Chen *Fixed Point Theory and Applications* 2012, **2012**:17
http://www.fixedpointtheoryandapplications.com/content/2012/1/17
Fixed Point Theory and Applications
a SpringerOpen Journal

RESEARCH Open Access

Fixed point theory for the cyclic weaker Meir-Keeler function in complete metric spaces

Chi-Ming Chen

Correspondence: ming@mail.nhcue.edu.tw
Department of Applied Mathematics, National Hsinchu University of Education, No. 521, Nanda Rd., Hsinchu City 300, Taiwan

Abstract
In this article, we introduce the notions of cyclic weaker $\phi \circ \varphi$-contractions and cyclic weaker (ϕ, φ)-contractions in complete metric spaces and prove two theorems which assure the existence and uniqueness of a fixed point for these two types of contractions. Our results generalize or improve many recent fixed point theorems in the literature.
MSC: 47H10; 54C60; 54H25; 55M20.

Keywords: fixed point theory, weaker Meir-Keeler function, cyclic weaker $\phi \circ \varphi$-contraction, cyclic weaker (ϕ, φ)-contraction

1 Introduction and preliminaries

Throughout this article, by $\mathbb{R}^+$, $\mathbb{R}$ we denote the sets of all nonnegative real numbers and all real numbers, respectively, while $\mathbb{N}$ is the set of all natural numbers. Let (X, d) be a metric space, D be a subset of X and f: $D \to X$ be a map. We say f is contractive if there exists $\alpha \in [0,1)$ such that for all $x, y \in D$,

$$d(fx, fy) \le \alpha \cdot d(x, y).$$

The well-known Banach's fixed point theorem asserts that if $D = X$, f is contractive and (X, d) is complete, then f has a unique fixed point in X. It is well known that the Banach contraction principle [1] is a very useful and classical tool in nonlinear analysis. In 1969, Boyd and Wong [2] introduced the notion of Φ-contraction. A mapping f: $X \to X$ on a metric space is called Φ-contraction if there exists an upper semi-continuous function Φ: $[0, \infty) \to [0, \infty)$ such that

$$d(fx, fy) \le \Phi(d(x, y)) \quad \text{for all} \quad x, y \in X.$$

Generalization of the above Banach contraction principle has been a heavily investigated branch research. (see, e.g., [3,4]). In 2003, Kirk et al. [5] introduced the following notion of cyclic representation.

Definition 1 [5]*Let X be a nonempty set,* $m \in \mathbb{N}$ *and* f: $X \to X$ *an operator. Then* $X = \cup_{i=1}^m A_i$*is called a cyclic representation of X with respect to f if*

(1) A_i, $i = 1, 2,..., m$ *are nonempty subsets of* X;
(2) $f(A_1) \subset A_2, f(A_2) \subset A_3,..., f(A_{m-1}) \subset A_m, f(A_m) \subset A_1$.

Kirk et al. [5] also proved the below theorem.

Theorem 1 [5]*Let* (X, d) *be a complete metric space,* $m \in \mathbb{N}$, $A_1, A_2,..., A_m$, *closed nonempty subsets of* X *and* $X = \cup_{i=1}^{m} A_i$. *Suppose that f satisfies the following condition.*

$$d(fx, fy) \le \psi(d(x, y)), \quad \text{for all} \quad x \in A_i, \quad y \in A_{i+1}, \quad i \in \{1, 2, ..., m\},$$

where ψ: $[0, \infty) \to [0, \infty)$ *is upper semi-continuous from the right and* $0 \le \psi(t) < t$ *for* $t > 0$. *Then, f has a fixed point* $z \in \cap_{i=1}^{n} A_i$.

Recently, the fixed theorems for an operator f: $X \to X$ that defined on a metric space X with a cyclic representation of X with respect to f had appeared in the literature. (see, e.g., [6-10]). In 2010, Păcurar and Rus [7] introduced the following notion of cyclic weaker ϕ-contraction.

Definition 2 [7]*Let* (X, d) *be a metric space,* $m \in \mathbb{N}$, $A_1, A_2,...,A_m$ *closed nonempty subsets of* X *and* $X = \cup_{i=1}^{m} A_i$. *An operator* f: $X \to X$ *is called a cyclic weaker* ϕ-*contraction if*

(1) $X = \cup_{i=1}^{m} A_i$ *is a cyclic representation of* X *with respect to* f;

(2) *there exists a continuous, non-decreasing function* ϕ: $[0, \infty) \to [0, \infty)$ *with* $\phi(t) > 0$ *for* $t \in (0, \infty)$ *and* $\phi(0) = 0$ *such that*

$$d(fx, fy) \le d(x, y) - \varphi(d(x, y)),$$

for any $x \in A_i$, $y \in A_{i+1}$, $i = 1,2,...,m$ *where* $A_{m+1} = A_1$.

And, Păcurar and Rus [7] proved the below theorem.

Theorem 2 [7]*Let* (X, d) *be a complete metric space,* $m \in \mathbb{N}$, $A_1, A_2,..., A_m$ *closed nonempty subsets of* X *and* $X = \cup_{i=1}^{m} A_i$. *Suppose that f is a cyclic weaker* ϕ-*contraction. Then, f has a fixed point* $z \in \cap_{i=1}^{n} A_i$.

In this article, we also recall the notion of Meir-Keeler function (see [11]). A function φ: $[0, \infty) \to [0, \infty)$ is said to be a Meir-Keeler function if for each $\eta > 0$, there exists $\delta > 0$ such that for $t \in [0, \infty)$ with $\eta \le t < \eta + \delta$, we have $\varphi(t) < \eta$. We now introduce the notion of weaker Meir-Keeler function φ: $[0, \infty) \to [0,\infty)$, as follows:

Definition 3 *We call* φ: $[0, \infty) \to [0, \infty)$ *a weaker Meir-Keeler function if for each* $\eta > 0$, *there exists* $\delta > 0$ *such that for* $t \in [0, \infty)$ *with* $\eta \le t < \eta + \delta$, *there exists* $n_0 \in \mathbb{N}$ *such that* $\phi^{n_0}(t) < \eta$.

2 Fixed point theory for the cyclic weaker $\varphi \circ \phi$-contractions

The main purpose of this section is to present a generalization of Theorem 1. In the section, we let φ: $[0, \infty) \to [0, \infty)$ be a weaker Meir-Keeler function satisfying the following conditions:

(φ_1) $\varphi(t) > 0$ for $t > 0$ and $\varphi(0) = 0$;

(φ_2) for all $t \in (0, \infty)$, $\{\varphi^n(t)\}_{n\in\mathbb{N}}$ is decreasing;

(φ_3) for $t_n \in [0, \infty)$, we have that

(a) if $\lim_{n\to\infty} t_n = \gamma > 0$, then $\lim_{n\to\infty} \varphi(t_n) < \gamma$, and

(b) if $\lim_{n\to\infty} t_n = 0$, then $\lim_{n\to\infty} \varphi(t_n) = 0$.

And, let ϕ: $[0, \infty) \to [0, \infty)$ be a non-decreasing and continuous function satisfying

(ϕ_1) $\phi(t) > 0$ for $t > 0$ and $\phi(0) = 0$;
(ϕ_2) ϕ is subadditive, that is, for every $\mu_1, \mu_2 \in [0, \infty)$, $\phi(\mu_1 + \mu_2) \leq \phi(\mu_1) + \phi(\mu_2)$;
(ϕ_3) for all $t \in (0, \infty)$, $\lim_{n\to\infty} t_n = 0$ if and only if $\lim_{n\to\infty} \phi(t_n) = 0$.

We state the notion of cyclic weaker $\varphi \circ \phi$-contraction, as follows:

Definition 4 *Let* (X, d) *be a metric space,* $m \in \mathbb{N}$, $A_1, A_2, \ldots, A_m$ *nonempty subsets of* X *and* $X = \cup_{i=1}^m A_i$. *An operator* $f: X \to X$ *is called a cyclic weaker* $\varphi \circ \phi$*-contraction if*

(i) $X = \cup_{i=1}^m A_i$ *is a cyclic representation of* X *with respect to* f;
(ii) for any $x \in A_i$, $y \in A_{i+1}$, $i = 1, 2, \ldots, m$,

$$\varphi(d(fx, fy)) \leq \phi(\varphi(d(x, y))),$$

where $A_{m+1} = A_1$.

Theorem 3 *Let* (X, d) *be a complete metric space,* $m \in \mathbb{N}$, $A_1, A_2, \ldots, A_m$ *nonempty subsets of* X *and* $X = \cup_{i=1}^m A_i$. *Let* $f: X \to X$ *be a cyclic weaker* $\varphi \circ \phi$*-contraction. Then,* f *has a unique fixed point* $z \in \cap_{i=1}^m A_i$.

Proof. Given x_0 and let $x_{n+1} = fx_n = f^{n+1}x_0$, for $n \in \mathbb{N} \cup \{0\}$. If there exists $n_0 \in \mathbb{N} \cup \{0\}$ such that $x_{n_0+1} = x_{n_0}$, then we finished the proof. Suppose that $x_{n+1} \neq x_n$ for any $n \in \mathbb{N} \cup \{0\}$. Notice that, for any $n > 0$, there exists $i_n \in \{1,2,\ldots,m\}$ such that $x_{n-1} \in A_{i_n}$ and $x_n \in A_{i_n+1}$. Since $f: X \to X$ is a cyclic weaker $\varphi \circ \phi$-contraction, we have that for all $n \in \mathbb{N}$

$$\begin{aligned}\varphi(d(x_n, x_{n+1})) &= \varphi(d(fx_{n-1}, fx_n)) \\ &\leq \phi(\varphi(d(x_{n-1}, x_n))),\end{aligned}$$

and so

$$\begin{aligned}\varphi(d(x_n, x_{n+1})) &\leq \phi(\varphi(d(x_{n-1}, x_n))) \\ &\leq \phi(\phi(\varphi(d(x_{n-2}, x_{n-1})))) = \phi^2(\varphi((d(x_{n-2}, x_{n-1}))) \\ &\leq \cdots\cdots \\ &\leq \phi^n(\varphi(d(x_0, x_1))).\end{aligned}$$

Since $\{\varphi^n(\phi(d(x_0, x1)))\}_{n\in\mathbb{N}}$ is decreasing, it must converge to some $\eta \geq 0$. We claim that $\eta = 0$. On the contrary, assume that $\eta > 0$. Then by the definition of weaker Meir-Keeler function φ, there exists $\delta > 0$ such that for $x_0, x_1 \in X$ with $\eta \leq \phi(d(x_0, x_1)) < \delta + \eta$, there exists $n_0 \in \mathbb{N}$ such that $\phi^{n_0}(\varphi(d(x_0, x_1))) < \eta$. Since $\lim_{n\to\infty} \varphi^n (\phi(d(x_0, x_1))) = \eta$, there exists $p_0 \in \mathbb{N}$ such that $\eta \leq \varphi^p (\phi(d(x_0, x_1)) < \delta + \eta$, for all $p \geq p_0$. Thus, we conclude that $\phi^{p_0+n_0}(\varphi(d(x_0, x_1))) < \eta$. So we get a contradiction. Therefore $\lim_{n\to\infty} \varphi^n (\phi(d(x_0, x_1))) = 0$, that is,

$$\lim_{n\to\infty} \varphi(d(x_n, x_{n+1})) = 0.$$

Next, we claim that $\{x_n\}$ is a Cauchy sequence. We claim that the following result holds:

Claim: for each $\varepsilon > 0$, there is $n_0(\varepsilon) \in \mathbb{N}$ such that for all $p, q \geq n_0(\varepsilon)$,

$$\varphi(d(x_p, x_q)) < \varepsilon, \qquad (*)$$

We shall prove (*) by contradiction. Suppose that (*) is false. Then there exists some $\varepsilon > 0$ such that for all $n \in \mathbb{N}$, there are p_n, $q_n \in \mathbb{N}$ with $p_n > q_n \geq n$ satisfying:

(i) $\varphi(d(x_{p_n}, x_{q_n})) \geq \varepsilon$, and
(ii) p_n is the smallest number greater than q_n such that the condition (i) holds.

Since

$$\begin{aligned}\varepsilon &\leq \varphi(d(x_{p_n}, x_{q_n}))\\ &\leq \varphi(d(x_{p_n}, x_{p_n-1}) + d(x_{p_n-1}, x_{q_n}))\\ &\leq \varphi(d(x_{p_n}, x_{p_n-1})) + \varphi(d(x_{p_n-1}, x_{q_n}))\\ &\leq \varphi(d(x_{p_n}, x_{p_n-1})) + \varepsilon,\end{aligned}$$

hence we conclude $\lim_{p\to\infty}\varphi(d(x_{p_n}, x_{q_n})) = \varepsilon$. Since ϕ is subadditive and nondecreasing, we conclude

$$\begin{aligned}\varphi(d(x_{p_n}, x_{q_n})) &\leq \varphi(d(x_{p_n}, x_{q_n+1}) + d(x_{p_n+1}, x_{q_n}))\\ &\leq \varphi(d(x_{p_n}, x_{q_n+1})) + \varphi(d(x_{p_n+1}, x_{q_n})),\end{aligned}$$

and so

$$\begin{aligned}\varphi(d(x_{p_n}, x_{q_n})) - \varphi(d(x_{p_n}, x_{p_n+1})) &\leq \varphi(d(x_{p_n+1}, x_{q_n}))\\ &\leq \varphi(d(x_{p_n}, x_{p_n+1}) + d(x_{p_n}, x_{q_n}))\\ &\leq \varphi(d(x_{p_n}, x_{p_n+1})) + \varphi(d(x_{p_n}, x_{q_n})).\end{aligned}$$

Letting $n \to \infty$, we also have

$$\lim_{n\to\infty} \varphi(d(x_{p_n+1}, x_{q_n})) = \varepsilon.$$

Thus, there exists i, $0 \leq i \leq m - 1$ such that $p_n - q_n + i = 1 \bmod m$ for infinitely many n. If $i = 0$, then we have that for such n,

$$\begin{aligned}\varepsilon &\leq \varphi(d(x_{p_n}, x_{q_n}))\\ &\leq \varphi(d(x_{p_n}, x_{p_n+1}) + d(x_{p_n+1}, x_{q_{n+1}}) + d(x_{q_n+1}, x_{q_n}))\\ &\leq \varphi(d(x_{p_n}, x_{p_n+1})) + \varphi(d(x_{p_n+1}, x_{q_n+1})) + \varphi(d(x_{q_n+1}, x_{q_n}))\\ &= \varphi(d(x_{p_n}, x_{p_n+1})) + \varphi(d(fx_{p_n}, fx_{q_n})) + \varphi(d(x_{q_n+1}, x_{q_n}))\\ &\leq \varphi(d(x_{p_n}, x_{p_n+1})) + \phi(\varphi(d(x_{p_n}, x_{q_n}))) + \varphi(d(x_{q_n+1}, x_{q_n})).\end{aligned}$$

Letting $n \to \infty$. Then by, we have

$$\varepsilon \leq 0 + \lim_{n\to\infty} \phi(\varphi(d(x_{p_n}, x_{q_n}))) + 0 < \varepsilon,$$

a contradiction. Therefore $\lim_{n\to\infty}\varphi(d(x_{p_n}, x_{q_n})) = 0$, by the condition ($\phi_3$), we also have $\lim_{n\to\infty}d(x_{p_n}, x_{q_n}) = 0$. The case $i \neq 0$ is similar. Thus, $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists $\nu \in \cup_{i=1}^{m} A_i$ such that $\lim_{n\to\infty} x_n = \nu$. Now for all $i = 0, 1, 2,..., m - 1$, $\{fx_{mn-i}\}$ is a sequence in A_i and also all converge to ν. Since A_i is clsoed for all $i = 1, 2,..., m$, we conclude $\nu \in \cup_{i=1}^{m} A_i$, and also we conclude that $\cap_{i=1}^{m} A_i \neq \phi$. Since

$$\begin{aligned}\varphi(d(\nu, f\nu)) &= \lim_{n\to\infty} \varphi(d(fx_{mn}, f\nu))\\ &\leq \lim_{n\to\infty} \phi(\varphi(d(fx_{mn-1}, \nu))) = 0,\end{aligned}$$

hence $\phi(d(v, fv)) = 0$, that is, $d(v, fv) = 0$, v is a fixed point of f.

Finally, to prove the uniqueness of the fixed point, let μ be another fixed point of f. By the cyclic character of f, we have $\mu, \nu \in \cap_{i=1}^{n} A_i$. Since f is a cyclic weaker $\phi \circ \phi$-contraction, we have

$$\begin{aligned}\varphi(d(\nu,\mu)) = \varphi(d(\nu, f\mu)) &= \lim_{n\to\infty} \varphi(d(fx_{mn}, f\mu))\\ &\leq \lim_{n\to\infty} \phi(\varphi(d(fx_{mn-1}, \mu)))\\ &< \varphi(d(\nu,\mu)),\end{aligned}$$

and this is a contradiction unless $\phi(d(v, \mu)) = 0$, that is, $\mu = v$. Thus v is a unique fixed point of f.

Example 1 *Let* $X = \mathbb{R}^3$ *and we define* d: $X \times X \to [0,\infty)$ *by* $d(x,y) = |x_1\text{-}y_1| + |x_2\text{-}y_2| + |x_3\text{-}y_3|$, *for* $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3) \in X$, *and let* $A = \{(x, 0,0):x \in \mathbb{R}\}$, $B = \{(0,y,0):y \in \mathbb{R}\}$, $C = \{(0,0, z): z \in \mathbb{R}\}$ *be three subsets of X. Define* f: $A \cup B \cup C \to A \cup B \cup C$ *by*

$$\begin{aligned} f((x,0,0)) &= \left(0, \frac{1}{4}x, 0\right); \quad \text{for all } x \in \mathbb{R};\\ f((0,y,0)) &= \left(0,0,\frac{1}{4}y\right); \quad \text{for all } y \in \mathbb{R};\\ f((0,0,z)) &= \left(\frac{1}{4}z, 0, 0\right); \quad \text{for all } z \in \mathbb{R}.\end{aligned}$$

We define ϕ: $[0, \infty) \to [0, \infty)$ *by*

$$\phi(t) = \frac{1}{3}t \text{ for } t \in [0,\infty),$$

and ϕ: $[0, \infty) \to [0, \infty)$ *by*

$$\varphi(t) = \frac{1}{2}t \text{ for } t \in [0,\infty).$$

Then f is a cyclic weaker $\phi \circ \phi$*-contraction and* $(0, 0, 0)$ *is the unique fixed point.*

3 Fixed point theory for the cyclic weaker (ϕ, ϕ-contractions

The main purpose of this section is to present a generalization of Theorem 2. In the section, we let ϕ: $[0, \infty) \to [0, \infty)$ be a weaker Meir-Keeler function satisfying the following conditions:

(ϕ_1) $\phi(t) > 0$ for $t > 0$ and $\phi(0) = 0$;
(ϕ_2) for all $t \in (0, \infty)$, $\{\phi^n(t)\}_{n\in\mathbb{N}}$ is decreasing;
(ϕ_3) for $t_n \in [0, \infty)$, if $\lim_{n\to\infty} t_n = \gamma$, then $\lim_{n\to\infty} \phi(t_n) \leq \gamma$.

And, let ϕ: $[0, \infty) \to [0, \infty)$ be a non-decreasing and continuous function satisfying $\phi(t) > 0$ for $t > 0$ and $\phi(0) = 0$.

We now state the notion of cyclic weaker (ϕ, ϕ)-contraction, as follows:

Definition 5 *Let* (X, d) *be a metric space,* $m \in \mathbb{N}$, $A_1, A_2,..., A_m$ *nonempty subsets of* X *and* $X = \cup_{i=1}^{m} A_i$. *An operator* f: $X \to X$ *is called a cyclic weaker* (ϕ,ϕ)*-contraction if*

(i) $X = \cup_{i=1}^{m} A_i$ *is a cyclic representation of X with respect to f;*

(ii) for any $x \in A_i$, $y \in A_{i+1}$, $i = 1, 2,..., m$,

$$d(fx, fy) \le \phi(d(x, y)) - \varphi(d(x, y)),$$

where $A_{m+1} = A_1$.

Theorem 4 *Let (X, d) be a complete metric space, $m \in \mathbb{N}$, $A_1, A_2,..., A_m$ nonempty subsets of X and $X = \cup_{i=1}^{m} A_i$. Let $f: X \to X$ be a cyclic weaker (ϕ, φ)-contraction. Then f has a unique fixed point $z \in \cap_{i=1}^{m} A_i$.*

Proof. Given x_0 and let $x_{n+1} = fx_n = f^{n+1} x_0$, for $n \in \mathbb{N} \cup \{0\}$. If there exists $n \in \mathbb{N} \cup \{0\}$ such that $x_{n_0+1} = x_{n_0}$, then we finished the proof. Suppose that $x_{n+1} \ne x_n$ for any $n \in \mathbb{N} \cup \{0\}$. Notice that, for any $n > 0$, there exists $i_n \in \{1,2,...,m\}$ such that $x_{n-1} \in A_{i_n}$ and $x_n \in A_{i_n+1}$. Since $f: X \to X$ is a cyclic weaker (ϕ, φ)-contraction, we have that $n \in \mathbb{N}$

$$\begin{aligned} d(x_n, x_{n+1}) &= d(fx_{n-1}, fx_n) \\ &\le \phi(d(x_{n-1}, x_n)) - \varphi(d(x_{n-1}, x_n)) \\ &\le \phi(d(x_{n-1}, x_n)), \end{aligned}$$

and so

$$\begin{aligned} d(x_n, x_{n+1}) &\le \phi(d(x_{n-1}, x_n)) \\ &\le \phi(\phi(d(x_{n-2}, x_{n-1})) = \phi^2(d(x_{n-2}, x_{n-1})) \\ &\le \cdots\cdots \\ &\le \phi^n(d(x_0, x_1)). \end{aligned}$$

Since $\{\phi^n (d(x_0, x_1))\}_{n\in\mathbb{N}}$ is decreasing, it must converge to some $\eta \ge 0$. We claim that $\eta = 0$. On the contrary, assume that $\eta > 0$. Then by the definition of weaker Meir-Keeler function ϕ, there exists $\delta > 0$ such that for $x_0, x_1 \in X$ with $\eta \le d(x_0, x_1) < \delta + \eta$, there exists $n_0 \in \mathbb{N}$ such that $\phi^{n_0}(d(x_0, x_1)) < \eta$. Since $\lim_{n\to\infty} \phi^n (d(x_0, x_1)) = \eta$, there exists $p_0 \in \mathbb{N}$ such that $\eta \le \phi^p (d(x_0, x_1)) < \delta + \eta$, for all $p \ge p_0$. Thus, we conclude that $\phi^{p_0+n_0}(d(x_0, x_1)) < \eta$. So we get a contradiction. Therefore $\lim_{n\to\infty} \phi^n(d(x_0, x_1)) = 0$, that is,

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0.$$

Next, we claim that $\{x_n\}$ is a Cauchy sequence. We claim that the following result holds:

Claim: For every $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that if $p, q \ge n$ with $p-q = 1 \bmod m$, then $d(x_p, x_q) < \varepsilon$.

Suppose the above statement is false. Then there exists $\epsilon > 0$ such that for any $n \in \mathbb{N}$, there are $p_n, q_n \in \mathbb{N}$ with $p_n > q_n \ge n$ with $p_n - q_n = 1 \bmod m$ satisfying

$$d(x_{q_n}, x_{p_n}) \ge \varepsilon.$$

Now, we let $n > 2m$. Then corresponding to $q_n \ge n$ use, we can choose p_n in such a way, that it is the smallest integer with $p_n > q_n \ge n$ satisfying $p_n - q_n = 1 \bmod m$ and $d(x_{q_n}, x_{p_n}) \ge \varepsilon$. Therefore $d(x_{q_n}, x_{p_n - m}) \le \varepsilon$ and

$$\begin{aligned} \varepsilon &\le d(x_{q_n}, x_{p_n}) \\ &\le d(x_{q_n}, x_{p_n-m}) + \sum_{i=1}^{m} d(x_{p_{n-i}}, x_{p_{n-i+1}}) \\ &< \varepsilon + \sum_{i=1}^{m} d(x_{p_{n-i}}, x_{p_{n-i+1}}). \end{aligned}$$

Letting $n \to \infty$, we obtain that

$$\lim_{n\to\infty} d(x_{q_n}, x_{p_n}) = \varepsilon.$$

On the other hand, we can conclude that

$$\begin{aligned}\varepsilon &\le d(x_{q_n}, x_{p_n})\\ &\le d(x_{q_n}, x_{q_{n+1}}) + d(x_{q_{n+1}}, x_{p_{n+1}}) + d(x_{x_{p_{n+1}}}, _{p_n})\\ &\le d(x_{q_n}, x_{q_{n+1}}) + d(x_{q_{n+1}}, x_{q_n}) + d(x_{q_n}, x_{p_n}) + d(x_{p_n}, x_{p_{n+1}}) + d(x_{x_{p_{n+1}}}, _{pn}).\end{aligned}$$

Letting $n \to \infty$, we obtain that

$$\lim_{n\to\infty} d(x_{q_{n+1}}, x_{p_{n+1}}) = \varepsilon.$$

Since x_{q_n} and x_{p_n} lie in different adjacently labeled sets A_i and A_{i+1} for certain $1 \le i \le m$, by using the fact that f is a cyclic weaker (ϕ, ϕ)-contraction, we have

$$d(x_{q_{n+1}}, x_{p_{n+1}}) = d(fx_{q_n}, fx_{p_n}) \le \phi(d(x_{q_n}, x_{p_n})) - \varphi(d(x_{q_n}, x_{p_n})).$$

Letting $n \to \infty$, by using the condition ϕ_3 of the function ϕ, we obtain that

$$\varepsilon \le \varepsilon - \varphi(\varepsilon),$$

and consequently, ϕ (ϵ) = 0. By the definition of the function ϕ, we get $\epsilon = 0$ which is contraction. Therefore, our claim is proved.

In the sequel, we shall show that $\{x_n\}$ is a Cauchy sequence. Let $\varepsilon > 0$ be given. By our claim, there exists $n_1 \in \mathbb{N}$ such that if p, $q \ge n_1$ with $p - q = 1$ mod m, then

$$d(x_p, x_q) \le \frac{\varepsilon}{2}.$$

Since $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$, there exists $n_2 \in \mathbb{N}$such that

$$d(x_n, x_{n+1}) \le \frac{\varepsilon}{2m},$$

for any $n \ge n_2$.

Let p, $q \ge \max\{n_1, n_2\}$ and $p > q$. Then there exists $k \in \{1, 2,..., m\}$ such that $p - q = k$ mod m. Therefore, $p - q + j = 1$ mod m for $j = m - k + 1$, and so we have

$$\begin{aligned}d(x_q, x_p) &\le d(x_q, x_{p+j}) + d(x_{p+j}, x_{p+j-1}) + \cdots + d(x_{p+1}, x_p)\\ &\le \frac{\varepsilon}{2} + j \times \frac{\varepsilon}{2m}\\ &\le \frac{\varepsilon}{2} + m \times \frac{\varepsilon}{2m} = \varepsilon.\end{aligned}$$

Thus, $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists $\nu \in \cup_{i=1}^m A_i$ such that $\lim_{n\to\infty} x_n = \nu$. Since $X = \cup_{i=1}^m A_i$ is a cyclic representation of X with respect to f, the sequence $\{x_n\}$ has infinite terms in each A_i for $i \in \{1,2,...,m\}$. Now for all $i = 1,2,..., m$, we may take a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ with $x_{n_k} \in A_{i-1}$ and also all converge to ν. Since

$$\begin{aligned}d(x_{n_{k+1}}, f\nu) &= d(fx_{n_k}, f\nu)\\ &\le \phi(d(x_{n_k}, \nu)) - \varphi(d(x_{n_k}, \nu))\\ &\le \phi(d(x_{n_k}, \nu)).\end{aligned}$$

Letting $k \to \infty$, we have

$$d(\nu, f\nu) \leq 0,$$

and so $\nu = f\nu$.

Finally, to prove the uniqueness of the fixed point, let μ be the another fixed point of f. By the cyclic character of f, we have $\mu, \nu \in \cap_{i=1}^{n} A_i$. Since f is a cyclic weaker (ϕ, φ)-contraction, we have

$$\begin{aligned} d(\nu, \mu) &= d(\nu, f\mu) \\ &= \lim_{n\to\infty} d(x_{n_{k+1}}, f\mu) \\ &= \lim_{n\to\infty} d(fx_{n_k}, f\mu) \\ &\leq \lim_{n\to\infty} [\phi(d(x_{n_k}, \mu)) - \varphi(d(x_{n_k}, \mu))] \\ &\leq d(\nu, \mu) - \varphi(d(\nu, \mu)), \end{aligned}$$

and we can conclude that

$$\varphi(d(\nu, \mu)) = 0.$$

So we have $\mu = \nu$. We complete the proof.

Example 2 *Let $X = [-1,1]$ with the usual metric. Suppose that $A_1 = [-1,0] = A_3$ and $A_2 = [0,1] = A_4$. Define f: $X \to X$ by $f(x) = \frac{-x}{6}$ for all $x \in X$, and let ϕ, φ: $[0,\infty) \to [0,\infty)$ be $\phi(t) = \frac{1}{2}, \varphi(t) = \frac{t}{4}$. Then f is a cyclic weaker (ϕ, φ)-contraction and 0 is the unique fixed point.*

Example 3 *Let $X = \mathbb{R}^+$ with the metric $d: X \times X \to \mathbb{R}^+$ given by*

$$d(x, y) = \max\{x, y\}, \quad for \quad x, y \in X.$$

Let $A_1 = A_2 = \ldots = A_m = \mathbb{R}^+$. Define f: $X \to X$ by

$$f(x) = \frac{x^2}{77} \quad for \quad x \in X,$$

and let ϕ, φ: $[0, \infty) \to [0,\infty)$ be $\varphi(t) = \frac{t^3}{2(t+2)}$ and

$$\phi(t) = \begin{cases} \frac{2t^3}{3t+8}, & if\ t \geq 1; \\ \frac{t^2}{2}, & if\ t < 1. \end{cases}$$

Then f is a cyclic weaker (ϕ, φ)-contraction and 0 is the unique fixed point.

Example 4 *Let $X = \mathbb{R}^3$ and we define d: $X \times X \to [0, \infty)$ by*

$$d(x, y) = \max\left\{|x_1 - y_1|, |x_2 - y_2|, |x_3 - y_3|,\right\}$$

for $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3) \in X$, and let $A = \{(x,0,0): x \in [0,1]\}$, $B = \{(0,y,0): y \in [0,1]\}$, $C = \{(0,0, z): z \in [0,1]\}$ be three subsets of X.

Define f: $A \cup B \cup C \to A \cup B \cup C$ *by*

$$f((x,0,0)) = \left(0, \frac{1}{8}x^2, 0\right); \quad \textit{for all } x \in [0,1];$$

$$f((0,y,0)) = \left(0, 0, \frac{1}{8}y^2\right); \quad \textit{for all } y \in [0,1];$$

$$f((0,0,z)) = \left(\frac{1}{8}z^2, 0, 0\right); \quad \textit{for all } z \in [0,1].$$

We define ϕ: $[0, \infty) \to [0,\infty)$ *by*

$$\phi(t) = \frac{t^2}{t+1} \textit{ for } t \in [0, \infty),$$

and ϕ: $[0, \infty) \to [0,\infty)$ *by*

$$\varphi(t) = \frac{t^2}{t+2} \textit{ for } t \in [0, \infty).$$

Then f is a cyclic weaker (ϕ, ϕ)*-contraction and* (0,0,0) *is the unique fixed point.*

Acknowledgements
The authors would like to thank referee(s) for many useful comments and suggestions for the improvement of the article.

Competing interests
The authors declare that they have no competing interests.

Received: 14 November 2011 Accepted: 16 February 2012 Published: 16 February 2012

References

1. Banach, S: Sur les operations dans les ensembles abstraits et leur application aux equations integerales. Fund Math. **3**, 133–181 (1922)
2. Boyd, DW, Wong, SW: On nonlinear contractions. Proc Am Math Soc. **20**, 458–464 (1969). doi:10.1090/S0002-9939-1969-0239559-9
3. Aydi, H, Karapinar, E, Shatnawi, W: Coupled fixed point results for (ψ - φ)-weakly contractive condition in ordered partial metric spaces. Comput Math Appl. **62**(12):4449–4460 (2011). doi:10.1016/j.camwa.2011.10.021
4. Karapinar, E: Weak φ-contraction on partial metric spaces and existence of fixed points in partially ordered sets. Mathematica Aeterna. **1**(4):237–244 (2011)
5. Kirk, WA, Srinivasan, PS, Veeramani, P: Fixed points for mappings satisfying cyclical contractive conditions. Fixed Point Theory. **4**(1):79–89 (2003)
6. Rus, IA: Cyclic representations and fixed points. Ann T Popoviciu, Seminar Funct Eq Approx Convexity. **3**, 171–178 (2005)
7. Păcurar, M, Rus, IA: Fixed point theory for cyclic φ-contractions. Nonlinear Anal. **72**(3-4):2683–2693 (2010)
8. Karapinar, E: Fixed point theory for cyclic weaker ⊠-contraction. Appl Math Lett. **24**(6):822–825 (2011). doi:10.1016/j.aml.2010.12.016
9. Karapinar, E, Sadarangani, K: Corrigendum to "Fixed point theory for cyclic weaker ⊠-contraction". Appl Math Lett (2011, in press). [Appl. Math. Lett. Vol.24(6), 822-825.]
10. Karapinar, E, Sadarangani, K: Fixed point theory for cyclic (⊠ -ψ)-contractions. Fixed Point Theory Appl. **2011**, 69 (2011). doi:10.1186/1687-1812-2011-69
11. Meir, A, Keeler, E: A theorem on contraction mappings. J Math Anal Appl. **28**, 326–329 (1969). doi:10.1016/0022-247X(69)90031-6

doi:10.1186/1687-1812-2012-17
Cite this article as: Chen: **Fixed point theory for the cyclic weaker Meir-Keeler function in complete metric spaces.** *Fixed Point Theory and Applications* 2012 **2012**:17.