$$

Define

$$\varphi(t) := \frac{\phi(t)}{t\phi'(t)} \quad \textit{for all } t \in \mathbb{R}_+.$$

Then:

(i) *Let*

$$\alpha \in \left(0, \frac{4}{\tilde{p}\tilde{q}c_1 \log_2 c_0}\right). \tag{1.20}$$

If Ω *satisfies the cancellation condition* (1.2), *then*

$$\|\mu_{\Omega,\rho,\phi,\alpha,q} f\|_{L^p(\mathbb{R}^n)} \le C\|\Omega\|_{L^1(S^{n-1})}\|f\|_{\dot{F}^\alpha_{pq}(\mathbb{R}^n)} \tag{1.21}$$

for all $f \in \mathcal{S}_\infty(\mathbb{R}^n)$.

(ii) *Let*

$$\alpha \in \left(-\min\left\{\frac{4\beta}{c_1 \log_2 c_0 \cdot \tilde{p}\tilde{q}}, \frac{\rho}{\log_2 c_0}\right\}, 0\right).$$

If ϕ *satisfies the following additional condition*:

$$\varphi(t) \text{ or } t\phi'(t) \text{ is monotonic on } \mathbb{R}_+, \tag{1.22}$$

and Ω *satisfies*

$$Z_\Omega := \sup_{\xi' \in S^{n-1}} \int_{S^{n-1}} \frac{|\Omega(y')|}{|y' \cdot \xi'|^\beta}\, d\sigma\left(y'\right) < +\infty, \tag{1.23}$$

for some $0 < \beta \le 1$, *then*

$$\|\mu_{\Omega,\rho,\phi,\alpha,q} f\|_{L^p(\mathbb{R}^n)} \le CZ_\Omega\|f\|_{\dot{F}^\alpha_{pq}(\mathbb{R}^n)} \tag{1.24}$$

for all $f \in \mathcal{S}_\infty(\mathbb{R}^n)$.

(iii) *Let* $\alpha = 0$. *If* $\Omega \in L\log L(S^{n-1})$ *and it satisfies the cancellation condition* (1.2), *then*

$$\|\mu_{\Omega,\rho,\phi,\alpha,q} f\|_{L^p(\mathbb{R}^n)} \le C\|\Omega\|_{L\log L(S^{n-1})}\|f\|_{\dot{F}^\alpha_{pq}(\mathbb{R}^n)} \tag{1.25}$$

for all $f \in \mathcal{S}_\infty(\mathbb{R}^n)$.

In any case, by density we can extend (1.21), (1.24) *and* (1.25) *and have them for all* $f \in \dot{F}^\alpha_{pq}(\mathbb{R}^n)$.

Note that (1.18) is referred to as the doubling condition. Thanks to the useful conversation with Professor XX Tao and Miss S He in the Zhejiang University of Science and Technology, we could improve our results.

We state our main result in full generality. Theorem 3 is almost a direct consequence of the next theorem.

Theorem 4 *Suppose that we are given* $\Omega \in L^1(S^{n-1})$, $\phi \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ *and parameters* p, q, α, γ, ρ *satisfying*

$$1 < p, q < \infty, \qquad \rho > 0, \qquad \gamma > \frac{1}{2}\max\{\tilde{p}, \tilde{q}\},$$

in addition to (1.18) *and* (1.19) *in Theorem* 3. *Then*:

(i) *Assume that*

$$\alpha \in \left(0, \frac{4}{c_1 \log_2 c_0} \cdot \frac{1/\tilde{p} - 1/(2\gamma)}{1 - 1/\gamma} \cdot \frac{1/\tilde{q} - 1/(2\gamma)}{1 - 1/\gamma}\right). \tag{1.26}$$

If $b \in \Delta_\gamma(\mathbb{R}_+)$ *and* Ω *satisfies the cancellation condition* (1.2), *then*

$$\left\| \mu^{(b)}_{\Omega,\rho,\phi,\alpha,q} f \right\|_{L^p(\mathbb{R}^n)} \le C \|\Omega\|_{L^1(S^{n-1})} \|b\|_{\Delta_\gamma} \|f\|_{\dot{F}^\alpha_{pq}(\mathbb{R}^n)} \tag{1.27}$$

for all $f \in \mathcal{S}_\infty(\mathbb{R}^n)$.

(ii) *Assume* $\alpha \in (-\min\{\frac{2\beta}{c_1 \log_2 c_0} \cdot \frac{1/\tilde{p}-1/(2\gamma)}{1-1/\gamma} \cdot \frac{1/\tilde{q}-1/(2\gamma)}{1-1/\gamma}, \frac{\rho}{\log_2 c_0}\}, 0)$ *for some* $\beta \in (0,1]$. *If* $b \in \Delta_{\max(\gamma,2)}$ *and*

$$W_\Omega := \sup_{\xi' \in S^{n-1}} \sqrt{\int_{S^{n-1} \times S^{n-1}} \frac{|\Omega(y')\Omega(z')|}{|(y'-z') \cdot \xi'|^\beta} \, d\sigma(y') \, d\sigma(z')} < +\infty, \tag{1.28}$$

then

$$\left\| \mu^{(b)}_{\Omega,\rho,\phi,\alpha,q} f \right\|_{L^p(\mathbb{R}^n)} \le C W_\Omega \|b\|_{\Delta_{\max(\gamma,2)}} \|f\|_{\dot{F}^\alpha_{pq}(\mathbb{R}^n)} \tag{1.29}$$

for all $f \in \mathcal{S}_\infty(\mathbb{R}^n)$.

(iii) *Assume* $\alpha = 0$. *If* $b \in \Delta_{\max(\gamma,2)}$, $\Omega \in L \log L(S^{n-1})$ *and it satisfies the cancellation condition* (1.2), *then*

$$\left\| \mu^{(b)}_{\Omega,\rho,\phi,\alpha,q} f \right\|_{L^p(\mathbb{R}^n)} \le C \|\Omega\|_{L \log L(S^{n-1})} \|b\|_{\Delta_{\max(\gamma,2)}} \|f\|_{\dot{F}^\alpha_{pq}(\mathbb{R}^n)} \tag{1.30}$$

for all $f \in \mathcal{S}_\infty(\mathbb{R}^n)$.

In any case, by density we can extend (1.27), (1.29) *and* (1.30) *and have them for all* $f \in \dot{F}^\alpha_{pq}(\mathbb{R}^n)$.

Theorem 3(i) and (iii) are direct consequences of Theorem 4. Indeed, assuming (1.20) and choosing $\gamma \gg 1$, we have (1.26). So, to obtain (i) we can apply Theorem 4 for such γ with $b \equiv 1$. Theorem 3(iii) is a direct conseuqence of Theorem 4(iii). Note that in Theorems 3(ii) and 4(ii), the conditions of α is slightly improved.

Our strategy is to employ the Littlewood-Paley decomposition as Ding *et al.* did in [20]. However, we distort things via the sequence $\{a_k\}_{k\in\mathbb{Z}}$. We rely upon the modified Littlewood-Paley decomposition for the proof of Theorem 4, which we shall describe now. Let $\{a_k\}_{k\in\mathbb{Z}}$ be a lacunary sequence of positive numbers in the sense that $a_{k+1}/a_k \ge a > 1$ ($k \in \mathbb{Z}$). A sequence $\{\Phi_k\}_{k\in\mathbb{Z}}$ of $C^\infty(\mathbb{R}^n)$-functions is said to be a partition of unity adapted to $\{a_k\}_{k\in\mathbb{Z}}$ if

$$\operatorname{supp} \widehat{\Phi}_k \subset \{\xi \in \mathbb{R}^n; a_{k-1} \le |\xi| \le a_{k+1}\} \quad (k \in \mathbb{Z}),$$
$$\sum_{k\in\mathbb{Z}} \widehat{\Phi}_k(\xi) = 1 \quad (\xi \in \mathbb{R}^n \setminus \{0\}),$$

and

$$\left|\xi^\beta \partial^\beta \widehat{\Phi}_k(\xi)\right| \le C_\beta$$

for any multiindex β.

Denote by $\mathcal{P}$ the set of all polynomials in $\mathbb{R}^n$. Let $1 < p,q < \infty$ and $\alpha \in \mathbb{R}$. For $f \in \mathcal{S}'(\mathbb{R}^n)/\mathcal{P}$, we define the norm $\|f\|_{\dot{F}^{\alpha,\{\Phi_k\}_{k\in\mathbb{Z}}}_{pq}(\mathbb{R}^n)}$ by

$$\|f\|_{\dot{F}^{\alpha,\{\Phi_k\}_{k\in\mathbb{Z}}}_{pq}(\mathbb{R}^n)} = \left\| \left(\sum_{k\in\mathbb{Z}} a_k^{\alpha q} |\Phi_k * f|^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)}. \tag{1.31}$$

We admit that Proposition 1 below is true and we prove Theorem 4 first. We postpone the proof of Proposition 1 until the end of the paper.

Proposition 1 *Let $\alpha \neq 0$ and $1 < p,q < \infty$. Let $\{a_k\}_{k\in\mathbb{Z}}$ be a lacunary sequence of positive numbers with $a_{k+1}/a_k \geq a > 1$ $(k \in \mathbb{Z})$. If $\|f\|_{\dot{F}^{\alpha,\{\Phi_k\}_{k\in\mathbb{Z}}}_{pq}(\mathbb{R}^n)}$ and $\|f\|_{\dot{F}^{\alpha,\{\Psi_k\}_{k\in\mathbb{Z}}}_{pq}(\mathbb{R}^n)}$ are equivalent for any two partitions of unity, $\{\Phi_k\}_{k\in\mathbb{Z}}$ and $\{\Psi_k\}_{k\in\mathbb{Z}}$, adapted to $\{a_k\}_{k\in\mathbb{Z}}$, then there exists $C_0 > a$ such that*

$$\frac{a_{k+1}}{a_k} \leq C_0 \quad (k \in \mathbb{Z}),$$

and, in this case, $\|f\|_{\dot{F}^{\alpha,\{\Phi_k\}_{k\in\mathbb{Z}}}_{pq}(\mathbb{R}^n)}$ is equivalent to the usual homogeneous Triebel-Lizorkin space norm $\|f\|_{\dot{F}^{\alpha}_{pq}(\mathbb{R}^n)}$.

In Sections 3-5, we shall prove Theorems 3 and 4 as well as Proposition 1, respectively.

2 A strategy of the proof of Theorem 4

2.1 A setup

For $t > 0$, a function b on $\mathbb{R}_+$ and a homogeneous kernel Ω on $\mathbb{R}^n$, assume

$$\int_{B(t)\setminus B(t/2)} \left| b(|x|)\Omega(x') \right| dx < \infty.$$

For $\rho > 0$ and a nice function ϕ, we define the family $\{\sigma_t; t \in \mathbb{R}_+\}$ of measures and the maximal operator σ^* on $\mathbb{R}^n$ by

$$\int_{\mathbb{R}^n} f(x)\, d\sigma_t(x) = \frac{1}{t^\rho} \int_{B(t)\setminus B(t/2)} f\bigl(\phi(|x|)x'\bigr) \frac{b(|x|)\Omega(x')}{|x|^{n-\rho}}\, dx, \tag{2.1}$$

$$\sigma^* f(x) = \sup_{t>0} \bigl| |\sigma_t| * f(x) \bigr| \quad \bigl(x \in \mathbb{R}^n\bigr). \tag{2.2}$$

Note that the mapping $x \in \mathbb{R}^n \setminus \{0\} \mapsto \phi(|x|)x' \in \mathbb{R}^n \setminus \overline{B(\inf\phi)}$ is a C^1-diffeomorphism, since $\phi \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ satisfies (1.18) and (1.19). Therefore, if we consider the measure $\sigma_t^\dagger$ by

$$\int_{\mathbb{R}^n} f(x)\, d\sigma_t^\dagger(x) = \frac{1}{t^\rho} \int_{B(t)\setminus B(t/2)} f(x) \frac{b(|x|)\Omega(x')}{|x|^{n-\rho}}\, dx,$$

then the above diffeomorphism induces σ_t. So, as regards the absolute value of σ_t, we have

$$\int_{\mathbb{R}^n} f(x)\, d|\sigma_t|(x) = \frac{1}{t^\rho} \int_{B(t)\setminus B(t/2)} f\bigl(\phi(|x|)x'\bigr) \frac{|b(|x|)\Omega(x')|}{|x|^{n-\rho}}\, dx.$$

Denote by $\|\sigma_t\|$ the total mass of σ_t. A direct consequence of this alternative definition of $|\sigma_t|$ is that we have

$$\|\sigma_t\| \le C\|b\|_{\Delta_1}\|\Omega\|_{L^1}. \tag{2.3}$$

If we use (2.1), then we can write

$$\tilde{\mu}^{(b)}_{\Omega,\alpha,\rho,q}(f)(x) = \left(\int_0^\infty \left|\sigma_t * f(x)\right|^q \frac{dt}{t\phi(t)^{q\alpha}}\right)^{1/q} \quad \left(x \in \mathbb{R}^n\right). \tag{2.4}$$

Lemma 2.1 *Let* $\Omega \in L^1(S^{n-1})$.

(1) *For all admissible parameters,*

$$\left|\widehat{\sigma}_t(\xi)\right| \le 2^{n-\rho}\|\Omega\|_{L^1(S^{n-1})}\|b\|_{\Delta_1} \quad \left(t > 0, \xi \in \mathbb{R}^n\right). \tag{2.5}$$

(2) *If in addition* Ω *satisfies* (1.2), *then we have*

$$\left|\widehat{\sigma}_t(\xi)\right| \le 2\|\Omega\|_{L^1(S^{n-1})}\|b\|_{\Delta_1}\phi(t)|\xi| \quad \left(t > 0, \xi \in \mathbb{R}^n\right). \tag{2.6}$$

Proof

(1) From the definition of the Fourier transform, we have an expression of $\widehat{\sigma}_t(\xi)$:

$$\widehat{\sigma}_t(\xi) = \frac{1}{t^\rho}\int_{B(t)\setminus B(t/2)} \frac{b(|y|)\Omega(y')}{|y|^{n-\rho}} e^{-i\phi(|y|)y'\cdot\xi}\,dy. \tag{2.7}$$

From (2.7) we get (2.5).

(2) Using the cancellation property (1.2) of Ω, we have another expression of $\widehat{\sigma}_t(\xi)$:

$$\widehat{\sigma}_t(\xi) = \frac{1}{t^\rho}\int_{B(t)\setminus B(t/2)} \frac{b(|y|)\Omega(y')}{|y|^{n-\rho}} \left(e^{-i\phi(|y|)y'\cdot\xi} - 1\right)dy. \tag{2.8}$$

From the monotonicity of ϕ, (1.18) and (2.8) we obtain

$$\begin{aligned}\left|\widehat{\sigma}_t(\xi)\right| &\le \frac{1}{t^\rho}\int_{t/2}^t \left(\int_{S^{n-1}} \left|\Omega\left(y'\right)\right| d\sigma\left(y'\right)\right)|\xi| \cdot \left|\phi(r)b(r)\right| r^{\rho-1}\,dr \\ &\le \|\Omega\|_{L^1(S^{n-1})}\phi(t)|\xi| \int_{t/2}^t \left|b(r)\right|\frac{dr}{r} \le 2\|\Omega\|_{L^1(S^{n-1})}\|b\|_{\Delta_1}\phi(t)|\xi|.\end{aligned}$$

So we are done. □

As for the maximal operator σ^* given by (2.2), we invoke the following lemma in [21, Lemma 3.2]: We define the directional Hardy-Littlewood maximal function of F for a fixed vector $y' \in S^{n-1}$ by

$$M_{y'}F(x) = \sup_{r>0} \frac{1}{2r}\int_{-r}^r \left|f\left(x - ty'\right)\right|dt.$$

By the orthogonal decomposition $\mathbb{R}^n = H \oplus \mathbb{R}y'$, we can prove that $M_{y'}$ is bounded on $L^p(\mathbb{R}^n)$ for all $1 < p < \infty$ and that the bound is uniform over y'. By combining the Hölder inequality and the change of variables to polar coordinates, we can prove the following.

Lemma 2.2 *Let $\gamma > 1$. Then there exists $C > 0$ such that*

$$\sigma^*(f)(x) \le C\|b\|_{\Delta_\gamma}\|\Omega\|_{L^1(S^{n-1})}^{1/\gamma}\left(\int_{S^{n-1}}\left|\Omega(y')\right|M_{y'}\left(|f|^{\gamma'}\right)(x)\,d\sigma(y')\right)^{1/\gamma'} \tag{2.9}$$

for all $x \in \mathbb{R}^n$.

Thanks to Lemma 2.2 and the Minkowski inequality, for $p > \gamma'$ there exists $C > 0$ such that

$$\left\|\sigma^*(f)\right\|_{L^p(\mathbb{R}^n)} \le C\|b\|_{\Delta_\gamma}\|\Omega\|_{L^1(S^{n-1})}\|f\|_{L^p(\mathbb{R}^n)}. \tag{2.10}$$

From the monotonicity, (1.18) and (2.6) we get, for $\alpha \in \mathbb{R}$, $k \in \mathbb{Z}$,

$$\left(\int_{2^k}^{2^{k+1}}\left|\widehat{\sigma}_t(\xi)\right|^2\frac{dt}{t\phi(t)^{2\alpha}}\right)^{1/2} \le 2\|\Omega\|_{L^1(S^{n-1})}\|b\|_{\Delta_1}|\xi|\frac{\phi(2^k)}{\phi(2^k)^\alpha}. \tag{2.11}$$

Using (1.18) and (1.19), we have the following.

Lemma 2.3 *For any $0 \le \beta < 1$,*

$$\left|\widehat{\sigma}_t(\xi)\right| \le CW_\Omega\|b\|_{\Delta_2}\frac{1}{(|\xi|\phi(t))^{\beta/2}} \tag{2.12}$$

for $\xi \in \mathbb{R}^n$. W_Ω is the quantity defined in (1.28).

For a precise proof, see the proof of [21, Lemma 2.4].

2.2 Properties of ϕ

We denote $a_j := 1/\phi(2^{-j})$ and $a := 2^{1/\|\varphi\|_{L^\infty(\mathbb{R}_+)}} > 1$. Then $\{a_j\}_{j\in\mathbb{Z}}$ is also a lacunary sequence of the same lacunarity as $\{\phi(2^j)\}_{j\in\mathbb{Z}}$. From the assumption (1.18), it follows that

$$\phi\left(2^k t\right) \le c_0^k\phi(t) \tag{2.13}$$

for $k \in \mathbb{N}$. It is easily seen from (1.19) that $\{\phi(2^j)\}_{j\in\mathbb{Z}}$ is a lacunary sequence of positive numbers satisfying

$$\phi\left(2^k t\right) = \phi(t)\exp\left(\int_t^{2^k t}\left(\log\phi(s)\right)'\,ds\right) \ge 2^{k/\|\varphi\|_{L^\infty(\mathbb{R}_+)}}\phi(t) = a^k\phi(t) \tag{2.14}$$

for $k \in \mathbb{N}$ and $t > 0$. See *e.g.* [21, Lemma 2.8] for details.

Note also that, for $\phi \in C^1(\mathbb{R}_+,\mathbb{R}_+)$ satisfying (1.18), the condition (1.19) implies

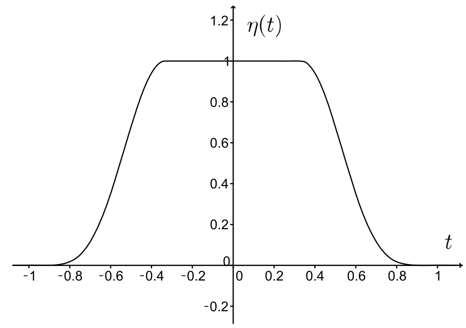
$$\phi(2t) \ge C_1\phi(t) \quad (t > 0) \tag{2.15}$$

for some $C_1 > 1$. Indeed, assuming (1.18), there exists $s \in [t, 2t]$

$$\phi(2t) - \phi(t) = t\phi'(s) \ge c_1\frac{t}{s}\phi(s) \ge \frac{c_1}{c_0}\phi(t)$$

by the mean value theorem, proving (2.15).

Figure 1 The graph of η.



If in addition ϕ is concave, then (2.15) implies (1.19). Indeed,

$$\phi'(2t) \geq \frac{\phi(2t) - \phi(t)}{t} \geq (C_1 - 1)\frac{\phi(t)}{t} \quad (t > 0).$$

2.3 Construction of partition of unity

For our purpose, we introduce a partition of unity and a characterization of the homogeneous Triebel-Lizorkin spaces associated to ϕ satisfying (1.18) and (1.19).

Take a nonincreasing $C^\infty(\mathbb{R})$-function η such that $\chi_{[-1/a,1/a]}(t) \leq \eta(t) \leq \chi_{[-1,1]}(t)$ for all $t \in \mathbb{R}$ (see Figure 1).

We define functions ψ_j on $\mathbb{R}^n$ by

$$\psi_j(\xi) = \eta\left(\frac{|\xi|}{a_{j+1}}\right) - \eta\left(\frac{|\xi|}{a_j}\right) \quad \bigl(\xi \in \mathbb{R}^n\bigr). \tag{2.16}$$

Then observe that

$$\psi_j(\xi) = \begin{cases} 0, & 0 \leq |\xi| \leq a_j/a, |\xi| \geq a a_{j+1}, \\ 1, & a a_j \leq t \leq a_{j+1}, \end{cases} \tag{2.17}$$

and that

$$\operatorname{supp} \psi_j \subset \bigl\{a_j/a \leq |\xi| \leq a a_{j+1}\bigr\}, \tag{2.18}$$

$$\operatorname{supp} \psi_j \cap \operatorname{supp} \psi_\ell = \emptyset, \quad \text{for } |j - \ell| \geq 2, \tag{2.19}$$

$$\sum_{j\in\mathbb{Z}} \psi_j(\xi) = 1 \quad \bigl(\mathbb{R}^n \setminus \{0\}\bigr). \tag{2.20}$$

That is, $\{\psi_j\}_{j\in\mathbb{Z}}$ is a smooth partition of unity adapted to $\{a_j\}_{j\in\mathbb{Z}}$.

Let Ψ_j be defined on $\mathbb{R}^n$ by $\widehat{\Psi}_j(\xi) = \psi_j(\xi)$ for $\xi \in \mathbb{R}^n$. By Proposition 1, we have

$$\left\| \left(\sum_{j=-\infty}^{\infty} \bigl| a_j^\alpha \Psi_j * f \bigr|^q \right)^{1/q} \right\|_{L^p} \approx \|f\|_{\dot{F}^\alpha_{pq}(\mathbb{R}^n)} \tag{2.21}$$

if $a_{j+1}/a_j \leq b$ ($j \in \mathbb{Z}$) for some $b \geq a$.

This condition is satisfied in our case, *i.e.* $a_{j+1}/a_j = \phi(2^{-j})/\phi(2^{-j-1}) \leq c_1$.

2.4 A reduction by using the scaling invariance

Now, using the definition of $\mu^{(b)}_{\Omega,\rho,\phi,\alpha,q}(f)(x)$ and the triangle inequality, via change of variables $y \mapsto 2^k y$, we obtain

$$\begin{aligned}
&\mu^{(b)}_{\Omega,\rho,\phi,\alpha,q}(f)(x) \\
&\quad = \left(\int_0^\infty \left|\frac{1}{t^\rho\phi(t)^\alpha}\int_{B(t)}\frac{b(|y|)\Omega(y')}{|y|^{n-\rho}}f\bigl(x-\phi\bigl(|y|\bigr)y'\bigr)\,dy\right|^q\frac{dt}{t}\right)^{1/q} \\
&\quad = \left(\int_0^\infty \left|\sum_{k=0}^\infty\frac{1}{t^\rho\phi(t)^\alpha}\int_{B(2^{-k}t)\setminus B(2^{-k-1}t)}\frac{b(|y|)\Omega(y')}{|y|^{n-\rho}}f\bigl(x-\phi\bigl(|y|\bigr)y'\bigr)\,dy\right|^q\frac{dt}{t}\right)^{1/q} \\
&\quad \le \sum_{k=0}^\infty\left(\int_0^\infty \left|\frac{1}{t^\rho\phi(t)^\alpha}\int_{B(2^{-k}t)\setminus B(2^{-k-1}t)}\frac{b(|y|)\Omega(y')}{|y|^{n-\rho}}f\bigl(x-\phi\bigl(|y|\bigr)y'\bigr)\,dy\right|^q\frac{dt}{t}\right)^{1/q} \\
&\quad = \sum_{k=0}^\infty\frac{1}{2^{\rho k}}\left(\int_0^\infty \left|\frac{1}{t^\rho\phi(2^k t)^\alpha}\int_{B(t)\setminus B(t/2)}\frac{b(2^k|y|)\Omega(y')}{|y|^{n-\rho}}f\bigl(x-\phi\bigl(2^k|y|\bigr)y'\bigr)\,dy\right|^q\frac{dt}{t}\right)^{1/q}.
\end{aligned}$$

Hence

$$\begin{aligned}
&\mu^{(b)}_{\Omega,\rho,\phi,\alpha,q}(f)(x) \\
&\quad \le \sum_{k=0}^\infty\frac{1}{2^{\rho k}} \\
&\qquad \times\left(\int_0^\infty \left|\frac{1}{t^\rho\phi(2^k t)^\alpha}\int_{B(t)\setminus B(t/2)}\frac{b(2^k|y|)\Omega(y')}{|y|^{n-\rho}}f\bigl(x-\phi\bigl(2^k|y|\bigr)y'\bigr)\,dy\right|^q\frac{dt}{t}\right)^{1/q}. \qquad (2.22)
\end{aligned}$$

So, in the case $\alpha \ge 0$ we have

$$\begin{aligned}
&\mu^{(b)}_{\Omega,\rho,\phi,\alpha,q}(f)(x) \\
&\quad \le \sum_{k=0}^\infty\frac{1}{2^{(\rho+\alpha/\|\varphi\|_\infty)k}} \\
&\qquad \times\left(\int_0^\infty \left|\frac{1}{t^\rho\phi(t)^\alpha}\int_{B(t)\setminus B(t/2)}\frac{b(2^k|y|)\Omega(y')}{|y|^{n-\rho}}f\bigl(x-\phi\bigl(2^k|y|\bigr)y'\bigr)\,dy\right|^q\frac{dt}{t}\right)^{1/q}.
\end{aligned}$$

So, in the case $0 > \alpha > -\rho/\log c_0$, from (2.22), we have

$$\begin{aligned}
&\mu^{(b)}_{\Omega,\rho,\phi,\alpha,q}(f)(x) \\
&\quad \le \sum_{k=0}^\infty\frac{1}{2^{(\rho+\alpha\log_2 c_0)k}} \\
&\qquad \times\left(\int_0^\infty \left|\frac{1}{t^\rho\phi(t)^\alpha}\int_{B(t)\setminus B(t/2)}\frac{b(2^k|y|)\Omega(y')}{|y|^{n-\rho}}f\bigl(x-\phi\bigl(2^k|y|\bigr)y'\bigr)\,dy\right|^q\frac{dt}{t}\right)^{1/q}.
\end{aligned}$$

Notice that b and $b(2^k\cdot)$ satisfy the same condition due to the scaling invariance of Δ_γ. Likewise ϕ and $\phi(2^k\cdot)$ satisfy the same conditions (1.18) and (1.19) with constants independent of k. Hence, for our purpose, it is sufficient to consider the modified operator given by

$$\tilde{\mu}^{(b)}_{\Omega,\rho,\phi,\alpha,q}(f)(x) := \left(\int_0^\infty \left|\frac{1}{t^\rho \phi(t)^\alpha}\int_{B(t)\setminus B(t/2)} \frac{b(|y|)\Omega(y')}{|y|^{n-\rho}} f\big(x-\phi(|y|)y'\big)\,dy\right|^q \frac{dt}{t}\right)^{1/q}$$

for $x \in \mathbb{R}^n$.

Now we proceed to the proof of Theorem 4. Let

$$\tilde{\mu}^{(b)}_{\Omega,\rho,\phi,\alpha,q,j} f(x) := \left(\sum_{k=-\infty}^{\infty}\int_{2^k}^{2^{k+1}} \left|\Psi_{j-k} * \sigma_t * f(x)\right|^q \frac{dt}{t\phi(t)^{q\alpha}}\right)^{1/q} \quad (x \in \mathbb{R}^n) \tag{2.23}$$

for each j. Using the partition of unity (2.16) and the triangle inequality, we then have

$$\begin{aligned}
\tilde{\mu}^{(b)}_{\Omega,\rho,\phi,\alpha,q} f(x) &= \left(\int_0^\infty \left|\sum_{j\in\mathbb{Z}} \Psi_j * \sigma_t * f(x)\right|^q \frac{dt}{t\phi(t)^{q\alpha}}\right)^{1/q} \\
&= \left(\sum_{k\in\mathbb{Z}}\int_{2^k}^{2^{k+1}} \left|\sum_{j\in\mathbb{Z}} \Psi_{j-k} * \sigma_t * f(x)\right|^q \frac{dt}{t\phi(t)^{q\alpha}}\right)^{1/q} \\
&\le \sum_{j\in\mathbb{Z}}\left(\sum_{k\in\mathbb{Z}}\int_{2^k}^{2^{k+1}} \left|\Psi_{j-k} * \sigma_t * f(x)\right|^q \frac{dt}{t\phi(t)^{q\alpha}}\right)^{1/q} \\
&\le \sum_{j\in\mathbb{Z}} \tilde{\mu}^{(b)}_{\Omega,\rho,\phi,\alpha,q,j} f(x).
\end{aligned} \tag{2.24}$$

Next, we treat the L^p-estimate of $\tilde{\mu}^{(b)}_{\Omega,\rho,\phi,\alpha,q,j} f$.

Let us set

$$\alpha(j) := \begin{cases} \alpha/c_1, & j \ge 0, \\ \alpha \log_2 c_0, & j < 0. \end{cases}$$

In Section 4 we plan to distinguish three cases to prove.

Lemma 2.4 *Assume either one of the following three conditions*:

1. $1 < q < r < \gamma q < \infty$ (*see Figure* 2).
2. $1 < q' < r' < \gamma q' < \infty$ (*see Figure* 3).
3. $1 < q = r < \infty$.

If $\Omega \in L^1(S^{n-1})$, *then we have*

$$\left\|\tilde{\mu}^{(b)}_{\Omega,\rho,\phi,\alpha,q,j} f\right\|_{L^r(\mathbb{R}^n)} \le C 2^{-\alpha(j)j} \|\Omega\|_{L^1(S^{n-1})} \|f\|_{\dot{F}^{\alpha}_{rq}(\mathbb{R}^n)}. \tag{2.25}$$

However, in case 3, we just interpolate cases 1 and 2. So we concentrate on cases 1 and 2 in Section 4.

Note that cases 1-3 do not cover all the cases as the above images show.

We also need to prove the following.

Figure 2 **1/*q*-1/*r* graph with $\gamma = \sqrt{3}$.**

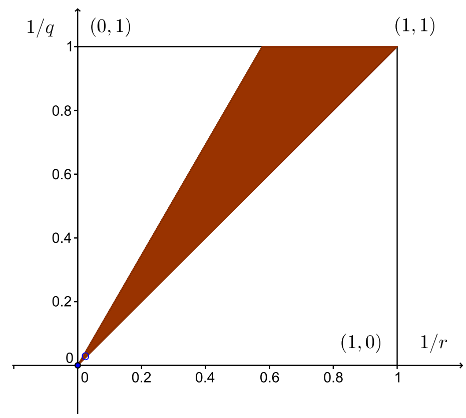
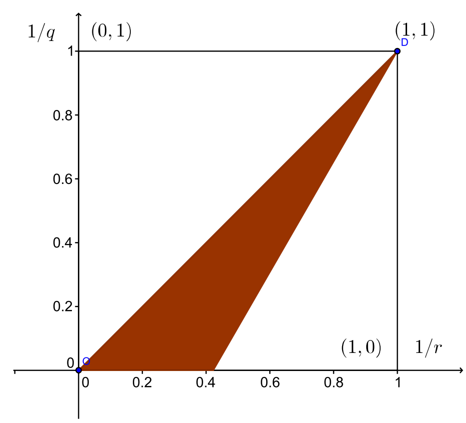


Figure 3 **1/*q*-1/*r* graph with $\gamma = \sqrt{3}$.**



Lemma 2.5 *Let ϕ satisfy the same conditions* (1.18) *and* (1.19). *Assume that $\Omega \in L^1(S^{n-1})$ satisfies the cancellation condition* (1.2). *Then*

$$\left\| \tilde{\mu}^{(b)}_{\Omega,\rho,\phi,\alpha,2,j} f \right\|_{L^2(\mathbb{R}^n)} \le C 2^{(\alpha(-j)/\alpha - \alpha(j))j} \|\Omega\|_{L^1(S^{n-1})} \|b\|_{\Delta_\gamma} \|f\|_{\dot{F}^{\alpha}_{2,2}}. \tag{2.26}$$

By using the strong decay of (2.25), interpolate (2.25) and (2.26) to have (2.25) again for any admissible p and q. Thus, in conclusion, (2.24) is summable over j by virtue of (2.25).

3 Proof of Theorem 4

In this section, we prove Theorem 4. One can obtain Theorem 4 by observing carefully the proof of [6, Theorem 6], but for the sake of completeness we shall give its detailed proof, modifying their one.

3.1 Proof of Lemma 2.4

Here we do not need the cancellation property of Ω and hence we can consider its absolute value of σ_t.

(1) In the case $q < r < \gamma q$, let

$$J := \left\| \tilde{\mu}^{(b)}_{\Omega,\rho,\phi,\alpha,q,j} f \right\|^q_{L^r(\mathbb{R}^n)} = \left\| \left(\sum_{k \in \mathbb{Z}} \int_{2^k}^{2^{k+1}} |\Psi_{j-k} * \sigma_t * f|^q \frac{dt}{t\phi(t)^{\alpha q}} \right)^{1/q} \right\|^q_{L^r(\mathbb{R}^n)}.$$

Let us set $s = (r/q)' = r/(r-q)$. By the duality $L^{q/r}$-L^s, we can take a nonnegative function $h \in L^s(\mathbb{R}^n)$ with $\|h\|_{L^s(\mathbb{R}^n)} = 1$ such that

$$J = \int_{\mathbb{R}^n} \sum_{k\in\mathbb{Z}} \left\{ \int_{2^k}^{2^{k+1}} \left|\Psi_{j-k} * \sigma_t * f(x)\right|^q \frac{dt}{t\phi(t)^{\alpha q}} \right\} h(x)\,dx.$$

Denote by $\|\sigma_t\|$ the total mass of σ_t. By the Hölder inequality

$$\begin{aligned} J &= \sum_{k\in\mathbb{Z}} \int_{2^k}^{2^{k+1}} \left\{ \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} \Psi_{j-k} * f(x-y)\,d\sigma_t(y) \right|^q h(x)\,dx \right\} \frac{dt}{t\phi(t)^{\alpha q}} \\ &\le \sum_{k\in\mathbb{Z}} \int_{2^k}^{2^{k+1}} \left\{ \int_{\mathbb{R}^n} \left[\int_{\mathbb{R}^n} \left|\Psi_{j-k} * f(x-y)\right|^q d|\sigma_t|(y) \right] \|\sigma_t\|^{q/q'} h(x)\,dx \right\} \frac{dt}{t\phi(t)^{\alpha q}}. \end{aligned}$$

By virtue of (2.3), we have

$$\begin{aligned} J &\le C\|\Omega\|_{L^1(S^{n-1})}^q \|b\|_{\Delta_1} \\ &\quad \times \sum_{k\in\mathbb{Z}} \int_{2^k}^{2^{k+1}} \left\{ \int_{\mathbb{R}^n} \left[\int_{\mathbb{R}^n} \left|\Psi_{j-k} * f(y)\right|^q d|\sigma_t|(x-y) \right] h(x)\,dx \right\} \frac{dt}{t\phi(t)^{\alpha q}} \\ &= C\|\Omega\|_{L^1(S^{n-1})}^q \|b\|_{\Delta_1} \\ &\quad \times \sum_{k\in\mathbb{Z}} \int_{2^k}^{2^{k+1}} \left\{ \int_{\mathbb{R}^n} \left|\Psi_{j-k} * f(y)\right|^q \left(\int_{\mathbb{R}^n} h(x)\,d|\sigma_t|(x-y) \right) dy \right\} \frac{dt}{t\phi(t)^{\alpha q}}. \end{aligned}$$

Since $1 < q < r < \gamma q$, we have $s > \gamma'$. So, by (2.10) and Hölder's inequality, we conclude

$$\begin{aligned} J^{1/q} &\le C\|\Omega\|_{L^1(S^{n-1})} \|b\|_{\Delta_\gamma} \left(\int_{\mathbb{R}^n} \left(\sum_{k\in\mathbb{Z}} \int_{2^k}^{2^{k+1}} \left|\Psi_{j-k} * f(y)\right|^q \sigma^*(h)(y) \frac{dt}{t\phi(t)^{\alpha q}} \right) dy \right)^{1/q} \\ &\le C\|\Omega\|_{L^1(S^{n-1})} \|b\|_{\Delta_\gamma} \left(\int_{\mathbb{R}^n} \sum_{k\in\mathbb{Z}} \frac{1}{\phi(2^k)^{\alpha q}} \left|\Psi_{j-k} * f(y)\right|^q \sigma^*(h)(y)\,dy \right)^{1/q} \\ &\le C\|\Omega\|_{L^1(S^{n-1})} \|b\|_{\Delta_\gamma} \left(\int_{\mathbb{R}^n} \left(\sum_{k\in\mathbb{Z}} \frac{1}{\phi(2^k)^{\alpha q}} \left|\Psi_{j-k} * f(y)\right|^q \right)^{s'} dy \right)^{1/(s'q)} \|h\|_{L^s(\mathbb{R}^n)}^{1/q} \\ &= C\|\Omega\|_{L^1(S^{n-1})} \|b\|_{\Delta_\gamma} \left(\int_{\mathbb{R}^n} \left(\sum_{\ell\in\mathbb{Z}} \frac{1}{\phi(2^{j-\ell})^{\alpha q}} \left|\Psi_\ell * f(y)\right|^q \right)^{r/q} dy \right)^{1/r}. \end{aligned}$$

Thus, we have

$$\left\| \tilde{\mu}_{\Omega,\rho,\phi,\alpha,q,j}^{(b)} f \right\|_{L^r(\mathbb{R}^n)} \le C 2^{-\alpha(j)j} \|b\|_{\Delta_\gamma} \|f\|_{\dot{F}^\alpha_{rq}}. \tag{3.1}$$

(2) In case $1 < r < q$ and $r' < \gamma q'$, it follows that $r' > q'$. By duality, there is a sequence of functions $g_k(x,t)$ such that

$$\left(\int_{\mathbb{R}^n} \left(\sum_{k\in\mathbb{Z}} \int_{2^k}^{2^{k+1}} \left|g_k(x,t)\right|^{q'} \frac{dt}{t} \right)^{r'/q'} dx \right)^{1/r'} = 1$$

and such that

$$\left\| \left(\sum_{k\in\mathbb{Z}} \int_{2^k}^{2^{k+1}} \left|\Psi_{j-k} * \sigma_t * f\right|^q \frac{dt}{t\phi(t)^{\alpha q}} \right)^{1/q} \right\|_{L^r(\mathbb{R}^n)}$$
$$= \int_{\mathbb{R}^n} \sum_{k\in\mathbb{Z}} \left\{ \int_{2^k}^{2^{k+1}} \left(\Psi_{j-k} * \sigma_t * f(x)\right) g_k(x,t) \frac{dt}{t\phi(t)^{\alpha}} \right\} dx.$$

Then we have

$$\int_{\mathbb{R}^n} \sum_{k\in\mathbb{Z}} \int_{2^k}^{2^{k+1}} \left(\Psi_{j-k} * \sigma_t * f(x)\right) g_k(x,t) \frac{dt}{t\phi(t)^{\alpha}}\, dx$$
$$\leq \int_{\mathbb{R}^n} \sum_{k\in\mathbb{Z}} \left\{ \int_{2^k}^{2^{k+1}} \left(\int_{\mathbb{R}^n} \left|\Psi_{j-k} * f(y)\right| d|\sigma_t|(x-y) \right) \left|g_k(x,t)\right| \frac{dt}{t\phi(t)^{\alpha}} \right\} dx$$
$$\leq \int_{\mathbb{R}^n} \sum_{k\in\mathbb{Z}} \left\{ \int_{2^k}^{2^{k+1}} \left|\Psi_{j-k} * f(y)\right| \left(\int_{\mathbb{R}^n} \left|g_k(x,t)\right| d|\sigma_t|(x-y) \right) \frac{dt}{t\phi(t)^{\alpha}} \right\} dx$$
$$\leq C \int_{\mathbb{R}^n} \sum_{k\in\mathbb{Z}} \left\{ \int_{2^k}^{2^{k+1}} \frac{1}{\phi(2^k)^{\alpha}} \left|\Psi_{j-k} * f(y)\right| \left(\int_{\mathbb{R}^n} \left|g_k(x,t)\right| d|\sigma_t|(x-y) \right) \frac{dt}{t} \right\} dy.$$

By using the Hölder inequality for sequences, we have

$$\int_{\mathbb{R}^n} \left(\sum_{k\in\mathbb{Z}} \int_{2^k}^{2^{k+1}} \left(\Psi_{j-k} * \sigma_t * f(x)\right) g_k(x,t) \frac{dt}{t\phi(t)^{\alpha}} \right) dx$$
$$\leq C \int_{\mathbb{R}^n} \left(\sum_{k\in\mathbb{Z}} \frac{1}{\phi(2^k)^{\alpha q}} \left|\Psi_{j-k} * f(y)\right|^q \right)^{1/q}$$
$$\times \left(\sum_{k\in\mathbb{Z}} \int_{2^k}^{2^{k+1}} \left(\int_{\mathbb{R}^n} \left|g_k(x,t)\right| d|\sigma_t|(x-y) \right)^{q'} \frac{dt}{t} \right)^{1/q'} dy.$$

By the properties of ϕ and Proposition 1, we conclude

$$\int_{\mathbb{R}^n} \left(\sum_{k\in\mathbb{Z}} \int_{2^k}^{2^{k+1}} \left(\Psi_{j-k} * \sigma_t * f(x)\right) g_k(x,t) \frac{dt}{t\phi(t)^{\alpha}} \right) dx$$
$$\leq C 2^{-\alpha(j)j} \|b\|_{\Delta_\gamma} \left(\int_{\mathbb{R}^n} \left(\sum_{k\in\mathbb{Z}} \frac{1}{\phi(2^k)^{\alpha q}} \left|\Psi_k * f(y)\right|^q \right)^{r/q} dy \right)^{1/r}$$
$$\times \left(\int_{\mathbb{R}^n} \left(\sum_{k\in\mathbb{Z}} \int_{2^k}^{2^{k+1}} \left(\int_{\mathbb{R}^n} \left|g_k(x,t)\right| d|\sigma_t|(x-y) \right)^{q'} \frac{dt}{t} \right)^{r'/q'} dy \right)^{1/r'}$$
$$= C 2^{-\alpha(j)j} \|b\|_{\Delta_\gamma} \|f\|_{\dot{F}^{\alpha}_{rq}}$$
$$\times \left(\int_{\mathbb{R}^n} \left(\sum_{k\in\mathbb{Z}} \int_{2^k}^{2^{k+1}} \left(\int_{\mathbb{R}^n} \left|g_k(x,t)\right| d|\sigma_t|(x-y) \right)^{q'} \frac{dt}{t} \right)^{r'/q'} dy \right)^{1/r'}.$$

In the same way as in [6, p.705], using (2.10), we can check

$$\left(\int_{\mathbb{R}^n}\left(\sum_{k\in\mathbb{Z}}\int_{2^k}^{2^{k+1}}\left(\int_{\mathbb{R}^n}\left|g_k(x,t)\right|d|\sigma_t|(x-y)\right)^{q'}\frac{dt}{t}\right)^{r'/q'}dy\right)^{1/r'}$$
$$\leq C\|\Omega\|_{L^1(S^{n-1})}\left(\int_{\mathbb{R}^n}\left(\sum_{k\in\mathbb{Z}}\int_{2^k}^{2^{k+1}}\left|g_k(x,t)\right|^{q'}\frac{dt}{t}\right)^{r'/q'}dx\right)^{1/r'},$$

if $(\frac{r'}{q'})' > \gamma'$. Hence we have, for $1 < q' < r' < \gamma q'$,

$$\left\|\tilde{\mu}^{(b)}_{\Omega,\rho,\phi,\alpha,q,j}f\right\|_{L^r(\mathbb{R}^n)} \leq C2^{-\alpha(j)j}\|f\|_{\dot{F}^{\alpha}_{rq}}. \tag{3.2}$$

So we are done.

3.2 Proof of Lemma 2.5

By virtue of the Plancherel theorem and the Fubini theorem, we have

$$\begin{aligned}\left\|\tilde{\mu}^{(b)}_{\Omega,\rho,\phi,\alpha,2,j}f\right\|_{L^2}^2 &= \sum_{k=-\infty}^{\infty}\int_{\mathbb{R}^n}\left(\int_{2^k}^{2^{k+1}}\left|\Psi_{j-k}*\sigma_t*f(x)\right|^2\frac{dt}{t\phi(t)^{2\alpha}}\right)dx\\ &= C\sum_{k=-\infty}^{\infty}\int_{\mathbb{R}^n}\left(\int_{2^k}^{2^{k+1}}\left|\widehat{\sigma_t}(\xi)\right|^2\frac{dt}{t\phi(t)^{2\alpha}}\right)\left|\hat{f}(\xi)\right|^2\psi_{j-k}(\xi)^2\,d\xi.\end{aligned} \tag{3.3}$$

By (2.11), (3.3) and the support property of ψ_{j-k}, we have

$$\begin{aligned}\left\|\tilde{\mu}^{(b)}_{\Omega,\rho,\phi,\alpha,2,j}f\right\|_{L^2}^2 &\leq C\|b\|_{\Delta_1}\sum_{k=-\infty}^{\infty}\int_{\mathbb{R}^n}\frac{|\xi|^2}{\phi(2^k)^{2\alpha-2}}\left|\hat{f}(\xi)\right|^2\psi_{j-k}(\xi)^2\,d\xi\\ &\leq C\|b\|_{\Delta_1}\sum_{\ell=-\infty}^{\infty}\int_{\mathbb{R}^n}\frac{1}{\phi(2^{-\ell})^2\phi(2^{j-\ell})^{2\alpha-2}}\left|\hat{f}(\xi)\right|^2\psi_{\ell}(\xi)^2\,d\xi.\end{aligned}$$

For $j \leq 0$, from (2.14), it follows that $\phi(2^{j-\ell}) \leq C2^{j/c_1}\phi(2^{-\ell})$, and for $j \geq 0$, from (2.13) we get $\phi(2^{j-\ell}) \leq 2^{j\log_2 c_0}\phi(2^{-\ell})$. Likewise, we have $\phi(2^{-\ell}) \leq 2^{-\alpha(j)j}\phi(2^{j-\ell})$.

We need to control the integrand; first of all,

$$\frac{1}{\phi(2^{j-l})^{2\alpha-2}} = \frac{\phi(2^{j-l})^2}{\phi(2^{j-l})^{2\alpha}}.$$

When $j \geq 0$, we use

$$\phi\left(2^{j-\ell}\right)^2 \leq C2^{2j\log_2 c_0}\phi\left(2^{-\ell}\right)^2$$

and

$$\left(\frac{1}{\phi(2^{j-\ell})}\right)^{2\alpha} \leq C\left(\frac{2^{-\alpha(j)j}}{\phi(2^{-\ell})}\right)^{2\alpha} \leq C\frac{2^{-2(\alpha/c_1)j}}{\phi(2^{-\ell})^{2\alpha}}.$$

When $j \leq 0$, we use

$$\phi\left(2^{j-\ell}\right)^2 \leq C2^{2j/c_1}\phi\left(2^{-\ell}\right)^2$$

and

$$\left(\frac{1}{\phi(2^{j-\ell})}\right)^{2\alpha} \le C\left(\frac{2^{-\alpha(j)j}}{\phi(2^{-\ell})}\right)^{2\alpha} \le C\frac{2^{-2\alpha(\log_2 c_0)j}}{\phi(2^{-\ell})^{2\alpha}}.$$

So, if $j \ge 0$, we have

$$\begin{aligned}\left\|\tilde{\mu}^{(b)}_{\Omega,\rho,\phi,\alpha,2,j}f\right\|^2_{L^2} &\le C2^{2(\log_2 c_0-\alpha/c_1)j}\sum_{\ell=-\infty}^{\infty}\int_{\mathbb{R}^n}\frac{1}{\phi(2^{-\ell})^{2\alpha}}\left|\hat{f}(\xi)\right|^2\psi_\ell(\xi)^2\,d\xi \\ &\le C2^{2(\log_2 c_0-\alpha/c_1)j}\|f\|^2_{\dot{F}^{\alpha}_{2,2}}.\end{aligned}$$

Hence, after incorporating a similar estimate for $j \le 0$, we get (2.26).

3.3 Interpolation and the conclusion of the proof of (i)

Let

$$\alpha \in \left(0, \frac{1}{c_1\log_2 c_0}\cdot\frac{1/\tilde{p}-1/(2\gamma)}{1/2-1/(2\gamma)}\cdot\frac{1/\tilde{q}-1/(2\gamma)}{1/2-1/(2\gamma)}\right).$$

By interpolating (2.26) and (2.25), we claim that there exists $\delta > 0$ such that

$$\left\|\tilde{\mu}^{(b)}_{\Omega,\rho,\phi,\alpha,q,j}f\right\|_{L^p(\mathbb{R}^n)} \le C2^{-\delta|j|}\|f\|_{\dot{F}^{\alpha}_{pq}}. \tag{3.4}$$

When $p = q = 2$, then (3.4) is correct by virtue of (2.25) ($j \ge 0$) and (2.26) ($j < 0$). We check next the case $p \ne 2$ and $q \ne 2$. For $j \ge 0$, by (2.25) we may take $\delta = \alpha(j) = \alpha/c_1$. For $j \le -1$, we take $1 < r_1, r_2 < \infty$ and $0 < \theta_1, \theta_2 < 1$ satisfying

$$\frac{1}{p} = \frac{\theta_1}{2} + \frac{1-\theta_1}{r_1}, \tag{3.5}$$

$$\frac{1}{q} = \frac{\theta_2}{2} + \frac{1-\theta_2}{r_2}. \tag{3.6}$$

Note that we have

$$(p-2)(r_1-2) > 0, \qquad (q-2)(r_2-2) > 0.$$

We choose $1 < r_1, r_2 < \infty$ so that

$$\tilde{p} < \tilde{r}_1 < 2\gamma, \qquad \tilde{q} < \tilde{r}_2 < 2\gamma$$

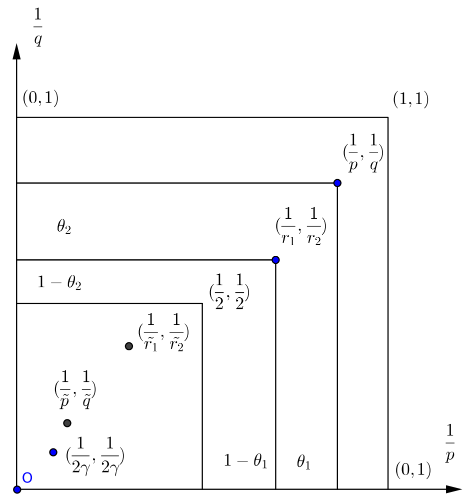
and then determine θ_1, θ_2 by (3.5), (3.6). As in the Figure 4, we can arrange that

$$\alpha < \frac{\theta_1\theta_2}{c_1\log_2 c_0} < \frac{1}{c_1\log_2 c_0}\cdot\frac{1/\tilde{p}-1/(2\gamma)}{1/2-1/(2\gamma)}\cdot\frac{1/\tilde{q}-1/(2\gamma)}{1/2-1/(2\gamma)}. \tag{3.7}$$

We shall see that this choice is possible. Recall that $\tilde{p}, \tilde{q} < 2\gamma$. Then some arithmetic shows that

$$\theta_1 = \frac{1/p-1/r_1}{1/2-1/r_1} = \frac{1/p'-1/r_1'}{1/2-1/r_1'} = \frac{1/\tilde{p}-1/\tilde{r}_1}{1/2-1/\tilde{r}_1}$$

Figure 4 The relation between (1/*p*, 1/*q*) and (1/r_1, 1/r_2).



and that

$$\theta_2 = \frac{1/q - 1/r_2}{1/2 - 1/r_2} = \frac{1/q' - 1/r_2'}{1/2 - 1/r_2'} = \frac{1/\tilde{q} - 1/\tilde{r}_2}{1/2 - 1/\tilde{r}_2}.$$

Assuming that $\tilde{p}, \tilde{q} > 2$, we conclude that the parameters θ_1 and θ_2 are increasing on $(2, \infty)$ with respect to $\tilde{r}_1$ and $\tilde{r}_2$ as functions in $\tilde{r}_1$ and $\tilde{r}_2$, respectively. Hence

$$\theta_1\theta_2 = \frac{1/\tilde{p} - 1/\tilde{r}_1}{1/2 - 1/\tilde{r}_1} \cdot \frac{1/\tilde{q} - 1/\tilde{r}_2}{1/2 - 1/\tilde{r}_2} < \frac{1/\tilde{p} - 1/(2\gamma)}{1/2 - 1/(2\gamma)} \cdot \frac{1/\tilde{q} - 1/(2\gamma)}{1/2 - 1/(2\gamma)}.$$

Therefore, since

$$0 < \alpha < \frac{1}{c_1 \log_2 c_0} \cdot \frac{1/\tilde{p} - 1/(2\gamma)}{1/2 - 1/(2\gamma)} \cdot \frac{1/\tilde{q} - 1/(2\gamma)}{1/2 - 1/(2\gamma)},$$

and $\tilde{p}, \tilde{q} < 2\gamma$, we get (3.7) by choosing r_1 sufficiently near 2γ if $p > 2$ and r_1' sufficiently near 2γ if $1 < p < 2$, and by choosing r_2 similarly according to $q > 2$ or $1 < q < 2$.

Now, interpolating (2.26) and (2.25) with $r = r_1$, $q = 2$, we get

$$\left\| \tilde{\mu}^{(b)}_{\Omega,\rho,\phi,\alpha,2,j} f \right\|_{L^p(\mathbb{R}^n)} \le C 2^{(\theta_1(1/c_1 - \alpha \log_2 c_0) - (1-\theta_1)\alpha \log_2 c_0)j} \|f\|_{\dot{F}^{\alpha}_{p2}}. \tag{3.8}$$

We then interpolate (2.25) and (3.8) with $r = p$, $q = r_2$. As a consequence, we have

$$\left\| \tilde{\mu}^{(b)}_{\Omega,\rho,\phi,\alpha,q,j} f \right\|_{L^p(\mathbb{R}^n)} \le C 2^{\{\theta_2(\theta_1(1/c_1 - \alpha \log_2 c_0) - (1-\theta_1)\alpha \log_2 c_0) - (1-\theta_2)\alpha \log_2 c_0\}j} \|f\|_{\dot{F}^{\alpha}_{pq}}.$$

An arithmetic together with (3.7) shows that

$$\theta_2\big(\theta_1(1/c_1 - \alpha \log_2 c_0) - (1-\theta_1)\alpha \log_2 c_0\big) - (1-\theta_2)\alpha \log_2 c_0 = \theta_1\theta_2/c_1 - \alpha \log_2 c_0 > 0.$$

Thus, taking $\delta = \min\{\alpha/c_1, \theta_1\theta_2/c_1 - \alpha \log_2 c_0\}$, we obtain the desired estimate (3.4).

In the case $p = 2$ or $q = 2$, we can get the desired estimate more simply, by applying interpolation once.

Thus by (2.24) and (3.4) we obtain

$$\|\tilde{\mu}^{(b)}_{\Omega,\rho,\phi,\alpha,q}f\|_{L^p(\mathbb{R}^n)} \le \sum_{j\in\mathbb{Z}} \|\tilde{\mu}^{(b)}_{\Omega,\rho,\phi,\alpha,q,j}f\|_{L^p(\mathbb{R}^n)} \le C\|f\|_{\dot{F}^{\alpha}_{pq}}. \tag{3.9}$$

This completes the proof of Theorem 4(i).

3.4 The proof of (ii)

Below we shall prove Theorem 4(ii). By the Schwarz inequality, we have

$$\begin{aligned}
|\widehat{\sigma}_t(\xi)| &= \frac{1}{t^\rho}\left|\int_{t/2<|y|\le t} e^{-i\phi(|y|)y'\cdot\xi}\frac{b(|y|)\Omega(y')}{|y|^{n-\rho}}\,dy\right| \\
&= \frac{1}{t^\rho}\left|\int_{t/2}^{t}\left(\int_{S^{n-1}}\Omega(y')e^{-i\phi(r)y'\cdot\xi}\,d\sigma(y')\right)b(r)r^{\rho-1}\,dr\right| \\
&\le \left(\int_{t/2}^{t}|b(r)|^2\frac{dr}{r}\right)^{1/2}\left(\int_{t/2}^{t}\left|\int_{S^{n-1}}\Omega(y')e^{-i\phi(r)y'\cdot\xi}\,d\sigma(y')\right|^2\frac{dr}{r}\right)^{1/2} \\
&\le C\|b\|_{\Delta_2}\left(\int_{t/2}^{t}\left|\int_{S^{n-1}}\Omega(y')e^{-i\phi(r)y'\cdot\xi}\,d\sigma(y')\right|^2\frac{dr}{r}\right)^{1/2}.
\end{aligned}$$

Recall

$$W_\Omega = \sup_{\xi\in\mathbb{R}^n\setminus\{0\}}\left(\int_{S^{n-1}\times S^{n-1}}\frac{|\Omega(y')\Omega(z')|}{|(y'-z')\cdot\xi'|^\beta}\,d\sigma(y')\,d\sigma(z')\right)^{1/2}. \tag{3.10}$$

Then, by (2.12) and the doubling condition of ϕ, we have

$$\left(\int_{2^k}^{2^{k+1}}|\widehat{\sigma}_t(\xi)|^2\frac{dt}{t\phi(t)^{2\alpha}}\right)^{1/2} \le \frac{CW_\Omega\|b\|_{\Delta_2}}{|\xi|^{\beta/2}\phi(2^k)^{\beta/2}\phi(2^k)^\alpha}. \tag{3.11}$$

By (3.3), (3.11) and the support property of ψ_{j-k}, we have

$$\begin{aligned}
\|\tilde{\mu}^{(b)}_{\Omega,\rho,\phi,\alpha,2,j}f\|^2_{L^2(\mathbb{R}^n)} &\le CW_\Omega^2\|b\|^2_{\Delta_2}\sum_{k=-\infty}^{\infty}\int_{\mathbb{R}^n}\frac{|\hat{f}(\xi)|^2\psi_{j-k}(|\xi|)^2}{|\xi|^\beta\phi(2^{k-1})^\beta\phi(2^{k+1})^{2\alpha}}\,d\xi \\
&\le CW_\Omega^2\|b\|^2_{\Delta_2}\sum_{\ell=-\infty}^{\infty}\int_{\mathbb{R}^n}\left(\frac{\phi(2^{-\ell})}{\phi(2^{j-\ell})}\right)^\beta\phi(2^{j-\ell})^{-2\alpha}|\hat{f}(\xi)|^2\psi_\ell(|\xi|)^2\,d\xi.
\end{aligned}$$

As in the case (i), we have

$$\phi(2^{-\ell}) \le 2^{-j/c_1}\phi(2^{j-\ell})$$

for $j \ge 0$, and

$$\phi(2^{-\ell}) \le 2^{-j\log_2 c_0}\phi(2^{j-\ell})$$

for $j \le 0$. Similarly, we have

$$\phi(2^{j-\ell}) \le 2^{j/c_1}\phi(2^{-\ell})$$

for $j \le 0$ and

$$\phi\left(2^{j-\ell}\right) \le c_0^{j-1} = 2^{j\log_2 c_0}\phi\left(2^{-\ell}\right)$$

for $j \ge 0$. So, as in the L^2-estimate in (i), we obtain

$$\left\| \tilde{\mu}^{(b)}_{\Omega,\rho,\phi,\alpha,2,j} f \right\|_{L^2(\mathbb{R}^n)} \le \begin{cases} C2^{-(\beta/(2c_1)+\alpha\log_2 c_0)j} W_\Omega \|b\|_{\Delta_2} \|f\|_{\dot{F}^\alpha_{2,2}} & \text{if } j \ge 1, \\ C2^{-((\beta\log_2 c_0)/2+\alpha/c_1)j} W_\Omega \|b\|_{\Delta_2} \|f\|_{\dot{F}^\alpha_{2,2}} & \text{if } j \le 0. \end{cases} \tag{3.12}$$

As for the L^p-estimate, since $\alpha < 0$, we use $\phi(2^{j-\ell}) \le c_0^j \phi(2^{-\ell})$ for $j \ge 0$ and $\phi(2^{j-\ell}) \le 2^{j/c_1}\phi(2^{-\ell})$ for $j \le 0$. Hence we get, as in the L^p-estimate in (i), for any $1 < q, r < \infty$ with $\tilde{r} < \gamma\tilde{q}$ and $j \in \mathbb{Z}$

$$\left\| \tilde{\mu}^{(b)}_{\Omega,\rho,\phi,\alpha,q,j} f \right\|_{L^r(\mathbb{R}^n)} \le \begin{cases} C2^{-(\alpha/c_1)j} W_\Omega \|b\|_{\Delta_2} \|f\|_{\dot{F}^\alpha_{rq}} & \text{for } j \le 0, \\ C2^{-(\alpha\log_2 c_0)j} W_\Omega \|b\|_{\Delta_2} \|f\|_{\dot{F}^\alpha_{rq}} & \text{for } j \ge 0. \end{cases} \tag{3.13}$$

It follows that, for

$$\alpha \in \left(-\frac{2\beta}{c_1 \log_2 c_0} \cdot \frac{1/\tilde{p} - 1/(2\gamma)}{1 - 1/\gamma} \cdot \frac{1/\tilde{q} - 1/(2\gamma)}{1 - 1/\gamma}, 0 \right),$$

there still exists $\delta > 0$ such that

$$\left\| \tilde{\mu}^{(b)}_{\Omega,\rho,\phi,\alpha,q,j} f \right\|_{L^p(\mathbb{R}^n)} \le C2^{-\delta|j|} \|f\|_{\dot{F}^\alpha_{pq}}, \tag{3.14}$$

by using (3.13) in the case $j \le 0$, and interpolating (3.12) and (3.13) in the case $j > 0$, as in the case (i).

Thus by (2.24) and (3.14) we obtain

$$\left\| \tilde{\mu}^{(b)}_{\Omega,\rho,\phi,\alpha,q} f \right\|_{L^p(\mathbb{R}^n)} \le \sum_{j\in\mathbb{Z}} \left\| \tilde{\mu}^{(b)}_{\Omega,\rho,\phi,\alpha,q,j} f \right\|_{L^p(\mathbb{R}^n)} \le C W_\Omega \|b\|_{\Delta_2} \|f\|_{\dot{F}^\alpha_{pq}}.$$

This completes the proof of Theorem 4(ii).

3.5 Proof of (iii)

We proceed to show (iii). Let $\Omega \in L\log L(S^{n-1})$. We normalize Ω to have $\|\Omega\|_{L\log L(S^{n-1})} = 1$. Then, as in [8, pp.698-699], there is a subset $\Lambda \subset \mathbb{N} \cup \{0\}$ and a sequence of functions $\{\Omega_m; m \in \Lambda\}$ satisfying $0 \in \Lambda$ and the following conditions:

$$\int_{S^{n-1}} \Omega_m\left(y'\right) d\sigma\left(y'\right) = 0; \tag{3.15}$$

$$\Omega\left(x'\right) = \sum_{m\in\Lambda} \Omega_m\left(x'\right); \tag{3.16}$$

$$\|\Omega_0\|_{L^2(S^{n-1})} + \sum_{m\in\Lambda} m\|\Omega_m\|_{L^1(S^{n-1})} \le C\|\Omega\|_{L\log L(S^{n-1})}. \tag{3.17}$$

Indeed, we just let

$$\Lambda = \left\{m \in \mathbb{N} : \sigma\left\{2^{m-1} < |\Omega| \le 2^m\right\} > 2^{-4m}\right\}$$

and define

$$\Omega_m(x) = \Omega(x)\chi_{\{2^{m-1}<|\Omega|\le 2^m\}}(x) - \frac{1}{\sigma(S^{n-1})}\int_{2^{m-1}<|\Omega(y)|\le 2^m}\Omega(y)\,d\sigma(y),$$

$$\Omega_0(x) = \Omega(x) - \sum_{m\in\Lambda\cap\mathbb{N}}\Omega_m(x).$$

For details we refer to [22].

Now for $m \in \Lambda$, by observing the proof of the case (i), we choose θ_1 and θ_2 very close to

$$4\frac{1/\tilde{p} - 1/(2\gamma)}{1 - 1/\gamma}\cdot\frac{1/\tilde{q} - 1/(2\gamma)}{1 - 1/\gamma}$$

so that $\delta = \alpha/c_1$ for small $\alpha > 0$. For large m, setting $\alpha = 1/m$, we obtain

$$\left\|\mu^{(b)}_{\Omega_m,\rho,\phi,\alpha,q,j}f\right\|_{L^p(\mathbb{R}^n)} \le C2^{-|j|/m}\|\Omega_m\|_{L^1(S^{n-1})}\|f\|_{\dot{F}^{1/m}_{pq}}, \quad j \in \mathbb{Z}. \tag{3.18}$$

Next, from $\Omega_m \in L^2(S^{n-1})$ it follows that Ω_m satisfies the condition in Theorem 4(ii) for any $\beta < 1/2$. Fix $0 < \beta < 1/2$ and $\alpha_0 > 0$ with

$$\alpha_0 < \left(0, \min\left\{\frac{2\beta}{c_1\log_2 c_0}\cdot\frac{1/\tilde{p} - 1/(2\gamma)}{1 - 1/\gamma}\cdot\frac{1/\tilde{q} - 1/(2\gamma)}{1 - 1/\gamma}, \frac{\rho}{\log_2 c_0}\right\}\right).$$

Let also

$$\delta_0 = \min\left\{\frac{\alpha_0}{c_1}, \frac{\beta\theta_1\theta_2}{2c_1} + \alpha_0\log_2 c_0\right\}$$

in the proof of the case (ii). Then we obtain

$$\left\|\mu^{(b)}_{\Omega_m,\rho,\phi,-\alpha_0,q,j}f\right\|_{L^p(\mathbb{R}^n)} \le C2^{-\delta_0|j|}\|\Omega_m\|_{L^2(S^{n-1})}\|f\|_{\dot{F}^{-\alpha_0}_{pq}}, \quad j \in \mathbb{Z}. \tag{3.19}$$

Since $\frac{\alpha_0}{1/m+\alpha_0} + (1 - \frac{\alpha_0}{1/m+\alpha_0}) = 1$ and $\frac{1}{m}\cdot\frac{\alpha_0}{1/m+\alpha_0} - \alpha_0(1 - \frac{\alpha_0}{1/m+\alpha_0}) = 0$, an interpolation between (3.18) and (3.19) yields

$$\begin{aligned}
&\left\|\mu^{(b)}_{\Omega_m,\rho,\phi,0,q,j}f\right\|_{L^p(\mathbb{R}^n)} \\
&\quad\le C2^{-(\alpha_0/(1+m\alpha_0)+\delta_0/(1+m\alpha_0))|j|}\|\Omega_m\|^{\alpha_0/(1/m+\alpha_0)}_{L^1(S^{n-1})}\|\Omega_m\|^{1/(1+m\alpha_0)}_{L^2(S^{n-1})}\|f\|_{\dot{F}^0_{pq}} \\
&\quad\le C2^{-|j|/m}2^{4/\alpha_0}\|f\|_{\dot{F}^0_{pq}}, \quad j \in \mathbb{Z}.
\end{aligned}$$

Thus, summing up the above estimate, we obtain

$$\left\|\mu^{(b)}_{\Omega_m,\rho,\phi,0,q}f\right\|_{L^p(\mathbb{R}^n)} \le \frac{C}{1 - 2^{-1/m}}\|f\|_{\dot{F}^0_{pq}} \le Cm\|f\|_{\dot{F}^0_{pq}}. \tag{3.20}$$

Combining (3.20) with (3.16) and (3.17) and the definition of $\mu^{(b)}_{\Omega,0,\rho,q}$, we obtain the desired estimate

$$\big\|\mu^{(b)}_{\Omega,\rho,\phi,0,q}f\big\|_{L^p(\mathbb{R}^n)} \le C\|\Omega\|_{L\log L(S^{n-1})}\|f\|_{\dot{F}^0_{pq}}.$$

Thus, we are done.

4 Proof of Theorem 3

Here we shall relax the condition on α by taking advantage of a new condition on ϕ. We use the notations in the proof of Theorem 4, by setting $b(t) \equiv 1$ and $\gamma = \infty$. Using (1.18) and (1.19), we apply Theorem 4(i) and we obtain the conclusion of Theorem 3(i).

We go to the proof of (ii). First

$$\begin{aligned}\widehat{\sigma_t}(\xi) &= \frac{1}{t^\rho}\int_{B(t)\setminus B(t/2)}\frac{\Omega(y')}{|y|^{n-\rho}}e^{-i\phi(|y|)y'\cdot\xi}\,dy \\ &= \int_{S^{n-1}}\Omega\big(y'\big)\frac{1}{t^\rho}\left(\int_{t/2}^{t}e^{-i\phi(|y|)y'\cdot\xi}r^{\rho-1}\,dr\right)d\sigma\big(y'\big). \end{aligned} \tag{4.1}$$

With a change of variables we get

$$\begin{aligned}B(t,\xi) &:= \frac{1}{t^\rho}\int_{t/2}^{t}e^{-i\phi(|y|)y'\cdot\xi}r^{\rho-1}\,dr \\ &= \frac{1}{t^\rho}\int_{\phi(t/2)}^{\phi(t)}e^{-isy'\cdot\xi}\frac{\phi^{-1}(s)^\rho}{\phi^{-1}(s)\phi'(\phi^{-1}(s))}\,ds \\ &= \frac{1}{t^\rho}\int_{\phi(t/2)}^{\phi(t)}e^{-isy'\cdot\xi}\frac{\phi^{-1}(s)^\rho}{s}\cdot\frac{\phi(\phi^{-1}(s))}{\phi^{-1}(s)\phi'(\phi^{-1}(s))}\,ds \\ &= \frac{1}{t^\rho}\int_{\phi(t/2)}^{\phi(t)}e^{-isy'\cdot\xi}\frac{\phi^{-1}(s)^\rho}{s}\cdot\varphi\big(\phi^{-1}(s)\big)\,ds. \end{aligned} \tag{4.2}$$

Suppose now that $t\phi'(t)$ is increasing on $\mathbb{R}_+$. Then $\phi^{-1}(s)\phi'(\phi^{-1}(s))$ is also increasing. So by applying the second mean value theorem to the real part of the expression (4.2), we see that there exists u with $\phi(t/2) < u < \phi(t)$ such that

$$\operatorname{Re}B(t,\xi) = \frac{1}{t^\rho\phi^{-1}(\phi(t/2))\phi'(\phi^{-1}(\phi(t/2))}\int_{\phi(t/2)}^{u}\operatorname{Re}\big(e^{-isy'\cdot\xi}\big)\phi^{-1}(s)^\rho\,ds.$$

Since $\phi^{-1}(s)^\rho$ is increasing, we have

$$\begin{aligned}\big|\operatorname{Re}B(t,\xi)\big| &\le \frac{\phi^{-1}(u)^\rho}{t^\rho\phi^{-1}(\phi(t/2))\phi'(\phi^{-1}(\phi(t/2))|y'\cdot\xi|} \\ &\le \frac{\phi^{-1}(\phi(t))^\rho}{t^\rho}\cdot\frac{\phi(\phi^{-1}(\phi(t/2)))}{\phi^{-1}(\phi(t/2))\phi'(\phi^{-1}(\phi(t/2))}\cdot\frac{1}{\phi(t/2)|y'\cdot\xi|} \\ &= \frac{\phi^{-1}(\phi(t))^\rho}{t^\rho}\cdot\frac{\phi(t/2)}{t/2\cdot\phi'(t/2)}\cdot\frac{1}{\phi(t/2)|y'\cdot\xi|} \\ &= \frac{\phi^{-1}(\phi(t))^\rho\varphi(t/2)}{t^\rho}\cdot\frac{1}{\phi(t/2)|y'\cdot\xi|} \le C\frac{c_0\|\varphi\|_\infty}{\phi(t)|y'\cdot\xi|}.\end{aligned}$$

After estimating $\operatorname{Im} B(t,\xi)$ in a similar manner, we obtain

$$\left|B(t,\xi)\right| \le C\frac{c_0\|\varphi\|_\infty}{\phi(t)|y'\cdot\xi|}. \tag{4.3}$$

In the case $t\phi'(t)$ is decreasing or $\varphi(t)$ is monotonic, we get the same estimate (4.3) in a similar way. Clearly, we have $|B(t,\xi)| \le 1/\rho$, and hence for any $0 < \beta \le 1$, $|B(t,\xi)| \le \frac{C}{(\phi(t)|y'\cdot\xi|)^\beta}$. By (4.1) we get

$$\left|\widehat{\sigma}_t(\xi)\right| \le C\left(\int_{S^{n-1}} \frac{|\Omega(y')|}{|y'\cdot\xi'|^\beta}\,d\sigma\left(y'\right)\right)\frac{1}{(\phi(t)|\xi|)^\beta}. \tag{4.4}$$

Now the rest of the proof is the same as that of the case (i).

This completes the proof of Theorem 3.

5 Proof of Proposition 1

The part is an appendix of the present paper, where we prove Proposition 1. Let $\psi \in \mathcal{S}(\mathbb{R}^n)$ (see Figure 5) be chosen so that

$$\chi_{B(1)} \le \psi \le \chi_{B(a^{1/3})}.$$

Define

$$\varphi_k(\xi) = \psi\left(a_k^{-1}\xi\right) - \psi\left(a_{k-1}^{-1}\xi\right) \quad \left(\xi \in \mathbb{R}^n\right). \tag{5.1}$$

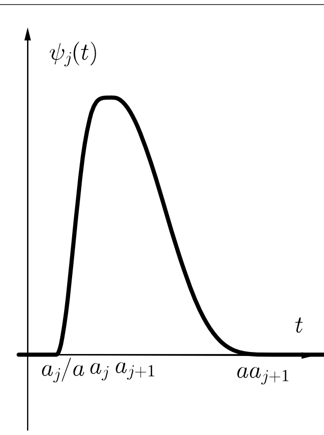
Notice that $\operatorname{supp}\varphi_k \subset \{\xi \in \mathbb{R}^n; a_{k-1} \le |\xi| \le a^{1/3}a_k\}$ $(k \in \mathbb{Z})$ and that $\varphi_k(\xi) = 1$ on $\{a^{1/3}a_{k-1} \le |\xi| \le a_k\}$. Let

$$\Phi_k = \mathcal{F}^{-1}\varphi_k. \tag{5.2}$$

Then we see that $\{\Phi_k\}_{k\in\mathbb{Z}}$ is a partition of unity adapted to $\{a_k\}_{k\in\mathbb{Z}}$. Similarly, taking ψ so that

$$\chi_{B(a^{-1/3})} \le \psi \le \chi_{B(1)},$$

Figure 5 The graph of ψ_j.



and setting

$$\varphi_k(\xi) = \psi\left(a_{k+1}^{-1}\xi\right) - \psi\left(a_k^{-1}\xi\right) \quad \left(\xi \in \mathbb{R}^n\right),$$

we obtain another partition of unity $\{\Psi_k\}_{k\in\mathbb{Z}}$ adapted to $\{a_k\}_{k\in\mathbb{Z}}$ satisfying $\operatorname{supp}\widehat{\Psi}_k \subset \{\xi \in \mathbb{R}^n; a_k/a^{1/3} \le |\xi| \le a_{k+1}\}$ $(k \in \mathbb{Z})$ and $\widehat{\Psi}_k(\xi) = 1$ on $\{a_k \le |\xi| \le a_{k+1}/a^{1/3}\}$. Note that $\{a_k \le |\xi| \le a^{2/3}a_k\} \subset \{a_k \le |\xi| \le a_{k+1}/a^{1/3}\}$. Let us take a function $\Theta \in \mathcal{S}$ so that $\operatorname{supp}(\mathcal{F}\Theta) \subset B(a^{1/3}/2 - 1/2)$. Consider

$$f_k(x) = f_k(x_1, x_2, \dots, x_n) = \exp\left(i\frac{(a^{1/3} + a^{2/3})}{2}a_k x_1\right)\Theta(a_k x) \quad \left(x \in \mathbb{R}^n\right).$$

Then we have

$$\mathcal{F}f_k(\xi) = a_k^{-n}\mathcal{F}\Theta\left(\frac{\xi}{a_k} - \frac{(a^{1/3} + a^{2/3})}{2}\mathbf{e}_1\right) \quad \left(\xi \in \mathbb{R}^n\right),$$

where $\mathbf{e}_1 = (1, 0, \dots, 0)$. It follows that $\operatorname{supp}\mathcal{F}f_k \subset \{a^{1/3}a_k \le |\xi| \le a^{2/3}a_k\}$. Hence we have

$$\Phi_j * f_k(x) = \delta_{(j-1)k}f_k(x) \quad \text{and} \quad \Psi_j * f_k(x) = \delta_{jk}f_k(x),$$

where

$$\delta_{jk} = \begin{cases} 1 & (j = k), \\ 0 & (j \neq k) \end{cases}$$

for $j, k \in \mathbb{Z}$.

$$\|f_k\|_{F_{pq}^{\alpha,\{\Phi_k\}_{k\in\mathbb{Z}}}} = a_{k+1}^{\alpha}\|f_k\|_{L^p(\mathbb{R}^n)} = a_{k+1}^{\alpha}\left\|\Theta(a_k\cdot)\right\|_{L^p(\mathbb{R}^n)}$$

and

$$\|f_k\|_{F_{pq+}^{\alpha,\{\Psi_k\}_{k\in\mathbb{Z}}}} = a_k^{\alpha}\|f_k\|_{L^p(\mathbb{R}^n)} = a_k^{\alpha}\left\|\Theta(a_k\cdot)\right\|_{L^p(\mathbb{R}^n)}.$$

Since the two norms are assumed equivalent, we obtain

$$\frac{a_{k+1}}{a_k} \le C_0$$

for some $C_0 > 1$. Since $\frac{a_{k+1}}{a_k} \ge a$, we have $C_0 > a$.

Thus we have proved the first part of our proposition. We proceed to the second part. Let $\{a_k\}_{k\in\mathbb{Z}}$ be a lacunary sequence of positive numbers with $1 < a \le a_{k+1}/a_k \le C_0$ $(k \in \mathbb{Z})$, and let $\{\Phi_k\}_{k\in\mathbb{Z}}$ be a partition of unity adapted to $\{a_k\}_{k\in\mathbb{Z}}$.

Now we can define the classical homogeneous Triebel-Lizorkin spaces as follows: Let $\psi \in \mathcal{S}(\mathbb{R}^n)$ be chosen so that $\chi_{B(a^{2/3})} \le \psi \le \chi_{B(a)}$. Define

$$\varphi_k(\xi) = \psi\left(a^{-k}\xi\right) - \psi\left(a^{-k+1}\xi\right) \quad \left(\xi \in \mathbb{R}^n\right).$$

Notice that $\varphi_k(\xi) = 1$ on $\{a^k \le |\xi| \le a^{k+\frac{2}{3}}\}$.

Define

$$\|f\|_{\dot{F}^{\alpha}_{pq}} = \left\| \left(\sum_{j=-\infty}^{\infty} \left| a^{\alpha j} \mathcal{F}^{-1} \varphi_j * f \right|^q \right)^{1/q} \right\|_{L^p}.$$

Let us prove

$$\|f\|_{\dot{F}^{\alpha,\{\Phi_k\}_{k\in\mathbb{Z}}}_{pq}} \le C\|f\|_{\dot{F}^{\alpha}_{pq}}.$$

For each $k \in \mathbb{Z}$, we choose $m_k \in \mathbb{Z}$ so that

$$a^{m_k} \le a_k < a^{m_k+1}.$$

Combining with $a^{m_k+1} \le a a_k \le a_{k+1}$, we get $m_{k+1} \ge m_k + 1$. And combining with $a_{k+1}/a_k \le C_0$, we have $m_{k+1} - m_k \le 1 + \log_a C_0$. Furthermore we have

$$\Phi_k = \Phi_k * \sum_{l=m_{k-1}}^{m_{k+1}+1} \varphi_l.$$

Consequently, we obtain

$$\begin{aligned}
\|f\|_{\dot{F}^{\alpha,\{\Phi_k\}_{k\in\mathbb{Z}}}_{pq}} &= \left\| \left(\sum_{k\in\mathbb{Z}} a_k^{\alpha q} |\Phi_k * f|^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)} \\
&= \left\| \left(\sum_{k\in\mathbb{Z}} a_k^{\alpha q} \left| \Phi_k * \sum_{l=m_{k-1}}^{m_{k+1}+1} \varphi_l * f \right|^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)} \\
&\le \left\| \left(\sum_{k\in\mathbb{Z}} a_k^{\alpha q} \left| \Phi_k * \sum_{l=m_{k-1}}^{m_{k+1}+1} \varphi_l * f \right|^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)}.
\end{aligned}$$

We now invoke the Plancherel-Polya-Nikolskij inequality: We have

$$\left| \sum_{l=m_{k-1}}^{m_{k+1}+1} \varphi_l * f(x) \right| \le C\left(1 + \left|a^{m_{k+1}+1}(x-y)\right|\right)^n M\left[\sum_{l=m_{k-1}}^{m_{k+1}+1} \varphi_l * f \right](y).$$

Using Plancherel's theorem, the assumption $|\xi^\beta \partial^\beta \widehat{\Phi}_k(\xi)| \le C_\beta$ for all β and that $\operatorname{supp} \widehat{\Phi}_k \subset \{a_{k-1} \le |\xi| \le a_{k+1}\}$, we get

$$\begin{aligned}
&\int_{\mathbb{R}^n} \left(1 + \left|a^{m_{k+1}+1}x\right|\right)^n \left|\Phi_k(x)\right| dx \\
&\quad\le C \int_{\mathbb{R}^n} \left(1 + |a_{k+1}x|\right)^n \left|\Phi_k(x)\right| dx \\
&\quad\le C\left(\int_{\mathbb{R}^n} \left(1 + |a_{k+1}x|\right)^{4n} \left|\Phi_k(x)\right|^2 dx \right)^{1/2} \left(\int_{\mathbb{R}^n} \left(1 + |a_{k+1}x|\right)^{-2n} dx \right)^{1/2} \\
&\quad\le C a_k^{-n/2} \left(\int_{\mathbb{R}^n} \left(\left|\widehat{\Phi}_k(\xi)\right|^2 + a_{k+1}^{4n} \left|\nabla^n \widehat{\Phi}_k(\xi)\right|^2 \right) d\xi \right)^{1/2}
\end{aligned}$$

$$\le C{a_k}^{-n/2}\left(\int_{a_{k-1}\le|\xi|\le a_{k+1}}\left(\left|\widehat{\Phi}_k(\xi)\right|^2+a_{k+1}^{4n}|\xi|^{-4n}\right)d\xi\right)^{1/2}$$
$$\le C.$$

Hence, it follows that

$$\|f\|_{\dot F_{pq}^{\alpha,\{\Phi_k\}_{k\in\mathbb{Z}}}}\le C\left\|\left(\sum_{k\in\mathbb{Z}}a_k^{\alpha q}M\left[\left|\sum_{l=m_{k-1}}^{m_{k+1}+1}\varphi_l*f\right|\right]^q\right)^{1/q}\right\|_{L^p(\mathbb{R}^n)}.$$

By the Fefferman-Stein vector-valued maximal inequality (see [23]), we obtain

$$\|f\|_{\dot F_{pq}^{\alpha,\{\Phi_k\}_{k\in\mathbb{Z}}}}\le C\left\|\left(\sum_{k\in\mathbb{Z}}a_k^{\alpha q}\left|\sum_{l=m_{k-1}}^{m_{k+1}+1}\varphi_l*f\right|^q\right)^{1/q}\right\|_{L^p(\mathbb{R}^n)}.$$

If we use $(a_1+a_2+\cdots+a_N)^q\le N^q(a_1^q+a_2^q+\cdots+a_N^q)$, then we obtain

$$\|f\|_{\dot F_{pq}^{\alpha,\{\Phi_k\}_{k\in\mathbb{Z}}}}\le C\left\|\left(\sum_{k\in\mathbb{Z}}a_k^{\alpha q}(m_{k+1}-m_{k-1}+2)^q\sum_{l=m_{k-1}}^{m_{k+1}+1}|\varphi_l*f|^q\right)^{1/q}\right\|_{L^p(\mathbb{R}^n)}.$$

Noting $m_{k+1}-m_k\ge 1$, $m_{k+1}-m_k\le 1+\log_a C_0$ and that $a^{m_{k-1}+1}\le a^{m_k}\le a_k<a^{m_k+1}\le a^{m_{k-1}+1+[\log_a C_0]}$, we conclude

$$\|f\|_{\dot F_{pq}^{\alpha,\{\Phi_k\}_{k\in\mathbb{Z}}}}\le C\left\|\left(\sum_{k\in\mathbb{Z}}a_k^{\alpha q}\sum_{l=m_{k-1}}^{m_{k-1}+2+[2\log_a C_0]}|\varphi_l*f|^q\right)^{1/q}\right\|_{L^p(\mathbb{R}^n)}\le C\|f\|_{F_{pq}^{\alpha}}.$$

Let us prove the reverse inequality. For each $k\in\mathbb{Z}$, we can choose $\ell_k\in\mathbb{Z}$ so that

$$a_{\ell_k}\le a^{k-1}\le a^{k+1}\le a_{\ell_k+3}.$$

Then we have

$$\varphi_k=\varphi_k(\Phi_{\ell_k}+\Phi_{\ell_k+1}+\Phi_{\ell_k+2}+\Phi_{\ell_k+3}).$$

Notice that

$$\sup_{l\in\mathbb{Z}}\sharp\{k:\ell_k=l\}\le 3\log_a C_0$$

because $a_{l+3}/a_l\le C_0^3$. Thus, it follows that

$$\begin{aligned}\|f\|_{\dot F_{pq}^{\alpha}}&=\left\|\left(\sum_{k=-\infty}^{\infty}\left|a^{\alpha k}\mathcal{F}^{-1}\varphi_k*f\right|^q\right)^{1/q}\right\|_{L^p}\\&=\left\|\left(\sum_{k=-\infty}^{\infty}\left|a^{\alpha k}\mathcal{F}^{-1}\varphi_k*\left(\mathcal{F}^{-1}\Phi_{\ell_k}+\cdots+\mathcal{F}^{-1}\Phi_{\ell_k+3}\right)*f\right|^q\right)^{1/q}\right\|_{L^p}\\&\le C\left\|\left(\sum_{k=-\infty}^{\infty}M\left[\left|a^{\alpha k}\left(\mathcal{F}^{-1}\Phi_{\ell_k}+\cdots+\mathcal{F}^{-1}\Phi_{\ell_k+3}\right)*f\right|\right]^q\right)^{1/q}\right\|_{L^p}.\end{aligned}$$

Again by the Fefferman-Stein vector-valued maximal inequality (see [23]), we obtain

$$\begin{aligned} \|f\|_{\dot{F}^{\alpha}_{pq}} &\le C\left\| \left(\sum_{k=-\infty}^{\infty} \left| a^{\alpha k} \left(\mathcal{F}^{-1}\Phi_{\ell_k} + \cdots + \mathcal{F}^{-1}\Phi_{\ell_k+3} \right) * f \right|^q \right)^{1/q} \right\|_{L^p} \\ &\le C\left\| \left(\sum_{k=-\infty}^{\infty} \left(\left| a^{\alpha}_{\ell_k} \mathcal{F}^{-1}\Phi_{\ell_k} * f \right| + \cdots + \left| a^{\alpha}_{\ell_k+3} \mathcal{F}^{-1}\Phi_{\ell_k+3} * f \right| \right)^q \right)^{1/q} \right\|_{L^p} \\ &\le C\|f\|_{\dot{F}^{\alpha,\{\Phi_k\}_{k\in\mathbb{Z}}}_{pq}}. \end{aligned}$$

This completes the proof of our proposition.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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