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Asymptotic Stability of Landau Solutions to Navier–Stokes System

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Abstract

It is known that the three-dimensional Navier–Stokes system for an incompressible fluid in the whole space has a one parameter family of explicit stationary solutions that are axisymmetric and homogeneous of degree -1. We show that these solutions are asymptotically stable under any L^2-perturbation.

1. Introduction

The initial value problem for the Navier–Stokes system describing a motion of a viscous incompressible fluid in the whole three-dimensional space has the form

$$u_t - \Delta u + (u \cdot \nabla)u + \nabla p = F, \quad (x,t) \in \mathbb{R}^3 \times (0,\infty) \tag{1.1}$$

$$\operatorname{div} u = 0, \tag{1.2}$$

$$u(x,0) = u_0(x). \tag{1.3}$$

Here, the velocity $u = (u_1, u_2, u_3)$ and the scalar pressure p are unknown. Moreover, u_0 and F denote a given initial velocity and a given external force, respectively.

It is well known, since the pioneering work of Leray [18], that for each $u_0 \in \big(L^2(\mathbb{R}^3)\big)^3$ satisfying $\operatorname{div} u_0 = 0$ and for $F \equiv 0$, problem (1.1) possesses a weak solution, satisfying a suitable energy inequality (see the monograph [28] for analogous results with nonzero F). The uniqueness and the regularity of weak solutions still remain open. In [18], Leray posed a question whether a weak solution $u = u(x,t)$ tends to zero in $L^2(\mathbb{R}^3)$ as $t \to \infty$, which was affirmatively solved

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by KATO [12] in the case of strong solutions and MASUDA [21] for weak solutions satisfying a strong energy inequality. Next, SCHONBEK [24] obtained decay rates for the L^2-norm of weak solutions using elementary properties of the Fourier transform. The ideas from [24] were developed and generalized by WIEGNER [30]. We refer the reader to monographs [17,28] for results on the existence of weak and strong solutions to (1.1)–(1.3) and to the review article [7] for a discussion of recent results of the large time behavior of solutions.

If $F \equiv 0$ in problem (1.1)–(1.3), the L^2-decay of weak solutions can be understood as the global asymptotic stability in $L^2(\mathbb{R}^3)$ of the trivial stationary solution $(u, p) = (0, 0)$. In this work, we address analogous questions on the global asymptotic stability of the family of stationary solutions to (1.1)–(1.2) given by the following explicit formulas

$$\begin{aligned} v_c^1(x) &= 2\frac{c|x|^2 - 2x_1|x| + cx_1^2}{|x|(c|x| - x_1)^2}, \quad & v_c^2(x) &= 2\frac{x_2(cx_1 - |x|)}{|x|(c|x| - x_1)^2}, \\ v_c^3(x) &= 2\frac{x_3(cx_1 - |x|)}{|x|(c|x| - x_1)^2}, & p_c(x) &= 4\frac{cx_1 - |x|}{|x|(c|x| - x_1)^2}, \end{aligned} \tag{1.4}$$

where $|x| = \sqrt{x_1^2 + x_2^2 + x_3^2}$ and c is an arbitrary constant such that $|c| > 1$. The functions v_c and p_c defined in (1.4) satisfy (1.1) with $F \equiv 0$ in the pointwise sense for every $x \in \mathbb{R}^3 \backslash \{0\}$. On the other hand, if one treats them as a distributional or generalized solution to (1.1) in the whole $\mathbb{R}^3$, they correspond to the very singular external force $F = (b(c)\delta_0, 0, 0)$, where the parameter $b \neq 0$ depends on c and δ_0 stands for the Dirac measure. Indeed, in [5, Proposition 2.1.] (see also [1, p. 206]), it was shown that for every test function $\varphi \in C_c^\infty(\mathbb{R}^3)$ the following equalities hold true

$$\int_{\mathbb{R}^3} v_c(x) \cdot \nabla\varphi(x)\, \mathrm{d}x = 0$$

and

$$\int_{\mathbb{R}^3} \left(\nabla v_c^k \cdot \nabla\varphi - v_c^k v_c \cdot \nabla\varphi - p_c \frac{\partial}{\partial x_k}\varphi \right) \mathrm{d}x = \begin{cases} b(c)\varphi(0) & \text{if } k = 1, \\ 0 & \text{if } k = 2, 3, \end{cases}$$

where

$$b(c) = \frac{8\pi c}{3(c^2 - 1)} \left(2 + 6c^2 - 3c(c^2 - 1) \log\left(\frac{c+1}{c-1} \right) \right). \tag{1.5}$$

In particular, the function $b = b(c)$ is decreasing on $(-\infty, -1)$ and $(1, +\infty)$. Moreover, $\lim_{c\to 1} b(c) = +\infty$, $\lim_{c\to -1} b(c) = -\infty$ and $\lim_{|c|\to\infty} b(c) = 0$.

These explicit stationary solutions to (1.1)–(1.2) were first calculated by LANDAU [15] and now they can be found in standard textbooks (see for example [16, p. 82] and [1, p. 206]). Let us also recall that the stationary solutions (1.4) were also independently found by SQUIRE [26] and discussed in [5,29] from a slightly different point of view. The main idea of Landau's calculation is that if we impose the additional axi-symmetry requirement, the stationary Navier–Stokes system

$$-\Delta u + (u\cdot\nabla)u + \nabla p = 0, \qquad \operatorname{div} u = 0, \tag{1.6}$$

reduces to a system of ODEs which can be solved explicitly in terms of elementary functions. Moreover, Šverák [27] proved recently that even if we drop the requirement of axi-symmetry, then the Landau solutions (1.4) are still the only solutions of (1.6) which are invariant under the natural scaling. More precisely, he proved that if $u : \mathbb{R}^3\backslash\{0\} \to \mathbb{R}^3$ is a non-trivial smooth solution of (1.6) satisfying $\lambda u(\lambda x) = u(x)$ for all $x \in \mathbb{R}^3\backslash\{0\}$ and each $\lambda > 0$, then $(u, p) = (v_c, p_c)$ is given by formulas (1.4) (modulo a rotation of $\mathbb{R}^3$).

The goal of this work is to show that problem (1.1)–(1.3) has a weak solution for every initial datum of the form $u_0 = v_c + w_0$, where $w_0 \in L^2(\mathbb{R}^3)$ and the external force $F = (b(c)\delta_0, 0, 0)$ with $b(c)$ defined in (1.5), provided $|c|$ is sufficiently large. Moreover, this solution converges, as $t \to \infty$, towards the stationary solution (1.4). In other words, we show that the flow described by the Landau solution is, in some sense, asymptotically stable under any L^2-perturbation.

The existence and stability of stationary solutions corresponding to nontrivial external forces are well understood in the case of bounded domains, see for example [8]. For related results in exterior domains, we refer the reader to [10,11] and to the references therein. The existence and the stability of stationary solutions in L^p with $p \geqq n$, where n is the dimension of the space, is obtained in [25], under the condition that the Reynolds number is sufficiently small, and in [5,6,13,14,31] under the assumption that the external force is sufficiently small. The stability of small stationary solutions of (1.1)–(1.3) in $L^p(\mathbb{R}^3)$ with $p < 3$ has been studied recently in [2,3].

Notation. In this work, the usual norm of the Lebesgue space $L^p(\mathbb{R}^3)$ is denoted by $\|\cdot\|_p$ for any $p \in [1,\infty]$. $C_c^\infty(\mathbb{R}^3)$ denotes the set of smooth and compactly supported functions. Here, we work with the Sobolev space $H^1(\mathbb{R}^3) = \{f \in L^2(\mathbb{R}^3) : \nabla f \in L^2(\mathbb{R}^3)\}$ and with its homogeneous counterpart $\dot H^1(\mathbb{R}^3) = \{f \in L^1_{loc}(\mathbb{R}^3) : \nabla f \in L^2(\mathbb{R}^3)\}$. We use the following notation for the Banach spaces of divergence free vector fields: $L^p_\sigma(\mathbb{R}^3) = \{u \in \left(L^p(\mathbb{R}^3)\right)^3 : \operatorname{div} u = 0\}$ and $\dot H^1_\sigma(\mathbb{R}^3) = \{u \in \left(\dot H^1(\mathbb{R}^3)\right)^3 : \operatorname{div} u = 0\}$ supplemented with usual norms. The constants (always independent of x and t) will be denoted by the same letter C, even if they vary from line to line.

2. Results and comments

We denote by $u = u(x,t)$ a solution of the Navier–Stokes system (1.1)–(1.3) with the external force $F = b(c)\delta_0$, where $b(c)$ is defined in (1.5), and the initial datum $u_0 = v_c + w_0$, where v_c is the singular stationary solution (1.4) and $w_0 \in L^2_\sigma(\mathbb{R}^3)$. Then the functions $w(x,t) = u(x,t) - v_c(x)$ and $\pi(x) = p(x) - p_c(x)$

satisfy the initial value problem

$$w_t - \Delta w + (w \cdot \nabla)w + (w \cdot \nabla)v_c + (v_c \cdot \nabla)w + \nabla\pi = 0, \tag{2.1}$$

$$\operatorname{div} w = 0, \tag{2.2}$$

$$w(x,0) = w_0(x). \tag{2.3}$$

The goal of this work is to show the existence of a global-in-time weak solution to problem (2.1)–(2.3) in a usual energy space (see (2.7) below) and to study its convergence in $L^2_\sigma(\mathbb{R}^3)$ as $t \to \infty$ zero. As in the classical work by LERAY [18], these solutions satisfy a suitable energy inequality. Here, however, in the proof of the L^2-decay of solutions to (2.1)–(2.3), we need a strong energy inequality, introduced by MASUDA [21] for the Navier–Stokes system (1.1)–(1.3).

In our analysis, the crucial role is played by the Hardy-type inequality

$$\left| \int_{\mathbb{R}^3} w \cdot (w \cdot \nabla) v_c \, \mathrm{d}x \right| \leqq K(c) \|\nabla \otimes w\|_2^2, \tag{2.4}$$

which is valid for all $w \in \dot{H}^1(\mathbb{R}^3)$. Here, the function $K = K(c) > 0$ satisfies $\lim_{|c|\to 1} K(c) = +\infty$ and $\lim_{|c|\to +\infty} K(c) = 0$ (see Theorem 3.2, below); hence, there exists $c_0 > 1$ such that

$$K(c) < 1 \quad \text{for all} \in \mathbb{R} \text{ satisfying} \quad |c| \geqq c_0 > 1. \tag{2.5}$$

In the next section, we deduce inequality (2.4) from the classical Hardy inequality

$$\int_{\mathbb{R}^3} \frac{|w(x)|^2}{|x|^2} \, \mathrm{d}x \leqq 4 \int_{\mathbb{R}^3} |\nabla w(x)|^2 \, \mathrm{d}x \ \text{ for all } w \in \dot{H}(\mathbb{R}^3) \tag{2.6}$$

which proof can be found for example in [18, Ch. I. 6].

First, we state the counterpart of the Leray result on the existence of weak solutions to the initial value problem (2.1)–(2.2).

Theorem 2.1. *Assume that* $c_0 > 1$ *satisfies* (2.5). *For each* $c \in \mathbb{R}$ *such that* $|c| > c_0$, *every* $w_0 \in L^2_\sigma(\mathbb{R}^3)$, *and every* $T > 0$ *problem* (2.1)–(2.3) *has a weak solution in the energy space*

$$X_T = L^\infty_w \left([0,T], L^2_\sigma(\mathbb{R}^3) \right) \cap L^2 \left([0,T], \dot{H}^1_\sigma(\mathbb{R}^3) \right), \tag{2.7}$$

which satisfies the strong energy inequality

$$\|w(t)\|_2^2 + 2(1 - K(c)) \int_s^t \|\nabla \otimes w(\tau)\|_2^2 \, \mathrm{d}\tau \leqq \|w(s)\|_2^2 \tag{2.8}$$

for almost all $s \geqq 0$, *including* $s = 0$ *and all* $t \geqq s$.

Recall that, following a classical approach, a function $w \in X_T$ is a weak solution of problem (2.1)–(2.3) if

$$(w(t), \varphi(t)) + \int_s^t [(\nabla w, \nabla \varphi) + (w \cdot \nabla w, \varphi) + (w \cdot \nabla v_c, \varphi) + (v_c \cdot \nabla w, \varphi)] \, \mathrm{d}\tau$$
$$= (w(s), \varphi(s)) + \int_s^t (w, \varphi_\tau) \, \mathrm{d}\tau \tag{2.9}$$

for all $t \geqq s \geqq 0$ and all $\varphi \in C([0, \infty), H^1_\sigma(\mathbb{R}^3)) \cap C^1([0, \infty), L^2_\sigma(\mathbb{R}^3))$, where $(\cdot, \cdot)$ is the standard L^2-inner product. Note that each term in (2.9) containing the singular function v_c is convergent due to the Hardy inequality (2.6), see calculations in (3.6)–(3.7), below.

The proof of Theorem 2.1 follows the well-known argument which we recall in Sect. 3. Here, we only recall that the most general result on the existence of weak solutions to the Navier–Stokes system in the exterior domain satisfying the strong energy inequality was proved by Miyakawa and Sohr [20].

The decay in $L^2(\mathbb{R}^3)$ of weak solutions from Theorem 2.1 is the main result of this work.

Theorem 2.2. *Every weak solution $w = w(x, t)$ to problem (2.1)–(2.3) satisfying the strong energy inequality (2.8) has the property:* $\lim_{t\to\infty} \|w(t)\|_2 = 0$.

Under additional assumptions on initial data, we find also the decay rate of $\|w(t)\|_2$.

Corollary 2.3. *Under the assumptions of Theorem 2.2 if, moreover, $w_0 \in L^p(\mathbb{R}^3) \cap L^2_\sigma(\mathbb{R}^3)$ for some $p \in (\frac{6}{5}, 2)$, then there exists $C > 0$ such that*

$$\|w(t)\|_2 \leqq C t^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{2})} \tag{2.10}$$

for all $t > 0$.

3. Hardy-type inequality and existence of weak solutions

First, we prove elementary pointwise estimates of the components of the matrix ∇v_c.

Lemma 3.1. *Let $|c| > 1$. There exist functions $K_{j,k} : (-\infty, -1) \cup (1, \infty) \to (0, \infty)$ for every $j, k \in \{1, 2, 3\}$ such that for all $x \in \mathbb{R}^3 \setminus \{0\}$, we have*

$$\left|\partial_{x_j} v_c^k(x)\right| \leqq \frac{K_{j,k}(c)}{|x|^2}. \tag{3.1}$$

Moreover, functions $K_{j,k} = K_{j,k}(c)$ have the following properties: $\lim_{|c|\to 1} K_{j,k}(c) = +\infty$ *and* $\lim_{|c|\to+\infty} K_{j,k}(c) = 0$ *for all $j, k \in \{1, 2, 3\}$.*

Proof. It follows from the explicit formula for v_c and p_c (see (1.4)) that

$$v_c^1(x) = \frac{1}{2}p(x)x_1 + \frac{2}{c|x| - x_1}, \quad v_c^2(x) = \frac{1}{2}p(x)x_2, \quad v_c^3(x) = \frac{1}{2}p(x)x_3, \tag{3.2}$$

and

$$\nabla p(x) = \frac{4}{|x|^3(c|x| - x_1)^3}\begin{pmatrix} (c^2-2)|x|^3 + 3c|x|^2x_1 - 3c^2|x|x_1^2 + cx_1^3 \\ cx_2(2|x|^2 - 3c|x|x_1 + x_1^2) \\ cx_3(2|x|^2 - 3c|x|x_1 + x_1^2) \end{pmatrix}. \tag{3.3}$$

Moreover, using the expression for p_c from (1.4) and the notation $s = x_1/|x|$, we obtain

$$|p_c(x)| \leqq \frac{4}{|x|^2}\left|\sup_{s\in[-1,1]} \frac{cs-1}{(c-s)^2}\right| = k_p(c)\frac{1}{|x|^2},$$

where $k_p(c) = \frac{4}{|c|-1}$. In the same way by (3.3), we have

$$|x_i\partial_{x_1}p_c(x)| \leqq \frac{4}{|x|^2}\left|\sup_{s\in[-1,1]} \frac{cs^3 - 3c^2s^2 + 3cs + c^2 - 2}{(c-s)^3}\right| = k_{i,1}(c)\frac{1}{|x|^2}$$

and

$$|x_i\partial_{x_2}p_c(x)| \leqq \frac{4c}{|x|^2}\left|\sup_{s\in[-1,1]} \frac{s^2 - 3cs + 2}{(c-s)^3}\right| = k_{i,2}(c)\frac{1}{|x|^2},$$

where $k_{i,1} = \frac{8}{1-|c|}$ and $k_{i,2} = \frac{12c}{(|c|-1)^2}$ for $i \in \{1,2,3\}$. Now, using the representation of v_c in terms of p_c from (3.2), we proceed in an analogous way to estimate all coefficients of the matrix $\{\partial_{x_j}v_c^k(x)\}_{j,k=1}^3$. □

The following theorem is the immediate consequence of Lemma 3.1 and of the classical Hardy inequality (2.6).

Theorem 3.2. (Hardy-type inequality) *There exists a function* $K : (-\infty,-1) \cup (1,\infty) \to (0,\infty)$ *with the following properties:*

$$\lim_{|c|\to 1} K(c) = +\infty \quad \textit{and} \quad \lim_{|c|\to+\infty} K(c) = 0$$

such that for all vector fields $w \in \dot{H}^1(\mathbb{R}^3)$, *we have* $w\cdot(w\cdot\nabla)v_c \in L^1(\mathbb{R}^3)$ *together with the inequality*

$$\left|\int_{\mathbb{R}^3} w\cdot(w\cdot\nabla)v_c \,\mathrm{d}x\right| \leqq K(c)\|\nabla\otimes w\|_2^2. \tag{3.4}$$

Proof. Applying Lemma 3.1, we get

$$H(w) \equiv \left| \int_{\mathbb{R}^3} w \cdot (w \cdot \nabla) v_c \, \mathrm{d}x \right| \leqq \sum_{j,k=1}^{3} \int_{\mathbb{R}^3} |w_j w_k| |\partial_{x_j} v_c^k| \, \mathrm{d}x \leqq \int_{\mathbb{R}^3} \frac{\tilde{K}(c)}{|x|^2} \sum_{j,k=1}^{3} |w_j| |w_k| \, \mathrm{d}x,$$

where $\tilde{K}(c) = \max_{j,k \in \{1,2,3\}} K_{j,k}(c)$. Using the elementary inequality $a \cdot b \leqq (a^2 + b^2)/2$, we obtain

$$H(w) \leqq \frac{1}{2} \int_{\mathbb{R}^3} \frac{\tilde{K}(c)}{|x|^2} \left(\sum_{j,k=1}^{3} |w_j|^2 + \sum_{j,k=1}^{3} |w_k|^2 \right) \mathrm{d}x = 3\tilde{K}(c) \int_{\mathbb{R}^3} \frac{|w|^2}{|x|^2} \, \mathrm{d}x.$$

Finally, from the classical Hardy inequality (2.6), we have $H(w) \leqq K(c) \|\nabla \otimes w\|_2^2$, where $K(c) = 12\tilde{K}(c)$, which completes the proof of Theorem 3.2. □

Now, we are in a position to sketch the construction of weak solutions to problem (2.1)–(2.3).

Proof of Theorem 2.1. This is the standard reasoning based on the Galerkin method. Since $H^1_\sigma(\mathbb{R}^3)$ is separable, there exists a sequence $g_1, \ldots, g_m, \ldots$ which is free and total in $H^1_\sigma(\mathbb{R}^3)$. For each m, we define an approximate solution $w_m = \sum_{i=1}^m d_{im}(t) g_i$, which satisfies the following system of ordinary differential equations

$$\begin{aligned} &\left(w_m'(t), g_j\right) + \left(\nabla w_m(t), g_j\right) + \left((w_m(t) \cdot \nabla) w_m(t), g_j\right) + \left((w_m(t) \cdot \nabla) v_c, g_j\right) \\ &\quad + \left((v_c \cdot \nabla) w_m(t), g_j\right) + \left(\nabla \pi, g_j\right) = 0 \quad \text{for} \quad j = 1, \ldots, m, \end{aligned} \tag{3.5}$$

where $(f, g) = \int_{\mathbb{R}^3} f(x) \cdot g(x) \, \mathrm{d}x$.

Let us prove that both terms in (3.5) containing the singular functions ∇v_c and v_c are convergent. First, using the estimates from Lemma 3.1 as in the proof of Theorem 3.2, we obtain

$$\begin{aligned} \left((w_m(t) \cdot \nabla) v_c, g_j\right) &\leqq \sum_{k,\ell=1}^{3} \int_{\mathbb{R}^3} |w_m^k g_j^\ell| |\partial_{x_k} v_c^\ell| \, \mathrm{d}x \\ &\leqq \frac{1}{2} \int_{\mathbb{R}^3} \frac{\tilde{K}(c)}{|x|^2} \left(\sum_{k,\ell=1}^{3} |w_m^k|^2 + |g_j^\ell|^2 \right) \mathrm{d}x. \end{aligned} \tag{3.6}$$

Each term on the right-hand side of (3.6) is finite due to the Hardy inequality (2.6). Next, using the explicit formulas (1.4), we immediately obtain $| \cdot | v_c \in \left(L^\infty(\mathbb{R}^3)\right)^3$; hence the Schwarz inequality implies

$$\left((v_c \cdot \nabla) w_m(t), g_j\right) \leqq \| | \cdot | v_c \|_\infty \left\| | \cdot |^{-1} g_j \right\|_2 \|\nabla w_m\|_2. \tag{3.7}$$

The right-hand side of this inequality is finite because $|\cdot|^{-1}g_j \in L^2(\mathbb{R}^3)$ by the Hardy inequality (2.6), again.

Now, we obtain a *priori* estimate of the sequence $\{w_m\}_{m=1}^{\infty}$ by multiplying (3.5) by d_{jm} and adding the resulting equations for $j = 1, 2, \ldots, m$. Taking into account that div $w_m = 0$, we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w_m(t)\|_2^2 + \|\nabla \otimes w_m(t)\|_2^2 + ((w_m(t)\cdot\nabla)v_c, w_m(t)) = 0.$$

Consequently, using inequality (3.4) and integrating from s to t, we obtain the estimate

$$\|w_m(t)\|_2^2 + 2\,(1 - K(c)) \int_s^t \|\nabla w_m(s)\|_2^2 \,\mathrm{d}s \leqq \|w(s)\|_2^2.$$

Now, repeating the classical reasoning from for example [28, Ch. III. Thm. 3.1], we obtain the existence of a weak solution in the energy space X_T defined in (2.7), which satisfies strong energy inequality (2.8). □

4. Linearized equation

In the proof of the L^2-decay of weak solutions to problem (2.1)–(2.3), we use properties of solutions to the linearized Cauchy problem

$$z_t - \Delta z + (z\cdot\nabla)v_c + (v_c\cdot\nabla)z + \nabla\pi = 0, \quad (x,t) \in \mathbb{R}^3 \times (0,\infty), \tag{4.1}$$

$$\operatorname{div} z = 0, \tag{4.2}$$

$$z(x,0) = z_0(x), \qquad x \in \mathbb{R}^3. \tag{4.3}$$

Let us first recall that the Leray projector on divergence-free vector fields is defined by the formula $\mathbb{P}v = v - \nabla\Delta^{-1}(\nabla\cdot v)$ for sufficiently smooth vectors $v = (v_1(x), v_2(x), v_3(x))$. To give a meaning to $\mathbb{P}$, it suffices to use the Riesz transforms R_j which are the pseudo-differential operators defined in the Fourier variables as $\widehat{R_k f}(\xi) = \frac{i\xi}{|\xi|}\widehat{f}(\xi)$. Here, the Fourier transform of an integrable function v is given by $\widehat{v}(\xi) = (2\pi)^{-3/2}\int_{\mathbb{R}^3} \mathrm{e}^{-ix\cdot\xi} v(x)\,\mathrm{d}x$. Applying these well-known operators we define $(\mathbb{P}v)_j = v_j + \sum_{k=1}^{3} R_j R_k v_k$.

Using the Leray projector $\mathbb{P}$, we can formally transform system (4.1)–(4.2) into

$$z_t - \Delta z + \mathbb{P}\left((z\cdot\nabla)v_c\right) + \mathbb{P}\left((v_c\cdot\nabla)z\right) = 0.$$

Now, for simplicity, let us denote the linear operator

$$\mathcal{L}z = -\Delta z + \mathbb{P}\big((z\cdot\nabla)v_c\big) + \mathbb{P}\big((v_c\cdot\nabla)z\big) \tag{4.4}$$

and its adjoint operator in $L^2_\sigma(\mathbb{R}^3)$ given by the formula

$$\mathcal{L}^* z = -\Delta z + (\nabla v_c)^T z - (v_c\cdot\nabla)z. \tag{4.5}$$

In the following, we study these operators via the corresponding sesquilinear forms which defined for all $z, v \in H^1_\sigma(\mathbb{R}^3)$ as follows:

$$a_{\mathcal{L}}(z,v) = \int_{\mathbb{R}^3} \nabla z \cdot \nabla v \,\mathrm{d}x + \int_{\mathbb{R}^3} (z\cdot\nabla)v_c \cdot v\,\mathrm{d}x + \int_{\mathbb{R}^3} (v_c\cdot\nabla)z\cdot v\,\mathrm{d}x \tag{4.6}$$

and

$$a_{\mathcal{L}^*}(z,v) = \int_{\mathbb{R}^3} \nabla z \cdot \nabla v \,\mathrm{d}x + \int_{\mathbb{R}^3} (\nabla v_c)^T z \cdot v\,\mathrm{d}x - \int_{\mathbb{R}^3} (v_c\cdot\nabla)z\cdot v\,\mathrm{d}x. \tag{4.7}$$

Our goal is to show that both operators $-\mathcal{L}$ and $-\mathcal{L}^*$ (in fact, their closures in $L^2_\sigma(\mathbb{R}^3)$) are infinitesimal generators of analytic semigroups of linear operators on $L^2_\sigma(\mathbb{R}^3)$, provided condition (2.5) is satisfied. Here, we use the following abstract criterion.

Proposition 4.1. *Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{V} \subset \mathcal{H}$ be a dense subspace. Assume that $\mathcal{V}$ is a Hilbert space with the inner product $(\cdot,\cdot)_{\mathcal{V}}$, and with the norm $\|\cdot\|_{\mathcal{V}}$ such that for a constant $C > 0$, we have $\|x\|_{\mathcal{H}} \leqq C\|x\|_{\mathcal{V}}$ for all $x \in \mathcal{V}$. Let $a(x,y)$ be a bounded sesquilinear form on $\mathcal{V}$, which defines an operator $A : \mathcal{D}(A) \to \mathcal{H}$ as follows*

$$\mathcal{D}(A) = \{z \in \mathcal{V} : |a(z,v)| \leqq C\|v\|_{\mathcal{H}}, v \in \mathcal{V}\}, \quad (Az,v)_{\mathcal{H}} = a(z,v).$$

Suppose that for some $\alpha > 0$ and $\lambda_0 \in \mathbb{R}$ we have

$$\alpha\|z\|^2_{\mathcal{V}} \leqq \mathrm{Re}\, a(z,z) + \lambda_0\|z\|^2_{\mathcal{H}}. \tag{4.8}$$

Then $-A$ is the infinitesimal generator of a strongly continuous semigroup of linear operators on $\mathcal{H}$ which is holomorphic in a sector $S_\varepsilon = \{s \in \mathbb{C} : |\mathrm{Arg}\, s| < \varepsilon\}$ for some $\varepsilon > 0$.

The result stated in Proposition 4.1 is essentially due to Lions [19]. Its proof is a combination of theorems from [19] and [23] and we do not include it here, because this is more or less standard reasoning. A detailed proof can be found for example either in [9, Prop. 1.1] or in [22, Prop. 1.51].

Now, we apply Proposition 4.1 to study operator $\mathcal{L}$ and $\mathcal{L}^*$.

Theorem 4.2. *Assume that $|c| \geqq c_0$, where c_0 is defined in (2.5). Then the operators $-\mathcal{L}$ and $-\mathcal{L}^*$ defined in (4.4) and (4.5) are infinitesimal generators of strongly continuous semigroups of linear operators on $L^2_\sigma(\mathbb{R}^3)$ which are holomorphic in a sector $\{s \in \mathbb{C} : |\mathrm{Arg}\ s| < \varepsilon\}$ for a certain $\varepsilon = \varepsilon(c) > 0$.*

Proof. We apply Proposition 4.1 with $\mathcal{H} = L^2_\sigma(\mathbb{R}^3)$ and $\mathcal{V} = H^1_\sigma(\mathbb{R}^3)$. To show that the sesquilinear forms $a_{\mathcal{L}}$ and $a_{\mathcal{L}^*}$ are bounded on $\mathcal{V}$, it suffices to follow estimates from (3.6) and (3.7).

Condition (4.8) for the sesquilinear form $a_{\mathcal{L}}$ defined in (4.6) results immediately the following inequality

$$\alpha\|\nabla\otimes z\|^2_2 \leqq a_{\mathcal{L}}(z,z) \tag{4.9}$$

for a certain $\alpha > 0$ and all $z \in H^1_\sigma(\mathbb{R}^3)$. Here, we would like to recall that $\int_{\mathbb{R}^3}(v_c \cdot \nabla)z \cdot z \, \mathrm{d}x = 0$ for $\operatorname{div} v_c = 0$. Hence, estimate (4.9) is a consequence of Hardy-type inequality (3.4):

$$a_{\mathcal{L}}(z,z) = \|\nabla \otimes z\|_2^2 + \int_{\mathbb{R}^3}(z \cdot \nabla)v_c \cdot z \, \mathrm{d}x \geqq (1 - K(c)) \, \|\nabla \otimes z\|_2^2, \tag{4.10}$$

where $K(c) < 1$ for $|c| \geqq c_0 > 1$ by (2.5). Using Proposition 4.1 we complete the proof that the operator $-\mathcal{L}$ generates a holomorphic semigroup of linear operators on $L^2_\sigma(\mathbb{R}^3)$.

An analogous argument applies to the adjoint operator $-\mathcal{L}^*$, where by Lemma 3.1, we get

$$\begin{aligned} a_{\mathcal{L}^*}(z,z) &= \|\nabla \otimes z\|_2^2 + \int_{\mathbb{R}^3} (\nabla v_c)^T z \cdot z \, \mathrm{d}x \\ &\geqq \|\nabla \otimes z\|_2^2 - \int_{\mathbb{R}^3} \sum_{j,k=1}^{3} |\partial_{x_j} v_c^k||z_j||z_k| \geqq (1 - K(c)) \, \|\nabla \otimes z\|_2^2. \end{aligned}$$

Applying Proposition 4.1, we complete the proof of Theorem 4.2. □

The following corollaries describe typical properties of generators of analytic semigroups. We state them for the operator $\mathcal{L}$, however, they are obviously valid for the adjoint operator $\mathcal{L}^*$, as well.

Corollary 4.3. *Under the assumptions of Theorem 4.2, the following inequality*

$$\|\nabla \otimes z\|_2 \leqq (1 - K(c)) \, \|\mathcal{L}^{1/2} z\|_2 \tag{4.11}$$

holds true for all $z \in \dot{H}^1_\sigma(\mathbb{R}^3)$.

Proof. By the definition of a square root of nonnegative operators, we have $\|\mathcal{L}^{1/2}z\|_2^2 = a_{\mathcal{L}}(z,z)$. Hence to complete this proof, it suffices to recall inequality (4.10). □

Corollary 4.4. *Under the assumptions of Theorem 4.2,*

$$\|\mathrm{e}^{-t\mathcal{L}} z_0\|_2 \leqq \|z_0\|_2 \tag{4.12}$$

for all $z_0 \in L^2_\sigma(\mathbb{R}^3)$ *and* $t > 0$.

Proof. Multiplying equation (4.1) by z and integrating over $\mathbb{R}^3$, we easily obtain energy equality

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|z(t)\|_2^2 + \|\nabla \otimes z(t)\|_2^2 + \int_{\mathbb{R}^3} (z \cdot \nabla)v_c \cdot z \, \mathrm{d}x = 0,$$

because $\int_{\mathbb{R}^3}(v_c \nabla)z \cdot z \, \mathrm{d}x = 0$ by the condition $\operatorname{div} v_c = 0$. Hence, the Hardy-type inequality (3.4) yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|z(t)\|_2^2 + (1 - K(c)) \, \|\nabla \otimes z(t)\|_2^2 \leqq 0, \tag{4.13}$$

where $1 - K(c) \geqq 0$ by (2.5). Now, it is sufficient to integrate from 0 to t to obtain the inequality (4.12). □

Corollary 4.5. *There exists a constant $C > 0$ such that for all $z_0 \in L^2_\sigma(\mathbb{R}^3)$ the following inequalities*

$$\|\mathcal{L}e^{-t\mathcal{L}}z_0\|_2 \leqq Ct^{-1}\|z_0\|_2 \tag{4.14}$$

and

$$\|\mathcal{L}^{1/2}e^{-t\mathcal{L}}z_0\|_2 \leqq Ct^{-\frac{1}{2}}\|z_0\|_2 \tag{4.15}$$

hold true for all $t > 0$.

Proof. Inequality (4.14) is the well-known property of analytic semigroups of linear operators (see for example [23, Theorem 5.2] for more details). Using properties of a square root of a nonnegative operator, the Schwarz inequality, inequality (4.14) and Corollary 4.4, we obtain

$$\|\mathcal{L}^{\frac{1}{2}}e^{-t\mathcal{L}}\psi\|_2^2 = |(\mathcal{L}e^{-t\mathcal{L}}\psi, e^{-t\mathcal{L}}\psi)| \leqq \|\mathcal{L}e^{-t\mathcal{L}}\psi\|_2\|e^{-t\mathcal{L}}\psi\|_2 \leqq Ct^{-1}\|\psi\|_2^2$$

for all $t > 0$. □

Corollary 4.6. *Under the assumptions of Theorem 4.2, for all $z_0 \in L^2_\sigma(\mathbb{R}^3)$*

$$\lim_{t\to\infty} \|e^{-t\mathcal{L}}z_0\|_2 = 0. \tag{4.16}$$

Proof. Let $z_0 \in L^2_\sigma(\mathbb{R}^3)$. Since the range of the operator $\mathcal{L}$ is a dense subspace of $L^2_\sigma(\mathbb{R}^3)$, for every $\varepsilon>0$ there exists a function $\varphi \in \text{Range}(\mathcal{L})$ such that $\|\varphi - z_0\|_2 < \varepsilon$. Consequently, applying Corollary 4.5 and Corollary 4.4, we obtain

$$\begin{aligned}\|e^{-t\mathcal{L}}z_0\|_2 &\leqq \|e^{-t\mathcal{L}}(z_0 - \varphi)\|_2 + \|e^{-t\mathcal{L}}\varphi\|_2 \leqq \varepsilon + \|\mathcal{L}e^{-t\mathcal{L}}\psi\|_2\\ &\leqq \varepsilon + Ct^{-1}\|\psi\|_2,\end{aligned}$$

where $\psi \in \mathcal{D}(\mathcal{L})$. Hence, $\limsup_{t\to\infty} \|e^{-t\mathcal{L}}z_0\|_2 \leqq \varepsilon$. Since $\varepsilon > 0$ is arbitrarily small, we complete the proof. □

Corollary 4.7. *Under the assumptions of Theorem 4.2, for all $z_0 \in L^2_\sigma(\mathbb{R}^3)$*

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \|e^{-s\mathcal{L}}z_0\|_2 \, ds = 0. \tag{4.17}$$

Proof. Substituting $s = t\tau$, we get

$$\frac{1}{t}\int_0^t \|e^{-s\mathcal{L}}z_0\|_2 = \int_0^1 \|e^{-t\tau\mathcal{L}}z_0\|_2 \, d\tau.$$

Now, the desired result follows from Corollaries 4.4 and 4.6 combined with the Lebesgue dominated convergence theorem. □

We conclude this section by showing the decay estimates of the semigroup $e^{-t\mathcal{L}}$.

Proposition 4.8. (Hypercontractivity) *Assume that* $|c| \geqq c_0 > 1$*, where* c_0 *satisfies* (2.5)*. For each* $p \in (\frac{6}{5}, 2)$ *there exists a constant* $C = C(p) > 0$ *such that for every* $z_0 \in L^p_\sigma(\mathbb{R}^3)$

$$\|e^{-t\mathcal{L}} z_0\|_2 \leqq C t^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{2})} \|z_0\|_p \tag{4.18}$$

for all $t > 0$.

Proof. First, we consider the semigroup generated by the adjoint operator $\mathcal{L}^*$ defined in (4.5). For every $q \in (2, 6)$, using the Gagliardo–Nirenberg–Sobolev inequality, we obtain

$$\|e^{-t\mathcal{L}^*} z_0\|_q^q \leqq C \|e^{-t\mathcal{L}^*} z_0\|_2^{\frac{1}{2}(6-q)} \|\nabla e^{-t\mathcal{L}^*} z_0\|_2^{\frac{3}{2}(q-2)}.$$

Next, applying Corollaries 4.4 and 4.5 with $\mathcal{L}$ replaced by $\mathcal{L}^*$, we get

$$\begin{aligned}\|e^{-t\mathcal{L}^*} z_0\|_q^q &\leqq C \|z_0\|_2^{\frac{1}{2}(6-q)} \|(\mathcal{L}^*)^{1/2} e^{-t\mathcal{L}^*} z_0\|_2^{\frac{3}{2}(q-2)} \\ &\leqq C \left(t^{-\frac{3}{2}(\frac{1}{2}-\frac{1}{q})} \|z_0\|_2\right)^q \qquad \text{for all } \ t > 0.\end{aligned}$$

Hence, by a duality argument, we immediately deduce the inequality

$$\|e^{-t\mathcal{L}} z_0\|_2 \leqq C t^{-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{2}\right)} \|z_0\|_p \quad \text{for all } \ t > 0,$$

with $p = \frac{q}{q-1} \in (\frac{6}{5}, 2)$. □

5. Asymptotic stability of weak solutions

To show the decay of $\|w(t)\|_2$, we use the approach from [4] which involves the weak L^p-spaces. By this reason, let us recall the weak Marcinkiewicz L^p-spaces $(1 < p < \infty)$, denoted as usual by $L^{p,\infty} = L^{p,\infty}(\mathbb{R})$, which belong to the scale of the Lorentz spaces and contain measurable functions $f = f(x)$ satisfying the condition

$$|\{x \in \mathbb{R} : |f(x)| > \lambda\}| \leqq C\lambda^{-p} \tag{5.1}$$

for all $\lambda > 0$ and a constant C. One check that (5.1) is equivalent to

$$\int_E |f(x)|\,\mathrm{d}x \leqq \widetilde{C}|E|^{\frac{1}{q}}$$

for every measurable set E with a finite measure, another constant $\widetilde{C}$, and $\frac{1}{q}+\frac{1}{p} = 1$. This fact allows us to define the norm in $L^{p,\infty}$

$$\|f\|_{p,\infty} = \sup\left\{|E|^{-1+\frac{1}{q}} \int_E |f(x)|\,\mathrm{d}x : E \in \mathcal{B}\right\} \tag{5.2}$$

where $\mathcal{B}$ is the collection of all Borel sets with a finite and positive measure. Recall the well-known imbedding $L^p \subset L^{p,\infty}$ being the consequence of the Markov inequality $|\{x \in \mathbb{R} : |f(x)| > \lambda\}| \leqq \lambda^{-p} \int_{\mathbb{R}} |f(x)|^p \, dx$. Moreover, the following inequalities hold true: *the weak Hölder inequality*:

$$\|fg\|_{r,\infty} \leqq \|f\|_{p,\infty}\|g\|_{q,\infty} \tag{5.3}$$

for every $1 < p \leqq \infty$ (here $L^{\infty,\infty} = L^{\infty}$), $1 < q < \infty$ and $1 < r < \infty$ satisfying $\frac{1}{r} = \frac{1}{q} + \frac{1}{p}$, and *the weak Young inequality*

$$\|f * g\|_{r,\infty} \leqq C\|f\|_{p,\infty}\|g\|_{q,\infty} \tag{5.4}$$

for every $1 < p < \infty$, $1 < q < \infty$ and $1 < r < \infty$ satisfying $1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. We refer reader to [4] for the proofs of the results stated above.

The following lemma is extracted from reasonings contained in [4] and its proof is based on properties of the weak L^p-spaces.

Lemma 5.1. *Assume that* $f \in L^1((0, +\infty))$. *For every* $\alpha \in (1, +\infty]$ *there exists a constant* $C > 0$ *such that*

$$\frac{1}{t}\int_0^t \left(|\cdot|^{-\frac{1}{2}} * \left(|\cdot|^{-\frac{1}{\alpha}} f^{\frac{3}{4}}\right)\right)(s)\, ds \leqq Ct^{-\frac{1}{4}-\frac{1}{\alpha}}\|f\|_1^{\frac{3}{4}}. \tag{5.5}$$

for all $t > 0$. *Here, for* $\alpha = +\infty$, *the quantity* $1/\alpha$ *should be replaced by* 0.

Proof. First, consider $1/\alpha = 0$. Using the definition of the norm (5.2) in the weak L^p-spaces, we have

$$L_1 = \frac{1}{t}\int_0^t \left(|\cdot|^{-\frac{1}{2}} * f^{\frac{3}{4}}\right)(s)\, ds \leqq t^{-\frac{1}{q}} \left\| |\cdot|^{-\frac{1}{2}} * f^{\frac{3}{4}} \right\|_{q,\infty}$$

for every $q \in (1, \infty)$ to be chosen later. Since, $\|g\|_{p,\infty} \leqq C\|g\|_p$ for all $p \in [1, \infty]$, the weak Young inequality (5.4) implies

$$L_1 \leqq Ct^{-\frac{1}{q}} \left\| |\cdot|^{-\frac{1}{2}} \right\|_{2,\infty} \|f\|_{\frac{3}{4}r}^{\frac{3}{4}},$$

where $1 + \frac{1}{q} = \frac{1}{2} + \frac{1}{r}$. Hence, however, we require $3r/4 = 1$; hence $q = 4$. Since the function $|\cdot|^{-\frac{1}{2}} \in L^{2,\infty}((0, +\infty))$, we complete the proof of (5.5) in case $\alpha = +\infty$.

For $\alpha \in (0, +\infty)$, applying an analogous argument involving the definition of the norm (5.2) in the weak L^p-spaces, the weak Young inequality (5.4), we obtain

$$\begin{aligned} L_2 = \frac{1}{t}\int_0^t \left(|\cdot|^{-\frac{1}{2}} * |\cdot|^{-\frac{1}{\alpha}} f^{\frac{3}{4}}\right)(s)\, ds &\leqq t^{-\frac{1}{p}} \left\| |\cdot|^{-\frac{1}{2}} * |\cdot|^{-\frac{1}{\alpha}} f^{\frac{3}{4}} \right\|_{p,\infty} \\ &\leqq t^{-\frac{1}{p}} \left\| |\cdot|^{-\frac{1}{2}} \right\|_{2,\infty} \left\| |\cdot|^{-\frac{1}{\alpha}} f^{\frac{3}{4}} \right\|_{q,\infty} \end{aligned}$$

for every $p, q \in (1, \infty)$ satisfying $1 + \frac{1}{p} = \frac{1}{2} + \frac{1}{q}$. Now, the weak Hölder inequality (5.3) gives us

$$L_2 \leqq Ct^{-\frac{1}{p}} \left\| |\cdot|^{-\frac{1}{2}} \right\|_{2,\infty} \left\| |\cdot|^{-\frac{1}{\alpha}} \right\|_{\alpha,\infty} \|f\|_{\frac{3}{4}r}^{\frac{3}{4}},$$

where $\frac{1}{q} = \frac{1}{r} + \frac{1}{\alpha}$. Assuming $3r/4 = 1$, we get $1/p = 1/4 + 1/\alpha$. □

Lemma 5.2. *There exists $C > 0$ such that for all $v, w \in H^1_\sigma(\mathbb{R}^3)$ and $\psi \in L^2_\sigma(\mathbb{R}^3)$ the following estimate*

$$\Big((w \cdot \nabla)v, e^{-t\mathcal{L}^*}\psi\Big) \leqq Ct^{-\frac{1}{2}}(\|w\|_2\|v\|_2)^{\frac{1}{4}}(\|\nabla w\|_2\|\nabla v\|_2)^{\frac{3}{4}}\|\psi\|_2 \tag{5.6}$$

holds true for all $t > 0$.

Proof. By inequalities (4.11) and (4.15) (with $\mathcal{L}$ replaced by $\mathcal{L}^*$), we have $\|\nabla e^{-t\mathcal{L}^*}\psi\|_2 \leqq Ct^{-\frac{1}{2}}\|\psi\|_2$. Hence, a direct calculation involving the integration by parts, the Hölder inequality and inequality (4.12) leads to

$$\begin{aligned} |\Big((w \cdot \nabla)v, e^{-t\mathcal{L}^*}\psi\Big)| = |\Big(v, w \cdot \nabla e^{-t\mathcal{L}^*}\psi\Big)| &\leqq \|v\|_4\|w \cdot \nabla e^{-t\mathcal{L}^*}\psi\|_{\frac{4}{3}} \\ &\leqq \|v\|_4\|w\|_4\|\nabla e^{-t\mathcal{L}}\psi\|_2 \leqq Ct^{-\frac{1}{2}}\|v\|_4\|w\|_4\|\psi\|_2. \end{aligned}$$

Hence, the proof is completed by the Sobolev inequality $\|v\|_4 \leqq C\|\nabla v\|_2^{\frac{3}{4}}\|v\|_2^{\frac{1}{4}}$, which holds true for all $v \in H^1_\sigma(\mathbb{R}^3)$. □

Proof of Theorem 2.2. Let w be a weak solution of system (2.1)–(2.3) in the space X_T defined in Theorem 2.1 which satisfies the strong energy inequality (2.8). First, we show that

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \|w(s)\|_2 \, ds = 0. \tag{5.7}$$

Observe that inequality (2.8) implies $\|w(\cdot)\|_2 \in L^\infty(0,\infty)$ and $\|\nabla w(\cdot)\|_2^2 \in L^1(0,\infty)$. Now, with an arbitrary $\psi \in C^\infty_{c,\sigma}(\mathbb{R}^3)$, we substitute $\varphi(\tau) = e^{-(s-\tau)\mathcal{L}^*}\psi$ into equation (2.9) (with $s = 0$ and t replaced by s) to obtain the following integral formulation of problem (2.1)–(2.3)

$$(w(s), \psi) = \Big(e^{-s\mathcal{L}}w_0, \psi\Big) + \int_0^s \Big((w \cdot \nabla w)(\tau), e^{-(s-\tau)\mathcal{L}^*}\psi\Big) \, d\tau. \tag{5.8}$$

Here, in calculations leading to (5.8), one should transform the last term on the right-hand side of (2.9) in the following way:

$$\begin{aligned} \int_0^s (w, \varphi_\tau) \, d\tau = \int_0^s \Big(w, \mathcal{L}^*\varphi\Big) \, d\tau &= \int_0^s (\mathcal{L}w, \varphi) \, d\tau \\ &= \int_0^s [(\nabla w, \nabla\varphi) + (w \cdot \nabla v_c, \varphi) + (v_c \cdot \nabla w, \varphi)] \, d\tau, \end{aligned}$$

because $\operatorname{div}\varphi = 0$.

Hence, applying Lemma 5.2 to estimate the nonlinear term in (5.8) and the L^2-duality argument, we get

$$\begin{aligned} \|w(s)\|_2 &\leqq \|e^{-s\mathcal{L}}w_0\|_2 + C\int_0^s (s-\tau)^{-\frac{1}{2}}\|w(\tau)\|_2^{\frac{1}{2}}\|\nabla w(\tau)\|_2^{\frac{3}{2}} \, d\tau \\ &\leqq \|e^{-s\mathcal{L}}w_0\|_2 + C\sup_{\tau>0}\|w(\tau)\|_2^{\frac{1}{2}}\int_0^s (s-\tau)^{-\frac{1}{2}}\|\nabla w(\tau)\|_2^{\frac{3}{2}} \, d\tau \end{aligned}$$

since $\|w(\cdot)\|_2 \in L^\infty(0,\infty)$. Integrating from 0 to t and multiplying by $1/t$, we obtain

$$\frac{1}{t}\int_0^t \|w(s)\|_2 \,\mathrm{d}s \leqq \frac{1}{t}\int_0^t \|\mathrm{e}^{-s\mathcal{L}}w_0\|_2 \,\mathrm{d}s + \frac{C}{t}\int_0^t \left(|\cdot|^{-\frac{1}{2}} * \|\nabla w(\cdot)\|_2^{\frac{3}{2}}\right)(s)\,\mathrm{d}s.$$

Now, since $\|\nabla w\|_2^2 \in L^1((0,+\infty))$, we apply Lemma 5.1 with $1/\alpha = 0$ to get the estimate

$$\frac{1}{t}\int_0^t \|w(s)\|_2 \,\mathrm{d}s \leqq \frac{1}{t}\int_0^t \|\mathrm{e}^{-s\mathcal{L}}w_0\|_2 \,\mathrm{d}s + Ct^{-\frac{1}{4}}, \tag{5.9}$$

which proves (5.7) by Corollary 4.7.

Next, notice that, by the strong energy inequality (2.8), $\|w(t)\|_2$ is a non-increasing function of t for almost all $t \geqq 0$. Hence, for $t > 0$ we obtain

$$\|w(t)\|_2 = \frac{1}{t}\|w(t)\|_2 \int_0^t \mathrm{d}s \leqq \frac{1}{t}\int_0^t \|w(s)\|_2 \,\mathrm{d}s. \tag{5.10}$$

The proof is completed by (5.7). □

Proof of Corollary 2.3. Using the decay estimate from Proposition 4.8, we have

$$\frac{1}{t}\int_0^t \|\mathrm{e}^{-s\mathcal{L}}w_0\|_2 \,\mathrm{d}s \leqq Ct^{-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{2}\right)}\|w_0\|_p$$

for each $p \in (\frac{6}{5}, 2)$ and all $t > 0$. Applying this inequality in (5.9) and recalling (5.10), we complete the proof of the corollary in the case of $p \in [\frac{3}{2}, 2)$.

Now, we notice that for $p \in (\frac{6}{5}, \frac{3}{2})$ inequality (5.9) implies $\|w(t)\|_2 \leqq Ct^{-\frac{1}{4}}$ for all $t > 0$. Hence, repeating the reasoning from the proof of Theorem 2.2 and applying Lemma 5.1 with $\alpha = 8$, we get the estimate

$$\begin{aligned}\frac{1}{t}\int_0^t \|w(s)\|_2 \,\mathrm{d}s &\leqq \frac{1}{t}\int_0^t \|\mathrm{e}^{-s\mathcal{L}}w_0\|_2 \,\mathrm{d}s \\ &\quad + \frac{C}{t}\int_0^t \left(|\cdot|^{-\frac{1}{2}} * |\cdot|^{-\frac{1}{8}}\|\nabla w(\cdot)\|_2^{\frac{3}{2}}\right)(s)\,\mathrm{d}s \\ &\leqq Ct^{-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{2}\right)}\|w_0\|_p + Ct^{-\frac{3}{8}},\end{aligned}$$

which proves decay estimate (2.10) for $p \in [\frac{4}{3}, \frac{3}{2})$. Repeating this procedure finitely many times, we complete the proof for each $p \in (\frac{6}{5}, 2)$. □

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