Published for SISSA by Springer
Received: *March 10, 2012*
Accepted: *June 11, 2012*
Published: *July 16, 2012*

On $\mathcal{N} = 2$ truncations of IIB on $T^{1,1}$

Nick Halmagyi,[a,b] **James T. Liu**[c] **and Phillip Szepietowski**[d]

[a] *Laboratoire de Physique Théorique et Hautes Energies, Université Pierre et Marie Curie, CNRS UMR 7589, F-75252 Paris Cedex 05, France*
[b] *Center for the Fundamental Laws of Nature Harvard University, Cambridge, MA 02138, U.S.A.*
[c] *Michigan Center for Theoretical Physics, Randall Laboratory of Physics, The University of Michigan, Ann Arbor, MI 48109-1040, U.S.A.*
[d] *Department of Physics, University of Virginia, Box 400714, Charlottesville, VA 22904, U.S.A.*

E-mail: halmagyi@lpthe.jussieu.fr, jimliu@umich.edu, pgs8b@virginia.edu

Abstract: We study the $\mathcal{N} = 4$ gauged supergravity theory which arises from the consistent truncation of IIB supergravity on the coset $T^{1,1}$. We analyze three $\mathcal{N} = 2$ subsectors and in particular we clarify the relationship between true superpotentials for gauged supergravity and certain fake superpotentials which have been widely used in the literature. We derive a superpotential for the general reduction of type I supergravity on $T^{1,1}$ and this together with a certain solution generating symmetry is tantamount to a superpotential for the baryonic branch of the Klebanov-Strassler solution.

Keywords: Flux compactifications, Gauge-gravity correspondence, Superstring Vacua, String Duality

ArXiv ePrint: 1111.6567

doi:10.1007/JHEP07(2012)098

JHEP07(2012)098

Contents

JHEP07(2012)098

1 Introduction

Starting with the work [1], the study of type IIB supergravity on the conifold has given rise to much progress in gauge/gravity duality. In particular, it provides an example of a gravity dual to a non-conformal, four dimensional field theory with minimal supersymmetry [2]. This background, known as the warped deformed conifold, can be used to model the local geometry of a flux compactification [3]. In the current work, following [4, 5], we study the gauged supergravity theory which arises from Kaluza-Klein reduction of IIB supergravity on the coset $T^{1,1}$.

The Kaluza-Klein reduction of ten and eleven dimensional supergravity to lower dimensional gauged supergravity theories has a rich history. In particular there has been much attention applied to the case of reduction on spheres down to maximally supersymmetric gauged supergravity [6, 7]. Another route to deriving lower dimensional gauged supergravity theories is to use a set of globally defined fundamental forms on the internal manifold which close under exterior derivative and wedge product. This technique has been used for nearly Kähler manifolds [8], cosets [4, 5, 9, 10], Sasaki-Einstein manifolds [11–15] and also more general flux backgrounds in [16–18]. Additionally, recent progress has been made exploring the fermion sector of these reductions [19–22].

The current work synthesizes aspects of the Kaluza-Klein reduction of IIB supergravity on $T^{1,1}$ performed in [4, 5] that retains just the singlet sector under the global symmetries of $T^{1,1}$. In fact similar reductions (restricted to just the scalar sector) were employed to derive the warped deformed conifold solution [2, 23] (and used in many other scenarios as well [24–29]), where a one-dimensional action was derived and a superpotential found from which one can compute the scalar potential. This superpotential was then used to facilitate the supersymmetry analysis and thus bypass using ten dimensional spinors directly. In more recent work [4, 5], it was found that there exists a supersymmetric Kaluza-Klein reduction on $T^{1,1}$ down to five dimensional $\mathcal{N} = 4$ gauged supergravity (generalizing the work on Sasaki-Einstein manifolds [12–15]) from which all these one dimensional models can be obtained by additional reduction on $\mathbb{R}^{1,3}$ and some further truncation of the fields.

The advantages of performing a rigorous supersymmetric reduction, thus including higher form fields and not just the scalar sector, are manyfold. It allows for a simple yet rigorous analysis of supersymmetric solutions, it allows one to consider solutions with non-trivial profiles for form fields relevant for AdS/CMT [30, 31], and it also helps to characterize which gauged supergravity theories can be obtained from string theory.

One goal of the current work is to develop the $\mathcal{N} = 2$ five dimensional gauged supergravity theories which are relevant for studying the physics of the warped deformed conifold solution and its relatives. One such $\mathcal{N} = 2$ theory is obtained by truncating to

JHEP07(2012)098

modes which are even under a particular $\mathbb{Z}_2$ symmetry $\mathcal{I}$ which will be explained in section 4.1. Within this $\mathcal{I}$ invariant truncation there exists a superpotential W_{KS} which has been known for some time [25]. But as we will see, W_{KS} is in fact a *fake* superpotential even though the theory is supersymmetric. It was essentially noticed in [32] that from W_{KS} one can derive a solution for fluxes on the warped deformed conifold which are known from ten dimensional analysis [33] to be non-supersymmetric. The analysis we perform resolves this seeming discrepancy since we can identify precisely how W_{KS} fails to be a true superpotential of the theory. We can then characterize which fluxes are in fact supersymmetric on the warped deformed conifold.

While there have been superpotentials provided for the solution of [34, 35] and also [2], it has been an open problem for some time to provide a superpotential for the interpolating solution of [36]. In section 4.3 we study the sector of the $\mathcal{N} = 4$ theory corresponding to retaining just (g_{MN}, ϕ, F_3) which we will call the NS-sector truncation.[1] Importantly, this sector retains $\mathcal{I}$-even and $\mathcal{I}$-odd modes, and we derive a superpotential for this truncation. Using the TST duality transformation of [37], from any solution of the NS-truncation one can generate a family of solutions which lie within the $\mathcal{N} = 4$ theory. As such, our superpotential can be considered a superpotential for the baryonic branch of the warped deformed conifold.

The organization of the rest of this paper is as follows. In the following section we lay the ground work for the $\mathcal{N} = 4$ reduction of IIB on $T^{1,1}$. In addition we analyze the duality group of the $\mathcal{N} = 4$ theory, notably finding the embedding of the $\mathrm{SL}(2, \mathbb{R})$ of IIB supergravity within the $\mathcal{N} = 4$ scalar coset. In section 3 we review some relevant material on five-dimensional $\mathcal{N} = 2$ gauged supergravity coupled to vector and hyper multiplets. We also include a discussion on the existence of superpotentials, real and fake, and their relation to solutions of the BPS domain wall equations. In section 4 we provide the relevant details of three truncations of the $\mathcal{N} = 4$ theory to $\mathcal{N} = 2$ gauged supergravity. We analyze the conditions imposed by supersymmetry and present superpotentials for each truncation. Finally, in section 5 we conclude with some remarks on the pitfalls and advantages of superpotential techniques. By studying a specific solution on the warped deformed conifold we detail precisely the way in which solutions found from a fake superpotential can end up being, in fact, non-supersymmetric. Additionally, we remark on potential future work towards understanding relations between the current work and solution generating techniques such as the TST transformation in string theory.

For sake of clarity we have relegated many important details of the $\mathcal{N} = 2$ truncations to appendices A, B and C. Specifically, for each truncation we include a detailed description of the scalar coset manifolds and the coordinate transformations which lift the coset coordinates to IIB supergravity fields. In addition we present the reduction of the IIB fermion variations, which we find to be consistent with the scalar coset structure, as expected. Finally, appendix D summarizes some differences in convention between the present work and refs. [4, 22] concerning the $T^{1,1}$ reduction.

[1] We are abusing notation here since we keep F_3 and not H_3. But these are related by S-duality.

2 $\mathcal{N}=4$ gauged supergravity from IIB on $T^{1,1}$

The consistent truncation of IIB supergravity on $T^{1,1}$ was performed in [4, 5], and the resulting theory is described by gauged $\mathcal{N}=4$ supergravity in five dimensions coupled to three vector multiplets. Since this is the starting point for the further $\mathcal{N}=2$ truncations, we first review this construction, establish notation and derive the action of the IIB $\mathrm{SL}(2,\mathbb{R})$ symmetry on the gauged supergravity theory.

The bosonic field content of IIB supergravity consists of the metric, IIB axi-dilaton $\tau=a+ie^{-\phi}$, three-forms F_3^i $(i=1,2)$ and RR five-form $\tilde{F}_5$. The ten dimensional metric is reduced according to

$$ds_{10}^2=e^{2u_3-2u_1}ds_5^2+e^{2u_1+2u_2}E_1'\overline{E}_1'+e^{2u_1-2u_2}E_2'\overline{E}_2'+e^{-6u_3-2u_1}E_5E_5, \tag{2.1}$$

where

$$\begin{aligned}
E_1&=\frac{1}{\sqrt{6}}(\sigma_1+i\sigma_2), & E_2&=\frac{1}{\sqrt{6}}(\Sigma_1+i\Sigma_2),\\
E_1'&=E_1\,, & E_2'&=E_2+v\overline{E}_1,\\
E_5&=g_5+A_1, & g_5&=\frac{1}{3}(\sigma_3+\Sigma_3),
\end{aligned} \tag{2.2}$$

and the SU(2)-invariant one forms satisfy $d\sigma_i=\frac{1}{2}\epsilon_{ijk}\sigma_j\wedge\sigma_k$ and $d\Sigma_i=\frac{1}{2}\epsilon_{ijk}\Sigma_j\wedge\Sigma_k$. This follows from writing $T^{1,1}$ as U(1) bundled over $\mathbb{P}^1\times\mathbb{P}^1$. In particular, the U(1) structure may be described by the invariant forms

$$J_1=\frac{i}{2}E_1\wedge\bar{E}_1,\qquad J_2=\frac{i}{2}E_2\wedge\bar{E}_2,\qquad \Omega=E_1\wedge E_2. \tag{2.3}$$

The reduction of the metric yields three real five-dimensional scalars (u_1,u_2,u_3), one complex scalar v, and a U(1) gauge field A_1 with field strength $F_2=dA_1$.

We adopt a mixed notation with respect to [22] and [4] for the IIB forms which makes the $\mathrm{SL}(2,\mathbb{R})$ invariance explicit. The differences in notation are summarized in appendix D. For the three-forms, we expand the two form potentials as

$$B_2^i=b_2^i+b_1^i\wedge E_5+c_0^iJ_++e_0^iJ_-+2\mathrm{Re}\,(b_0^i\Omega), \tag{2.4}$$

where $J_\pm=J_1\pm J_2$, and write

$$F_3^i=dB_2^i+j_0^iJ_-\wedge E_5, \tag{2.5}$$

where j_0^i are the charges coming from topological flux on the $S^3\subset T^{1,1}$. Explicitly, for the three forms, we have

$$F_3^i=g_3^i+g_2^i\wedge E_5+(g_1^i+h_1^i)\wedge J_1+(g_1^i-h_1^i)\wedge J_2+j_0^iJ_-\wedge E_5+2\mathrm{Re}\,[f_1^i\wedge\Omega+f_0^i\Omega\wedge E_5], \tag{2.6}$$

where

$$\begin{aligned}
g_3^i&=db_2^i-b_1^i\wedge F_2, & g_2^i&=db_1^i, & g_1^i&=dc_0^i-2b_1^i\equiv Dc_0^i,\\
h_1^i&=de_0^i-j_0^iA_1\equiv De_0^i, & f_1^i&=db_0^i-3ib_0^iA_1\equiv Db_0^i, & f_0^i&=3ib_0^i\,.
\end{aligned} \tag{2.7}$$

The three-forms contribute two $\mathrm{SL}(2,\mathbb{R})$ doublets of real scalars (c_0^i, e_0^i), one doublet complex scalar b_0^i, one doublet of U(1) gauge fields b_1^i with field strength $g_2^i = db_1^i$ and one doublet two-form potential b_2^i. Alternatively, one may define the complex three-form field strength

$$\frac{1}{\sqrt{\tau_2}} G_3 = v_i F_3^i = \frac{1}{\sqrt{\tau_2}} \left(F_3^2 - \tau F_3^1\right) , \tag{2.8}$$

where we have introduced the $\mathrm{SL}(2,\mathbb{R})$ vielbein v_i. However, we will always use a notation that leaves the $\mathrm{SL}(2,\mathbb{R})$ structure explicit.

The five-form field strength can be expanded in the basis

$$\begin{aligned} \widetilde{F}_5 = (1+*)[e^Z J_1 \wedge J_2 \wedge E_5 + K_1 \wedge J_1 \wedge J_2 + K_{21} \wedge J_1 \wedge E_5 \\ + K_{22} \wedge J_2 \wedge E_5 + 2\mathrm{Re}\,(L_2 \wedge \Omega \wedge E_5)]. \end{aligned} \tag{2.9}$$

The Bianchi identity $d\tilde{F}_5 = \frac{1}{2}\epsilon_{ij} F_3^i \wedge F_3^j$ yields the constraints

$$\begin{aligned} e^Z &= Q - 6i\epsilon_{ij}(b_0^i \bar{b}_0^j - \bar{b}_0^i b_0^j) + \epsilon_{ij}(j_0^i e_0^j - j_0^j e_0^i) , \\ K_1 &= Dk + 2\epsilon_{ij}[b_0^i D\bar{b}_0^j + \bar{b}_0^i D b_0^j] - \epsilon_{ij} e_0^i h_1^j , \\ K_{21} &= Dk_{11} + \frac{1}{4}\epsilon_{ij} g_1^i \wedge g_1^j + \frac{1}{2}\epsilon_{ij} g_1^i \wedge h_1^j , \\ K_{22} &= Dk_{12} + \frac{1}{4}\epsilon_{ij} g_1^i \wedge g_1^j - \frac{1}{2}\epsilon_{ij} g_1^i \wedge h_1^j , \end{aligned} \tag{2.10}$$

where the covariant derivatives are defined as

$$\begin{aligned} Dk &= dk - QA_1 - 2k_{11} - 2k_{12} - \epsilon_{ij} j_0^i e_0^j A_1 , \\ Dk_{11} &= dk_{11} - \epsilon_{ij} j_0^i b_2^j , \\ Dk_{12} &= dk_{12} + \epsilon_{ij} j_0^i b_2^j . \end{aligned} \tag{2.11}$$

The charge Q comes from mobile D3-branes. The five-form contributes one real scalar k, two one-forms (k_{11}, k_{12}) and a complex two-form L_2.

In summary, the reduction of IIB supergravity on $T^{1,1}$ yields $\mathcal{N} = 4$ supergravity coupled to three vector multiplets. The scalar manifold is

$$\mathcal{M}_{sc} = \frac{\mathrm{SO}(5, n_v)}{\mathrm{SO}(5) \times \mathrm{SO}(n_v)} \times \mathrm{SO}(1,1), \tag{2.12}$$

with $n_v = 3$. As shown in [4, 5], the SO(1,1) is parameterized by u_3, while the remaining $5 \times n_v = 5 \times 3$ scalars are

$$(u_1, u_2, c_0^i, e_0^i, k, \tau, \bar{\tau}, v, \bar{v}, b_0^i, \bar{b}_0^i). \tag{2.13}$$

Along with the scalars, there are a total of nine-vectors: a singlet vector A_1, along with $5 + n_v = 8$ additional vectors transforming in the vector representation of $\mathrm{SO}(5, n_v)$. The latter eight vectors correspond to the potentials

$$(b_1^i, k_{11}, k_{12}, b_2^i, L_2, \bar{L}_2), \tag{2.14}$$

where the two-form potentials b_2^i and L_2 are dual to vectors in five dimensions.

2.1 Duality transformations

In ungauged supergravity with a scalar manifold given by a coset $\widehat{G}/\widehat{H}$, the duality group is given by global $\widehat{G}$ transformations. These transformations act on the coset on the right, say, and are compensated by the left action of a local $\widehat{H}$ transformation which brings the coset element back to a canonical form. After gauging, only a subgroup of $\widehat{G}$ transformations remain symmetries of the theory. It is clear for $\mathcal{N}=4$ theories that the commutant of the gauge group G in $\mathrm{SO}(5,n_v)$ is a symmetry of the theory. But, in addition, there could be further symmetries. There is currently no understanding in general of how large the symmetry group is or how to compute it for a given gauged supergravity theory. To perform an analysis of the duality group, the embedding tensor formalism (see e.g. [38]) is quite useful since it facilitates the embedding of the gauge group into the scalar manifold in a covariant way.

As reviewed above, the $T^{1,1}$ reduction yields $\mathcal{N}=4$ supergravity coupled to three $\mathcal{N}=4$ vector multiplets, with the scalar manifold [4, 5]

$$\mathcal{M}_{sc} = \frac{\mathrm{SO}(5,3)}{\mathrm{SO}(5)\times\mathrm{SO}(3)} \times \mathrm{SO}(1,1). \tag{2.15}$$

The field content combined with the embedding tensor [38] completely specify the $\mathcal{N}=4$ supergravity. In [4, 5] the embedding tensor (f_{MNP}, ξ_{MN}) was shown to be

$$\begin{aligned}
f_{123} &= -f_{128} = f_{137} = f_{178} = 2\,,\\
\xi_{23} &= -\xi_{28} = \xi_{37} = \xi_{78} = -Q/\sqrt{2}\,,\\
\xi_{45} &= -3\sqrt{2}\,,\\
\xi_{36} &= \xi_{68} = \sqrt{2}\, j_0^2\,,\\
\xi_{26} &= \xi_{67} = \sqrt{2}\, j_0^1\,,
\end{aligned} \tag{2.16}$$

and permutations. From this we find that the gauge group G is generated by

$$\begin{aligned}
g_0 &= 2\sqrt{3}\, t_{45} + \sqrt{2}Q\,(t_{37}+t_{78}+t_{23}-t_{28}) + \sqrt{2}j_0^2(t_{36}+t_{68}) + \sqrt{2}j_0^1(t_{26}+t_{67})\,,\\
g_1 &= t_{13}-t_{18}\,,\\
g_2 &= t_{12}-t_{17}\,,\\
g_3 &= t_{37}+t_{78}+t_{23}-t_{28}\,,
\end{aligned} \tag{2.17}$$

where

$$(t_{MN})_P{}^Q = \delta^Q_{[M}\eta_{N]P} \tag{2.18}$$

are the standard generators of $\mathrm{SO}(5,3)$ and $\eta = \mathrm{diag}\{-1,-1,-1,-1,-1,+1,+1,+1\}$.

We find that the commutant of G inside $\mathrm{SO}(5,3)$ is in general given by the following two elements

$$\begin{aligned}
t_{45}:&\quad v \to e^{i\beta} v,\ M_0 \to e^{-i\beta} M_0,\ N_0 \to e^{-i\beta} N_0\,,\\
t_{37}+t_{28}+t_{23}-t_{28}:&\quad k \to k+\beta\,,
\end{aligned} \tag{2.19}$$

In addition there are two more elements

$$\begin{aligned} t_{26}+t_{67}: &\ \ e_0^1 \to e_0^1+\beta,\ \ k \to k+\beta e_0^2\,, \\ t_{36}+t_{68}: &\ \ e_0^2 \to e_0^2+\beta,\ \ k \to k+\beta e_0^1 \end{aligned} \tag{2.20}$$

generating symmetries which are broken by the terms in the scalar potential

$$V_{sc} \sim j_0^2 e_0^1 - j_0^1 e_0^2. \tag{2.21}$$

This is clearly not the full duality group since for example we at least expect to find the action of the $\mathrm{SL}(2,\mathbb{R})$ symmetry of IIB supergravity. It turns out that this $\mathrm{SL}(2,\mathbb{R})$ lives inside the normalizer of G in $\mathrm{SO}(5,3)$. The normalizer is ten dimensional, but by explicit computation we find that the only elements which are symmetries of the scalar potential are the realization of the $\mathrm{SL}(2,\mathbb{R})$ symmetry of IIB supergravity. We find these to be generated by

$$\begin{aligned} h &= 2(t_{27}-t_{38})\,, \\ e &= t_{28}-t_{78}+t_{23}+t_{37}\,, \\ f &= t_{28}+t_{78}-t_{23}+t_{37}\,, \end{aligned} \tag{2.22}$$

satisfying

$$[e,f]=h, \quad [h,e]=2e, \quad [h,f]=-2f\,. \tag{2.23}$$

With general charges j_0^i, the whole symmetry is broken, but with $j_0^2=0$ ($j_0^1=0$) the symmetry generated by e (f) survives as a symmetry of the scalar potential. When $j_0^1=j_0^2$ the full $\mathrm{SL}(2,\mathbb{R})$ is a symmetry of the theory.

It is interesting that non-trivial duality symmetries are found outside the commutator of the gauge group inside $\mathrm{SO}(5,3)$. In ref. [39] the gauged supergravity was studied which arises from compactification of IIB supergravity on the orbifold $S^5/\mathbb{Z}_n$. There it was found that the commutator of the gauge group $G=\mathrm{SU}(2)\times\mathrm{U}(1)$, inside $\mathrm{SO}(5,2n)$ was $\mathrm{SU}(1,n)$. This result is at odds with the discrete duality group found in [40] which does not quite fit inside $\mathrm{SU}(1,n)$. It is expected that the discrete duality group is a symmetry of the dual field theory at finite N and this should be enhanced to the continuous group in the limit of large N. (See [41] for a derivation of this fact for $N=4$ SYM in four dimensions.) What we have found here is an example of duality symmetries which lie outside the commutator of the gauge group inside $\mathrm{SO}(5,n_v)$ and it would be interesting to explore if the duality group found in [39] can be extended by considering the normalizer of the gauge group in $\mathrm{SO}(5,2n)$.

3 Preliminaries on $\mathcal{N}=2$ gauged supergravity

Before examining the various truncations of the $\mathcal{N}=4$ theory, we first review some of the salient features of $\mathcal{N}=2$ gauged supergravity. In general, $\mathcal{N}=2$ supergravity may be coupled to vector, tensor and hypermultiplets. However, we will not consider tensor multiplets, as they will not appear in any of the truncations. As is well known, the bosonic

JHEP07(2012)098

field content of this theory consists of the metric $g_{\mu\nu}$, n_v+1 vectors A^I_μ (with $I=0,\ldots,n_v$), n_v vector multiplet scalars ϕ^x living on a very special manifold and $4n_h$ hyperscalars q^X on a quaternionic manifold.

The bosonic $\mathcal{N}=2$ Lagrangian is

$$\begin{aligned}\mathcal{L} = R &- \frac{1}{2}g_{xy}D_\mu\phi^x D^\mu\phi^y - \frac{1}{2}g_{XY}D_\mu q^X D^\mu q^Y - V \\ &-\frac{1}{4}G_{IJ}F^I_{\mu\nu}F^{J\,\mu\nu} + \frac{1}{24}c_{IJK}\epsilon^{\mu\nu\rho\lambda\sigma}F^I_{\mu\nu}F^J_{\rho\lambda}A^K_\sigma\,,\end{aligned} \tag{3.1}$$

and the fermionic supersymmetry transformations are (for the gravitino, gauginos and hyperinos)

$$\begin{aligned}\delta\psi_{\mu\,i} &= \left[D_\mu + \frac{i}{24}X_I({\gamma_\mu}^{\nu\rho} - 4\delta^\nu_\mu\gamma^\rho)F_{I\,\nu\rho}\right]\epsilon_i + \frac{i}{6}X^I(P_I)_i{}^j\epsilon_j\,,\\ \delta\lambda^x_i &= \left(-\frac{i}{2}\gamma\cdot D\phi^x - \frac{1}{4}g^{xy}\partial_y X^I\gamma^{\mu\nu}F_{I\,\mu\nu}\right)\epsilon_i - g^{xy}\partial_y X^I(P_I)_i{}^j\epsilon_j\,,\\ \delta\zeta^A &= f^{iA}_X\left(-\frac{i}{2}\gamma\cdot Dq^X + \frac{1}{2}X^I K^X_I\right)\epsilon_i\,.\end{aligned} \tag{3.2}$$

The covariant derivatives are

$$D_\mu\phi^x = \partial_\mu\phi^x + A^I_\mu K^x_I(\phi^x) \tag{3.3}$$

for the vector multiplet scalars and

$$D_\mu q^X = \partial_\mu q^X + A^I_\mu K^X_I(q^X) \tag{3.4}$$

for the hypermultiplet scalars, where we have fixed the gauge coupling $g=1$. The Killing vectors $K^x_I(\phi^x)$ and $K^X_I(q^X)$ correspond to the gauging of the isometries of the very special manifold and quaternionic manifold, respectively.

The vector multiplet scalars are given in terms of the n_v+1 constrained scalars $X^I = X^I(\phi^x)$ subject to the very special geometry constraint

$$\frac{1}{6}c_{IJK}X^I X^J X^K = 1. \tag{3.5}$$

Additionally, the scalar metric for the vector multiplet scalars is determined by

$$\begin{aligned}G_{IJ} &= X_I X_J - c_{IJK}X^K\,,\\ X_I &= \frac{1}{2}c_{IJK}X^J X^K\,,\\ g_{xy} &= \partial_x X^I\partial_y X^J G_{IJ}\,.\end{aligned} \tag{3.6}$$

The Killing prepotentials $(P_I)_i{}^j = P^r_I(i\sigma^r)_i{}^j$ are determined by the Killing vectors and depend only on the hyperscalars. They satisfy

$$\iota_{K_I}\Omega^r = dP^r_I + \epsilon^{rst}\omega^s P^t_I\,, \tag{3.7}$$

JHEP07(2012)098

where ω^s is the SU(2) connection, or in co-ordinates

$$K_I^X \Omega^r_{XY} = \nabla_Y P_I^r \,. \tag{3.8}$$

Here Ω^r are the triplet of covariantly constant two-forms on the quaternion manifold. While this is a differential equation for the Killing prepotentials, one can solve for them algebraically by using the fact [42] that P_I^r are eigenfunctions of the Laplacian

$$\nabla^X \nabla_X P_I^r = -4n_h P_I^r \,. \tag{3.9}$$

We then see that

$$P_I^r = -\frac{1}{4n_h} \nabla^X \left(K_\Lambda^Y \Omega^r_{XY} \right) \tag{3.10}$$

is a solution to (3.8). Note that the Killing prepotentials are unique only up to a local SU(2) gauge transformation. Finally, the scalar potential couples the hypermultiplet scalars to the vector multiplet scalars and is given by

$$V = 2g^{xy} \partial_x X^I \partial_y X^J P_I^r P_J^r - \frac{4}{3} P^r P^r + \frac{1}{2} g_{XY} K^X K^Y \,, \tag{3.11}$$

where $P^r = X^I P_I^r$. For convenience, we will often denote P^r as an SU(2) vector, namely $\vec{P} = (P^1, P^2, P^3)$.

3.1 Real and fake $\mathcal{N} = 2$ superpotentials

As we will discuss in the following subsection, the construction of BPS solutions to gauged supergravity is often based on solving first order equations constructed from the $\mathcal{N} = 2$ superpotential. In the absence of hypermatter, where a rigid U(1) is gauged in SU(2), the Killing prepotentials are all aligned, say in the $r = 3$ direction. In this case, the superpotential is given by $W = X^I P_I^3$, and the scalar potential is determined in the usual manner by

$$V = 2g^{xy} \partial_x W \partial_y W - \frac{4}{3} W^2 \,, \tag{3.12}$$

in perfect agreement with (3.11)

It is often assumed that a superpotential will continue to exist when hypermatter is included. However, comparing the actual potential (3.11) with the expression (3.12) indicates a couple of differences. Firstly, the gauging of isometries of the quaternion manifold gives rise to an additional contribution $\frac{1}{2} g_{XY} K^X K^Y$ to the potential. Secondly, the first term of (3.11) only agrees with the first term of (3.12) for rigid P_I^r, since W was obtained by aligning P_I^r along $r = 3$. Nevertheless, it is possible to come close by defining a superpotential [43]

$$W = \sqrt{P^r P^r} \,. \tag{3.13}$$

and using the relation

$$V = 2g^{\Lambda\Sigma} \partial_\Lambda W \partial_\Sigma W - \frac{4}{3} W^2 \,, \tag{3.14}$$

where Λ, Σ run over both vector multiplet and hypermultiplet scalars. But in order for this relation to work, a further constraint on the phase of P^r must hold off-shell:

$$\partial_x Q^r = 0 \,, \tag{3.15}$$

where

$$P^r = W Q^r \,. \tag{3.16}$$

This condition is essentially a requirement that any U(1) component that is being gauged inside SU(2) must be rigid as a function of the vector multiplet scalars. This condition will hold if, e.g., the gauging of SU(2) is aligned with $r = 3$. However, this is a special case, and we will find explicit examples below where this constraint is in fact not satisfied off-shell.

Even when a particular gauging does not admit a superpotential, in some cases it is nevertheless possible to find a *fake* superpotential that reproduces the correct scalar potential using the relation (3.14). In this case, one can still write down first order equations for domain wall solutions. However, there is no guarantee that such solutions are actually supersymmetric; only examination of the true Killing spinor equations obtained from (3.2) will indicate whether the BPS conditions are satisfied or not. In practice, most solutions obtained in this fashion are supersymmetric. However, we are not aware of a general principle governing the existence of a fake superpotential nor determining when the resulting solution is supersymmetric.

An alternate approach to obtaining BPS solutions in the absence of a true superpotential is to nevertheless use the square-root superpotential (3.13) to derive a set of first order equations. In general, the result of solving this system may not satisfy the true equations of motion. However, once we impose the constraint (3.15), the background is then guaranteed to be a solution to the equations of motion as well as BPS. In fact, all BPS domain wall solutions may be obtained in this fashion. We explore this in a bit more detail below.

3.2 BPS domain-wall equations

A particularly interesting class of solutions in gauged supergravity are BPS domain walls. The domain wall ansatz is given by the five-dimensional metric

$$ds_5^2 = dr^2 + a(r)^2 \eta_{\mu\nu} dx^\mu dx^\nu \,, \tag{3.17}$$

and is supported by scalar fields that depend only on r. The vector fields vanish because of the isometry. Given this ansatz, it was shown in [43] that the BPS equations are given by

$$\frac{1}{a}\frac{da(r)}{dr} = \pm\frac{1}{3} W \,, \tag{3.18}$$

$$\frac{d\phi^\Lambda}{dr} = \mp 2 g^{\Lambda\Sigma} \partial_\Sigma W \,, \tag{3.19}$$

$$\partial_x Q^r = 0 \,. \tag{3.20}$$

The curious equation here is (3.20) which is not a standard BPS flow equation but is equivalent to the constraint encountered above in (3.15).

It is worthwhile to formally analyze the constraint (3.20) a little further. Recalling that $\vec{Q} = \vec{P}/|\vec{P}|$, we find that this constraint is equivalent to

$$\vec{P} \times (\vec{P} \times \partial_x \vec{P}) = 0 \qquad \Rightarrow \qquad \vec{P} \times \partial_x \vec{P} = 0 \,. \tag{3.21}$$

Moreover multiplying this expression by $G_{IK}\partial_x X^K$ and using the special geometry relation [44]

$$G_{IK}\partial^x X^K \partial_x X^J = \delta_I{}^J - X_I X^J \,, \tag{3.22}$$

we see that

$$0 = G_{IK}\partial^x X^K \vec{P} \times \partial_x \vec{P} = \vec{P} \times \vec{P}_I \,, \tag{3.23}$$

As a result, the constraint implies that

$$\vec{P} \times \vec{P}_I = 0. \tag{3.24}$$

We now conclude that the only way to satisfy (3.20) is to have $\vec{P}$ identically zero or to have every nonzero $\vec{P}_I$ lie along the same direction in SU(2), with possibly an arbitrary number of the $\vec{P}_I$ vanishing. An equivalent statement is to say that all cross products between any two prepotentials must vanish

$$\vec{P}_I \times \vec{P}_J = 0 \,. \tag{3.25}$$

This demonstrates that the square-root superpotential (3.13) can be used to obtain BPS domain wall solutions when combined with the constraint that all prepotentials are parallel in SU(2) space. This constraint was observed in [43] at fixed points of the domain wall flow. However, here we have shown that the parallel constraint must hold along all points of the supersymmetric flow. Additionally, this constraint was discussed in [45], however it was not recognized as a necessary condition of the BPS equations.

4 The truncations to $\mathcal{N} = 2$ gauged supergravity

It is generally useful to restrict our attention to $\mathcal{N} = 2$ subsectors of the full theory when looking for BPS solutions. This is because we may then apply the well-studied flow equations (3.18) and (3.19) along with all its associated machinery. Starting from $\mathcal{N} = 4$ supergravity coupled to three vector multiplets, the truncation to $\mathcal{N} = 2$ proceeds by removing the massive $\mathcal{N} = 2$ gravitino multiplet. Since the $\mathcal{N} = 4$ gravity multiplet reduces to a gravity multiplet coupled to a gravitino and a vector multiplet, and each $\mathcal{N} = 4$ vector reduces to a vector multiplet and a hypermultiplet, the decomposition gives four vector multiplets and three hypermultiplets. However, the massive gravitino multiplet will eat two vector multiplets, so upon truncation we are limited to at most two vector multiplets and three hypermultiplets [5].

Compared to the reduction on a generic Sasaki-Einstein manifold, the reduction on $T^{1,1}$ yields one additional $\mathcal{N} = 4$ vector multiplet, denoted the Betti vector multiplet in [5]. Furthermore, ref. [5] considered two truncations to $\mathcal{N} = 2$. The first retains the $\mathcal{N} = 2$ Betti hypermultiplet, and gives rise to a total of one vector multiplet and three hypermultiplets, with field content

$$\begin{array}{rl} & \underline{\text{Betti-hyper truncation}} \\ \text{gravity + vector:} & (g_{\mu\nu}; A_1, k_{11} + k_{22}; u_3)\,, \\ \text{3 hypers:} & (u_1, k, e_0^i, \tau, \bar{\tau}, b_0^i, \bar{b}_0^i, v, \bar{v})\,. \end{array} \tag{4.1}$$

The second truncation retains the $\mathcal{N}=2$ Betti vector multiplet, and yields two vector multiplets and two hypermultiplets

$$\begin{array}{rl} & \underline{\text{Betti-vector truncation}} \\ \text{gravity + 2 vectors:} & (g_{\mu\nu}; A_1, k_{11}, k_{12}; u_2, u_3)\,, \\ \text{2 hypers:} & (u_1, k, \tau, \bar{\tau}, b_0^i, \bar{b}_0^i)\,. \end{array} \tag{4.2}$$

We will examine both of these truncations below.

Of course, it is possible to further truncate away the entire Betti multiplet, leaving the universal $\mathcal{N}=2$ Sasaki-Einstein system

$$\begin{array}{rl} & \underline{\text{Sasaki-Einstein truncation}} \\ \text{gravity + vector:} & (g_{\mu\nu}; A_1, k_{11}+k_{12}; u_3)\,, \\ \text{2 hypers:} & (u_1, k, \tau, \bar{\tau}, b_0^i, \bar{b}_0^i)\,. \end{array} \tag{4.3}$$

If desired, the universal hypermultiplet may be truncated away, leaving

$$\begin{array}{rl} & \underline{\text{Massive vector truncation}} \\ \text{gravity + vector:} & (g_{\mu\nu}; A_1, k_{11}+k_{12}; u_3)\,, \\ \text{2 hypers:} & (u_1, k, b_0^{m^2=21}, \bar{b}_0^{m^2=21})\,. \end{array} \tag{4.4}$$

Alternatively, we may also keep only the universal hypermultiplet

$$\begin{array}{rl} & \underline{\text{Universal hyper truncation}} \\ \text{gravity:} & (g_{\mu\nu}; A_1 + \frac{1}{3}(k_{11}+k_{12}))\,, \\ \text{hyper:} & (\tau, \bar{\tau}, b_0^{m^2=-3}, \bar{b}_0^{m^2=-3})\,. \end{array} \tag{4.5}$$

Finally, all matter may be removed, leaving pure $\mathcal{N}=2$ supergravity

$$\begin{array}{rl} & \underline{\text{Pure sugra truncation}} \\ \text{gravity:} & (g_{\mu\nu}; A_1 + \frac{1}{3}(k_{11}+k_{12}))\,. \end{array} \tag{4.6}$$

In addition to the above family of truncations, it is possible to truncate IIB supergravity to the NSNS sector before reducing. Equivalently, we keep only fields arising from (g_{MN}, ϕ, F_3), where we have considered an S-duality rotated basis for convenience in relating our results to the conifold. The resulting NS truncation retains two vector multiplets and two hypermultiplets

$$\begin{array}{rl} & \underline{\text{NS truncation}} \\ \text{gravity + 2 vectors:} & (g_{\mu\nu}; A_1, b_1^2, b_2^2; \phi + 4u_1, u_3)\,, \\ \text{2 hypers:} & (\phi - 4u_1, u_2, c_0^2, e_0^2, b_0^2, \bar{b}_0^2, v, \bar{v})\,. \end{array} \tag{4.7}$$

As we show below, this is distinct from the Betti-vector truncation, even though they both result in two vector multiplets and two hypermultiplets. The NS truncation is related to the baryonic branch of Klebanov-Strassler through a TST transformation [37].

In the following sub-sections we present the details of the Betti-hyper, Betti-vector and the NS truncations. The theories are determined by the geometry of the special Kähler and quaternionic scalar coset manifolds. Along with some background information on the truncations, we provide only the particular Killing vectors which are gauged in each model as well as the form of the prepotentials. This is the most relevant information necessary to construct the superpotential and discuss the BPS flow equations. Additional information for each truncation will be relegated to the appendices. For completeness, we present the reduction of the IIB fermion supersymmetry variations in the appendices as well. As a consistency check we have verified that the Killing vectors and prepotentials determined from the coset and the fermion reductions are in agreement.

4.1 Betti-hyper truncation

We first consider the Betti-hyper truncation, which includes what is known as the Betti-hypermultiplet [5]. In total, it contains three $\mathcal{N}=2$ hypermultiplets and one vector multiplet. This field space has a critical point corresponding to the Klebanov-Strassler solution and thus this truncation is of particular interest. The supergravity theory is known to admit a superpotential [22, 25], but as we will discuss, this is not in fact a genuine superpotential but rather a *fake* superpotential.

The field content of the Betti-Hyper truncation is obtained from the $\mathcal{N}=4$ theory by restricting to the modes which are invariant under the $\mathcal{I}$ symmetry:

$$\mathcal{I} = \Omega_p \cdot (-1)^{F_L} \cdot \sigma\,, \tag{4.8}$$

where

$$\begin{aligned} \Omega_p \cdot (-1)^{F_L}:\ & (g,\phi,B_{(2)},C_{(0)},C_{(2)},C_{(4)}) \to (g,\phi,-B_{(2)},C_{(0)},-C_{(2)},C_{(4)})\,,\\ \sigma:\ & (J_+,,J_-,\Omega_{\rm R},\Omega_{\rm I}) \to (J_+,,-J_-,-\Omega_{\rm R},-\Omega_{\rm I})\,. \end{aligned} \tag{4.9}$$

The surviving field content is given in (4.1), and additional details of the truncation are presented in appendix A.

4.1.1 Killing vectors

The Killing vectors, which can be read off from the covariant derivatives in section 2 or from the supersymmetry variations in appendix A, are

$$\begin{aligned} K_0 &= -(Q+\epsilon_{ij}j_0^i e_0^j)\partial_k - (3ib_0^i\partial_{b_0^i} + c.c) + \left(\frac{3}{2}(1+\rho^2)\partial_\rho + c.c.\right) - j_0^i\partial_{e_0^i}\,,\\ K_1 &= 4\partial_k\,. \end{aligned} \tag{4.10}$$

The corresponding Killing prepotentials can be obtained either from the gravitino variation (A.27) or by explicitly constructing the SU(2)-connection ω^r and the triplet of two-forms Ω^r on the hypermultiplet moduli space and then using (3.10). In principle these two methods should only agree up to a local SU(2) transformation, but in fact we found them to agree precisely:

$$\begin{aligned} P_0 &= -i\left[\left(\frac{3}{2\rho_2}(1+|\rho|^2)-\frac{1}{2}e^{-4u_1}e^Z\right)\sigma_3\right. \\ &\quad -\frac{i}{2\rho}e^{-2u_1}v_i((\bar\rho-i)^2\bar f_0^i-(\bar\rho+i)^2 f_0^i+i(1-i\bar\rho)(1+i\bar\rho)j_0^i)\sigma_+ \\ &\quad \left.+\frac{i}{2\rho}e^{-2u_1}\bar v_i((\rho+i)^2 f_0^i-(\rho-i)^2\bar f_0^i-i(1+i\rho)(1-i\rho)j_0^i)\sigma_-\right], \\ P_1 &= -2ie^{-4u_1}\sigma_3\,. \end{aligned} \tag{4.11}$$

Note that $P_I^+=\overline{(P_I^-)}$.

4.1.2 The superpotential

Much of the motivation of the current work is to understand the origin in gauged supergravity of the superpotential first written down in [25]:

$$W_{KS} = -\frac{1}{2}e^{-4u_1+4u_3}e^Z+2e^{-4u_1-2u_3}+\frac{3}{2\rho}(1+|\rho|^2)e^{4u_3}\,. \tag{4.12}$$

Due to the particular form of the Killing prepotentials, namely that $P_1^1=P_1^2=0$, the only non-trivial way to solve the algebraic prepotential constraint (3.25) is to set also $P_0^1=P_0^2=0$. This amounts to the condition

$$3\frac{(1+i\bar\rho)}{(1-i\bar\rho)}v_i b_0^i+3\frac{(1-i\bar\rho)}{(1+i\bar\rho)}v_i\bar b_0^i = v_i j_0^i\,. \tag{4.13}$$

Evaluated on this constraint, one finds

$$\sqrt{P^rP^r}|_{\partial_x Q^r=0}=P^3|_{\partial_x Q^r=0}\,, \tag{4.14}$$

and thus the scalar potential can be obtained from the superpotential using the simple potential from superpotential relation (3.14), so long as all quantities are subject to the constraint $P_0^1=P_0^2=0$. What is particular interesting in this model is the non-trivial fact that

$$W_{KS}=P^3 \tag{4.15}$$

recreates the scalar potential using (3.14), even without imposing any constraints. As a result, P^3 plays the even more powerful role of a fake superpotential for this truncation.

In fact, nonsupersymmetric solutions of the KS system have been studied in [32]; in their analysis certain solutions to the "BPS" equations from the superpotential were shown to correspond to $(3,0)$ flux on the deformed conifold, which is known to be non-supersymmetric. From our analysis we can directly check that these non-supersymmetric solutions do not satisfy the constraint (4.13). Therefore they do not satisfy the true BPS equations and are explicitly non-supersymmetric. We will elaborate on this point in section 5.1.

4.2 Betti-vector truncation

We now turn to the Betti-vector truncation. Compared to the universal Sasaki-Einstein truncation, this keeps an additional $\mathcal{N}=2$ vector multiplet as opposed to the additional hypermultiplet of the Betti-hyper truncation, for a total of two hypermultiplets and two vector multiplets. Details of this truncation are given in appendix B. In particular, we have the following three Killing vectors

$$\begin{aligned} K_0 &= -(3ib_0^i\partial_{b_0^i} + c.c.) - Q\partial_k\,,\\ K_1 &= 2\partial_k\,,\\ K_2 &= 2\partial_k\,, \end{aligned} \tag{4.16}$$

and the prepotentials

$$\begin{aligned} P_0 &= -i\left[\left(3-\frac{1}{2}e^{-4u_1}e^Z\right)\sigma_3 - 2ie^{-2u_1}v_if_0^i\sigma_+ + 2ie^{-2u_1}\bar{v}_i\bar{f}_0^i\sigma_-\right]\,,\\ P_1 &= -ie^{-4u_1}\sigma_3\,,\\ P_2 &= -ie^{-4u_1}\sigma_3\,. \end{aligned} \tag{4.17}$$

Similar to the Betti-hyper truncation, the prepotentials (P_1, P_2) are particularly simple. This again appears to be the key to constructing a fake superpotential from the P^3 term. From $P^r \equiv X^I P_I^r$, where X^I are given by

$$X^0 = e^{4u_3}, \qquad X^1 = e^{2u_2-2u_3}, \qquad X^2 = e^{-2u_2-2u_3}\,, \tag{4.18}$$

we find

$$W_{BV} = -\frac{1}{2}e^{-4u_1+4u_3}e^Z + e^{-4u_1-2u_2-2u_3} + e^{-4u_1+2u_2-2u_3} + 3e^{4u_3}\,.$$

As in the Betti-hyper truncation this superpotential acts as a fake superpotential. However, to our knowledge, the solution space of this has not been analyzed. Of course, the fake superpotential must be supplemented with the prepotential constraint (3.25), which in this case takes on the particularly simple form

$$v_i f_0^i = 0\,, \tag{4.19}$$

and which is equivalent to two real constraints.

4.3 NS-sector truncation

We now consider the NS-sector truncation. This particular truncation on $T^{1,1}$ has not been previously worked out explicitly. However, its consistency is obvious from ten dimensions. We set the RR axion, the five-form and, for simplicity, the NSNS-three form to zero.[2] The resulting field content is listed in (4.7), and the details of the truncation are given in appendix C.

In [37] this sector was shown to be related via a TST transformation to the baryonic branch of the Klebanov-Strassler theory. In the following we determine a superpotential for this sector which in essence is then a superpotential on the baryonic branch. However, we note that a fake superpotential in this sector has not been found.

[2] By S-duality this is related to a setup where only the NSNS-fields are non-vanishing.

4.3.1 Killing vectors

Again, the Killing vectors can be determined from either the covariant derivatives in section 2 or the fermion variations in appendix C. They are

$$
\begin{aligned}
K_0 &= -(3ib_0\partial_{b_0} + c.c.) + (3iv\partial_v + c.c.) - P\partial_{e_0}\,, \\
K_1 &= 2\partial_{c_0}\,, \\
K_2 &= 0\,.
\end{aligned} \tag{4.20}
$$

The prepotentials, which can be computed from these Killing vectors on the scalar manifold or simply read off from the gravitino variation (C.25), are

$$
\begin{aligned}
P_0 &= -i\left[\left(3 - \frac{1}{2}e^{\phi/2-2u_1}(e^{-2u_2}((1+|v|^2)j_0^2 + 2ivf_0 - 2i\bar{v}\bar{f}_0) - e^{2u_2}P)\right)\sigma_3\right. \\
&\qquad \left. - \left(3\bar{v} + 2ie^{\phi/2-2u_1}\left(f_0 - \frac{i}{2}\bar{v}P\right)\right)\sigma_+ - \left(3v - 2ie^{\phi/2-2u_1}\left(\bar{f}_0 + \frac{i}{2}vP\right)\right)\sigma_-\right]\,, \\
P_1 &= -i\Big[e^{\phi/2-2u_1}(e^{-2u_2}(1-|v|^2) + e^{2u_2})\sigma_3 - 2\bar{v}e^{\phi/2-2u_1}\sigma_+ - 2ve^{\phi/2-2u_1}\sigma_-\Big]\,, \\
P_2 &= 0\,,
\end{aligned} \tag{4.21}
$$

where in the above, and for the remainder of this section, we have suppressed the upper $\mathrm{SL}(2,\mathbb{R})$ index on the fields from the RR three-form and have set $j_0^2 = P$.

4.3.2 The superpotential

Curiously, we were not able to find a fake superpotential in this sector. This seems to be related to the fact that P_0 and P_1 are both non-trivial in all three components and so there is no natural SU(2) direction for the prepotentials to lie. This is in contrast to the previous two truncations, which naturally fell into the 3-direction. One could argue that these prepotentials can be rotated by an SU(2) transformation into the same form as in (4.11). However, due to the nontrivial dependence of P_1 on the hyper-scalars this rotation is field dependent and does not yield a suitable fake superpotential. The key to constructing a fake superpotential from prepotentials seems to be related to the fact that theories which admit such a fake superpotential admit a *rigid* rotation of all non-trivial prepotentials into one direction. However, a rigorous demonstration of this statement has not been established.

Nevertheless, we may find the closest possibility for a superpotential in this sector by computing $W = \sqrt{P^r P^r}$ and explicitly imposing the algebraic prepotential constraints (3.25) off-shell. In this case we find two independent constraints on the fields. The first is

$$
\mathrm{Im}\,(vb_0) = 0, \tag{4.22}
$$

which can be solved by setting

$$
b_0 = \alpha\bar{v}, \tag{4.23}
$$

where α is a real function. The second constraint is more complicated and the detailed form is not illuminating. It however fixes the coefficient α to be such that

$$
b_0 = \frac{\left(2P + 3e^{2u_1-2u_2-\phi/2}(1-|v|^2-e^{4u_2})\right)}{6(1+|v|^2+e^{4u_2})}\bar{v}. \tag{4.24}
$$

Once this identification has been made, the superpotential defined by

$$W = \sqrt{\vec{P} \cdot \vec{P}}, \tag{4.25}$$

can be used in the standard fashion and becomes

$$\begin{aligned} W = &\sqrt{1 + \frac{1}{4} e^{-4u_2}(1 - |v|^2 - e^{4u_2})^2} \\ &\times \left[2e^{-4u_1 - 2u_3} - Pe^{\phi/2 - 2u_1 + 4u_3} \left(\frac{1 - |v|^2 - e^{4u_2}}{1 + |v|^2 + e^{4u_2}} \right) + 6e^{4u_3 - 2u_2} \left(\frac{|v|^2 + e^{4u_2}}{1 + |v|^2 + e^{4u_2}} \right) \right]. \end{aligned} \tag{4.26}$$

It can be checked that once the constraint (4.24) is imposed, this expression for W gives the potential, which is also subject to (4.24), through the standard potential from superpotential relation (3.14). A version of this superpotential, as well as the constraint (4.24), has been previously derived in [46] in the context of a string dual to $\mathcal{N} = 1$ SQCD.[3] In [46], Hamilton-Jacobi techniques are used to derive the superpotential in an effective one-dimensional scalar theory. This is somewhat different in philosophy to our analysis, where (4.26) is highlighted as a true superpotential within a genuine five-dimensional supergravity.

Note that the NS truncation includes the Maldacena-Nunez solution [35]. In fact, substituting in the ansatz for the IIB fields, the expression (4.26) reproduces the superpotential shown in [25]. Moreover, we can verify that the more generic ansatz of [37] obeys the BPS flow equations derived from this superpotential. Therefore, via the TST transformation detailed in [37], this superpotential in fact describes the baryonic branch of the Klebanov-Strassler theory.

5 Discussion

The coset reduction of IIB supergravity on $T^{1,1}$ naturally yields five-dimensional gauged $\mathcal{N} = 4$ supergravity. We have analyzed three particular $\mathcal{N} = 2$ truncations of this reduction that are relevant to the conifold solution and its relatives. In particular, we have highlighted the difference between fake and real superpotentials and demonstrated the importance of the prepotential constraint (3.25) as a necessary condition for the supersymmetry of the solutions.

5.1 Fake superpotentials and the warped deformed conifold

There is a particularly relevant class of solutions within the Betti-hyper truncation which correspond to taking the ten-dimensional IIB background to be a warped product of $\mathbb{R}^{1,3}$ and the Ricci-flat metric on the deformed conifold. We can solve this system explicitly using the fake superpotential (4.12). In particular, this amounts to specifying the fields coming from the metric to take the form of the deformed conifold metric and solving the flow equations with the fake superpotential (4.12). This is a particularly nice example to

[3] We would like to thank I. Papadimitriou calling [46] to our attention.

study in the context of fake superpotentials as there exists a known non-supersymmetric solution to the flow equations derived from (4.12), found in [32].

In order to make the connection with previous solutions as transparent as possible we define the flux of the NS and RR three forms to be $j_0^1 = R$ and $j_0^2 = P$, respectively, and make the following KS-like parametrization for the other scalars in the three forms:

$$\begin{aligned} b_0^1 &= -\frac{R}{3}\left(\tilde{F} - \frac{1}{2}\right) - i\frac{P}{6}(f_{KS} - k_{KS}), \qquad & e_0^1 &= \frac{P}{3}(f_{KS} + k_{KS}), \\ b_0^2 &= -\frac{P}{3}\left(F_{KS} - \frac{1}{2}\right) + i\frac{R}{6}(\tilde{f} - \tilde{k}), \qquad & e_0^2 &= -\frac{R}{3}(\tilde{f} + \tilde{k}). \end{aligned} \tag{5.1}$$

The functions f_{KS}, k_{KS}, and F_{KS} are the standard functions in the KS ansatz, and the tilde-ed functions $\tilde{f}$, $\tilde{k}$, and $\tilde{F}$ are their S-dual analogs. Assuming a vanishing axion, $a = 0$, the equations reduce to two decoupled systems for $\{f_{KS}, k_{KS}, F_{KS}\}$ and $\{\tilde{f}, \tilde{k}, \tilde{F}\}$ and the solution is given by [32]:

$$\begin{aligned} f_{KS}(t) &= \frac{(-t\coth t + 1)}{2\sinh t}(-1 + \cosh t) \\ &\quad + C_1\left(-t + \frac{1}{2}\sinh t + \frac{t}{2(1+\cosh t)} + \frac{1}{2}\tanh\frac{t}{2}\right) - \frac{C_2}{1+\cosh t} + C_3, \\ k_{KS}(t) &= \frac{(-t\coth t + 1)}{2\sinh t}(1 + \cosh t) \\ &\quad + C_1\left(-t - \frac{1}{2}\sinh t - \frac{t}{2(-1+\cosh t)} + \frac{1}{2}\coth\frac{t}{2}\right) - \frac{C_2}{1-\cosh t} + C_3, \\ F_{KS}(t) &= \frac{1}{2} - \frac{t}{2\sinh t} + \frac{1}{2}C_1\left(\cosh t - \frac{t}{\sinh t}\right) + \frac{C_2}{\sinh t}, \end{aligned} \tag{5.2}$$

where C_1, C_2, and C_3 are integration constants. Additionally, the solution for the tilde-ed functions is exactly the same, but with different integration constants $\widetilde{C}_1$, $\widetilde{C}_2$, and $\widetilde{C}_3$.

The solution to the "KS" system (i.e. with $R = 0$) has already been solved in [32], yielding the above solution. The only non-singular solution in this sector is with $C_1 = C_2 = C_3 = 0$ which reduces exactly to the Klebanov-Strassler solution. In [32], it was also noted that the solution with $C_1 = 1$ and $C_2 = C_3 = 0$ corresponds to a background with $(0,3)$-flux which breaks supersymmetry by arguments from string theory [33]. In the present context we can verify explicitly that this solution is not supersymmetric by evaluating the two constraints $P_0^1 = 0$ and $P_0^2 = 0$. The explicit form of the constraints is not important. However we find that $P_0^1 \propto \widetilde{C}_1$ and $P_0^2 \propto C_1$. This means that solutions with C_1 or $\widetilde{C}_1$ non-vanishing are not supersymmetric. In particular, we see that the non-supersymmetric solution found in [32] is due to the superpotential (4.12) being a fake superpotential. In this case, solving the first order flow equations is insufficient in itself in guaranteeing supersymmetry, and the algebraic prepotential conditions must also be checked.

In fact, there is a subtlety in obtaining non-supersymmetric solutions using the fake superpotential. Ordinarily, solving the first order BPS equations will ensure a solution to the bosonic equations of motion. However, if the prepotential conditions are not satisfied, there is at least a possibility that the system may not solve the full set of equations of motion. In the present case, there would be a concern that the fluxes $j_0^1 = R$ and $j_0^2 = P$ along with non-trivial scalar profiles for e_0^i as well as the complex charged scalars b_0^i may source the graviphoton A_1. However, we have checked that the source for A_1 vanishes regardless of the choice of integration constants C_i and $\widetilde{C}_i$. Hence the solution is valid in both the supersymmetric and non-supersymmetric cases.

Note that since both F_3 and H_3 are nonzero, the five-form is sourced so that in addition to the flux term in the original KS solution, which is encoded in e^Z, the scalar k is, in general, non-zero as well. The explicit form of k is not so illuminating. However it vanishes for the non-singular solution when all integration constants are set to zero.

The notion of non-supersymmetric flux on warped Calabi-Yau backgrounds has been generalized in [47] to include $\mathrm{SU}(3) \times \mathrm{SU}(3)$ structure backgrounds. It would be interesting to connect those ideas to the existence of a fake superpotential in five dimensions for some more general truncation than those considered in this work.

5.2 Superpotential for the baryonic branch of the warped deformed conifold

One distinguishing feature of the baryonic branch of the warped deformed conifold is that away from the origin it breaks the $\mathbb{Z}_2$ symmetry which we call $\mathcal{I}$. The NS truncation we considered includes $\mathbb{Z}_2$ odd and even modes and within this theory there is a line of half-BPS solutions [36]. A very neat observation of [37] is that one can perform a certain TST transformation on this family of solutions and connect it to the family which is dual to the baryonic branch of the warped deformed conifold. Physically this latter solution space is more interesting since the whole family is dual to quantum field theory.

In principle it is possible to make a five dimensional domain wall ansatz and then perform the TST transformation on the full theory off-shell. This is quite an unwieldy operation, but it would interesting to work out a way to characterize this transformation covariantly in terms of the scalar cosets of the NS truncation.

One motivation for uncovering a superpotential for the baryonic branch is to study perturbation of the warped deformed conifold along the lines of [28, 29]. For those works the superpotential used only included $\mathbb{Z}_2$-even modes. But using the superpotential computed in this work, it should be possible to include $\mathbb{Z}_2$-odd modes in the NS sector and then use the TST transformation to map them to genuine perturbations of the warped deformed conifold.

Acknowledgments

P.S. would like to thank Ibrahima Bah and Alberto Faraggi for useful discussions. This work was supported in part by the US Department of Energy under grants DE-FG02-95ER40899 and DE-FG02-97ER41027. The work of N.H. was support by NSF grant PHY-0804450 and by the grant ANR-07-CEXC-006.

JHEP07(2012)098

A Details of the Betti-hyper truncation

Here we present some additional details of the Betti-hyper truncation. This truncation gives rise to $\mathcal{N}=2$ gauged supergravity coupled to one vector multiplet and three hypermultiplets. The bosonic fields in the gravity and vector multiplet are $(g_{\mu\nu}; A_1, k_{11}+k_{22}; u_3)$, and the 12 scalars in the hypermultiplet are $(u_1, k, e_0^i, \tau, \bar\tau, b_0^i, \bar b_0^i, v, \bar v)$.

A.1 Bosonic sector

The full Lagrangian is

$$\mathcal{L} = \mathcal{L}_{\rm gr} + \mathcal{L}_{\rm hyp} + \mathcal{L}_{\rm vec} + \mathcal{L}_{\rm g,kin} + \mathcal{L}_{\rm CS} + \mathcal{L}_{\rm pot}, \tag{A.1}$$

where the individual components are given below.

A.1.1 Hypermultiplet sector

The hypermultplet kinetic terms are

$$\begin{aligned}
\mathcal{L}_{\rm hyp} =& -e^{-4u_1}\mathcal{M}_{ij}\left[\frac{1}{2}e^{-4u_2}\hat g_{11}^i\wedge *\hat g_{11}^j + \frac{1}{2}e^{4u_2}\hat g_{12}^i\wedge *\hat g_{12}^j + 2(\hat f_1^i\wedge *\hat{\bar f}_1^j + \hat{\bar f}_1^i\wedge *\hat f_1^j)\right] \\
&-8du_1\wedge *du_1 - 4du_2\wedge *du_2 - 12du_3\wedge *du_3 - e^{-4u_2}(d|v|\wedge *d|v| + |v|^2 D\theta\wedge *D\theta) \\
&-\frac{1}{2}e^{-8u_1}K_1\wedge *K_1 - \frac{1}{2}d\phi\wedge *d\phi - \frac{1}{2}e^{2\phi}da\wedge *da,
\end{aligned} \tag{A.2}$$

with the relation

$$e^{2u_2} = \frac{1}{\cosh y}\,, \quad |v| = \tanh y\,, \tag{A.3}$$

where

$$\begin{aligned}
\hat g_{11}^i &= De_0, \\
\hat g_{12}^i &= (1+|v|^2)De_0^i - 4{\rm Im}\,(vDb_0^i), \\
\hat f_1^i &= Db_0^i - \frac{i}{2}\bar v De_0^i, \\
De_0^i &= de_0^i - j_0^i A_1, \\
Db_0^i &= db_0^i - 3ib_0^i A_1, \\
D\theta &= d\theta + 3A_1,
\end{aligned} \tag{A.4}$$

and

$$\mathcal{M} = e^{\phi}\begin{pmatrix} a^2 + e^{-2\phi} & -a \\ -a & 1 \end{pmatrix}. \tag{A.5}$$

Following [48], the generators of the solvable subalgebra of SO(4,3) may be taken as

$$\begin{aligned}
&H_1 = e_{11} - e_{55}\,, && H_2 = e_{22} - e_{66}\,, && H_3 = e_{33} - e_{77}\,, \\
&E_1{}^2 = -e_{21} + e_{56}\,, && E_1{}^3 = -e_{31} + e_{57}\,, && E_2{}^3 = -e_{32} + e_{67}\,, \\
&V^{12} = e_{16} - e_{25}\,, && V^{13} = e_{17} - e_{35}\,, && V^{23} = e_{27} - e_{36}\,, \\
&U_1^1 = e_{14} + e_{45}\,, && U_1^2 = e_{24} + e_{46}\,, && U_1^3 = e_{34} + e_{47}\,.
\end{aligned} \tag{A.6}$$

Using these, the metric on the hyperscalar coset is

$$
\begin{aligned}
-\frac{1}{8}\mathrm{Tr}dM\wedge *dM^{-1} = & \frac{1}{4}\big(d\phi_1^2+d\phi_2^2+d\phi_3^2\big) \\
&+\frac{1}{2}e^{-\phi_1+\phi_2}dx_4^2+\frac{1}{2}e^{\phi_3-\phi_1}\big(dx_5+x_4dx_6\big)^2+\frac{1}{2}e^{-\phi_2+\phi_3}dx_6^2 \\
&+\frac{1}{2}e^{\phi_1+\phi_2}\big(dx_7+x_{10}dx_{11}\big)^2+\frac{1}{2}e^{\phi_1+\phi_3}\big(dx_8-x_6dx_7+x_{10}(dx_{12}-x_6dx_{11})\big)^2 \\
&+\frac{1}{2}e^{\phi_2+\phi_3}\big(dx_9-x_4dx_8+(x_5+x_4x_6)(dx_7+x_{10}dx_{11})+(x_{11}-x_4x_{10})dx_{12}\big)^2 \\
&+\frac{1}{2}e^{\phi_1}dx_{10}^2+\frac{1}{2}e^{\phi_2}\big(dx_{11}-x_4dx_{10}\big)^2+\frac{1}{2}e^{\phi_3}\big(dx_{12}-x_5dx_{10}-x_6dx_{11}\big)^2,
\end{aligned}
\tag{A.7}
$$

where

$$
M = L^T L, \tag{A.8}
$$

and

$$
L = e^{\frac{\phi_1}{2}H_1}\cdot e^{\frac{\phi_2}{2}H_2}\cdot e^{\frac{\phi_3}{2}H_3}\cdot e^{x_4 {E_1}^2}\cdot e^{x_5 {E_1}^3}\cdot e^{x_6 {E_2}^3}\cdot e^{x_7 V_{12}}\cdot e^{x_8 V_{13}}\cdot e^{x_9 V_{23}}\cdot e^{x_{10}U_1^1}\cdot e^{x_{11}U_2^1}\cdot e^{x_{12}U_3^1}\,. \tag{A.9}
$$

The supergravity fields and the coset fields are related by the coordinate transformations

$$
\begin{aligned}
\phi_1 &= 2x-2\pi i\,, \\
\phi_2 &= -4u_1-\phi\,, \\
\phi_3 &= -4u_1+\phi\,, \\
x_4 &= e_0^1+2b_{0i}^1\,, \\
x_5 &= e_0^2-ae_0^1+2b_{0i}^2\,, \\
x_6 &= a\,, \\
x_7 &= e_0^1(1-\chi^2)-2b_{0i}^1(1+\chi^2)\,, \\
x_8 &= e_0^2(1-\chi^2)-2b_{0i}^2(1+\chi^2)\,, \\
x_9 &= k-4b_{0r}^1b_{0r}^2-2(e_0^1b_{0i}^2-e_0^2b_{0i}^1)+4\chi b_{0r}^1(e_0^2+2b_{0i}^2)-\chi^2(e_0^1+2b_{0i}^1)(e_0^2+2b_{0i}^2)\,, \\
x_{10} &= \sqrt{2}\chi\,, \\
x_{11} &= \sqrt{2}\big(-2b_{0r}^1+\chi(e_0^1+2b_{0i}^1)\big)\,. \\
x_{12} &= \sqrt{2}\big(-2b_{0r}^2+\chi(e_0^2+2b_{0i}^2)\big)\,.
\end{aligned}
\tag{A.10}
$$

In these coordinates, we note that

$$
\rho = \chi + ie^{-x} \tag{A.11}
$$

is an $\mathrm{SL}(2,\mathbb{R})$ factor within the coset which descends from the scalar v by the identification

$$
v = -\frac{(i-\rho)(i-\bar{\rho})}{1+|\rho|^2}. \tag{A.12}
$$

A.1.2 Vector multiplet sector

The vector multiplet kinetic terms and Chern-Simons terms are

$$\begin{aligned}
\mathcal{L}_{\rm vec} &= -12 du_3 \wedge * du_3, && \text{(A.13)}\\
\mathcal{L}_{\rm g,kin} &= -\frac{1}{2} e^{-8u_3} F_2 \wedge * F_2 - e^{4u_3} K_2 \wedge * K_2, && \text{(A.14)}\\
\mathcal{L}_{\rm CS} &= -A \wedge K_2 \wedge K_2. && \text{(A.15)}
\end{aligned}$$

This and the supersymmetry variations lead to the identification of the constrained scalars as

$$X^0 = e^{4u_3}, \qquad X^1 = e^{-2u_3}, \tag{A.16}$$

with $c_{011} = 2$. The field strengths are given by

$$F^0 = F_2, \qquad F^1 = -K_2. \tag{A.17}$$

A.1.3 Scalar potential

The scalar potential has several contributions which we distinguish for clarity:

$$\begin{aligned}
\mathcal{L}_{\rm pot} &= -\big(V_{\rm gr} + V_{F_{(3)}} + V_{F_{(5)}}\big), && \text{(A.18)}\\
V_{\rm gr} &= -12 e^{-4u_1-2u_2+2u_3}\big(1+|v|^2+e^{4u_2}\big) + 9|v|^2 e^{-4u_2+8u_3}\\
&\quad +2e^{-8u_1-4u_3}\big(e^{4u_2}+e^{-4u_2}(1-|v|^2)^2+2|v|^2\big), && \text{(A.19)}\\
V_{F_{(3)}} &= \frac{1}{2} e^{-4u_1+8u_3}\mathcal{M}_{ij}\Big(e^{-4u_2}\hat{j}^i_{01}\hat{j}^j_{01} + e^{4u_2}\hat{j}^i_{02}\hat{j}^j_{02} + 2(\hat{f}^i_0\bar{\hat{f}}^j_0 + \bar{\hat{f}}^i_0\hat{f}^j_0)\Big), && \text{(A.20)}\\
V_{F_{(5)}} &= \frac{1}{2} e^{2Z} e^{-8u_1+8u_3}\,, && \text{(A.21)}
\end{aligned}$$

where

$$\begin{aligned}
e^{4u_2} &= 1-|v|^2,\\
\hat{j}^i_{01} &= (1+|v|^2) j^i_0 - 4\,\mathrm{Im}\,(f^i_0 v),\\
\hat{j}^i_{02} &= -j^i_0,\\
\hat{f}^i_0 &= f^i_0 - \frac{i}{2} j^i_0 \bar{v},\\
e^Z &= Q - \frac{2i}{3}\epsilon_{ij}(f^i_0\bar{f}^j_0 - \bar{f}^i_0 f^j_0) + \epsilon_{ij}(j^i_0 e^j_0 - j^j_0 e^i_0).
\end{aligned} \tag{A.22}$$

A.2 Fermion variations

The supersymmetry variations of the KS-sector have been worked out in [22], where the fermions were organized according to mass eigenstates of the fluctuations on the AdS_5 background solution. However, in terms of $\mathcal{N}=2$ gauged supergravity, they are more naturally organized into variations appropriate for three hypermultiplets and one vector multiplet. This is accomplished by defining the following linear combinations of the AdS_5

mass eigenstates as the three hyperini and one gaugino

$$\begin{aligned}
\zeta^1 &= -\lambda^c, \\
\zeta^2 &= -\left(\frac{1+|\rho|^2}{1+\bar{\rho}^2}\right)\psi^{m=-3/2}, \\
\zeta^3 &= -\frac{1}{15}(2\psi^{m=11/2} - 3\psi^{m=-9/2}), \\
\xi^1 &= \frac{1}{5}(\psi^{m=11/2} + \psi^{m=-9/2}),
\end{aligned} \tag{A.23}$$

where $\rho = \chi + ie^{-x}$ is the $\mathrm{SL}(2,\mathbb{R})$ scalar descending from v.

Furthermore we define a phase rotated supersymmetry parameter ε' as

$$\varepsilon = \left(\frac{\rho+i}{\bar{\rho}-i}\right)^{1/2}\varepsilon'. \tag{A.24}$$

We similarly rotate the ζ^i, ξ^1 and the gravitino,

$$\begin{aligned}
\zeta^i &= \left(\frac{\rho+i}{\bar{\rho}-i}\right)^{1/2}\zeta^{i\prime}, \\
\xi^1 &= \left(\frac{\rho+i}{\bar{\rho}-i}\right)^{1/2}\xi^{1\prime}, \\
\psi_\alpha &= \left(\frac{\rho+i}{\bar{\rho}-i}\right)^{1/2}\psi'_\alpha.
\end{aligned} \tag{A.25}$$

With these identifications, the supersymmetry transformations are

$$\begin{aligned}
\delta\zeta^{1\prime} &= \left(-\frac{i}{2}\gamma\cdot\partial\phi - \frac{1}{2}e^{\phi}\gamma\cdot\partial a\right)\varepsilon' + \frac{ie^{-2u_1}}{4\tau_2}\bar{v}_i\Big[i(1+\bar{\rho}^2)\big(\gamma\cdot De_0^i - ie^{4u_3}j_0^i\big) \\
&\quad + (\bar{\rho}-i)^2\big(\gamma\cdot f_1^i - ie^{4u_3}f_0^i\big) - (\bar{\rho}+i)^2\big(\gamma\cdot\bar{f}_1^i - ie^{4u_3}\bar{f}_0^i\big)\Big](\varepsilon')^c, \\
\delta\zeta^{2\prime} &= \left(\frac{1}{2\rho_2}\gamma\cdot D\rho + \frac{3i}{2\rho_2}e^{4u_3}(1+\rho^2)\right)\varepsilon' - \frac{1}{2}e^{-2u_1}v_i\Big[\big(\gamma\cdot De_0^i - ie^{4u_3}j_0^i\big) \\
&\quad - i\frac{(\rho-i)(\bar{\rho}-i)}{1+|\rho|^2}\big(\gamma\cdot f_1^i - ie^{4u_3}f_0^i\big) + i\frac{(\rho+i)(\bar{\rho}+i)}{1+|\rho|^2}\big(\gamma\cdot\bar{f}_1^i - ie^{4u_3}\bar{f}_0^i\big)\Big](\varepsilon')^c, \\
\delta\zeta^{3\prime} &= \left[-\frac{i}{2}\gamma\cdot\partial u_1 - \frac{1}{8}e^{-4u_1}\gamma\cdot K_1 - \frac{i}{2}e^{-4u_1-2u_3} + \frac{i}{8}e^{-4u_1+4u_3}e^Z\right]\varepsilon' \\
&\quad - \frac{e^{-2u_1}}{16\tau_2}v_i\Big[(\bar{\rho}-i)^2\big(i\gamma\cdot f_1^i - e^{4u_3}f_0^i\big) - (\bar{\rho}+i)^2\big(i\gamma\cdot\bar{f}_1^i - e^{4u_3}\bar{f}_0^i\big) \\
&\qquad\qquad - (1+\bar{\rho}^2)\big(\gamma\cdot De_0^i + ie^{4u_3}j_0^i\big)\Big](\varepsilon')^c, \\
\delta\xi^{1\prime} &= \left[\frac{i}{2}\gamma\cdot\partial u_3 + \frac{1}{24}e^{-4u_3}\gamma\cdot(F_2 + e^{6u_3}K_2) + \frac{i}{6}e^{-4u_1-2u_3} - \frac{i}{4\rho_2}(1+|\rho|^2)e^{4u_3}\right. \\
&\quad \left. + \frac{i}{12}e^{-4u_1+4u_3}e^Z\right]\varepsilon' + \frac{e^{-2u_1+4u_3}}{12\rho_2}v_i\Big[(\bar{\rho}-i)^2 f_0^i - (\bar{\rho}+i)^2\bar{f}_0^i + i(1+\bar{\rho}^2)j_0^i\Big](\varepsilon')^c,
\end{aligned} \tag{A.26}$$

JHEP07(2012)098

where $D\rho \equiv d\rho - \frac{3}{2}(1+\rho^2)A_1$. The gravitino variation is

$$\begin{aligned}
\delta\psi'_\alpha = &\Big[\mathcal{D}_\alpha + \frac{i}{24}(\gamma_\alpha{}^{\beta\gamma} - 4\delta\alpha^\beta\gamma^\gamma)(e^{-4u_3}F_{\beta\gamma} - 2e^{2u_3}K_{2\,\beta\gamma}) \\
&+ \frac{1}{6}\gamma_\alpha\left(2e^{-4u_1-2u_3} + \frac{3}{2\tau_2}(1+|\rho|^2)e^{4u_3} - \frac{1}{2}e^{-4u_1+4u_3}e^Z\right)\Big]\varepsilon' \\
&-\frac{i}{12\rho_2}e^{-2u_1+4u_3}v_i\gamma_\alpha\left[(\bar\rho-i)^2 f_0^i - (\bar\rho+i)^2\bar f_0^i + i(1+\bar\rho^2)j_0^i\right](\varepsilon')^c, \qquad \text{(A.27)}
\end{aligned}$$

where the supercovariant derivative acts as

$$\begin{aligned}
\mathcal{D}_\alpha\varepsilon' = &\left(\nabla_\alpha - \frac{3i}{2}A_\alpha - \frac{i}{4}e^{-4u_1}K_{1\,\alpha} + \frac{i}{4}e^\phi\partial_\alpha a - \frac{i}{2\rho_2}\partial_\alpha\rho_1\right)\varepsilon' \\
&+ \frac{1}{4\rho_2}e^{-2u_1}v_i\Big((\bar\rho-i)^2 f_{1\,\alpha}^i - (\bar\rho+i)^2\bar f_{1\,\alpha}^i + i(1+\bar\rho^2)D_\alpha e_0^i\Big)(\varepsilon')^c. \qquad \text{(A.28)}
\end{aligned}$$

Here we have written the terms from the three-forms using the $\mathrm{SL}(2,\mathbb{R})$ vielbein v_i where $v_1 = -(ae^{\phi/2} + ie^{-\phi/2})$ and $v_2 = e^{\phi/2}$, such that the complex three-form takes the form

$$\frac{1}{\sqrt{\tau_2}}G_3 = v_i F_3^i. \qquad \text{(A.29)}$$

B Details of the Betti-vector truncation

The Betti-vector truncation yields $\mathcal{N}=2$ gauged supergravity coupled to two vector multiples and two hypermultiplets. The bosonic fields in the gravity and vector multiplets are $(g_{\mu\nu}; A_1, k_{11}, k_{12}; u_2, u_3)$, and the eight scalars in the hypermultiplet are $(u_1, k, \tau, \bar\tau, b_0^i, \bar b_0^i)$.

B.1 Bosonic sector

The full Lagrangian is

$$\mathcal{L} = \mathcal{L}_{\rm gr} + \mathcal{L}_{\rm hyp} + \mathcal{L}_{\rm vec} + \mathcal{L}_{\rm g,kin} + \mathcal{L}_{\rm CS} + \mathcal{L}_{\rm pot}, \qquad \text{(B.1)}$$

where the individual components are given below.

B.1.1 Hypermultiplet sector

The hypermultplet kinetic terms are

$$\begin{aligned}
\mathcal{L}_{\rm hyp} = &-4e^{-4u_1+\phi}\mathcal{M}_{ij}f_1^i\wedge *\bar f_1^j - 8du_1\wedge *du_1 - 4du_2\wedge *du_2 - 12du_3\wedge *du_3 \\
&-\frac{1}{2}e^{-8u_1}K_1\wedge *K_1 - \frac{1}{2}d\phi\wedge *d\phi - \frac{1}{2}e^{2\phi}da\wedge *da\,, \qquad \text{(B.2)}
\end{aligned}$$

where

$$\begin{aligned}
f_1^i &= Db_0^i, \\
K_1 &= Dk + 2\epsilon_{ij}[b_0^i D\bar b_0^j + \bar b_0^i Db_0^j], \\
Dk &= dk - QA_1 - 2k_{11} - 2k_{12}. \qquad \text{(B.3)}
\end{aligned}$$

Using the conventions of [48], the metric on the coset

$$\mathcal{M}_{\rm hyp} = \frac{\rm SO(4,2)}{\rm SO(4)\times SO(2)} \tag{B.4}$$

is

$$\begin{aligned} -\frac{1}{8}{\rm Tr}dM\wedge *dM^{-1} &= \frac{1}{4}\big(d\phi_1^2+d\phi_2^2\big)+\frac{1}{2}e^{-\phi_1+\phi_2}dx_1^2+\frac{1}{2}e^{\phi_1}\big(dx_3^2+dx_4^2\big) \\ &\quad +\frac{1}{2}e^{\phi_2}\big((d(x_5-x_1x_4)+x_4dx_1)^2+(d(x_6-x_1x_3)+x_3dx_1)^2\big) \\ &\quad +\frac{1}{2}e^{\phi_1+\phi_2}\big(dx_2+x_3dx_6+x_4dx_5\big)^2, \end{aligned} \tag{B.5}$$

which is related to $\mathcal{L}_{\rm hyp}$ by the field redefinitions

$$\begin{aligned} -\phi-4u_1 &= \phi_1, \\ \phi-4u_1 &= \phi_2, \\ a &= x_1, \\ k &= x_2+\frac{1}{2}x_1(x_3x_6+x_4x_5), \\ 2\sqrt{2}b_0^1 &= x_4-ix_3, \\ 2\sqrt{2}b_0^2 &= x_5-x_1x_4-i(x_6-x_1x_3)\,. \end{aligned} \tag{B.6}$$

B.1.2 Vector multiplet sector

The scalars in the vector multiplets have

$$\mathcal{L}_{\rm vec} = -12du_3\wedge *du_3-\frac{1}{4}d(4u_1+\phi)\wedge *d(4u_1+\phi). \tag{B.7}$$

The gauge kinetic terms are

$$\mathcal{L}_{\rm g,kin} = -\frac{1}{2}e^{-8u_3}F_2\wedge *F_2-\frac{1}{2}e^{-4u_2+4u_3}K_{21}\wedge *K_{21}-\frac{1}{2}e^{4u_2+4u_3}K_{22}\wedge *K_{22}. \tag{B.8}$$

There is also the Chern-Simons coupling

$$\mathcal{L}_{\rm CS} = -A_1\wedge K_{21}\wedge K_{22}. \tag{B.9}$$

From $\mathcal{L}_{\rm vec}$ we see that the two real scalars in the vector multiplets are u_3 and u_2 and they parameterize the manifold

$$\mathcal{M}_v = {\rm SO(1,1)\times SO(1,1)}. \tag{B.10}$$

The special geometry data for this case is given by the constrained scalars

$$X^0 = e^{4u_3}, \qquad X^1 = e^{2u_2-2u_3}, \qquad X^2 = e^{-2u_2-2u_3}, \tag{B.11}$$

with $c_{012}=1$, and the vector field strengths are given by

$$F^0 = F_2, \qquad F^1 = -K_{21}, \qquad F^2 = K_{22}. \tag{B.12}$$

JHEP07(2012)098

B.1.3 Scalar potential

The scalar potential has several contributions which we distinguish for clarity:

$$\begin{aligned}\mathcal{L}_{\text{pot}} &= -\big(V_{\text{gr}} + V_{F_{(3)}} + V_{F_{(5)}}\big), && \text{(B.13)}\\ V_{\text{gr}} &= -12e^{-4u_1-2u_2+2u_3}\big(1+e^{4u_2}\big) + 2e^{-8u_1-4u_3}\big(e^{4u_2}+e^{-4u_2}\big), && \text{(B.14)}\\ V_{F_{(3)}} &= 2e^{-4u_1+8u_3}\mathcal{M}_{ij}(f_0^i\overline{f}_0^j + \overline{f}_0^i f_0^j), && \text{(B.15)}\\ V_{F_{(5)}} &= \frac{1}{2}e^{2Z}e^{-8u_1+8u_3}, && \text{(B.16)}\end{aligned}$$

where

$$e^Z = Q - \frac{2i}{3}\epsilon_{ij}(f_0^i\bar{f}_0^j - \bar{f}_0^i f_0^j). \tag{B.17}$$

B.2 Fermion variations

The supersymmetry variations were worked out in [22]. We again organize the fermions into linear combinations appropriate to the $\mathcal{N}=2$ multiplet identifications as opposed to the mass eigenstates. In particular, we define

$$\begin{aligned}\zeta^1 &= -\lambda^c,\\ \zeta^2 &= \frac{1}{2}(\psi^{m=11/2} + \psi^{m=-9/2}),\\ \xi^1 &= -\psi^{m=-1/2},\\ \xi^2 &= -\frac{1}{15}(2\psi^{m=11/2} - 3\psi^{m=-9/2}),\end{aligned} \tag{B.18}$$

where ζ^i are the two hyperini and ξ^i are the gaugini. The supersymmetry transformations are then

$$\begin{aligned}\delta\zeta^1 &= \left(-\frac{i}{2}\gamma\cdot\partial\phi - \frac{1}{2}e^{\phi}\gamma\cdot\partial a\right)\varepsilon - e^{-2u_1}\bar{v}_i\left(i\gamma\cdot f_1^i + e^{4u_3}f_0^i\right)\varepsilon^c,\\ \delta\zeta^2 &= \left(-\frac{i}{2}\gamma\cdot\partial u_1 - \frac{1}{8}e^{-4u_1}\gamma\cdot K_1 - \frac{i}{4}e^{-4u_1-2u_3}\left(e^{-2u_2}+e^{2u_2}-\frac{1}{2}e^{6u_3}e^Z\right)\right)\varepsilon\\ &\quad -e^{-2u_1}v_i\left(i\gamma\cdot f_1^i + e^{4u_3}f_0^i\right)\varepsilon^c,\\ \delta\xi^1 &= \left(-\frac{i}{2}\gamma\cdot\partial u_2 + \frac{1}{16}e^{2u_3}\gamma\cdot(e^{-2u_2}K_{21} - e^{2u_2}K_{22}) - \frac{i}{4}e^{-4u_1-2u_3}(e^{-2u_2}-e^{2u_2})\right)\varepsilon,\\ \delta\xi^2 &= \left(-\frac{i}{2}\gamma\cdot\partial u_3 + \frac{1}{24}e^{-4u_3}\gamma\cdot\left(F_2 + \frac{1}{2}e^{-2u_2+6u_3}K_{21} + \frac{1}{2}e^{2u_2+6u_3}K_{22}\right) + \frac{i}{12}e^{-4u_1+4u_3}e^Z\right.\\ &\quad\left.+\frac{i}{12}(e^{-4u_1-2u_2-2u_3} + e^{-4u_1+2u_2-2u_3} - 6e^{4u_3})\right)\varepsilon - \frac{1}{3}e^{-2u_1+4u_3}v_i f_0^i\varepsilon^c.\end{aligned} \tag{B.19}$$

Finally, the gravitino variation is

$$\begin{aligned}\delta\psi_\alpha = \Bigg(&D_\alpha + \frac{i}{24}(\gamma_\alpha{}^{\beta\gamma} - 4\delta_\alpha^\beta\gamma^\gamma)(e^{-4u_3}F_{\beta\gamma} - e^{-2u_2+2u_3}K_{1\,\beta\gamma} - e^{2u_2+2u_3}K_{2\,\beta\gamma})\\ &+\frac{1}{6}\gamma_\alpha\left(e^{-4u_1-2u_2-2u_3} + e^{-4u_1+2u_2-2u_3} + 3e^{4u_3} - \frac{1}{2}e^{-4u_1+4u_3}e^Z\right)\Bigg)\varepsilon\\ &-\frac{i}{3}\gamma_\alpha e^{-2u_1+4u_3}v_i f_0^i\varepsilon^c,\end{aligned} \tag{B.20}$$

JHEP07(2012)098

where the covariant derivative acts on the supersymmetry parameter as

$$D_\alpha\varepsilon \equiv \left(\nabla_\alpha - \frac{3i}{2}A_\alpha - \frac{i}{4}e^{-4u_1}K_{1\,\alpha} + \frac{i}{4}e^{\phi}\partial_\alpha a\right)\varepsilon + e^{-2u_1}v_i f^i_\alpha \varepsilon^c. \tag{B.21}$$

C Details of the NS truncation

The final $\mathcal{N}=2$ truncation is to the NS sector of IIB supergravity. The resulting truncation has two vector multiplets and two hypermulitplets. The bosonic fields in the gravity and vector multiplets are $(g_{\mu\nu}; A_1, b_1^2, b_2^2; \phi+4u_1, u_3)$ and the eight scalars in the hypermultiplet are $(\phi-4u_1, u_2, c_0^2, e_0^2, b_0^2, \bar{b}_0^2, v, \bar{v})$.

C.1 Bosonic sector

The full Lagrangian is

$$\mathcal{L} = \mathcal{L}_{\rm gr} + \mathcal{L}_{\rm hyp} + \mathcal{L}_{\rm vec} + \mathcal{L}_{\rm g,kin} + \mathcal{L}_{\rm pot}\,. \tag{C.1}$$

The individual components are given below.

C.1.1 Hypermultiplet sector

The hypermulitplet kinetic terms are

$$\begin{aligned}\mathcal{L}_{\rm hyp} \;=\; & -\frac{1}{2}e^{-4(u_1+u_2)+\phi}\hat{g}_{11}\wedge *\hat{g}_{11} - \frac{1}{2}e^{-4(u_1-u_2)+\phi}\hat{g}_{12}\wedge *\hat{g}_{12} - 4e^{-4u_1+\phi}\hat{f}_1\wedge *\hat{\bar{f}}_1 \\ & -\frac{1}{4}d(4u_1-\phi)\wedge *d(4u_1-\phi) - 4du_2\wedge *du_2 - e^{-4u_2}Dv\wedge *D\bar{v},\end{aligned} \tag{C.2}$$

where

$$\begin{aligned}\hat{g}_{11} &= (1-|v|^2)Dc_0 + (1+|v|^2)De_0 - 4\mathrm{Im}\,(vDb_0),\\ \hat{g}_{12} &= Dc_0 - De_0,\\ \hat{f}_1 &= Db_0 + \frac{i}{2}\bar{v}(Dc_0 - De_0)\,.\end{aligned} \tag{C.3}$$

Additionally, note that since we have set the NS three form to zero we are suppressing the $SL(2,\mathbb{R})$ indices from the RR three-form in this truncation. Using the conventions of [48], the metric on the coset

$$\mathcal{M}_{\rm hyp} = \frac{SO(4,2)}{SO(4)\times SO(2)} \tag{C.4}$$

is

$$\begin{aligned}-\frac{1}{8}\mathrm{Tr}dM\wedge *dM^{-1} \;=\; & \frac{1}{4}\left(d\phi_1^2 + d\phi_2^2\right) + \frac{1}{2}e^{-\phi_1+\phi_2}dx_1^2 + \frac{1}{2}e^{\phi_1}\left(dx_3^2 + dx_4^2\right)\\ & +\frac{1}{2}e^{\phi_2}\left((d(x_5 - x_1x_4) + x_4dx_1)^2 + (d(x_6 - x_1x_3) + x_3dx_1)^2\right)\\ & +\frac{1}{2}e^{\phi_1+\phi_2}\left(dx_2 + x_3dx_6 + x_4dx_5\right)^2.\end{aligned} \tag{C.5}$$

This is related to $\mathcal{L}_{\rm hyp}$ by the field redefinitions

$$\begin{aligned}
\phi - 4u_1 &= \phi_2, \\
-4u_2 &= \phi_1, \\
\sqrt{2}\, v &= x_4 - ix,3 \\
2\sqrt{2} b_0 &= x_6 - x_1x_3 - i(x_5 - x_1x_4), \\
c_0 - e_0 &= -x_1, \\
c_0 + e_0 &= x_2 + \frac{1}{2}x_1(x_3^2 + x_4^2)\,.
\end{aligned} \tag{C.6}$$

C.1.2 Vector multiplet sector

The scalars in the vector multiplet have kinetic terms

$$\mathcal{L}_{\rm vec} = -12du_3 \wedge *du_3 - \frac{1}{4}d(4u_1 + \phi) \wedge *d(4u_1 + \phi). \tag{C.7}$$

We see that the two real scalars in the vector multiplets are u_3 and $4u_1 + \phi$ and they parameterize the manifold

$$\mathcal{M}_v = \mathrm{SO}(1,1) \times \mathrm{SO}(1,1). \tag{C.8}$$

In terms of A_1, b_1 and b_2, the gauge kinetic terms are

$$\mathcal{L}_{\rm g,kin} = -\frac{1}{2}e^{-8u_3}F_2 \wedge *F_2 - \frac{1}{2}e^{4u_1-4u_3+\phi}g_3 \wedge *g_3 - \frac{1}{2}e^{4u_1+4u_3+\phi}g_2 \wedge *g_2. \tag{C.9}$$

We may integrate out the tensor field by dualizing $\hat{g}_3 = db_2$ into a vector field. This is done by adding

$$\Delta\mathcal{L} = \tilde{b}_1 \wedge d\hat{g}_3 \tag{C.10}$$

to the Lagrangian. This results in

$$\mathcal{L}_{\rm g,kin} = -\frac{1}{2}e^{-8u_3}F_2 \wedge *F_2 - \frac{1}{2}e^{4u_1+4u_3+\phi}g_2 \wedge *g_2 - \frac{1}{2}e^{-4u_1+4u_3-\phi}\tilde{g}_2 \wedge *\tilde{g}_2, \tag{C.11}$$

along with a Chern-Simons term

$$\mathcal{L}_{\rm CS} = \tilde{g}_2 \wedge b_1 \wedge F_2, \tag{C.12}$$

where $\tilde{g}_2 = d\tilde{b}_1$.

The special geometry data for this case is very similar to the Betti-vector sector and is given by the following constrained scalars

$$X^0 = e^{4u_3}, \qquad X^1 = e^{-2u_1-2u_3-\phi/2}, \qquad X^2 = e^{2u_1-2u_3+\phi/2}, \tag{C.13}$$

with $c_{012} = 1$. The vector field strengths are given by

$$F^0 = F_2, \qquad F^1 = -g_2, \qquad F^2 = \tilde{g}_2. \tag{C.14}$$

C.1.3 Scalar potential

The scalar potential has two contributions which we distinguish for clarity:

$$\begin{aligned}
\mathcal{L}_{\rm pot} &= -\big(V_{\rm gr} + V_{F_{(3)}}\big), && \text{(C.15)}\\
V_{\rm gr} &= -12e^{-4u_1-2u_2+2u_3}\big(1+|v|^2+e^{4u_2}\big) + 9|v|^2 e^{-4u_2+8u_3}\\
&\quad -2e^{-8u_1-4u_3}\big(e^{4u_2}+e^{-4u_2}(1-|v|^2)^2+2|v|^2\big), && \text{(C.16)}\\
V_{F_{(3)}} &= \frac{1}{2}e^{-4u_1+8u_3+\phi}\Big(8|\hat{f}_0|^2 + e^{4u_2}P^2 + e^{-4u_2}\big(P(|v|^2-1)+4\,\mathrm{Im}\,(\hat{f}_0 v)\big)^2\Big), && \text{(C.17)}
\end{aligned}$$

where

$$\hat{f}_0 = f_0 - \frac{i}{2}P\overline{v}. \tag{C.18}$$

C.2 Fermion variations

In order to reduce from $\mathcal{N}=4$ to $\mathcal{N}=2$, we restrict the transformation parameter ε. To do this we make the identification

$$\varepsilon = -i\sigma_2\varepsilon^c, \tag{C.19}$$

where the conjugation is defined by $\varepsilon^c = \gamma_0 C\varepsilon^*$. Additionally we make the same identification for all of the fermions. Given this identification, the components of

$$\varepsilon = \begin{bmatrix}\varepsilon_1\\ \varepsilon_2\end{bmatrix} \tag{C.20}$$

satisfy the symplectic-Majorana condition, $\varepsilon_1 = -\varepsilon_2^c$, which can be expressed as

$$\varepsilon^i \equiv \epsilon^{ij}\varepsilon_j = \varepsilon_i^c. \tag{C.21}$$

We can then identify the two hyperini as

$$\begin{aligned}
\zeta^1 &= \lambda - 2(\psi^{(5)}+\psi^{(7)}),\\
\zeta^2 &= \psi^{(5)} - \psi^{(7)},
\end{aligned} \tag{C.22}$$

and the gaugini are given by some linear combination of

$$\begin{aligned}
\chi^1 &= \lambda + 2(\psi^{(5)}+\psi^{(7)}),\\
\chi^2 &= \psi^{(5)} + \psi^{(7)} + 2\psi^{(9)}.
\end{aligned} \tag{C.23}$$

The $\mathcal{N}=2$ susy transformations are then

$$\begin{aligned}
\delta\zeta^1 = &\Big[-\frac{i}{2}\gamma\cdot\partial(\phi-4u_1) + \frac{1}{2}e^{\phi/2-2u_1}(e^{-2u_2}(1-|v|^2)+e^{2u_2})(\gamma\cdot g_1 + 2ie^{-2u_1-2u_3-\phi/2})\\
&+\frac{1}{2}e^{\phi/2-2u_1}(e^{-2u_2}(1+|v|^2)-e^{2u_2})(\gamma\cdot h_1 - iPe^{4u_3})\\
&+ie^{\phi/2-2u_1-2u_2}[v(\gamma\cdot f_1 - ie^{4u_3}f_0) - \bar{v}(\gamma\cdot\bar{f}_1 - ie^{4u_3}\bar{f}_0)]\Big]\,\varepsilon\\
&-\bar{v}e^{\phi/2-2u_1}\Big[(\gamma\cdot g_1 + 2ie^{-2u_1-2u_3-\phi/2}) - (\gamma\cdot h_1 - iPe^{4u_3}) + 2i(\gamma\cdot f_1 - ie^{4u_3}f_0)\Big]\varepsilon^c,
\end{aligned}$$

$$\begin{aligned}
\delta\zeta^2 =& \Big[-i\gamma\cdot\partial u_2 - \frac{1}{4}e^{\phi/2-2u_1}(e^{-2u_2}(1-|v|^2)+e^{2u_2})(\gamma\cdot g_1 + 2ie^{-2u_1-2u_3-\phi/2}) \\
& - \frac{1}{4}e^{\phi/2-2u_1}(e^{-2u_2}(1+|v|^2)-e^{2u_2})(\gamma\cdot h_1 - iPe^{4u_3}) \\
& - \frac{i}{2}e^{\phi/2-2u_1-2u_2}[v(\gamma\cdot f_1 - ie^{4u_3}f_0) - \bar{v}(\gamma\cdot\bar{f}_1 - ie^{4u_3}\bar{f}_0)]\Big]\varepsilon \\
& + \frac{i}{2}e^{2u_2}(\gamma\cdot\overline{Dv} + 3\bar{v}e^{4u_3})\varepsilon^c, \\
\delta\chi^1 =& \Big[-\frac{i}{2}\gamma\cdot\partial(\phi+4u_1) - \frac{1}{12}e^{-\phi/2-2u_1+2u_3}\gamma\cdot\tilde{g}_2 - \frac{1}{4}e^{\phi/2+2u_1+2u_3}\gamma\cdot g_2 \\
& - ie^{-4u_1-2u_3}(e^{-2u_2}(1-|v|^2)+e^{2u_2})\Big]\varepsilon + 2i\bar{v}e^{-4u_1-2u_3}\varepsilon^c, \\
\delta\chi^2 =& \Big[3i\gamma\cdot\partial u_3 + \frac{1}{4}e^{-4u_3}\gamma\cdot F_2 - \frac{1}{24}e^{-\phi/2-2u_1+2u_3}\gamma\cdot\tilde{g}_2 + \frac{1}{8}e^{\phi/2+2u_1+2u_3}\gamma\cdot g_2 \\
& + \frac{i}{2}\Big(e^{-4u_1-2u_3}(e^{-2u_2}(1-|v|^2)+e^{2u_2}) - 6e^{4u_3} \\
& + e^{\phi/2-2u_1+4u_3}[e^{-2u_2}((1+|v|^2)P + 2ivf_0 - 2i\bar{v}\bar{f}_0) - e^{2u_2}P]\Big)\Big]\varepsilon \\
& - \left[\bar{v}e^{-4u_1-2u_3} - 3\bar{v}e^{-2u_2+4u_3} + 2\left(f_0 - \frac{i}{2}\bar{v}P\right)e^{\phi/2-2u_1+4u_3}\right]\varepsilon^c,
\end{aligned} \tag{C.24}$$

along with

$$\begin{aligned}
\delta\psi_\alpha =& \Big[D_\alpha + \frac{i}{24}(\gamma_\alpha{}^{\beta\gamma} - 4\delta_\alpha^\beta\gamma^\gamma)\Big(e^{-4u_3}F_{\beta\gamma} + e^{-\phi/2-2u_1+2u_3}\tilde{g}_{2\,\beta\gamma} - e^{\phi/2+2u_1+2u_3}g_{2\,\beta\gamma}\Big) \\
& + \frac{1}{6}\gamma_\alpha\Big(e^{-4u_1-2u_3}(e^{-2u_2}(1-|v|^2)+e^{2u_2}) + 3e^{4u_3} \\
& - \frac{1}{2}e^{\phi/2-2u_1+4u_3}(e^{-2u_2}((1+|v|^2)P + 2ivf_0 - 2i\bar{v}\bar{f}_0) - e^{2u_2}P)\Big)\Big]\varepsilon \\
& + \frac{i}{6}\gamma_\alpha\left[(2e^{-4u_1-2u_3} + 3e^{-2u_2+4u_3})i\bar{v} - 2e^{\phi/2-2u_1+4u_3}\left(f_0 - \frac{i}{2}\bar{v}P\right)\right]\varepsilon^c.
\end{aligned} \tag{C.25}$$

D Field redefinitions and conventions

Here we make explicit the relations between our reduction ansatz and those presented in refs. [4] and [22]. Our ansatz follows the conventions of ref. [4] for the metric and the five-form ansatz. However, we have chosen a manifestly $\mathrm{SL}(2,\mathbb{R})$ covariant notation for the three-form. Our three-form ansatz is related to that of ref. [4] according to

$$\begin{aligned}
& b_2^1 = B_2 + \frac{1}{2}bF_2, \quad && b_1^1 = B_1, \quad && 3ib_0^1 = M_0, \quad && c_0^1 = b, \quad && e_0^1 = \tilde{b}, \quad && j_1^1 = 0, \\
& b_2^2 = C_2 + \frac{1}{2}cF_2, \quad && b_1^2 = C_1, \quad && 3ib_0^2 = N_0 + aM_0, \quad && c_0^2 = c, \quad && e_0^2 = \tilde{c}, \quad && j_0^2 = P.
\end{aligned} \tag{D.1}$$

Additionally, the conventions here are consistent with that of the three-form in ref. [22]. But for the metric and five-form the relations are given by

$$\begin{aligned} u_1 &= \frac{1}{2}(B_1 + B_2), \quad u_2 = \frac{1}{2}(B_1 - B_2), \quad u_3 = -\frac{1}{6}(B_1 + B_2) - \frac{1}{3}C, \quad v = \alpha, \\ e^Z &= 4 + \phi_0, \quad K_1 = \mathbb{A}_1, \quad K_{21} = p_{21}, \quad K_{22} = p_{22}. \end{aligned} \tag{D.2}$$

References

[1] I.R. Klebanov and E. Witten, *Superconformal field theory on three-branes at a Calabi-Yau singularity*, Nucl. Phys. **B 536** (1998) 199 [hep-th/9807080] [INSPIRE].

[2] I.R. Klebanov and M.J. Strassler, *Supergravity and a confining gauge theory: Duality cascades and χSB resolution of naked singularities*, JHEP **08** (2000) 052 [hep-th/0007191] [INSPIRE].

[3] S.B. Giddings, S. Kachru and J. Polchinski, *Hierarchies from fluxes in string compactifications*, Phys. Rev. **D 66** (2002) 106006 [hep-th/0105097] [INSPIRE].

[4] I. Bena, G. Giecold, M. Graña, N. Halmagyi and F. Orsi, *Supersymmetric Consistent Truncations of IIB on $T^{1,1}$*, JHEP **04** (2011) 021 [arXiv:1008.0983] [INSPIRE].

[5] D. Cassani and A.F. Faedo, *A Supersymmetric consistent truncation for conifold solutions*, Nucl. Phys. **B 843** (2011) 455 [arXiv:1008.0883] [INSPIRE].

[6] B. de Wit, H. Nicolai and N.P. Warner, *The Embedding of Gauged $N = 8$ Supergravity Into $d = 11$ Supergravity*, Nucl. Phys. **B 255** (1985) 29 [INSPIRE].

[7] H. Nastase, D. Vaman and P. van Nieuwenhuizen, *Consistent nonlinear KK reduction of 11d supergravity on $AdS_7 \times S^4$ and selfduality in odd dimensions*, Phys. Lett. **B 469** (1999) 96 [hep-th/9905075] [INSPIRE].

[8] A.K. Kashani-Poor, *Nearly Kähler Reduction*, JHEP **11** (2007) 026 [arXiv:0709.4482] [INSPIRE].

[9] T. House and E. Palti, *Effective action of (massive) IIA supergravity on manifolds with* SU(3) *structure*, Phys. Rev. **D 72** (2005) 026004 [hep-th/0505177] [INSPIRE].

[10] D. Cassani and A.K. Kashani-Poor, *Exploiting $N = 2$ in consistent coset reductions of type IIA*, Nucl. Phys. **B 817** (2009) 25 [arXiv:0901.4251] [INSPIRE].

[11] J.P. Gauntlett, S. Kim, O. Varela and D. Waldram, *Consistent supersymmetric Kaluza-Klein truncations with massive modes*, JHEP **04** (2009) 102 [arXiv:0901.0676] [INSPIRE].

[12] J.P. Gauntlett and O. Varela, *Universal Kaluza-Klein reductions of type IIB to $N = 4$ supergravity in five dimensions*, JHEP **06** (2010) 081 [arXiv:1003.5642] [INSPIRE].

[13] J.T. Liu, P. Szepietowski and Z. Zhao, *Consistent massive truncations of type-IIB supergravity on Sasaki-Einstein manifolds*, Phys. Rev. **D 81** (2010) 124028 [arXiv:1003.5374] [INSPIRE].

JHEP07(2012)098

[14] K. Skenderis, M. Taylor and D. Tsimpis, *A Consistent truncation of IIB supergravity on manifolds admitting a Sasaki-Einstein structure*, *JHEP* **06** (2010) 025 [arXiv:1003.5657] [INSPIRE].

[15] D. Cassani, G. Dall'Agata and A.F. Faedo, *Type IIB supergravity on squashed Sasaki-Einstein manifolds*, *JHEP* **05** (2010) 094 [arXiv:1003.4283] [INSPIRE].

[16] A. Donos, J.P. Gauntlett, N. Kim and O. Varela, *Wrapped M5-branes, consistent truncations and AdS/CMT*, *JHEP* **12** (2010) 003 [arXiv:1009.3805] [INSPIRE].

[17] E.Ó Colgáin and O. Varela, *Consistent reductions from $D = 11$ beyond Sasaki-Einstein*, *Phys. Lett.* **B 703** (2011) 180 [arXiv:1106.4781] [INSPIRE].

[18] T. Danckaert, J. Louis, D. Martinez-Pedrera, B. Spanjaard and H. Triendl, *The $\mathcal{N} = 4$ effective action of type IIA supergravity compactified on* SU(2)*-structure manifolds*, *JHEP* **08** (2011) 024 [arXiv:1104.5174] [INSPIRE].

[19] I. Bah, A. Faraggi, J.I. Jottar, R.G. Leigh and L.A. Pando Zayas, *Fermions and $D = 11$ Supergravity On Squashed Sasaki-Einstein Manifolds*, *JHEP* **02** (2011) 068 [arXiv:1008.1423] [INSPIRE].

[20] I. Bah, A. Faraggi, J.I. Jottar and R.G. Leigh, *Fermions and Type IIB Supergravity On Squashed Sasaki-Einstein Manifolds*, *JHEP* **01** (2011) 100 [arXiv:1009.1615] [INSPIRE].

[21] J.T. Liu, P. Szepietowski and Z. Zhao, *Supersymmetric massive truncations of IIB supergravity on Sasaki-Einstein manifolds*, *Phys. Rev.* **D 82** (2010) 124022 [arXiv:1009.4210] [INSPIRE].

[22] J.T. Liu and P. Szepietowski, *Supersymmetry of consistent massive truncations of IIB supergravity*, [arXiv:1103.0029] [INSPIRE].

[23] I.R. Klebanov and A.A. Tseytlin, *Gravity duals of supersymmetric* $\mathrm{SU}(N) \times \mathrm{SU}(N + M)$ *gauge theories*, *Nucl. Phys.* **B 578** (2000) 123 [hep-th/0002159] [INSPIRE].

[24] S.S. Gubser, A.A. Tseytlin and M.S. Volkov, *NonAbelian $4 - d$ black holes, wrapped five-branes and their dual descriptions*, *JHEP* **09** (2001) 017 [hep-th/0108205] [INSPIRE].

[25] G. Papadopoulos and A.A. Tseytlin, *Complex geometry of conifolds and five-brane wrapped on two sphere*, *Class. Quant. Grav.* **18** (2001) 1333 [hep-th/0012034] [INSPIRE].

[26] L.A. Pando Zayas and A.A. Tseytlin, *3-branes on resolved conifold*, *JHEP* **11** (2000) 028 [hep-th/0010088] [INSPIRE].

[27] L.A. Pando Zayas and A.A. Tseytlin, *3-branes on spaces with $R \times S^2 \times S^3$ topology*, *Phys. Rev.* **D 63** (2001) 086006 [hep-th/0101043] [INSPIRE].

[28] V. Borokhov and S.S. Gubser, *Non-supersymmetric deformations of the dual of a confining gauge theory*, *JHEP* **05** (2003) 034 [hep-th/0206098] [INSPIRE].

[29] I. Bena, M. Graña and N. Halmagyi, *On the Existence of Meta-stable Vacua in Klebanov-Strassler*, *JHEP* **09** (2010) 087 [arXiv:0912.3519] [INSPIRE].

[30] J.P. Gauntlett, J. Sonner and T. Wiseman, *Holographic superconductivity in M-theory*, *Phys. Rev. Lett.* **103** (2009) 151601 [arXiv:0907.3796] [INSPIRE].

[31] J.P. Gauntlett, J. Sonner and T. Wiseman, *Quantum Criticality and Holographic Superconductors in M-theory*, *JHEP* **02** (2010) 060 [arXiv:0912.0512] [INSPIRE].

JHEP07(2012)098

[32] S. Kuperstein and J. Sonnenschein, *Analytic non-supersymmtric background dual of a confining gauge theory and the corresponding plane wave theory of hadrons*, *JHEP* **02** (2004) 015 [hep-th/0309011] [INSPIRE].

[33] M. Graña and J. Polchinski, *Gauge/gravity duals with holomorphic dilaton*, *Phys. Rev.* **D 65** (2002) 126005 [hep-th/0106014] [INSPIRE].

[34] A.H. Chamseddine and M.S. Volkov, *Non-Abelian BPS monopoles in $N = 4$ gauged supergravity*, *Phys. Rev. Lett.* **79** (1997) 3343 [hep-th/9707176] [INSPIRE].

[35] J.M. Maldacena and C. Núñez, *Towards the large N limit of pure $N = 1$ super Yang-Mills Theory*, *Phys. Rev. Lett.* **86** (2001) 588 [hep-th/0008001] [INSPIRE].

[36] A. Butti, M. Graña, R. Minasian, M. Petrini and A. Zaffaroni, *The Baryonic branch of Klebanov-Strassler solution: a supersymmetric family of* SU(3) *structure backgrounds*, *JHEP* **03** (2005) 069 [hep-th/0412187] [INSPIRE].

[37] J. Maldacena and D. Martelli, *The Unwarped, resolved, deformed conifold: fivebranes and the baryonic branch of the Klebanov-Strassler theory*, *JHEP* **01** (2010) 104 [arXiv:0906.0591] [INSPIRE].

[38] J. Schön and M. Weidner, *Gauged $N = 4$ supergravities*, *JHEP* **05** (2006) 034 [hep-th/0602024] [INSPIRE].

[39] R. Corrado, M. Günaydin, N.P. Warner and M. Zagermann, *Orbifolds and flows from gauged supergravity*, *Phys. Rev.* **D 65** (2002) 125024 [hep-th/0203057] [INSPIRE].

[40] N. Halmagyi, C. Römelsberger and N.P. Warner, *Inherited duality and quiver gauge theory*, *Adv. Theor. Math. Phys.* **10** (2006) 159 [hep-th/0406143] [INSPIRE].

[41] K.A. Intriligator, *Bonus symmetries of $N = 4$ super Yang-Mills correlation functions via AdS duality*, *Nucl. Phys.* **B 551** (1999) 575 [hep-th/9811047] [INSPIRE].

[42] R. D'Auria and S. Ferrara, *On fermion masses, gradient flows and potential in supersymmetric theories*, *JHEP* **05** (2001) 034 [hep-th/0103153] [INSPIRE].

[43] A. Ceresole, G. dall'Agata, R. Kallosh and A. van Proeyen, *Hypermultiplets, domain walls and supersymmetric attractors*, *Phys. Rev.* **64** (2001) 104006 [arXiv:hep-th/0104056] [INSPIRE].

[44] K. Behrndt, M. Cvetič and W.A. Sabra, *Non-extreme black holes of five-dimensional $N = 2$ AdS supergravity*, *Nuclear Physics B* **553** (1999) 317 [arXiv:hep-th/9810227] [INSPIRE].

[45] K. Behrndt, C. Herrmann, J. Louis and S. Thomas, *Domain walls in five-dimensional supergravity with nontrivial hypermultiplets*, *JHEP* **01** (2001) 011 [hep-th/0008112] [INSPIRE].

[46] C. Hoyos-Badajoz, C. Núñez and I. Papadimitriou, *Remarks on the String dual to $N = 1$ SQCD*, *Phys. Rev.* **D 78** (2008) 086005 [arXiv:0807.3039] [INSPIRE].

[47] D. Lüst, F. Marchesano, L. Martucci and D. Tsimpis, *Generalized non-supersymmetric flux vacua*, *JHEP* **11** (2008) 021 [arXiv:0807.4540] [INSPIRE].

[48] H. Lü, C.N. Pope and K.S. Stelle, *M theory/heterotic duality: a Kaluza-Klein perspective*, *Nucl. Phys.* **B 548** (1999) 87 [hep-th/9810159] [INSPIRE].

JHEP07(2012)098