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Research Article

Exact Multiplicity of Positive Solutions for a Class of Second-Order Two-Point Boundary Problems with Weight Function

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An exact multiplicity result of positive solutions for the boundary value problems $u''+\lambda a(t)f(u)=0$, $t\in(0,1)$, $u'(0)=0$, $u(1)=0$ is achieved, where λ is a positive parameter. Here the function $f:[0,\infty)\to[0,\infty)$ is C^2 and satisfies $f(0)=f(s)=0$, $f(u)>0$ for $u\in(0,s)\cup(s,\infty)$ for some $s\in(0,\infty)$. Moreover, f is asymptotically linear and f'' can change sign only once. The weight function $a:[0,1]\to(0,\infty)$ is C^2 and satisfies $a'(t)<0$, $3(a'(t))^2<2a(t)a''(t)$ for $t\in[0,1]$. Using bifurcation techniques, we obtain the exact number of positive solutions of the problem under consideration for λ lying in various intervals in $\mathbb{R}$. Moreover, we indicate how to extend the result to the general case.

1. Introduction

Consider the problem

$$u''+\lambda a(t)f(u)=0,\quad t\in(0,1),$$
$$u'(0)=0,\qquad u(1)=0, \tag{1.1}$$

where $\lambda>0$ is a parameter and $a\in C^2[0,1]$ is a weight function.

The existence and multiplicity of positive solutions for ordinary differential equations have been studied extensively in many literatures, see, for example, [1–3] and references therein. Several different approaches, such as the Leray-Schauder theory, the fixed-point theory, the lower and upper solutions theory, and the shooting method etc has been applied in these literatures. In [4, 5], Ma and Thompson obtained the multiplicity results for a class of second-order two-point boundary value problems depending on a positive parameter λ by using bifurcation theory.

Exact multiplicity of positive solutions have been studied by many authors. See, for example, the papers by Korman et al. [6], Ouyang and Shi [7, 8], Shi [9], Korman and Ouyang [10, 11], Korman [12], Rynne [13], Bari and Rynne [14] (for 2mth-order problems), as well as Korman and Li [15]. In these papers, bifurcation techniques are used. The basic method of proving their results can be divided into three steps: proving positivity of solutions of the linearized problems; studying the direction of bifurcation; showing uniqueness of solution curves.

Ouyang and Shi [7] obtained the curves of positive solutions for the semilinear problem

$$\begin{gathered}\Delta u+\lambda f(u)=0, \quad \text{in } B^n, \\ u=0, \quad \text{on } \partial B^n,\end{gathered} \tag{1.2}$$

where B^n is the unit ball in R^n $(n \geq 1)$ and $f \in C^2(\mathbb{R}^+)$ $(\mathbb{R}^+=[0,\infty))$. In [7], the following two cases were considered:

(i) f'' does not change its sign on $\mathbb{R}^+$; (ii) f'' changes its sign only once on $\mathbb{R}^+$.

Korman and Ouyang [10] studied the problem

$$\begin{gathered}u''+\lambda f(t,u)=0, \quad t \in (-1,1), \\ u(-1)=0, \qquad u(1)=0\end{gathered} \tag{1.3}$$

under the conditions $f \in C^2([-1,1];\mathbb{R})$ and

$$f_{uu}(t,u)>0 \quad \text{for } t \in (-1,1),\ u \in (0,\infty). \tag{1.4}$$

They obtained a full description of the positive solution set of (1.3) and proved that all positive solutions of (1.3) lie on a single smooth solution curve bifurcating from the point $(0,0)$ and tending to $(0,\infty)$ in the (λ,u) plane. Condition (1.4) is very important to conclude the direction of bifurcation curve.

Of course a natural question is how about the structure of the positive solution set of (1.3) when f_{uu} changes its sign only once on $\mathbb{R}^+$?

It is extremely difficult to answer such a question in general. So we shift our study to the problem (1.1) in this paper. We are interested in discussing the exact multiplicity of positive solutions of (1.1) with a weight function a when f'' changes its sign only once on $\mathbb{R}^+$.

Suppose the following.

(H1) One has $f \in C^2[0,\infty)$ with $f(0)=f(s)=0$ for some $s \in (0,\infty)$ and $f(u)>0$ for $u \in (0,s) \cup (s,\infty)$.

(H2) f is *concave convex* that is, there exists $\theta>0$ such that

$$f''(u)<0, \quad \text{if } 0 \leq u<\theta; \qquad f''(u)>0, \quad \text{if } u>\theta. \tag{1.5}$$

(H3) The limits $f_0 = \lim_{u \to 0}(f(u)/u) \in (0, \infty)$ and $f_\infty = \lim_{u \to 0}(f(u)/u) \in (0, \infty)$.

(H4) $a \in C^2[0,1]$ satisfies $a(t) > 0$; $a'(t) < 0$ and $3(a'(t))^2 < 2a(t)a''(t)$, if $t \in [0,1]$.

In this paper, we obtain exactly two disjoint smooth curves of positive solutions of (1.1) under conditions (H1)–(H4). According to this, we can conclude the existence and exact numbers of positive solutions of (1.1) for λ lying in various intervals in $\mathbb{R}$.

Remark 1.1. Korman and Ouyang [10] obtained the unique positive solution curve of (1.3) under the condition (1.4). However they gave no information when f_{uu} can change sign. In [7], they did not treat the case that the equation contains a weight function.

On the other hand, suppose the following.

(H1′) One has $f \in C^2[0, \infty)$ with $f(0) = 0, f(u) > 0$, $u \in (0, \infty)$. There exists $s > 0$ such that $f'(u) < f(u)/u$, $u \in (0, s)$ and $f'(u) > f(u)/u$, $u \in (s, \infty)$.

Remark 1.2. If $a(t) > 0, t \in [0,1]$, then we know from the proof in [4] that the assumptions (H1′) and (H3) imply that the component of positive solutions from the trivial solution and the component from infinity are coincident. However, these two components are disjoint under the assumptions (H1) and (H3) (see [5]). Hence, the essential role is played by the fact of whether f possesses zeros in $\mathbb{R} \setminus \{0\}$. In Section 3, we prove that (1.1) has exactly two positive solution curves which are disjoint and have no turning point on them (Theorem 3.8) under Conditions (H1)–(H4). And (1.1) has a unique positive solution curve with only one turning point (Theorem 3.9) if (H1) is replaced by (H1′). The condition (H4) is used to prove the positivity of solutions of the linearized problems of (1.1) and the direction of bifurcation.

Our main tool is the following bifurcation theorem of Crandall and Rabinowitz.

Theorem 1.3 (see [16]). *Let X and Y be Banach spaces. Let $(\overline{\lambda}, \overline{u}) \in \mathbb{R} \times X$ and let F be a continuously differentiable mapping of an open neighborhood of $(\overline{\lambda}, \overline{u})$ into Y. Let the null-space $N(F_u(\overline{\lambda}, \overline{u})) = \operatorname{span}\{w\}$ be one dimensional and $\operatorname{codim} R(F_u(\overline{\lambda}, \overline{u})) = 1$. Let $F_\lambda(\overline{\lambda}, \overline{u}) \notin R(F_u(\overline{\lambda}, \overline{u}))$. If Z is a complement of span$\{w\}$ in X, then the solution of $F(\lambda, u) = F(\overline{\lambda}, \overline{u})$ near $(\overline{\lambda}, \overline{u})$ forms a curve $(\lambda(s), u(s)) = (\overline{\lambda} + \tau(s), \overline{u} + sw + z(s))$, where $s \to (\tau(s), z(s)) \in \mathbb{R} \times Z$ is a continuously differentiable function near $s = 0$ and $\tau(0) = \tau'(0) = z(0) = z'(0) = 0$.*

2. Notations and Preliminaries

Let $Y = C[0,1]$ with the norm

$$\|y\|_\infty = \max_{t \in [0,1]} |y(t)|, \tag{2.1}$$

and let

$$E = \left\{ y \in C^1[0,1] \mid y'(0) = y(1) = 0 \right\} \tag{2.2}$$

with the norm

$$\|y\|_E = \max\{\|y\|_\infty, \ \|y'\|_\infty\}. \tag{2.3}$$

Set

$$X = \left\{ y \in C^2[0,1] \mid y'(0) = y(1) = 0 \right\} \tag{2.4}$$

equipped with the norm

$$\|y\|_X = \max\{\|y\|_\infty, \ \|y'\|_\infty, \ \|y''\|_\infty\}. \tag{2.5}$$

Define the operator $L : X \to Y$,

$$Lu = -u'', \quad u \in X. \tag{2.6}$$

Then, $L^{-1} : Y \to E$ is a completely continuous operator.

Definition 2.1. For a nontrivial solution of (1.1), (λ, u) is *degenerate* if the linearized problem

$$\begin{gathered} w'' + \lambda a(t) f'(u) w = 0, \quad t \in (0,1), \\ w'(0) = 0, \qquad w(1) = 0 \end{gathered} \tag{2.7}$$

has a nontrivial solution; otherwise, it is *nondegenerate*.

Lemma 2.2. *Let (H1) and (H4) hold. For any degenerate positive solution* (λ, u) *of* (1.1), *the nontrivial solution* w *of* (2.7) *can be chosen as positive.*

Proof. The proof is motivated by Lemma 2.6 in [11].

Suppose to the contrary that w has zeros on (0,1). Without loss of generality, suppose that $w(0) > 0$. Note that w and u_t satisfy

$$w'' + \lambda a(t) f'(u) w = 0, \tag{2.8}$$

$$u_t'' + \lambda a(t) f'(u) u_t + \lambda a'(t) f(u) = 0, \tag{2.9}$$

respectively. We claim that w has at most one zero in (0,1). Otherwise, let $0 < \alpha < \beta < 1$ be the first two zeros of w. Then,

$$\begin{gathered} w(\alpha) = w(\beta) = 0, \quad w(t) < 0, \quad t \in (\alpha, \beta), \\ w'(\alpha) \le 0, \qquad w'(\beta) \ge 0. \end{gathered} \tag{2.10}$$

Multiplying (2.9) by $g(t)w$ and (2.8) by $g(t)u_t$, subtracting, and integrating over (α,β), we have

$$\int_\alpha^\beta u_t'' g w\,dt - \int_\alpha^\beta w'' g u_t + \lambda \int_\alpha^\beta a' g f w = 0 \tag{2.11}$$

with $g(t) > 0$ to be specified. We denote the left side of (2.11) by I and a constant $-w'(\beta)g(\beta)u'(\beta) + w'(\alpha)g(\alpha)u'(\alpha)$ by A. Integrating by parts,

$$\begin{aligned} I &= A + \int_\alpha^\beta u' g'' w\,dt + 2\int_\alpha^\beta u' g' w' dt + \lambda \int_\alpha^\beta a' g f w\,dt \\ &= A - \int_\alpha^\beta u' g'' w\,dt - 2\int_\alpha^\beta w g' u'' dt + \lambda \int_\alpha^\beta a' g f w\,dt \\ &= A - \int_\alpha^\beta u' g'' w\,dt + \lambda \int_\alpha^\beta (2g' a + a' g) f w\,dt. \end{aligned} \tag{2.12}$$

Let

$$2g' a + a' g = 0, \quad g'' < 0 \tag{2.13}$$

on $(0,1)$. From (2.10), (2.13), and $u'(t) < 0$, $t \in (0,1]$, we have

$$I = -w'(\beta)g(\beta)u'(\beta) + w'(\alpha)g(\alpha)u'(\alpha) - \int_\alpha^\beta u' g'' w\,dt > 0. \tag{2.14}$$

Note that the right side of (2.11) is zero, which is a contradiction.

Hence, w has at most one zero in (0,1). Suppose that there is one point γ such that $w(\gamma) = 0$. Then,

$$\begin{gathered} w(\gamma) = w(1) = 0, \quad w(t) < 0, \quad t \in (\gamma, 1), \\ w'(\gamma) \le 0, \qquad w'(1) \ge 0. \end{gathered} \tag{2.15}$$

Repeating the above proof on $(\gamma, 1)$, we can get similar contradiction.

Finally, integrating the differential equation in (2.13), we can choose

$$g(t) = a^{-1/2}(t). \tag{2.16}$$

In view of (H4), $g'' < 0$. So, the auxiliary function g exists. □

The following lemma is an important result in this paper.

Lemma 2.3. *Let (H1) and (H4) hold. Suppose that* (λ^*, u^*) *is a degenerate positive solution of* (1.1). *Then, the following are considered.*

(i) *All solutions of* (1.1) *near* (λ^*, u^*) *have the form* $(\lambda(s), u^* + sw + z(s))$ *for* $s \in (-\delta, \delta)$ *and some* $\delta > 0$, *where* $\lambda(0) = \lambda^*, \lambda'(0) = 0,\ z(0) = z'(0) = 0$.

(ii) *One has* $\lambda''(0) < 0$, *if* f *is concave convex;* $\lambda''(0) > 0$, *if* f *is convex concave.*

Proof. (i) The proof is standard. Let $F : X \to Y$ be such that $F(\lambda, u) = u'' + \lambda a(t) f(u)$. We will show that the conditions of Theorem 1.3 hold.

Since (λ^*, u^*) is a degenerate positive solution of (1.1), we denote the corresponding solution of (2.7) by w. From Lemma 2.2 and the theory of compact disturbing of a Fredholm operator, $N(F_u(\lambda^*, u^*)) = \text{span}\{w\}$ is one dimensional and $\text{codim} R(F_u(\lambda^*, u^*)) = 1$.

Now, we show that $F_\lambda(\lambda^*, u^*) \notin R(F_u(\lambda^*, u^*))$. Suppose to the contrary that $F_\lambda(\lambda^*, u^*) \in R(F_u(\lambda^*, u^*))$. Then, there is a $v \in X$ such that

$$v'' + \lambda^* a(t) f'(u^*) v = a(t) f(u^*), \tag{2.17}$$

$$v'(0) = v(1) = 0. \tag{2.18}$$

Note that w satisfies

$$w'' + \lambda^* a(t) f'(u^*) w = 0, \tag{2.19}$$

$$w'(0) = w(1) = 0. \tag{2.20}$$

Multiplying (2.17) by w and (2.19) by v, subtracting, and integrating on both sides, we obtain

$$\int_0^1 (v''w - w''v)\,dt = \int_0^1 a(t) f(u^*) w\,dt > 0. \tag{2.21}$$

However, the left side of (2.21) is equal to zero according to boundary conditions (2.18) and (2.20). This implies that $F_\lambda(\lambda^*, u^*) \notin I_m(F_u(\lambda^*, u^*))$. According to Theorem 1.3, the result (i) holds.

(ii) Substituting $\lambda = \lambda(s), \quad u = u^* + sw + z(s)$ into (1.1), we obtain

$$[u^* + sw + z(s)]'' + \lambda(s) a(t) f(u^* + sw + z(s)) = 0. \tag{2.22}$$

Since $f \in C^2[0, \infty)$, then, by the implicit function theorem, the solution curve near (λ^*, u^*) is also C^2. Differentiating (2.22) twice with respect to s, we have

$$u''_{ss} + \lambda''(s) a(t) f(u) + 2\lambda'(s) a(t) f'(u) u_s + \lambda(s) a(t) f''(u) u_s^2 + \lambda(s) a(t) f'(u) u_{ss} = 0. \tag{2.23}$$

Evaluating at $s = 0$, we obtain

$$u''_{ss} + \lambda''(0) a(t) f(u^*) + \lambda^* a(t) f''(u^*) w^2 + \lambda^* a(t) f'(u^*) u_{ss} = 0. \tag{2.24}$$

Multiplying (2.24) by w and (2.19) by u_{ss}, subtracting, and integrating, we get

$$\lambda''(0) = -\frac{\lambda^* \int_0^1 a(t) f''(u^*) w^3 dt}{\int_0^1 a(t) f(u^*) w\, dt}. \tag{2.25}$$

According to (H1), (H4), and Lemma 2.2, we see that $\int_0^1 a(t) f(u^*) w\, dt > 0$. Next, for the sign of $\lambda''(0)$, we consider the sign of $\int_0^1 a(t) f''(u^*) w^3 dt$.

We first prove that

$$\int_0^1 f''(u^*) u_t^{*2} w\, dt = 0. \tag{2.26}$$

Differentiating (1.1) and (2.19) with respect to t, we have

$$u_t^{*''} + \lambda a'(t) f(u^*) + \lambda a(t) f'(u^*) u_t^* = 0, \tag{2.27}$$

$$w_t'' + \lambda a'(t) f'(u^*) w + \lambda a(t) f''(u^*) u_t^* w + \lambda a(t) f'(u^*) w_t = 0. \tag{2.28}$$

Multiplying, (2.27) by $h(t) w_t$ and (2.28) by $h(t) u_t^*$, subtracting, and integrating over $(0, 1)$, we get

$$\int_0^1 \left(u_t^{*''} w_t h - w_t'' u_t^* h \right) dt + \lambda \int_0^1 a' \left[f(u^*) w_t - f'(u^*) w u_t^* \right] h\, dt = \lambda \int_0^1 a f''(u^*) u_t^{*2} w h\, dt \tag{2.29}$$

with $h(t) > 0$ to be specified. Integrating by parts on the left side of (2.29),

$$\begin{aligned} &\int_0^1 \left(u_t^{*''} w_t h - w_t'' u_t^* h \right) dt + \lambda \int_0^1 a' \left[f(u^*) w_t - f'(u^*) w u_t^* \right] h\, dt \\ &\quad = \lambda \int_0^1 (a h' + a' h) \left(f(u^*) w' - f'(u^*) w u^{*'} \right) dt. \end{aligned} \tag{2.30}$$

Let

$$a h' + a' h = 0. \tag{2.31}$$

From (2.29), we get

$$\int_0^1 a f''(u^*) u_t^{*2} w h\, dt = 0. \tag{2.32}$$

Solving the equation $a h' + a' h = 0$, we can choose the auxiliary function

$$h = a^{-1}. \tag{2.33}$$

Combining with (2.32), we obtain (2.26).

The following proof is motivated by the proof of Theorem 2.2 in [8].

Since $w > 0$, (2.26) implies that $f''(u^*(t))$ must change sign. If f is concave convex, then there exists $t_0 \in (0,1)$ such that

$$f''(u^*(t)) \geq 0 \quad \text{in } [0,t_0],$$
$$f''(u^*(t)) \leq 0 \quad \text{in } [t_0,1]. \tag{2.34}$$

Next, we claim that there exists $k > 0$, such that

$$kw \geq -u_t^* \quad \text{in } [0,t_0],$$
$$kw \leq -u_t^* \quad \text{in } [t_0,1]. \tag{2.35}$$

Let $x = kw + u_t^*$. Then, $x(0) = kw(0) + u_t^*(0) = kw(0) > 0$, and $x(1) = kw(1) + u_t^*(1) < 0$. So, x has at least one zero in $(0,1)$. Moreover, we can prove that x has only one zero in $(0,1)$. Note that kw satisfies

$$(kw)'' + \lambda^* a(t) f'(u^*)(kw) = 0. \tag{2.36}$$

We get

$$x'' + \lambda a(t) f'(u^*) x = -\lambda a'(t) f(u^*) \geq 0, \tag{2.37}$$

since $a'(t) < 0$ and $f(u^*) > 0$. Suppose that x has more than one zero in (0, 1). Let $t_1 < t_2$ be the last two zeros of x, then we say that

$$x(t_1) = x(t_2) = 0, \quad x(t) > 0, \quad \text{if } t \in (t_1,t_2),$$
$$x'(t_1) \geq 0, \qquad x'(t_2) \leq 0. \tag{2.38}$$

We first prove the above statement. On the contrary, suppose that

$$x(t_1) = x(t_2) = 0, \quad x(t) < 0, \quad \text{if } t \in (t_1,t_2) \cup (t_2,1]. \tag{2.39}$$

Consider the problem

$$q''(t) + \lambda a(t) f'(u^*) q(t) = 0, \quad q(t_1) = q(1) = 0. \tag{2.40}$$

Obviously, x is a subsolution and 0 is a supersolution of (2.40), respectively. Note that $x \leq 0$. By the strong maximum principle, we obtain that $x < 0$ on $(t_1,1)$. This contradicts $x(t_2) = 0$. Hence, the statement holds.

Now let us consider the claim related to k. Multiplying (2.36) by x and (2.37) by kw, subtracting, and integrating over (t_1, t_2), we get

$$-(kw)(t_2)x'(t_2) + (kw)(t_1)x'(t_1) = \lambda \int_{t_1}^{t_2} a'(t)f(u^*)(kw)dt < 0 \tag{2.41}$$

since $a'(t) < 0$. Note that the left side is nonnegative. Such a contradiction implies that x has only one zero in (0,1). By varying k such that $x(t_0) = kw(t_0) + u_t^*(t_0) = 0$, we can conclude the claim.

From the claim and $a'(t) < 0$, we have

$$\begin{aligned}
k^2 \int_0^1 a(t)f''(u^*)w^3 dt &= \int_0^{t_0} af''(u^*)\left(k^2w^2\right)w\,dt + \int_{t_0}^1 af''(u^*)\left(k^2w^2\right)w\,dt \\
&> \int_0^{t_0} af''(u^*)u_t^2 w\,dt + \int_{t_0}^1 af''(u^*)u_t^2 w\,dt \\
&> \int_0^{t_0} a(t_0)f''(u^*)u_t^2 w\,dt + \int_{t_0}^1 a(t_0)f''(u^*)u_t^2 w\,dt \\
&= a(t_0)\int_0^1 f''(u^*)u_t^2 w\,dt = 0.
\end{aligned} \tag{2.42}$$

Hence, $\lambda''(0) < 0$ from (2.25).

If f is convex concave, then $\lambda''(0) > 0$ with a similar proof. □

3. The Main Results and the Proofs

In this section we state our main results and proofs.

Definition 3.1. Define

$$\lambda_1^0 = \frac{\lambda_1}{f_0} \qquad \lambda_1^\infty = \frac{\lambda_1}{f_\infty}, \tag{3.1}$$

where λ_1 is the first eigenvalue of the corresponding linear problem

$$\begin{gathered}
\varphi'' + \lambda a(t)\varphi = 0, \quad t \in (0,1), \\
\varphi'(0) = 0, \qquad \varphi(1) = 0.
\end{gathered} \tag{3.2}$$

Remark 3.2. It is well known that the eigenvalues of (3.2) are given by

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_k < \lambda_{k+1} < \cdots, \qquad \lim_{k\to\infty} \lambda_k = \infty. \tag{3.3}$$

For each $k \in \mathbb{N}$, algebraic multiplicity of λ_k is equal to 1, and the corresponding eigenfunction φ_k has exactly $k-1$ simple zeros in $(0,1)$.

Definition 3.3 (see [7]). Let $f \in C^1[a,b]$. Then f is said to be *superlinear* (resp., *sublinear*) on $[a,b]$ if $f(u)/u \leq f'(u)$ (resp., $f(u)/u \geq f'(u)$) on $[a,b]$. And f is said to be *sup-sub* (resp., *sub-sup*) on $[a,b]$ if there exists $c \in (a,b)$ such that $f(u)$ is superlinear (resp., sublinear) on $[a,c]$, and superlinear (resp., sublinear) on $[c,b]$.

Lemma 3.4. *(i) Let $f \in C^1[0,\infty)$, $f(0)=0$, $f'(0)>0$, and (H4) hold. Suppose that $(\lambda_*,0)$ is a point where a bifurcation from the trivial solutions occurs and that Γ_1 is the corresponding positive solution bifurcation curve of* (1.1)*. If there exists $\delta_1>0$ such that f is superlinear (resp., sublinear) on $[0,\delta_1]$, then Γ_1 tends to the left (resp., the right) near $(\lambda_*,0)$.*

(ii) Let $f \in C^2[0,\infty)$, $f_\infty \in (0,\infty)$, and (H4) hold. Suppose that (λ_*,∞) is a point where a bifurcation from infinity occurs and that Γ_2 is the corresponding positive solution bifurcation curve of (1.1)*. If there exists $\delta_2>0$ such that f is superlinear (resp., sublinear) on $[\delta_2,\infty)$ and $f''(u) \geq 0(\not\equiv 0)$ (resp., $f''(u) \leq 0(\not\equiv 0)$) for $u>\delta_2$, then Γ_2 tends to the right (resp., the left) near (λ_*,∞).*

Proof. The proof is similar to that of Proposition 3.4 in [7], so we omit it. □

Lemma 3.5. *Let (H1)–(H4) hold, let $I \subset \mathbb{R}$ be a bounded and closed interval, and let $\lambda_1^0, \lambda_1^\infty \in I$. Suppose that (λ_n,u_n) are positive solutions of* (1.1)*. Then,*

(i) $\lambda_n \to \lambda_1^0$, *if* $\|u_n\|_E \to 0$,

(ii) $\lambda_n \to \lambda_1^\infty$, *if* $\|u_n\|_E \to \infty$.

Proof. Let $\zeta,\ \xi \in C(\mathbb{R},\mathbb{R})$ be such that

$$f(u) = f_0 u + \zeta(u), \qquad f(u) = f_\infty u + \xi(u). \tag{3.4}$$

Clearly,

$$\lim_{|u|\to 0} \frac{\zeta(u)}{u} = 0, \qquad \lim_{|u|\to\infty} \frac{\xi(u)}{u} = 0. \tag{3.5}$$

Let us consider

$$Lu - \lambda a(t) f_0 u = \lambda a(t) \zeta(u) \tag{3.6}$$

as a bifurcation problem from $u \equiv 0$. Note that (3.6) is the same as to (1.1). From Remark 3.2 and the standard bifurcation theorem from simple eigenvalues [17], we have (i).

Let us consider

$$Lu - \lambda a(t) f_\infty u = \lambda a(t) \xi(u) \tag{3.7}$$

as a bifurcation problem from infinity. Note that (3.7) is also the same as to (1.1). The proof of Theorem 1.1 in [5] ensures that (ii) is correct. □

Lemma 3.6. *Let (H1), (H4) hold. Suppose that* (λ, u) *is a positive solution of* (1.1). *Then,*

$$\max\{u(t) \mid t \in [0,1]\} \neq s. \tag{3.8}$$

Proof. Suppose to the contrary that

$$\max\{u(t) \mid t \in [0,1]\} = s. \tag{3.9}$$

By (1.1) and (H1), we have $u(0) = s$. Note that $f(s) = 0$. By the uniqueness of solutions of initial value problem, the problem

$$\begin{gathered} u'' + \lambda a(t) f(u) = 0, \quad t \in (0,1), \\ u'(0) = 0, \qquad u(0) = s \end{gathered} \tag{3.10}$$

has a unique solution $u(t) \equiv s$. This contradicts $u(1) = 0$. □

The following Lemma is an interesting and important result.

Lemma 3.7. *Let (H1)–(H4) hold. Suppose that* (λ_*, u_*) *is a positive solution of* (1.1), *then* (λ_*, u_*) *is nondegenerate.*

Proof. From conditions (H1)–(H3), we can check easily that

$$f'(u)u < f(u), \quad \forall u \in (0,s); \qquad f'(u)u > f(u), \quad \forall u \in (s,\infty). \tag{3.11}$$

In fact, let $G(u) = f(u) - uf'(u)$, then

$$G(0) = G(s) = 0, \tag{3.12}$$

since $f(0) = f(s) = f'(s) = 0$. Note that $G'(u) = -f''(u)u > 0$, if $u \in (0,\theta)$, and $G'(u) < 0$, if $u \in (\theta,\infty)$. This together with (3.12) implies that $\theta < s$ and (3.11).

Now, we give the proof in two cases.

Case I ($\max\{u_*(t) \mid t \in [0,\ 1]\} < s$). On the contrary, suppose that (λ_*, u_*) is a degenerate solution with $\max\{u_*(t) \mid t \in [0,1]\} < s$, then $u_*(t) < s$, for all $t \in [0,1]$. By (3.11), we get

$$u_*(t) f'(u_*(t)) < f(u_*(t)), \quad \forall t \in [0,1]. \tag{3.13}$$

Multiplying (1.1) by w_* and (2.7) by u_*, subtracting, and integrating, we have

$$0 = \int_0^1 (u_0'' w_* - w_*'' u_0)\, d\tau = \lambda \int_0^1 a(\tau)\left[f'(u_*)u_* - f(u_*)\right] w_* d\tau. \tag{3.14}$$

By Lemma 2.2, (3.13), and $a(t) > 0$, for all $t \in [0,1]$, the right side of (3.14) is negative. This is a contradiction.

Case II ($\max\{u_*(t) \mid t \in [0,1]\} > s$). On the contrary, suppose that (λ_*, u_*) is a degenerate solution with $\max\{u_*(t) \mid t \in [0,1]\} > s$. According to Lemmas 2.2 and 2.3, we know that all solutions of (1.1) near (λ_*, u_*) satisfy $(\lambda(s), u_* + sw_* + z(s))$ for $s \in (-\delta, \delta)$ and some $\delta > 0$, where $\lambda(0) = \lambda_*$, $\lambda'(0) = 0, z(0) = z'(0) = 0$. It follows that for λ close to λ_* we have two solutions $u^-(t,\lambda)$ and $u^+(t,\lambda)$ with $u^-(t,\lambda)$ strictly increasing in λ and $u^+(t,\lambda)$ with strictly decreasing in λ. We will show that the lower branch $u^-(t,\lambda)$ is strictly increasing for all $\lambda < \lambda_*$.

Note that $u_\lambda^-(t,\lambda) > 0$ for λ close to λ_* and all $t \in (0,1)$. Let λ^* be the largest λ where this inequality is violated; that is, $u_\lambda^-(t,\lambda^*) \geq 0$ and $u_\lambda^-(t_0,\lambda^*) = 0$ for some $t_0 \in (0,1)$. Differentiating (1.1) with respect to λ,

$$u_\lambda'' + \lambda^* a(t) f'(u) u_\lambda = -a(t) f(u) \leq 0, \quad u_\lambda'(0) = u_\lambda(1) = 0. \tag{3.15}$$

We can extend evenly a, u, and u_λ on $(-1,1)$, then we obtain

$$u_\lambda'' + \lambda^* a(t) f'(u) u_\lambda = -a(t) f(u) \leq 0, \quad u_\lambda(-1) = u_\lambda(1) = 0. \tag{3.16}$$

By the strong maximum principle, we conclude that $u_\lambda^-(t,\lambda^*) > 0$ for all $[0,1)$. This contradicts that $u_\lambda^-(t_0,\lambda^*) = 0$.

By Lemma 2.3, we get $\lambda''(0) < 0$ at every degenerate positive solution. Hence, there is no degenerate positive solution on the lower branch $u^-(t,\lambda)$. However, the lower branch has no place to go. In fact, there must exist some positive constant $\alpha \geq s$ such that $\max\{u(t) \mid t \in [0,1]\} > \alpha$ for any (λ, u) lying on $u^-(t,\lambda)$. Hence, the lower branch cannot go to the λ axis. And it also cannot go to the u axis, since (1.1) has only the trivial solution at $\lambda = 0$.

So, (λ_*, u_*) is nondegenerate. □

Our main result is the following.

Theorem 3.8. *Let (H1)–(H4) hold. Then the following are considered.*

(i) *All positive solutions of* (1.1) *lie on two continuous curves Σ_1 and Σ_2 without intersection. Σ_1 bifurcates from $(\lambda_1^0,\ 0)$ to infinity and $\mathrm{Proj}_{\mathbb{R}}\Sigma_1 = (\lambda_1^0, \infty)$; Σ_2 bifurcates from $(\lambda_1^\infty,\ \infty)$ to infinity and $\mathrm{Proj}_{\mathbb{R}}\Sigma_2 = (\lambda_1^\infty, \infty)$. There is no degenerate positive solution on these curves. For any $(\lambda, u) \in \Sigma_1$, $\|u\|_\infty < s$, and for any $(\lambda, u) \in \Sigma_2$, $\|u\|_\infty > s$.*

(ii) *Equation* (1.1) *has no positive solution for $0 \leq \lambda \leq \min\{\lambda_1^0,\ \lambda_1^\infty\}$ has exactly one positive solution for $\min\{\lambda_1^0,\ \lambda_1^\infty\} < \lambda \leq \max\{\lambda_1^0,\ \lambda_1^\infty\}$ but and has exactly two positive solutions for $\max\{\lambda_1^0,\ \lambda_1^\infty\} < \lambda < \infty$ (see Figure 1).*

Proof. (i) Since $f(0) = 0$ and $f_0 > 0$, then $\lambda_1^0 > 0$. From Lemma 3.5(i) and the standard Crandall and Rabinowitz theorem on local bifurcation from simple eigenvalues [17], λ_1^0 is the unique point where a bifurcation from the trivial solution occurs. Moreover, by Lemma 3.4, the curve bifurcates to the right. We denote this local curve by Σ_1 and continue Σ_1 to the right as long as it is possible. Meanwhile, by Lemma 3.6, there is no positive solution of (1.1) which has the maximum value s on $[0,1]$. So, if $(\lambda, u) \in \Sigma_1$, then $\|u\|_\infty < s$. From (1.1), we have

$$\left|u'(t)\right| = \int_0^t \lambda a(\tau) f(u(\tau)) d\tau \leq M \int_0^t \lambda a(\tau) d\tau, \tag{3.17}$$

where $M = \max\{f(r),\ \ 0 \leq r \leq s\}$. Obviously, there exists a constant C such that $\|u\|_E < C$ if λ is bounded. Hence, Σ_1 cannot blow up.

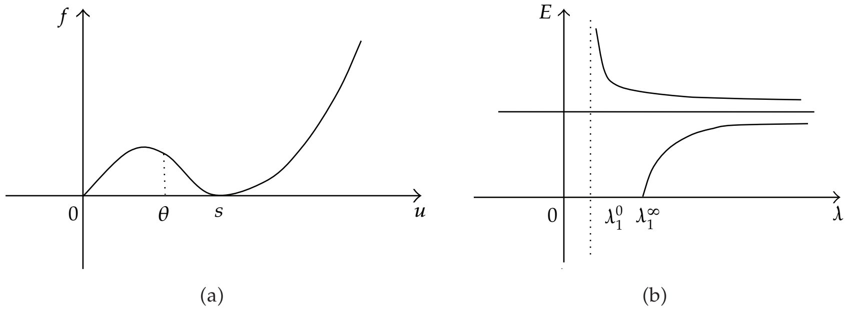


Figure 1

On the other hand, Lemma 3.7 and the implicit function theorem ensure that Σ_1 cannot stop at a finite point $(\widetilde{\lambda}, \widetilde{u})$.

From the above discussion, we see that Σ_1 can be extended continuously to infinity and $\mathrm{Proj}_{\mathbb{R}}\Sigma_1 = (\lambda_1^0, \infty)$. Meanwhile, the maximum values of all positive solutions of (1.1) are less than s.

Now, we consider positive solutions of (1.1), for which the maximum value on $[0,1]$ is greater than s.

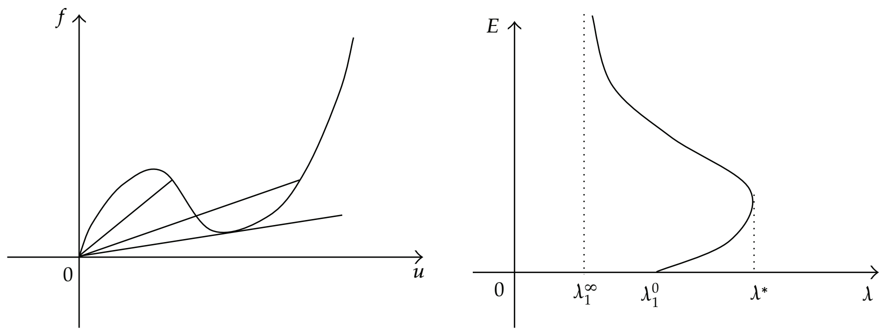
Let us return to consider (3.6) as the bifurcation problem from infinity. Note that (3.6) is also the same as to (1.1). Since $\lim_{|u|\to\infty}(\xi(u)/u) = 0$, by Theorem 1.6 and Corollary 1.8 in [18], there exists a subcontinuum $\mathfrak{D} \subset \mathbb{R}\times E$ of positive solutions of (3.6) which meets $(\lambda_1^\infty, \infty)$. Take $\Lambda \subset \mathbb{R}$ as an interval such that $\Lambda \cap \{\lambda_j/f_\infty \mid j \in \mathbb{N}\} = \{\lambda_1^\infty\}$ and $\mathcal{U}$ as a neighborhood of $(\lambda_1^\infty, \infty)$ whose projection on $\mathbb{R}$ lies in Λ and whose projection on E is bounded away from 0. Then, there exists a neighborhood $\mathcal{O} \subset \mathcal{U}$ such that any positive solution $(\lambda, u) \in \mathcal{O}$ of (1.1) satisfies $(\lambda, u) = (\lambda(\alpha), \alpha\varphi_1 + z(\alpha))$ for $\alpha \in (\delta, \infty)$ and some $\delta > 0$ and $|\lambda - \lambda_1^\infty| = o(1)$, $\|z\|_X = o(\alpha)$ at $\alpha = \infty$, where φ_1 denotes the normalized eigenvector of (3.2) corresponding to λ_1. So, $\|\varphi_1\| = 1$.

Hence, $\mathfrak{D} \cap \mathcal{O}$ is a continuous curve, and we denote it by Σ_2. It tends to the right from Lemma 3.4(ii). From Lemma 3.7 and the implicit function theorem, Σ_2 can be continued to a maximal interval of definition over the λ axis. We claim that $\Sigma_2 \setminus \{(\lambda_1^\infty, \infty)\}$ cannot blow up if λ is bounded. In fact, suppose that there exists a positive solutions sequence $\{(\lambda_n, u_n)\}$ of (1.1) and $\overline{\lambda} < \infty$ such that $\|u_n\|_E \to \infty$ as $\lambda_n \to \overline{\lambda}$. Then, by Lemma 3.5(ii), $\overline{\lambda} = \lambda_1^\infty$. This is a contradiction. On the other hand, the implicit function theorem implies that Σ_2 cannot stop at a finite point $(\widetilde{\lambda}, \widetilde{u})$. Thus, $\mathrm{Proj}_{\mathbb{R}}\Sigma_2 = (\lambda_1^\infty, \infty)$ and $\|u\|_\infty > s$ if $(\lambda, u) \in \Sigma_2$.

Finally, we show that both curves Σ_1 and Σ_2 are the only two positive solutions curves of (1.1). On the contrary, suppose that (λ, u) is a positive solution of (1.1) with $(\lambda, u) \notin \Sigma_1 \cup \Sigma_2$. Without loss of generality, assume that $\|u\|_\infty > s$. Note that (λ, u) is nondegenerate, so we can extend it to form a curve. We denote this curve by Σ and the corresponding maximal interval of definition by $I = (e_1, e_2)$. Since all positive solutions of (1.1) are nondegenerate, according to the implicit function theorem, we must have that

$$\lim_{\lambda\to e_1} \|u(t)\|_E = \infty. \tag{3.18}$$

It follows that $e_1 = \lambda_1^\infty$ from Lemma 3.5(ii). But all solutions near $(\lambda_1^\infty, \infty)$ can be parameterized by α for $\alpha > \delta$ and some $\delta > 0$; thus, $\Sigma = \Sigma_2$. This contradicts that $(\lambda, u) \notin \Sigma_2$.



(a) (b)

Figure 2

Similarly, we can show that every positive solution of (1.1), the maximum value on $[0,1]$ of which is less than s, lies on Σ_1.

(ii) The result (ii) is a corollary of (i). □

Next, we will give directly other theorems. Their proofs are similar to that of Theorem 3.8. So, we omit them.

Theorem 3.9. *Let* (H1′) *and (H2)–(H4) hold. Then, the following are considered.*

(i) *All positive solutions of* (1.1) *lie on a single continuous curve* Σ. *And* Σ *bifurcates from* $(\lambda_1^0,\ 0)$ *to the right to a unique degenerate positive solution* (λ^*, u^*) *of* (1.1), *then it tends to the left to* $(\lambda_1^\infty, \infty)$.

(ii) *Equation* (1.1) *has no positive solution for* $\lambda \in (0, \min\{\lambda_1^0,\ \lambda_1^\infty\}) \cup (\lambda^*, \infty)$, *and has exactly one positive solution for* $\lambda \in (\min\{\lambda_1^0,\ \lambda_1^\infty\}, \max\{\lambda_1^0,\ \lambda_1^\infty\}) \cup \{\lambda^*\}$, *and has exactly two positive solutions for* $\lambda \in (\max\{\lambda_1^0,\ \lambda_1^\infty\}, \lambda^*)$ *(see Figure 2).*

Remark 3.10. In fact, if we reverse the inequalities in (H1), (H1′), (H2), we will obtain corresponding results similar to Theorems 3.8 and 3.9.

Also using the method in this paper, we can obtain the exact numbers of positive solutions for the Dirichlet problem

$$u'' + \lambda a(t) f(u) = 0, \quad t \in (-1,1),$$
$$u(-1) = u(1) = 0, \tag{3.19}$$

where $\lambda > 0$ is a parameter. We assume that

(H4′) $a \in C^1[-1,1]$ with $a(t) > 0, a(-t) = a(t)$, and $a'(t) < 0$, for all $t \in (0,1)$.

Definition 3.11. Define

$$\tilde{\lambda}_1^0 = \frac{\tilde{\lambda}_1}{f_0}, \qquad \tilde{\lambda}_1^\infty = \frac{\tilde{\lambda}_1}{f_\infty}, \tag{3.20}$$

where $\tilde{\lambda}_1$ is the first eigenvalue of the corresponding linear problem of (3.19).

Theorem 3.12. *Let (H1′), (H2), (H3), and (H4′) hold. Then, the following are considered.*

(i) *All positive solutions of* (3.19) *lie on a single continuous curve* $\widetilde{\Sigma}$. *And* $\widetilde{\Sigma}$ *bifurcates from* $(\widetilde{\lambda}_1^0,\ 0)$ *to the right to a unique degenerate positive solution* $(\widetilde{\lambda}^*, \widetilde{u}^*)$ *of* (3.19), *then it tends to the left to* $(\widetilde{\lambda}_1^\infty, \infty)$.

(ii) *Equation* (1.1) *has no positive solution for* $\lambda \in (0, \min\{\widetilde{\lambda}_1^0,\ \widetilde{\lambda}_1^\infty\}) \cup (\widetilde{\lambda}^*, \infty)$ *but has exactly one positive solution for* $\lambda \in (\min\{\widetilde{\lambda}_1^0,\ \widetilde{\lambda}_1^\infty\}, \max\{\widetilde{\lambda}_1^0,\ \widetilde{\lambda}_1^\infty\}) \cup \{\widetilde{\lambda}^*\}$ *and has exactly two positive solutions for* $\lambda \in (\max\{\widetilde{\lambda}_1^0,\ \widetilde{\lambda}_1^\infty\}, \widetilde{\lambda}^*)$.

Theorem 3.13. *Let (H1), (H2), (H3), (H4′) hold. Then*

(i) *All positive solutions of* (3.19) *lie on two continuous curves* $\widetilde{\Sigma}_1$ *and* $\widetilde{\Sigma}_2$ *without intersection.* $\widetilde{\Sigma}_1$ *bifurcates from* $(\widetilde{\lambda}_1^0,\ 0)$ *to infinity and* $\operatorname{Proj}_{\mathbb{R}}\widetilde{\Sigma}_1 = (\widetilde{\lambda}_1^0, \infty)$; $\widetilde{\Sigma}_2$ *bifurcates from* $(\widetilde{\lambda}_1^\infty,\ \infty)$ *to infinity and* $\operatorname{Proj}_{\mathbb{R}}\widetilde{\Sigma}_2 = (\widetilde{\lambda}_1^\infty, \infty)$. *There is no degenerate positive solution on these curves. For any* $(\lambda, u) \in \widetilde{\Sigma}_1$, $\|u\|_\infty < s$, *and for any* $(\lambda, u) \in \widetilde{\Sigma}_2$, $\|u\|_\infty > s$.

(ii) *Equation* (3.19) *has no positive solution for* $0 \le \lambda \le \min\{\widetilde{\lambda}_1^0,\ \widetilde{\lambda}_1^\infty\}$, *and has exactly one positive solution for* $\min\{\widetilde{\lambda}_1^0,\ \widetilde{\lambda}_1^\infty\} < \lambda \le \max\{\widetilde{\lambda}_1^0,\ \widetilde{\lambda}_1^\infty\}$, *and has exactly two positive solutions for* $\max\{\widetilde{\lambda}_1^0,\ \widetilde{\lambda}_1^\infty\} < \lambda < \infty$.

Remark 3.14. Theorems 3.12 and 3.13 extend the main result Theorem 1 in [10], where $f'' > 0$ for $u > 0$.

4. Examples

In this section, we give some examples.

Example 4.1. Let

$$f(u) = \begin{cases} u(u-1)^2, & \text{for } u \in [0,2], \\ -32\ln u + 21u + 32\ln 2 - 40, & \text{for } u \in (2,\infty). \end{cases} \tag{4.1}$$

Then, f satisfies (H1), (H2), and (H3). Moreover, $s = 1$, $\theta = 2/3$, $f_0 = 1$, and $f_\infty = 21$.

Example 4.2. Let

$$f(u) = \begin{cases} u\left(u^2 - u + 2\right), & \text{for } u \in [0,1], \\ -4\ln u + 7u - 5, & \text{for } u \in (1,\infty). \end{cases} \tag{4.2}$$

Then, f satisfies (H1′), (H2), and (H3). Moreover, $s = 1/2$, $\theta = 1/3$, and $f_0 = 2$, $f_\infty = 7$.

Example 4.3. Let $a(t) = \beta(t) + c$. Here, $\beta \in C^2[0,1]$, $\beta''(t) > 0$, $\beta'(t) < 0$, for all $t \in [0,1]$, and c is a large enough constant. Then, a satisfies (H4). On the other hand, functions which satisfy (H4′) can be found easily.

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