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Variational inequality problems over split fixed point sets of strict pseudo-nonspreading mappings and quasi-nonexpansive mappings with applications

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Abstract

In this paper, we first establish a strong convergence theorem for a variational inequality problem over split fixed point sets of a finite family of strict pseudo-nonspreading mappings and a countable family of quasi-nonexpansive mappings. As applications, we establish a strong convergence theorem of split fixed point sets of a finite family of strict pseudo-nonspreading mappings and a countable family of strict pseudo-nonspreading mappings without semicompact assumption on the strict pseudo-nonspreading mappings. We also study the variational inequality problems over split common solutions of fixed points for a finite family of strict pseudo-nonspreading mappings, fixed points of a countable family of pseudo-contractive mappings (or strict pseudo-nonspreading mappings) and solutions of a countable family of nonlinear operators. We study fixed points of a countable family of pseudo-contractive mappings with hemicontinuity assumption, neither Lipschitz continuity nor closedness assumption is needed.

Keywords: hierarchical problems; split feasibility problem; fixed point problem; strict pseudo-nonspreading mappings; pseudo-contractive mappings

1 Introduction

The split feasibility problem (SFP) in finite dimensional Hilbert spaces was first introduced by Censor and Elfving [1] for modeling inverse problems which arise from phase retrievals and in medical image reconstruction. Since then, the split feasibility problem (SFP) has received much attention due to its applications in signal processing, image reconstruction, with particular progress in intensity-modulated radiation therapy, approximation theory, control theory, biomedical engineering, communications, and geophysics. For example, one can see [2–5].

Let C be a nonempty closed convex subset of a real Hilbert space H_1 with the inner product $\langle\cdot,\cdot\rangle$ and the norm $\|\cdot\|$.

Let f be a contraction on C, let $T : C \to C$ be a nonexpansive mapping and $\{\alpha_n\}$ be a sequence in $[0,1]$. In 2004, Xu [6] proved that under some condition on $\{\alpha_n\}$, the sequence

$\{x_n\}$ generated by

$$x_{n+1} = \alpha_n f x_n + (1 - \alpha_n) T x_n$$

strongly converges to x^* in $\mathrm{Fix}(T)$ which is the unique solution of the variational inequality

$$\langle (I - f)x^*, x - x^* \rangle \geq 0$$

for all $x \in \mathrm{Fix}(T)$.

Xu [7] studied the following minimization problem over the set of fixed point set of a nonexpansive operator T on a real Hilbert space H_1:

$$\min_{x \in \mathrm{Fix}(T)} \frac{1}{2} \langle Bx, x \rangle - \langle a, x \rangle,$$

where a is a given point in H_1 and B is a strongly positive bounded linear operator on H_1. In [7], Xu proved that the sequence $\{x_n\}$ defined by the following iterative method:

$$x_{n+1} = (I - \alpha_n B) T x_n + \alpha_n a$$

converges strongly to the unique solution of the minimization problem of a quadratic function.

Let H_1, H_2 be two real Hilbert spaces, $A : H_1 \to H_2$ be a bounded linear operator, $\{S_i\}_{i=1}^{\infty} : H_1 \to H_1$ be an infinite family of k_i-strictly pseudo-nonspreading mappings and $\{T_i\}_{i=1}^{N} : H_2 \to H_2$ be a finite family of ρ_i-strict pseudo-nonspreading mappings, $C = \bigcap_{i=1}^{\infty} \mathrm{Fix}(S_i)$ and $Q = \bigcap_{i=1}^{N} \mathrm{Fix}(T_i)$, Chang *et al.* [8] studied the following iterative sequence.

Let $\{x_n\} \subset H_1$ be defined by

$$\begin{cases} x_1 \in C \quad \text{chosen arbitrarily,} \\ y_n = [I - \gamma A^*(I - T_{n(\mathrm{mod}\, N)})A]x_n, \\ x_{n+1} = a_{0,n} y_n + \sum_{i=1}^{\infty} a_{i,n} S_{i,\beta} y_n \quad \text{for all } n \geq 1, \end{cases}$$

where $S_{i,\beta} = \beta I + (1 - \beta) S_i$, $\beta \in (0, 1)$ is a constant, I is an identity function on H_1.

Chang *et al.* [8] proved a weak convergence theorem to the solution of the following problem: Find $\bar{x} \in C$, $A\bar{x} \in Q$.

In addition, if there exists some positive integer m such that S_m is semicompact, then $\{x_n\}$, $\{y_n\}$ converge strongly to a point $\bar{x} \in \Gamma = \{x \in C, Ax \in Q\}$. In 2013, Tang [9] studied common solutions of fixed points of continuous pseudo-contractive mappings and zero points of the sum of monotone mappings. In 2014, Zegeye and Shahzad [10] and Yao *et al.* [11] studied common solutions of fixed points of Lipschitz continuous pseudo-contractive mappings and zero points of the sum of monotone mappings. Wangkeree and Nammanee [12] studied common solutions of fixed points of continuous pseudo-contractive mappings and solutions of a continuous monotone variational inequality.

In 2013, Cheng *et al.* [13] constructed a three-step iteration and obtained a convergence theorem for a countable family of uniformly Lipschitz and uniformly closed pseudo-contractive mappings. Deng [14] constructed another iteration and obtained a convergence theorem for a countable family of closed and Lipschitz pseudo-contractive mappings. Both the results of Cheng *et al.* [13] and Deng [14] used too strong conditions to study common fixed points of a countable family of pseudo-contractive mappings. The fixed point theorems of pseudo-contractive mappings in the literature assumed the Lipschitz continuous condition; see, for example, [15, 16].

Motivated and inspired by the above results, in this paper, we first establish a strong convergence theorem for a variational inequality problem over split fixed point sets of a finite family of strict pseudo-nonspreading mappings and a countable family of quasi-nonexpansive mappings. As applications, we establish a strong convergence theorem of split fixed point sets of a finite family of strict pseudo-nonspreading mappings and a countable family of strict pseudo-nonspreading mappings without semicompact assumption on the strict pseudo-nonspreading mappings. We also study the variational inequality problems over split common solutions of a fixed point for a finite family of strict pseudo-nonspreading mappings, fixed points of a countable family of pseudo-contractive mappings (or strict pseudo-nonspreading mappings) and solutions of a countable family of nonlinear operators. We study a fixed point of a countable family of pseudo-contractive mappings with hemicontinuity assumption, neither Lipschitz continuity nor closedness assumptions is needed.

2 Preliminaries

Throughout this paper, let $\mathbb{N}$ be the set of positive integers, and let $\mathbb{R}$ be the set of real numbers, H_1 be a (real) Hilbert space. Let $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ denote the inner product and the norm of H_1, respectively. Let C be a nonempty closed convex subset of H_1. We denote the strong convergence and the weak convergence of $\{x_n\}$ to $x \in H_1$ by $x_n \to x$ and $x_n \rightharpoonup x$, respectively.

Let $T: C \to H_1$ be a mapping, and let $\mathrm{Fix}(T) := \{x \in C : Tx = x\}$ denote the set of fixed points of T. A mapping $T: H_1 \to H_1$ is called

(i) pseudo-contractive if for each $x, y \in H_1$, we have $\langle Tx - Ty, x - y \rangle \le \|x - y\|^2$.
Note that the above inequality can be equivalently written as

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + \big\|(I - T)x - (I - T)y\big\|^2 \quad \text{for all } x, y \in H_1;$$

(ii) a k-strictly pseudo-contractive mapping if there exists a constant $k \in [0, 1)$ such that $\langle x - y, Tx - Ty \rangle \le \|x - y\|^2 - k\|(I - T)x - (I - T)y\|^2$ for all $x, y \in H_1$;

(iii) nonexpansive if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in H_1$;

(iv) firmly nonexpansive if $\|Tx - Ty\|^2 \le \|x - y\|^2 - \|(I - T)x - (I - T)y\|^2$ for every $x, y \in C$;

(v) quasi-nonexpansive if $\mathrm{Fix}(T) \ne \emptyset$ and $\|Tx - p\| \le \|x - p\|$ for all $x \in H_1$, $p \in \mathrm{Fix}(T)$;

(vi) nonspreading [17] if $2\|Tx - Ty\|^2 \le \|Tx - y\|^2 + \|Ty - x\|^2$ for all $x, y \in H_1$, that is, $\|Tx - Ty\|^2 \le \|x - y\|^2 + 2\langle x - Tx, y - Ty \rangle$ for all $x, y \in H_1$;

(vii) α-strictly pseudo-nonspreading [18] if there exists $\alpha \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + \alpha\big\|x - Tx - (y - Ty)\big\|^2 + 2\langle x - Tx, y - Ty \rangle$$

for all $x, y \in H_1$;

(viii) strongly monotone if there exists $\bar{\gamma} > 0$ such that $\langle x - y, Tx - Ty \rangle \ge \bar{\gamma}\|x - y\|^2$ for all $x, y \in H_1$;

(ix) Lipschitz continuous if there exists $L > 0$ such that $\|Tx - Ty\| \le L\|x - y\|$ for all $x, y \in H_1$;

(x) hemicontinuous ([19], p.204) if, for all $x, y \in H_1$, the mapping $g : [0,1] \to H_1$ defined by $g(t) = T(tx + (1 - t)y)$ is continuous, where H_1 has a weak topology;

(xi) α-inverse-strongly monotone if $\langle x - y, Vx - Vy\rangle \ge \alpha\|Tx - Ty\|^2$ for all $x, y \in H_1$ and $\alpha > 0$;

(xi) monotone if $\langle x - y, Tx - Ty\rangle \ge 0$ for all $x, y \in H_1$.

We also know that (i) if V is an α-inverse-strongly monotone mapping and $0 < \lambda \le 2\alpha$, then $I - \lambda V : C \to H_1$ is a nonexpansive mapping; (ii) if V is a monotone mapping, then $T = I - V : C \to H_1$ is a pseudo-contractive mapping.

Let $T : C \to H_1$ be a mapping. Then $p \in C$ is called an asymptotic fixed point of T [20] if there exists $\{x_n\} \subseteq C$ such that $x_n \rightharpoonup p$, and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. We denote by $\mathrm{Fix}(\hat{T})$ the set of asymptotic fixed points of T. A mapping $T : C \to H$ is said to be demiclosed if it satisfies $\mathrm{Fix}(T) = \mathrm{Fix}(\hat{T})$.

Let $B : H_1 \multimap H_1$ be a multivalued mapping. The effective domain of B is denoted by $D(B)$, that is, $D(B) = \{x \in H_1 : Bx \neq \emptyset\}$.

Then $B : H_1 \multimap H_1$ is called

(i) a monotone operator on H_1 if $\langle x - y, u - v\rangle \ge 0$ for all $x, y \in D(B)$, $u \in Bx$, and $v \in By$;

(ii) a maximal monotone operator on H_1 if B is a monotone operator on H_1 and its graph is not properly contained in the graph of any other monotone operator on H_1.

For a maximal monotone operator B on H_1 and $r > 0$, we may define a single-valued operator $J_r = (I + rB)^{-1} : H_1 \to D(B)$, which is called the resolvent of B for r. Let $B^{-1}0 = \{x \in H_1 : 0 \in Bx\}$.

The following lemmas are needed in this paper.

A mapping $T : H_1 \to H_1$ is said to be averaged if $T = (1 - \alpha)I + \alpha S$, where $\alpha \in (0,1)$ and $S : H_1 \to H_1$ is nonexpansive. In this case, we also say that T is α-averaged. A firmly nonexpansive mapping is $\frac{1}{2}$-averaged.

Lemma 2.1 [5, 21] *Let $T : C \to C$ be a mapping. Then the following are satisfied*:

(i) *T is nonexpansive if and only if the complement $(I - T)$ is $1/2$-ism.*

(ii) *If S is υ-ism, then, for $\gamma > 0$, γS is υ/γ-ism.*

(iii) *S is averaged if and only if the complement $I - S$ is υ-ism for some $\upsilon > 1/2$.*

(iv) *If S and T are both averaged, then the product (composite) ST is averaged.*

(v) *If the mappings $\{T_i\}_{i=1}^n$ are averaged and have a common fixed point, then $\bigcap_{i=1}^n \mathrm{Fix}(T_i) = \mathrm{Fix}(T_1 \cdots T_n)$.*

Lemma 2.2 [22] *Let $V : H_1 \to H_1$ be a $\bar{\gamma}$-strongly monotone and L-Lipschitz continuous operator with $\bar{\gamma} > 0$ and $L > 0$. Let $\theta \in H_1$ and $V_1 : H_1 \to H_1$ such that $V_1 x = Vx - \theta$. Then V_1 is a $\bar{\gamma}$-strongly monotone and L-Lipschitz continuous mapping. Furthermore, there exists a unique fixed point z_0 in C satisfying $z_0 = P_C(z_0 - Vz_0 + \theta)$. This point $z_0 \in C$ is also a unique solution of the hierarchical variational inequality*

$$\langle Vz_0 - \theta, q - z_0\rangle \ge 0, \quad \forall q \in C.$$

Lemma 2.3 [19] *Let $B : H_1 \multimap H_1$ be maximal monotone.*

(i) *For each $\beta > 0$, J_β^B is single-valued and firmly nonexpansive.*

(ii) *$\mathcal{D}(J_\beta^B) = H_1$ and $\mathrm{Fix}(J_\beta^B) = \{x \in \mathcal{D}(B) : 0 \in Bx\}$.*

Lemma 2.4 [23] *Let $\{a_n\}$ be a sequence of real numbers such that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $a_{n_i} < a_{n_i+1}$ for all $i \in \mathbb{N}$. Then there exists a nondecreasing sequence $\{m_k\} \subseteq \mathbb{N}$ such that $m_k \to \infty$ and the following properties are satisfied by all (sufficiently large) numbers $k \in \mathbb{N}$:*

$$a_{m_k} \le a_{m_k+1} \quad \text{and} \quad a_k \le a_{m_k+1}.$$

In fact, $m_k = \max\{j \le k : a_j < a_{j+1}\}$.

Lemma 2.5 [24] *Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of nonnegative real numbers, $\{\alpha_n\}$ be a sequence of real numbers in $[0,1]$ with $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\{u_n\}$ be a sequence of nonnegative real numbers with $\sum_{n=1}^{\infty} u_n < \infty$, $\{t_n\}$ be a sequence of real numbers with $\limsup t_n \le 0$. Suppose that $a_{n+1} \le (1-\alpha_n)a_n + \alpha_n t_n + u_n$ for each $n \in \mathbb{N}$. Then $\lim_{n\to\infty} a_n = 0$.*

The equilibrium problem is to find $z \in C$ such that

$$\text{(EP)} \quad g(z,y) \ge 0 \quad \text{for each } y \in C,$$

where $g : C \times C \to \mathbb{R}$ is a bifunction.

This problem includes fixed point problems, optimization problems, variational inequality problems, Nash equilibrium problems, minimax inequalities, and saddle point problems as special cases. (For example, one can see [25] and related literature.) The solution set of equilibrium problem (EP) is denoted by $\mathrm{EP}(g)$.

For solving the equilibrium problem, let us assume that the bifunction $g : C \times C \to \mathbb{R}$ satisfies the following conditions:

(A1) $g(x,x) = 0$ for each $x \in C$.
(A2) g is monotone, *i.e.*, $g(x,y) + g(y,x) \le 0$ for any $x, y \in C$.
(A3) For each $x, y, z \in C$, $\limsup_{t\downarrow 0} g(tz + (1-t)x, y) \le g(x,y)$.
(A4) For each $x \in C$, the scalar function $y \to g(x,y)$ is convex and lower semicontinuous.

We have the following result from Blum and Oettli [25].

Theorem 2.1 [25] *Let $g : C \times C \to \mathbb{R}$ be a bifunction which satisfies conditions* (A1)-(A4). *Then, for each $r > 0$ and each $x \in H_1$, there exists $z \in C$ such that*

$$g(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \ge 0$$

for all $y \in C$.

In 2005, Combettes and Hirstoaga [26] established the following important properties of a resolvent operator.

Theorem 2.2 [26] *Let $g : C \times C \to \mathbb{R}$ be a function satisfying conditions* (A1)-(A4). *For $r > 0$, define $T_r^g : H_1 \to C$ by*

$$T_r^g x = \left\{ z \in C : g(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \ge 0, \forall y \in C \right\}$$

for all $x \in H_1$. *Then the following hold*:

(i) T_r^g *is single-valued.*
(ii) T_r^g *is firmly nonexpansive, that is,* $\|T_r^g x - T_r^g y\|^2 \le \langle x - y, T_r^g x - T_r^g y\rangle$ *for all* $x, y \in H$.
(iii) $\{x \in H_1 : T_r^g x = x\} = \{x \in C : g(x,y) \ge 0, \forall y \in C\}$.
(iv) $\{x \in C : g(x,y) \ge 0, \forall y \in C\}$ *is a closed and convex subset of* C.

We call such T_r^g the resolvent of g for $r > 0$.

Lemma 2.6 [27] *Let* E *be a uniformly convex Banach space,* $r > 0$ *be a constant. Then there exists a continuous, strictly increasing and convex function* $g : [0, 2r) \to [0, \infty)$ *with* $g(0) = 0$ *such that*

$$\left\| \sum_{k=1}^{\infty} \alpha_k x_k \right\|^2 \le \sum_{k=1}^{\infty} \alpha_k \|x_k\|^2 - \alpha_i \alpha_j g\big(\|x_i - x_j\|\big)$$

for all $i, j \in \mathbb{N}$, $x_k \in B_r := \{z \in E : \|z\| \le r\}$, $\alpha_k \in \mathbb{N}$ *with* $\sum_{k=1}^{\infty} \alpha_k = 1$.

Takahashi *et al.* [28] showed the following result.

Lemma 2.7 [28] *Let* $g : C \times C \to \mathbb{R}$ *be a bifunction satisfying the conditions* (A1)-(A4). *Define* A_g *as follows*:

$$\text{(L4.1)} \quad A_g x = \begin{cases} \{z \in H_1 : g(x,y) \ge \langle y - x, z\rangle, \forall y \in C\}, & \forall x \in C; \\ \emptyset, & \forall x \notin C. \end{cases}$$

Then $\mathrm{EP}(g) = A_g^{-1}0$ *and* A_g *is a maximal monotone operator with the domain of* $A_g \subset C$. *Furthermore, for any* $x \in H_1$ *and* $r > 0$, *the resolvent* T_r^g *of* g *coincides with the resolvent of* A_g, *i.e.,* $T_r^g x = (I + rA_g)^{-1} x$.

Lemma 2.8 [18] *Let* H *be a real Hilbert space,* C *be a nonempty and closed convex subset of* H, *and* $T : C \to C$ *be a* k*-strictly pseudo-nonspreading mapping. Then the following hold*:

(i) *If* $\mathrm{Fix}(T) \neq \emptyset$, *then* $\mathrm{Fix}(T)$ *is closed and convex.*
(ii) T *is demiclosed.*

Lemma 2.9 [8] *Let* H *be a real Hilbert space,* C *be a nonempty and closed convex subset of* H, *and* $T : C \to C$ *be a* k*-strictly pseudo-nonspreading mapping and* $\mathrm{Fix}(T) \neq \emptyset$. *Let* $T_\gamma = \gamma I + (1 - \gamma)T$, $k \le \gamma < 1$. *Then* T_γ *is a quasi-nonexpansive mapping.*

3 Convergence theorems of hierarchical problems

Let C and Q be nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively. For each $i \in \mathbb{N}$, let G_i be a maximal monotone mapping on H_1 such that the domains of G_i are included in C, and let $J_{\lambda_i}^{G_i} = (I + \lambda_i G_i)^{-1}$ for each $\lambda_i > 0$. For $i \in \mathbb{N}$, $\kappa_i > 0$, let B_i be a κ_i-inverse-strongly monotone mapping of C into H_1. Let $A : H_1 \to H_2$ be a bounded linear operator and A^* be the adjoint of A, and let R be the spectral radius of the operator A^*A. Let $\{\theta_n\} \subset H_1$ be a sequence, and let V be a $\bar{\gamma}$-strongly monotone and

L-Lipschitz continuous operator with $\bar{\gamma} > 0$ and $L > 0$. Let $\{F_i\}_{i=1}^{\infty}$ and $\{W_i\}_{i=1}^{\infty}$ be countable families of quasi-nonexpansive mappings from C into itself with demiclosed property, and let $\{T_i\}_{i=1}^{N} : H_2 \to H_2$ be a finite family of ρ_i-strictly pseudo-nonspreading mappings with $\rho = \max\{\rho_i : i = 1, 2, \dots, N\} \in (0,1)$. Let $\{S_i\}_{i=1}^{\infty}$ be a countable family of δ_i-strictly pseudo-nonspreading mappings from C into itself with $\delta = \sup_{i\geq 1} \delta_i$, and let $\Psi_i : C \to C$ be a countable family of pseudo-contractive mappings. Throughout this paper, we use these notations and assumptions unless specified otherwise.

In this paper, we first study the variational inequality problem over split common solutions for a fixed point of a finite family of strict pseudo-nonspreading mappings, and a solution of a countable family of quasi-nonexpansive mappings.

Theorem 3.1 *Suppose that* $\Pi_1 := \{x \in C : x \in \bigcap_{i=1}^{\infty} \operatorname{Fix}(F_i)$ *and* $Ax \in \bigcap_{i=1}^{N} \operatorname{Fix}(T_i)\} \neq \emptyset$. *Take* $\mu \in \mathbb{R}$ *as follows*:

$$0 < \mu < \frac{2\bar{\gamma}}{L^2}.$$

Let $\{x_n\} \subset H$ *be defined by*

$$(3.1)\qquad \begin{cases} x_1 \in C \quad \textit{chosen arbitrarily}, \\ y_n = [I - bA^*(I - T_{n(\operatorname{mod} N)})A]x_n, \\ s_n = a_{n,0}y_n + \sum_{i=1}^{\infty} a_{n,i}F_i y_n, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)(\beta_n\theta_n + (1-\beta_n V)s_n) \end{cases}$$

for each $n \in \mathbb{N}$, $\{a_{n,i} : n, i \in \mathbb{N}\} \subset [0,1]$ *and* $\{\alpha_n, \beta_n\} \subset (0,1)$. *Assume the following*:

(i) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$;
(ii) $\lim_{n\to\infty} \beta_n = 0$ *and* $\sum_{n=1}^{\infty} \beta_n = \infty$;
(iii) $a_{n,0} + \sum_{i=1}^{\infty} a_{n,i} = 1$, $\liminf_{n\to\infty} a_{n,0}a_{n,i} > 0$, $\forall i \in \mathbb{N}$ *and* $0 < b < \frac{1-\rho}{\|A\|^2}$;
(iv) $\lim_{n\to\infty} \theta_n = \theta$ *for some* $\theta \in H$.

Then $\lim_{n\to\infty} x_n = \bar{x}$, *where* $\bar{x} = P_{\Pi_1}(\bar{x} - V\bar{x} + \theta)$. *This point* $\bar{x}$ *is also a unique solution of the following hierarchical variational inequality*:

$$\langle V\bar{x} - \theta, q - \bar{x}\rangle \geq 0, \quad \forall q \in \Pi_1.$$

Proof Take any $\bar{x} \in \Pi_1$ and let $\bar{x}$ be fixed. Then $\bar{x} \in \bigcap_{i=1}^{\infty} F_i\bar{x}$ and $A\bar{x} \in \bigcap_{i=1}^{N} T_i$. For each $n \in \mathbb{N}$, by Lemma 2.6, we have

$$\begin{aligned} \|s_n - \bar{x}\|^2 &= \left\| a_{n,0}y_n + \sum_{i=1}^{\infty} a_{n,i}F_i y_n - \bar{x} \right\|^2 \\ &\leq a_{n,0}\|y_n - \bar{x}\|^2 + \sum_{i=1}^{\infty} a_{n,i}\|F_i y_n - \bar{x}\|^2 - a_{n,0}a_{n,i}g\big(\|y_n - F_i y_n\|\big) \\ &\leq a_{n,0}\|y_n - \bar{x}\|^2 + \sum_{i=1}^{\infty} a_{n,i}\|y_n - \bar{x}\|^2 - a_{n,0}a_{n,i}g\big(\|y_n - F_i y_n\|\big) \\ &\leq \|y_n - \bar{x}\|^2 - a_{n,0}a_{n,i}g\big(\|y_n - F_i y_n\|\big) \\ &\leq \|y_n - \bar{x}\|^2. \end{aligned} \qquad (1)$$

In Theorem 3.1 in [8], Chang *et al.* showed that

$$\begin{aligned}\|y_n-\bar{x}\|^2 &= \big\|\big[I-bA^*(I-T_{n(\operatorname{mod} N)})A\big]x_n-\bar{x}\big\|^2\\ &\le \|x_n-\bar{x}\|^2-b\big(1-\rho-b\|A\|^2\big)\big\|(I-T_{n(\operatorname{mod} N)})Ax_n\big\|^2\\ &\le \|x_n-\bar{x}\|^2. \end{aligned}\tag{2}$$

Let $z_n=\beta_n\theta_n+(I-\beta_n V)s_n$ and $\tau=\bar{\gamma}-\frac{L^2\mu}{2}$, following the same argument as in the proof of Theorem 3.1 in [22], we have

$$\|x_{n+1}-\bar{x}\|\le \max\big\{\|x_n-\bar{x}\|,M\big\},$$

where $M=\max\{\frac{\|\theta_n-V\bar{x}\|}{\tau}:n\in\mathbb{N}\}$. By mathematical induction, we know

$$\|x_n-\bar{x}\|\le \max\big\{\|x_1-\bar{x}\|,M\big\}.$$

This implies that the sequence $\{x_n\}$ is bounded. Furthermore, $\{z_n\}$, $\{s_n\}$ and $\{F_ix_n\}$ are bounded for all $i\in\mathbb{N}$.

Following the same argument as in the proof of Theorem 3.1 in [22] again, we have

$$\begin{aligned}&\|x_{n+1}-\bar{x}\|^2-\|x_n-\bar{x}\|^2+(1-\alpha_n)\alpha_n\|s_n-x_n\|^2\\ &\quad\le 2(1-\alpha_n)\beta_n\langle\theta_n,x_n-\bar{x}\rangle-2(1-\alpha_n)\beta_n\langle Vs_n,x_n-\bar{x}\rangle\\ &\qquad+(1-\alpha_n)^2\big[\beta_n^2\|\theta_n-Vs_n\|^2+2\beta_n\|\theta_n-Vs_n\|\|s_n-x_n\|\big]. \end{aligned}\tag{3}$$

We will divide the proof into two cases as follows.

Case 1: There exists a natural number N such that $\|x_{n+1}-\bar{x}\|\le\|x_n-\bar{x}\|$ for each $n\ge N$. So, $\lim_{n\to\infty}\|x_n-\bar{x}\|$ exists. Hence, it follows from (3), (i) and (ii) that

$$\lim_{n\to\infty}\|s_n-x_n\|=0.\tag{4}$$

Following the same argument as in the proof of Theorem 3.1 in [22] again, we have

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0,\tag{5}$$

$$\lim_{n\to\infty}\|z_n-s_n\|=0\tag{6}$$

and

$$\lim_{n\to\infty}\|z_n-x_n\|=0.\tag{7}$$

By (1) and (2), we have

$$\|s_n-\bar{x}\|^2\le\|x_n-\bar{x}\|^2-b\big(1-\rho-b\|A\|^2\big)\big\|(I-T_{n(\operatorname{mod} N)})Ax_n\big\|^2-a_{n,0}a_{n,i}g\big(\|y_n-F_iy_n\|\big)$$

for all $i\in\mathbb{N}$.

Therefore,

$$\begin{aligned} & b\big(1-\rho-b\|A\|^2\big)\big\|(I-T_{n(\mathrm{mod}\,N)})Ax_n\big\|^2 + a_{n,0}a_{n,i}g\big(\|y_n-F_iy_n\|\big) \\ & \quad \le \|x_n-\bar{x}\|^2 - \|s_n-\bar{x}\|^2 \\ & \quad \le \|x_n-s_n\|\big(\|s_n-\bar{x}\| + \|x_n-\bar{x}\|\big) \end{aligned} \tag{8}$$

for all $i \in \mathbb{N}$.

Thus, by (4), (8), and condition (iii), we have

$$\lim_{n\to\infty}\big\|(I-T_{n(\mathrm{mod}\,N)})Ax_n\big\| = 0 \quad \text{and} \quad \lim_{n\to\infty} g\big(\|y_n-F_iy_n\|\big) = 0 \tag{9}$$

for all $i \in \mathbb{N}$.

From the properties of g, we conclude that

$$\lim_{n\to\infty}\|y_n-F_iy_n\| = 0 \tag{10}$$

for all $i \in \mathbb{N}$.

By Lemma 2.8 and the assumption, we have that $\Pi_1 := \{x \in C : x \in \bigcap_{i=1}^{\infty}\mathrm{Fix}(F_i)$ and $Ax \in \bigcap_{i=1}^{N}\mathrm{Fix}(T_i)\}$ is a nonempty closed convex subset of H. Hence, by Lemma 2.2, we can take $\bar{x}_0 \in \Pi_1$ such that

$$\bar{x}_0 = P_{\Pi_1}(\bar{x}_0 - V\bar{x}_0 + \theta).$$

This point $\bar{x}_0$ is also a unique solution of the hierarchical variational inequality

$$\langle V\bar{x}_0-\theta, q-\bar{x}_0\rangle \ge 0, \quad \forall q \in \Pi_1. \tag{11}$$

We show that

$$\limsup_{n\to\infty}\langle V\bar{x}_0-\theta, z_n-\bar{x}_0\rangle \ge 0.$$

Without loss of generality, there exists a subsequence $\{z_{n_k}\}$ of $\{z_n\}$ such that $z_{n_k} \rightharpoonup w$ for some $w \in H$ and

$$\limsup_{n\to\infty}\langle V\bar{x}_0-\theta, z_n-\bar{x}_0\rangle = \lim_{k\to\infty}\langle V\bar{x}_0-\theta, z_{n_k}-\bar{x}_0\rangle. \tag{12}$$

By (7), we have that

$$\lim_{n\to\infty}\|x_n-z_n\| = 0 \quad \text{and} \quad x_{n_k} \rightharpoonup w. \tag{13}$$

Since A is a bounded linear operator, this implies that $Ax_{n_k} \rightharpoonup Aw$. Also, by equation (10), we have

$$\lim_{n_k\to\infty}\big\|(I-T_{n_k(\mathrm{mod}\,N)})Ax_{n_k}\big\| = 0. \tag{14}$$

Hence there exist a positive integer $j \in \{1, 2, \ldots, N\}$ and a subsequence $\{n_{k_i}\}$ of $\{n_k\}$ with $n_{k_i} (\operatorname{mod} N) = j$ such that

$$\lim_{n_{k_i} \to \infty} \left\| (I - T_j) A x_{n_{k_i}} \right\| = 0. \tag{15}$$

Since $Ax_{n_{k_i}} \rightharpoonup Aw$, it follows from Lemma 2.8 that $I - T_j$ is demiclosed at 0. This implies that $Aw \in \operatorname{Fix}(T_j)$.

By equations (13) and (14), we have

$$y_{n_k} = x_{n_k} - \gamma A^*(I - T_j) A x_{n_{k_i}} \rightharpoonup w. \tag{16}$$

For each $i \in \mathbb{N}$, by F_i is demiclosed, equations (10) and (16), we have $w \in \operatorname{Fix}(F_i)$. Hence, $w \in \Pi_1$. Thus, from (11) and (12) we have

$$\limsup_{n \to \infty} \langle V\bar{x}_0 - \theta, z_n - \bar{x}_0 \rangle = \lim_{k \to \infty} \langle V\bar{x}_0 - \theta, z_{n_k} - \bar{x}_0 \rangle = \langle V\bar{x}_0 - \theta, w - \bar{x}_0 \rangle \geq 0. \tag{17}$$

Following the same argument as in the proof of Theorem 3.1 in [22] again, we have that

$$\begin{aligned} \|x_{n+1} - \bar{x}_0\|^2 \leq {} & \bigl[1 - 2(1 - \alpha_n)\beta_n \tau\bigr] \|x_n - \bar{x}_0\|^2 + 2(1 - \alpha_n)\beta_n \tau \biggl(\frac{\beta_n \tau \|x_n - \bar{x}_0\|^2}{2} \\ & + \frac{\langle \theta_n - \theta, z_n - \bar{x}_0 \rangle}{\tau} + \frac{\langle \theta - V\bar{x}_0, z_n - \bar{x}_0 \rangle}{\tau} \biggr). \end{aligned} \tag{18}$$

By (17), (18), assumptions, and Lemma 2.5, we know that $\lim_{n \to \infty} x_n = \bar{x}_0$, where

$$\bar{x}_0 = P_{\Pi_1}(\bar{x}_0 - V\bar{x}_0 + \theta).$$

Case 2: Suppose that there exists $\{n_i\}$ of $\{n\}$ such that $\|x_{n_i} - \bar{x}\| \leq \|x_{n_{i+1}} - \bar{x}\|$ for all $i \in \mathbb{N}$. By Lemma 2.4, there exists a nondecreasing sequence $\{m_j\}$ in $\mathbb{N}$ such that $m_j \to \infty$ and

$$\|x_{m_j} - \bar{x}\| \leq \|x_{m_j+1} - \bar{x}\| \quad \text{and} \quad \|x_j - \bar{x}\| \leq \|x_{m_j+1} - \bar{x}\|. \tag{19}$$

Following the same argument as in the proof of Theorem 3.1 in [22] again, we have

$$\lim_{j \to \infty} \|x_{m_{j+1}} - \bar{x}_0\| = 0. \tag{20}$$

By equations (19) and (20),

$$\lim_{j \to \infty} \|x_j - \bar{x}_0\| \leq \lim_{j \to \infty} \|x_{m_j+1} - \bar{x}_0\| = 0.$$

Thus, the proof is completed. □

Remark 3.1 Theorem 3.1 improves Theorem 3.1 in [29], Theorem 3.3 in [30] and Theorem 3.1 in [31] in the sense that our convergence is for the common fixed point of a countable family of quasi-nonexpansive mappings and for the split feasibility problem.

Applying Theorem 3.1, we study a variational inequality problem over split common solutions for a fixed point of a finite family of strict pseudo-nonspreading mappings and a common fixed point of countable families of mappings.

Theorem 3.2 *Suppose that* $\Pi_2 := \{x \in C : x \in \bigcap_{i=1}^{\infty} \operatorname{Fix}(S_i) \cap \bigcap_{i=1}^{\infty} \operatorname{Fix}(W_i)$ *and* $Ax \in \bigcap_{i=1}^{N} \operatorname{Fix}(T_i)\} \neq \emptyset$. *Take* $\mu \in \mathbb{R}$ *as follows*:

$$0 < \mu < \frac{2\bar{\gamma}}{L^2}.$$

Let $\{x_n\} \subset H$ *be defined by*

$$\begin{cases} x_1 \in C \quad \textit{chosen arbitrarily}, \\ y_n = [I - bA^*(I - T_{n(\operatorname{mod} N)})A]x_n, \\ s_n = a_{n,0}y_n + \sum_{i=1}^{\infty} b_{n,i}W_i y_n + \sum_{i=1}^{\infty} a_{n,i}S_{i,\gamma}y_n, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)(\beta_n\theta_n + (1-\beta_n V)s_n), \end{cases} \tag{3.2}$$

where $S_{i,\gamma} = \gamma I + (1-\gamma)S_i$, $i \geq 1$, $\gamma \in [\delta, 1)$ *for each* $n \in \mathbb{N}$, $\{\alpha_n, \beta_n\} \subset (0,1)$, $\{a_{n,i} : n, i \in \mathbb{N}\} \subset [0,1]$ *and* $\{b_{n,i} : n, i \in \mathbb{N}\} \subset [0,1]$. *Assume the following*:

(i) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$;
(ii) $\lim_{n\to\infty} \beta_n = 0$ *and* $\sum_{n=1}^{\infty} \beta_n = \infty$;
(iii) $a_{n,0} + \sum_{i=1}^{\infty} b_{n,i} + \sum_{i=1}^{\infty} a_{n,i} = 1$, $\liminf_{n\to\infty} a_{n,0}a_{n,i} > 0$, $\liminf_{n\to\infty} a_{n,0}b_{n,i} > 0$, $\forall i \in \mathbb{N}$ *and* $0 < b < \frac{1-\rho}{\|A\|^2}$;
(iv) $\lim_{n\to\infty} \theta_n = \theta$ *for some* $\theta \in H$.

Then $\lim_{n\to\infty} x_n = \bar{x}$, *where* $\bar{x} = P_{\Pi_2}(\bar{x} - V\bar{x} + \theta)$. *This point* $\bar{x}$ *is also a unique solution of the following hierarchical variational inequality*:

$$\langle V\bar{x} - \theta, q - \bar{x}\rangle \geq 0, \quad \forall q \in \Pi_2.$$

Proof By $\Pi_2 \neq \emptyset$, there exists $w \in C$ such that $w \in \bigcap_{i=1}^{\infty} \operatorname{Fix}(S_i) \bigcap_{i=1}^{\infty} \operatorname{Fix}(W_i) = \bigcap_{i=1}^{\infty} \operatorname{Fix}(S_{i,\gamma}) \bigcap_{i=1}^{\infty} \operatorname{Fix}(W_i)$. For each $i \in \mathbb{N}$, S_i is a δ_i-strictly pseudo-nonspreading mapping and $\gamma \in [\delta, 1)$, by Lemma 2.9, we have $S_{i,\gamma}$ is quasi-nonexpansive. For each $i \in \mathbb{N}$, by Lemma 2.8 and $\|x - S_{i,\gamma}x\| = (1-\gamma)\|x - S_i x\|$, we see that $S_{i,\gamma}$ is demiclosed.

For each $i \in \mathbb{N}$, let $F_{2i} = S_{i,\gamma}$ and $F_{2i-1} = W_i$ in Theorem 3.1, then F_i is quasi-nonexpansive with demiclosed property and $\Pi_2 = \Pi_1$. It follows from Theorem 3.1 that $\lim_{n\to\infty} x_n = \bar{x}$, where $\bar{x} = P_{\Pi_2}(\bar{x} - V\bar{x} + \theta)$. This point $\bar{x}$ is also a unique solution of the following hierarchical variational inequality:

$$\langle V\bar{x} - \theta, q - \bar{x}\rangle \geq 0, \quad \forall q \in \Pi_2.$$ □

Applying Theorem 3.1, we study split common solutions for common fixed points of a finite family of strict pseudo-nonspreading mappings, and common fixed points of a countable family of strict pseudo-nonspreading mappings. Our result improves Theorem 3.1 in [8].

Theorem 3.3 *Suppose that* $\Pi_3 := \{x \in C : x \in \bigcap_{i=1}^{\infty} \operatorname{Fix}(S_i)$ *and* $Ax \in \bigcap_{i=1}^{N} \operatorname{Fix}(T_i)\} \neq \emptyset$. *Take* $\mu \in \mathbb{R}$ *as follows*:

$$0 < \mu < \frac{2\bar{\gamma}}{L^2}.$$

Let $\{x_n\} \subset H$ *be defined by*

$$(3.3)\quad \begin{cases} x_1 \in C \quad \text{chosen arbitrarily}, \\ y_n = [I - bA^*(I - T_{n(\text{mod}\, N)})A]x_n, \\ s_n = s_n = c_{n,0}y_n + \sum_{i=1}^{\infty} c_{n,i}S_{i,\gamma}y_n, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)(\beta_n\theta_n + (1-\beta_n)s_n), \end{cases}$$

where $S_{i,\gamma} = \gamma I + (1-\gamma)S_i$, $i \geq 1$, $\gamma \in [\delta, 1)$ *for each* $n \in \mathbb{N}$, $\{\alpha_n, \beta_n\} \subset (0,1)$ *and* $\{c_{n,i} : n, i \in \mathbb{N}\} \subset [0,1]$. *Assume the following*:

(i) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$;
(ii) $\lim_{n\to\infty} \beta_n = 0$ *and* $\sum_{n=1}^{\infty} \beta_n = \infty$;
(iii) $c_{n,0} + \sum_{i=1}^{\infty} c_{n,i} = 1$, $\liminf_{n\to\infty} c_{n,0}c_{n,i} > 0$, $\forall i \in \mathbb{N}$ *and* $0 < b < \frac{1-\rho}{\|A\|^2}$;
(iv) $\lim_{n\to\infty} \theta_n = \theta$ *for some* $\theta \in H$.

Then $\lim_{n\to\infty} x_n = \bar{x}$, *where* $\bar{x} = P_{\Pi_3}(\theta)$.

Proof By $\Pi_3 \neq \emptyset$, there exists $w \in C$ such that $w \in \bigcap_{i=1}^{\infty} \text{Fix}(S_i) = \bigcap_{i=1}^{\infty} \text{Fix}(S_{i,\gamma})$. For each $i \in \mathbb{N}$, S_i is a δ_i-strictly pseudo-nonspreading mapping and $\gamma \in [\delta, 1)$, by Lemma 2.9, we have $S_{i,\gamma}$ is quasi-nonexpansive and $\text{Fix}(S_i) = \text{Fix}(S_{i,\gamma})$. For each $i \in \mathbb{N}$, by Lemma 2.8 and $\|x - S_{i,\gamma}x\| = (1-\gamma)\|x - S_i x\|$, we see that $S_{i,\gamma}$ is demiclosed.

For each $i \in \mathbb{N}$, let $F_i = S_{i,\gamma}$ and $V(x) = x$ for all $x \in C$ in Theorem 3.1, then F_i is quasi-nonexpansive with demiclosed property and $\Pi_3 = \Pi_1$. It follows from Theorem 3.1 that $\lim_{n\to\infty} x_n = \bar{x}$, where $\bar{x} = P_{\Pi_3}(\theta)$. □

Remark 3.2 Theorem 3.3 improves Theorem 3.1 in [8]. Indeed, Theorem 3.1 in [8] establishes a weak convergence. In Theorem 3.1 in [8], if there exists some positive integer m such that S_m is semicompact, then $\{x_n\}$ converges strongly to $\bar{x} \in \Pi_3$. However, Theorem 3.3 is a strong convergence theorem without semicompact assumption.

Let $\gamma = \delta = \delta_i = 0$ for all $i \in \mathbb{N}$ and $\rho = \rho_j = 0$ for all $j = 1, 2, \ldots, N$ in Theorem 3.2, we have the following theorem.

Theorem 3.4 *Suppose that for each* $i \in \mathbb{N}, S_i$ *is a nonspreading mapping, and let* $\{T_i\}_{i=1}^N : H_2 \to H_2$ *be a finite family of nonspreading mappings. Let* $\Pi_4 := \{x \in C : x \in \bigcap_{i=1}^{\infty} \text{Fix}(S_i) \cap \bigcap_{i=1}^{\infty} \text{Fix}(W_i)$ *and* $Ax \in \bigcap_{i=1}^{N} \text{Fix}(T_i)\} \neq \emptyset$. *Take* $\mu \in \mathbb{R}$ *as follows*:

$$0 < \mu < \frac{2\bar{\gamma}}{L^2}.$$

Let $\{x_n\} \subset H$ *be defined by*

$$(3.4)\quad \begin{cases} x_1 \in C \quad \text{chosen arbitrarily}, \\ y_n = [I - bA^*(I - T_{n(\text{mod}\, N)})A]x_n, \\ s_n = a_{n,0}y_n + \sum_{i=1}^{\infty} b_{n,i}W_i y_n + \sum_{i=1}^{\infty} a_{n,i}S_i y_n, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)(\beta_n\theta_n + (1-\beta_n V)s_n) \end{cases}$$

for each $n \in \mathbb{N}$, $\{\alpha_n, \beta_n\} \subset (0,1)$, $\{a_{n,i} : n, i \in \mathbb{N}\} \subset [0,1]$ *and* $\{b_{n,i} : n, i \in \mathbb{N}\} \subset [0,1]$. *Assume the following*:

(i) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$;

(ii) $\lim_{n\to\infty} \beta_n = 0$ *and* $\sum_{n=1}^{\infty} \beta_n = \infty$;

(iii) $a_{n,0} + \sum_{i=1}^{\infty} b_{n,i} + \sum_{i=1}^{\infty} a_{n,i} = 1$, $\liminf_{n\to\infty} a_{n,0}a_{n,i} > 0$, $\liminf_{n\to\infty} a_{n,0}b_{n,i} > 0$, $\forall i \in \mathbb{N}$ *and* $0 < b < \frac{1-\rho}{\|A\|^2}$;

(iv) $\lim_{n\to\infty} \theta_n = \theta$ *for some* $\theta \in H$.

Then $\lim_{n\to\infty} x_n = \bar{x}$, *where* $\bar{x} = P_{\Pi_4}(\bar{x} - V\bar{x} + \theta)$. *This point* $\bar{x}$ *is also a unique solution of the following hierarchical variational inequality*:

$$\langle V\bar{x} - \theta, q - \bar{x}\rangle \geq 0, \quad \forall q \in \Pi_4.$$

Proof For each $i \in \mathbb{N}$, S_i is a nonspreading mapping, hence S_i is a quasi-nonexpansive mapping and a δ_i-strictly pseudo-nonspreading mapping with $\delta_i = 0$. For each $i = 1, 2, \ldots, N$, T_i is a nonspreading mapping, hence T_i is a ρ_i-strictly pseudo-nonspreading mapping with $\rho_i = 0$.

Let $\gamma = \delta = \delta_i = 0$ for all $i \in \mathbb{N}$ and $\rho = \rho_j = 0$ for all $j = 1, 2, \ldots, N$ in Theorem 3.2. It follows from Theorem 3.2 and $\Pi_4 = \Pi_2$ that $\lim_{n\to\infty} x_n = \bar{x}$, where $\bar{x} = P_{\Pi_4}(\bar{x} - V\bar{x} + \theta)$. This point $\bar{x}$ is also a unique solution of the following hierarchical variational inequality:

$$\langle V\bar{x} - \theta, q - \bar{x}\rangle \geq 0, \quad \forall q \in \Pi_4. \qquad \square$$

(I) We study split common fixed points for a finite family of ρ_i-strictly pseudo-nonspreading mappings, zeros for a countable family of mappings which are the sum of two monotone mappings and common fixed points for a countable family of quasi-nonexpansive mappings:

$$\begin{cases} \text{Find } \bar{x} \in C \text{ such that} \\ \bar{x} \in \bigcap_{i=1}^{\infty}(G_i + B_i)^{-1}0 \cap \bigcap_{i=1}^{\infty} \operatorname{Fix}(W_i) \end{cases}$$

and

$$\begin{cases} \bar{u} = A\bar{x} \in H_2 \text{ such that} \\ \bar{u} \in \bigcap_{i=1}^{N} \operatorname{Fix}(T_i). \end{cases}$$

Theorem 3.5 *Suppose that* $\Pi_5 := \{x \in C : x \in \bigcap_{i=1}^{\infty}(B_i + G_i)^{-1}0 \bigcap_{i=1}^{\infty} \operatorname{Fix}(W_i)$ *and* $Ax \in \bigcap_{i=1}^{N} \operatorname{Fix}(T_i)\} \neq \emptyset$. *Take* $\mu \in \mathbb{R}$ *as follows*:

$$0 < \mu < \frac{2\bar{\gamma}}{L^2}.$$

Let $\{x_n\} \subset H$ *be defined by*

$$\begin{cases} x_1 \in C \quad \text{chosen arbitrarily}, \\ y_n = [I - bA^*(I - T_{n(\operatorname{mod} N)})A]x_n, \\ s_n = a_{n,0}y_n + \sum_{i=1}^{\infty} b_{n,i}W_i y_n + \sum_{i=1}^{\infty} a_{n,i}J_{r_i}^{G_i}(I - r_i B_i)y_n, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n)(\beta_n\theta_n + (1 - \beta_n V)s_n) \end{cases} \tag{3.5}$$

for each $n \in \mathbb{N}$, $\{\alpha_n, \beta_n\} \subset (0,1)$, $\{a_{n,i} : n, i \in \mathbb{N}\} \subset [0,1]$, $\{b_{n,i} : n, i \in \mathbb{N}\} \subset [0,1]$ *and* $\{r_i\} \subset (0,\infty)$. *Assume the following*:

(i) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$;

(ii) $\lim_{n\to\infty}\beta_n = 0$ *and* $\sum_{n=1}^{\infty}\beta_n = \infty$;

(iii) $a_{n,0} + \sum_{i=1}^{\infty} b_{n,i} + \sum_{i=1}^{\infty} a_{n,i} = 1$, $\liminf_{n\to\infty} a_{n,0}a_{n,i} > 0$, $\liminf_{n\to\infty} a_{n,0}b_{n,i} > 0$, $\forall i \in \mathbb{N}$ *and* $0 < b < \frac{1-\rho}{\|A\|^2}$;

(iv) $0 < r_i < 2\kappa_i$ *for all* $i \in \mathbb{N}$ *and* $\lim_{n\to\infty}\theta_n = \theta$ *for some* $\theta \in H$.

Then $\lim_{n\to\infty} x_n = \bar{x}$, *where* $\bar{x} = P_{\Pi_5}(\bar{x} - V\bar{x} + \theta)$. *This point* $\bar{x}$ *is also a unique solution of the following hierarchical variational inequality*:

$$\langle V\bar{x} - \theta, q - \bar{x}\rangle \geq 0, \quad \forall q \in \Pi_5.$$

Proof By Theorem 4.1 in [32] and Lemma 2.1(iv), (v), we have that

$$J_{r_i}^{G_{2,i}}(I - r_iB_{2,i}) \text{ is averaged}$$

for each $i \in \mathbb{N}$.

For each $i \in \mathbb{N}$, let $F_{2i} = W_i$ and let $F_{2i-1} = J_{r_i}^{G_i}(I - r_iB_i)$. Then F_i is a quasi-nonexpansive mapping and

$$\bigcap_{i=1}^{\infty}\operatorname{Fix}(F_i) = \bigcap_{i=1}^{\infty}\operatorname{Fix}\bigl(J_{r_i}^{G_i}(I - r_iB_i)\bigr) \cap \bigcap_{i=1}^{\infty}\operatorname{Fix}(W_i).$$

Thus, $\Pi_5 = \Pi_1$. By Theorem 3.1, we have that $\lim_{n\to\infty} x_n = \bar{x}$, where $\bar{x} = P_{\Pi_5}(\bar{x} - V\bar{x} + \theta)$. This point $\bar{x}$ is also a unique solution of the following hierarchical variational inequality:

$$\langle V\bar{x} - \theta, q - \bar{x}\rangle \geq 0, \quad \forall q \in \Pi_5,$$

and the proof is completed. □

Remark 3.3 Theorem 3.5 improves Theorem 10 in [33] in the sense that our convergence is for the fixed point of a countable family of quasi-nonexpansive mappings and for the split feasibility problem.

(II) We study split common fixed points for a finite family of ρ_i-strictly pseudo-nonspreading mappings, zeros for a countable family of mappings which are the sum of two monotone mappings and common fixed points of a countable family of δ_i-strictly pseudo-nonspreading mappings:

$$\begin{cases} \text{Find } \bar{x} \in C \text{ such that} \\ \bar{x} \in \bigcap_{i=1}^{\infty}(G_i + B_i)^{-1}0 \cap \bigcap_{i=1}^{\infty}\operatorname{Fix}(S_i) \end{cases}$$

and

$$\begin{cases} \bar{u} = A\bar{x} \in H_2 \text{ such that} \\ \bar{u} \in \bigcap_{i=1}^{N}\operatorname{Fix}(T_i). \end{cases}$$

Theorem 3.6 *Suppose that* $\Pi_6 := \{x \in C : x \in \bigcap_{i=1}^{\infty}\operatorname{Fix}(J_{r_i}^{G_i}(I - r_iB_i)) \bigcap_{i=1}^{\infty}\operatorname{Fix}(S_i)$ *and* $Ax \in \bigcap_{i=1}^{N}\operatorname{Fix}(T_i)\} \neq \emptyset$. *Take* $\mu \in \mathbb{R}$ *as follows*:

$$0 < \mu < \frac{2\bar{\gamma}}{L^2}.$$

Let $\{x_n\} \subset H$ *be defined by*

$$(3.6)\quad \begin{cases} x_1 \in C \quad \text{chosen arbitrarily}, \\ y_n = [I - bA^*(I - T_{n(\operatorname{mod} N)})A]x_n, \\ s_n = a_{n,0}y_n + \sum_{i=1}^{\infty} b_{n,i}S_{i,\gamma}y_n + \sum_{i=1}^{\infty} a_{n,i}J_{r_i}^{G_i}(I - r_iB_i)y_n, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n)(\beta_n\theta_n + (1 - \beta_n V)s_n) \end{cases}$$

for each $n \in \mathbb{N}$, $\{\alpha_n, \beta_n\} \subset (0,1)$, $\{a_{n,i} : n, i \in \mathbb{N}\} \subset [0,1]$, $\{b_{n,i} : n, i \in \mathbb{N}\} \subset [0,1]$ *and* $\{r_i\} \subset (0,\infty)$. *Assume the following*:

(i) $0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty} \alpha_n < 1$;

(ii) $\lim_{n\to\infty} \beta_n = 0$ *and* $\sum_{n=1}^{\infty} \beta_n = \infty$;

(iii) $a_{n,0} + \sum_{i=1}^{\infty} b_{n,i} + \sum_{i=1}^{\infty} a_{n,i} = 1$, $\liminf_{n\to\infty} a_{n,0}a_{n,i} > 0$, $\liminf_{n\to\infty} a_{n,0}b_{n,i} > 0$, $\forall i \in \mathbb{N}$ *and* $0 < b < \frac{1-\rho}{\|A\|^2}$;

(iv) $0 < r_i < 2\kappa_i$ *for all* $i \in \mathbb{N}$ *and* $\lim_{n\to\infty} \theta_n = \theta$ *for some* $\theta \in H$.

Then $\lim_{n\to\infty} x_n = \bar{x}$, *where* $\bar{x} = P_{\Pi_4}(\bar{x} - V\bar{x} + \theta)$. *This point* $\bar{x}$ *is also a unique solution of the following hierarchical variational inequality*:

$$\langle V\bar{x} - \theta, q - \bar{x}\rangle \ge 0, \quad \forall q \in \Pi_6.$$

Proof For each $i \in \mathbb{N}$, let $S_{i,\gamma} = \gamma S_i + (1-\gamma)I$. Then $S_{i,\gamma}$ is a quasi-nonexpansive mapping and $\operatorname{Fix} S_i = \operatorname{Fix} S_{i,\gamma}$ for each $i \in \mathbb{N}$. Then Theorem 3.6 follows immediately from Theorem 3.5. □

(III) We study split common fixed points for a finite family of ρ_i-strictly pseudo-nonspreading mappings, zeros for a countable family of monotone mappings and common fixed points for a countable family of δ_i-strictly pseudo-nonspreading mappings:

$$\begin{cases} \text{Find } \bar{x} \in C \text{ such that} \\ \bar{x} \in \bigcap_{i=1}^{\infty} B_i^{-1}0 \cap \bigcap_{i=1}^{\infty} \operatorname{Fix}(S_i) \end{cases}$$

and

$$\begin{cases} \bar{u} = A\bar{x} \in H_2, \bar{u}_i = A_i\bar{x} \text{ such that} \\ \bar{u} \in \bigcap_{i=1}^{N} \operatorname{Fix}(T_i). \end{cases}$$

Theorem 3.7 *Suppose that* $\Pi_7 := \{x \in C : x \in \bigcap_{i=1}^{\infty} B_i^{-1}0 \bigcap_{i=1}^{\infty} \operatorname{Fix}(S_i)$ *and* $Ax \in \bigcap_{i=1}^{N} \operatorname{Fix}(T_i)\} \neq \emptyset$. *Take* $\mu \in \mathbb{R}$ *as follows*:

$$0 < \mu < \frac{2\bar{\gamma}}{L^2}.$$

Let $\{x_n\} \subset H$ *be defined by*

$$(3.7)\quad \begin{cases} x_1 \in C \quad \text{chosen arbitrarily}, \\ y_n = [I - bA^*(I - T_{n(\operatorname{mod} N)})A]x_n, \\ s_n = a_{n,0}y_n + \sum_{i=1}^{\infty} b_{n,i}S_{i,\gamma}y_n + \sum_{i=1}^{\infty} a_{n,i}(I - r_iB_i)y_n, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n)(\beta_n\theta_n + (1 - \beta_n V)s_n), \end{cases}$$

where $S_{i,\gamma} = \gamma I + (1-\gamma)S_i$, $i \geq 1$, $\gamma \in [\delta, 1)$ *for each* $n \in \mathbb{N}$, $\{\alpha_n, \beta_n\} \subset (0,1)$, $\{a_{n,i} : n, i \in \mathbb{N}\} \subset [0,1]$, $\{b_{n,i} : n, i \in \mathbb{N}\} \subset [0,1]$ *and* $\{r_i\} \subset (0,\infty)$. *Assume the following*:

(i) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$;
(ii) $\lim_{n\to\infty} \beta_n = 0$ *and* $\sum_{n=1}^{\infty} \beta_n = \infty$;
(iii) $a_{n,0} + \sum_{i=1}^{\infty} b_{n,i} + \sum_{i=1}^{\infty} a_{n,i} = 1$, $\liminf_{n\to\infty} a_{n,0}a_{n,i} > 0$, $\liminf_{n\to\infty} a_{n,0}b_{n,i} > 0$, $\forall i \in \mathbb{N}$ *and* $0 < b < \frac{1-\rho}{\|A\|^2}$;
(iv) $0 < r_i < 2\kappa_i$ *for all* $i \in \mathbb{N}$ *and* $\lim_{n\to\infty} \theta_n = \theta$ *for some* $\theta \in H$.

Then $\lim_{n\to\infty} x_n = \bar{x}$, *where* $\bar{x} = P_{\Pi_4}(\bar{x} - V\bar{x} + \theta)$. *This point* $\bar{x}$ *is also a unique solution of the following hierarchical variational inequality*:

$$\langle V\bar{x} - \theta, q - \bar{x}\rangle \geq 0, \quad \forall q \in \Pi_7.$$

Proof For each $i \in \mathbb{N}$, S_i is a δ_i-strictly pseudo-nonspreading mapping and $\gamma \in [\delta, 1)$, by Lemma 2.9, we have $S_{i,\gamma}$ is quasi-nonexpansive. By Lemma 2.1, we see that $(I - r_iB_i)$ is averaged for each $i \in \mathbb{N}$.

For each $i \in \mathbb{N}$, let $F_{2i} = S_{i,\gamma}$ and let $F_{2i-1} = (I - r_iB_i)$ in Theorem 3.1. Then F_i is a quasi-nonexpansive mapping and

$$\bigcap_{i=1}^{\infty} \operatorname{Fix}(F_i) = \bigcap_{i=1}^{\infty} \operatorname{Fix}\bigl((I - r_iB_i)\bigr) \cap \bigcap_{i=1}^{\infty} \operatorname{Fix}(S_{i,\gamma}).$$

Thus, $\Pi_7 = \Pi_1$. Then Theorem 3.7 follows immediately from Theorem 3.1. □

(IV) Let $f_i : C \times C \to \mathbb{R}$ be a bifunction satisfying conditions (A1)-(A4). We study split common solutions for a countable family of equilibrium problems, common fixed points for a countable family of quasi-nonexpansive mappings and common fixed points of a finite family of ρ_i-strictly pseudo-nonspreading mappings.

That is,

$$\begin{cases} \text{Find } \bar{x} \in C \text{ such that} \\ \bar{x} \in \bigcap_{i=1}^{\infty} \operatorname{EP}(f_i) \cap \bigcap_{i=1}^{\infty} \operatorname{Fix}(W_i) \end{cases}$$

for all $i \in \mathbb{N}$, and

$$\begin{cases} \bar{u} = A\bar{x} \in H_2 \text{ such that} \\ \bar{u} \in \bigcap_{i=1}^{N} \operatorname{Fix}(T_i). \end{cases}$$

Theorem 3.8 *For each* $i \in \mathbb{N}$, *let* A_{f_i} *be as in Lemma* 2.7. *Suppose that* $\Pi_8 := \{x \in C : x \in \bigcap_{i=1}^{\infty} \operatorname{EP}(f_i) \cap \bigcap_{i=1}^{\infty} \operatorname{Fix}(W_i)$ *and* $Ax \in \bigcap_{i=1}^{N} \operatorname{Fix}(T_i)\} \neq \emptyset$. *Take* $\mu \in \mathbb{R}$ *as follows*:

$$0 < \mu < \frac{2\bar{\gamma}}{L^2}.$$

Let $\{x_n\} \subset H$ *be defined by*

$$\begin{cases} x_1 \in C \quad \textit{chosen arbitrarily}, \\ y_n = [I - bA^*(I - T_{n(\operatorname{mod} N)})A]x_n, \\ s_n = a_{n,0}y_n + \sum_{i=1}^{\infty} b_{n,i}W_iy_n + \sum_{i=1}^{\infty} a_{n,i}J_{r_i}^{A_{f_i}}y_n, \\ x_{n+1} = \alpha_nx_n + (1-\alpha_n)(\beta_n\theta_n + (1-\beta_nV)s_n) \end{cases} \tag{3.8}$$

for each $n \in \mathbb{N}$, $\{\alpha_n, \beta_n\} \subset (0,1)$, $\{a_{n,i} : n, i \in \mathbb{N}\} \subset [0,1]$, $\{b_{n,i} : n, i \in \mathbb{N}\} \subset [0,1]$ *and* $\{r_i\} \subset (0,\infty)$. *Assume the following*:

(i) $0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty} \alpha_n < 1$;
(ii) $\lim_{n\to\infty} \beta_n = 0$ *and* $\sum_{n=1}^{\infty} \beta_n = \infty$;
(iii) $a_{n,0} + \sum_{i=1}^{\infty} b_{n,i} + \sum_{i=1}^{\infty} a_{n,i} = 1$, $\liminf_{n\to\infty} a_{n,0}a_{n,i} > 0$, $\liminf_{n\to\infty} a_{n,0}b_{n,i} > 0$, $\forall i \in \mathbb{N}$ *and* $0 < b < \frac{1-\rho}{\|A\|^2}$;
(iv) $\lim_{n\to\infty} \theta_n = \theta$ *for some* $\theta \in H$.

Then $\lim_{n\to\infty} x_n = \bar{x}$, *where* $\bar{x} = P_{\Pi_8}(\bar{x} - V\bar{x} + \theta)$. *This point* $\bar{x}$ *is also a unique solution of the following hierarchical variational inequality*:

$$\langle V\bar{x} - \theta, q - \bar{x}\rangle \ge 0, \quad \forall q \in \Pi_8.$$

Proof For each $i \in \mathbb{N}$, by Lemma 2.7, we know that $\mathrm{EP}(f_i) = A_{f_i}^{-1}0$ and A_{f_i} is a maximal monotone operator with the domain of $A_{f_i} \subset C$. For each $i \in \mathbb{N}$, by Lemma 2.3, we know that $J_{\lambda_i}^{A_{f_i}}$ is firmly nonexpansive, so $J_{\lambda_i}^{A_{f_i}}$ is quasi-nonexpansive. For each $i \in \mathbb{N}$, let $F_{2i} = W_i$, and let $F_{2i-1} = J_{\lambda_i}^{A_{f_i}}$ in Theorem 3.1. Then F_i is a quasi-nonexpansive mapping and

$$\bigcap_{i=1}^{\infty} \mathrm{Fix}(F_i) = \bigcap_{i=1}^{\infty} \mathrm{Fix}\bigl(J_{\lambda_i}^{A_{f_i}}\bigr) \cap \bigcap_{i=1}^{\infty} \mathrm{Fix}(W_i).$$

$\Pi_9 = \Pi_1$. By Theorem 3.1, we have that $\lim_{n\to\infty} x_n = \bar{x}$, where $\bar{x} = P_{\Pi_8}(\bar{x} - V\bar{x} + \theta)$. This point $\bar{x}$ is also a unique solution of the following hierarchical variational inequality:

$$\langle V\bar{x} - \theta, q - \bar{x}\rangle \ge 0, \quad \forall q \in \Pi_8.$$

The proof is completed. □

Remark 3.4 Theorem 3.8 improves Theorem 3.1 in [34] and Theorem 15 in [33] in the sense that our convergence is for the split feasibility problem and for the common solutions of a countable family of equilibrium problems and fixed points for a countable family of quasi-nonexpansive mappings.

(V) Let $f_i : C \times C \to \mathbb{R}$ be a bifunction satisfying conditions (A1)-(A4). We study split common solutions for a countable family of equilibrium problems, common fixed points for a countable family of δ_i-strictly pseudo-nonspreading mappings and common fixed points of a finite family of ρ_i-strictly pseudo-nonspreading mappings:

$$\begin{cases} \text{Find } \bar{x} \in C \text{ such that} \\ \bar{x} \in \bigcap_{i=1}^{\infty} \mathrm{EP}(f_i) \cap \bigcap_{i=1}^{\infty} \mathrm{Fix}(S_i) \end{cases}$$

for all $i \in \mathbb{N}$, and

$$\begin{cases} \bar{u} = A\bar{x} \in H_2 \text{ such that} \\ \bar{u} \in \bigcap_{i=1}^{N} \mathrm{Fix}(T_i). \end{cases}$$

Theorem 3.9 *For each* $i \in \mathbb{N}$, *let* A_{f_i} *be as in Lemma* 2.7, *and suppose that* $\Pi_9 := \{x \in C : x \in \bigcap_{i=1}^{\infty} \mathrm{EP}(f_i) \cap \bigcap_{i=1}^{\infty} \mathrm{Fix}(S_i)$ *and* $Ax \in \bigcap_{i=1}^{N} \mathrm{Fix}(T_i)\} \neq \emptyset$. *Take* $\mu \in \mathbb{R}$ *as follows*:

$$0 < \mu < \frac{2\bar{\gamma}}{L^2}.$$

Let $\{x_n\} \subset H$ *be defined by*

$$(3.9)\qquad \begin{cases} x_1 \in C \quad \text{chosen arbitrarily}, \\ y_n = [I - bA^*(I - T_{n(\bmod N)})A]x_n, \\ s_n = a_{n,0}y_n + \sum_{i=1}^{\infty} b_{n,i}S_{i,\gamma}y_n + \sum_{i=1}^{\infty} a_{n,i}J_{r_i}^{A_{f_i}}y_n, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)(\beta_n\theta_n + (1-\beta_n V)s_n), \end{cases}$$

where $S_{i,\gamma} = \gamma I + (1-\gamma)S_i$, $i \geq 1$, $\gamma \in [\delta, 1)$ *for each* $n \in \mathbb{N}$, $\{\alpha_n, \beta_n\} \subset (0,1)$, $\{a_{n,i} : n, i \in \mathbb{N}\} \subset [0,1]$, $\{b_{n,i} : n, i \in \mathbb{N}\} \subset [0,1]$ *and* $\{r_i\} \subset (0,\infty)$. *Assume the following*:

(i) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$;

(ii) $\lim_{n\to\infty} \beta_n = 0$ *and* $\sum_{n=1}^{\infty} \beta_n = \infty$;

(iii) $a_{n,0} + \sum_{i=1}^{\infty} b_{n,i} + \sum_{i=1}^{\infty} a_{n,i} = 1$, $\liminf_{n\to\infty} a_{n,0}a_{n,i} > 0$, $\liminf_{n\to\infty} a_{n,0}b_{n,i} > 0$, $\forall i \in \mathbb{N}$ *and* $0 < b < \frac{1-\rho}{\|A\|^2}$;

(iv) $\lim_{n\to\infty} \theta_n = \theta$ *for some* $\theta \in H$.

Then $\lim_{n\to\infty} x_n = \bar{x}$, *where* $\bar{x} = P_{\Pi_9}(\bar{x} - V\bar{x} + \theta)$. *This point* $\bar{x}$ *is also a unique solution of the following hierarchical variational inequality*:

$$\langle V\bar{x} - \theta, q - \bar{x}\rangle \geq 0, \quad \forall q \in \Pi_9.$$

Proof For each $i \in \mathbb{N}$, let $F_{2i} = S_{i,\gamma}$ and $F_{2i-1} = J_{\lambda_i}^{A_{f_i}}$. Then F_i is a quasi-nonexpansive mapping and

$$\bigcap_{i=1}^{\infty} \operatorname{Fix}(F_i) = \bigcap_{i=1}^{\infty} \operatorname{Fix}\bigl(J_{\lambda_i}^{A_{f_i}}\bigr) \cap \bigcap_{i=1}^{\infty} \operatorname{Fix}(S_i).$$

$\Pi_9 = \Pi_1$. By Theorem 3.1, we have that $\lim_{n\to\infty} x_n = \bar{x}$, where $\bar{x} = P_{\Pi_9}(\bar{x} - V\bar{x} + \theta)$. This point $\bar{x}$ is also a unique solution of the following hierarchical variational inequality:

$$\langle V\bar{x} - \theta, q - \bar{x}\rangle \geq 0, \quad \forall q \in \Pi_9,$$

and the proof is completed. □

(VI) For each $i \in \mathbb{N}$, let $M_i : C \to H_1$ be a hemicontinuous monotone mapping. We study split common solutions for a countable family of variational inequality problems, common fixed points of a countable family of quasi-nonexpansive mappings and common fixed points of a finite family of ρ_i-strictly pseudo-nonspreading mappings:

$$\begin{cases} \text{Find } \bar{x} \in C \text{ such that} \\ \bar{x} \in \bigcap_{i=1}^{\infty} \operatorname{VI}(C, M_i) \cap \bigcap_{i=1}^{\infty} \operatorname{Fix}(W_i), \end{cases}$$

where

$$\operatorname{VI}(C, M_i) = \bigl\{z \in C : \langle y - z, M_i z\rangle \geq 0 \text{ for all } y \in C\bigr\}$$

and

$$\begin{cases} \bar{u} = A\bar{x} \in H_2 \text{ such that} \\ \bar{u} \in \bigcap_{i=1}^{N} \operatorname{Fix}(T_i). \end{cases}$$

For each $i \in \mathbb{N}$, let $f_i(x,y) = \langle y-x, M_i x\rangle$ for all $x \in C$ in Theorem 3.9, we have the following theorem.

Theorem 3.10 *For each $i \in \mathbb{N}$, let A_{f_i} be as in Lemma 2.7 and suppose that M_i is bounded on any line segment of C; that is, for each $x, y \in C$, there exists $\kappa_i(x,y) > 0$ such that $M_i(tx + (1-t)y) \leq \kappa_i(x,y)$ for all $t \in [0,1]$. Suppose that $\Pi_{10} := \{x \in C : x \in \bigcap_{i=1}^{\infty} \mathrm{VI}(C, M_i) \cap \bigcap_{i=1}^{\infty} \mathrm{Fix}(W_i)$ and $Ax \in \bigcap_{i=1}^{N} \mathrm{Fix}(T_i)\} \neq \emptyset$. Take $\mu \in \mathbb{R}$ as follows:*

$$0 < \mu < \frac{2\bar{\gamma}}{L^2}.$$

Let $\{x_n\} \subset H$ be defined by

$$\begin{cases} x_1 \in C \quad \text{chosen arbitrarily}, \\ y_n = [I - bA^*(I - T_{n(\operatorname{mod} N)})A]x_n, \\ s_n = a_{n,0}y_n + \sum_{i=1}^{\infty} b_{n,i} W_i y_n + \sum_{i=1}^{\infty} a_{n,i} J_{r_i}^{A_{f_i}}, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)(\beta_n \theta_n + (1-\beta_n V)s_n), \end{cases} \tag{3.10}$$

where $f_i(x,y) = \langle y-x, M_i x\rangle$ for all $x, y \in C$, for each $n \in \mathbb{N}$, $\{\alpha_n, \beta_n\} \subset (0,1)$, $\{a_{n,i} : n, i \in \mathbb{N}\} \subset [0,1]$, $\{b_{n,i} : n, i \in \mathbb{N}\} \subset [0,1]$, and $\{r_i\} \subset (0,\infty)$. Assume the following:

(i) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$;

(ii) $\lim_{n\to\infty} \beta_n = 0$ *and* $\sum_{n=1}^{\infty} \beta_n = \infty$;

(iii) $a_{n,0} + \sum_{i=1}^{\infty} b_{n,i} + \sum_{i=1}^{\infty} a_{n,i} = 1$, $\liminf_{n\to\infty} a_{n,0}a_{n,i} > 0$, $\liminf_{n\to\infty} a_{n,0}b_{n,i} > 0$, $\forall i \in \mathbb{N}$ *and* $0 < b < \frac{1-\rho}{\|A\|^2}$;

(iv) $\lim_{n\to\infty} \theta_n = \theta$ *for some* $\theta \in H$.

Then $\lim_{n\to\infty} x_n = \bar{x}$, where $\bar{x} = P_{\Pi_{10}}(\bar{x} - V\bar{x} + \theta)$. This point $\bar{x}$ is also a unique solution of the following hierarchical variational inequality:

$$\langle V\bar{x} - \theta, q - \bar{x}\rangle \geq 0, \quad \forall q \in \Pi_{10}.$$

Proof Since $M_i : C \to H$ is a hemicontinuous monotone mapping, it is sufficient to show that for each $i \in \mathbb{N}$, f_i satisfies conditions (A1)-(A4). Since M_i is monotone, we have

$$\langle y - x, M_i y - M_i x\rangle \geq 0$$

and

$$f_i(x,y) + f_i(y,x) = \langle y - x, M_i x - M_i y\rangle \leq 0.$$

Therefore (A2) is satisfied. Since $M_i : C \to H$ is a hemicontinuous mapping, for any $x, y \in H_1$, the mapping $g : [0,1] \to H_1$ defined by $g(t) = M_i(tx + (1-t)y)$ is continuous, where H_1 has a weak topology. Since M_i is bounded on any line segment of C and each $x, y, z \in H_1$,

$$\begin{aligned} g\big(tz + (1-t)x, y\big) &= \big\langle y - tz - (1-t)x, M_i\big(tz + (1-t)x\big)\big\rangle \\ &= \big\langle y - tz - (1-t)x - (y-x), M_i\big(tz + (1-t)x\big)\big\rangle \\ &\quad + \big\langle (y-x), M_i\big(tz + (1-t)x\big)\big\rangle, \end{aligned}$$

it is easy to see that $\lim_{t\downarrow 0} f_i(tz+(1-t)x,y) = \lim_{t\downarrow 0}\langle y-tz-(1-t)x, M_i(tz+(1-t)x)\rangle = \langle y-x, M_i(x)\rangle = f_i(x,y)$. This shows that condition (A3) is satisfied. It is easy to see that (A1) and (A4) are satisfied. Then Theorem 3.10 follows immediately from Theorem 3.8. □

Remark 3.5 (i) It is easy to see that if M_i is continuous on C, then M_i is hemicontinuous and bounded on any line segment of C.

(ii) It is easy to see that Theorem 3.10 is true if for each $i\in\mathbb{N}$, $M_i : C\to H$ is a κ_i-inverse strongly monotone mapping or is a continuous strongly monotone mapping. Theorem 3.10 improves Theorem 3.1 of Qing and Shang [35] in the sense that our convergence is for the split feasibility problem and for the common solutions of a countable family of variational inequality problems of hemicontinuous monotone mappings and fixed points for a countable family of quasi-nonexpansive mappings.

(VII) For each $i\in\mathbb{N}$, let $M_i : C\to H$ be a hemicontinuous monotone mapping. We study split common solutions for a countable family of variational inequality problems, common fixed points for a countable family of δ_i-strictly pseudo-nonspreading mappings and common fixed points of a finite family of ρ_i-strictly pseudo-nonspreading mappings:

$$\begin{cases} \text{Find } \bar{x}\in C \text{ such that} \\ \bar{x}\in\bigcap_{i=1}^{\infty}\mathrm{VI}(C,M_i)\cap\bigcap_{i=1}^{\infty}\mathrm{Fix}(S_i) \end{cases}$$

and

$$\begin{cases} \bar{u}=A\bar{x}\in H_2 \text{ such that} \\ \bar{u}\in\bigcap_{i=1}^{N}\mathrm{Fix}(T_i). \end{cases}$$

Applying Theorem 3.9 and following the same argument as in Theorem 3.10, we have the following result.

Theorem 3.11 *For each $i\in\mathbb{N}$, let A_{f_i} be as in Lemma 2.7 and suppose that M_i is bounded on any line segment of C. Suppose that $\Pi_{11} := \{x\in C : x\in\bigcap_{i=1}^{\infty}\mathrm{VI}(C,M_i)\cap\bigcap_{i=1}^{\infty}\mathrm{Fix}\,S_i$ and $Ax\in\bigcap_{i=1}^{N}\mathrm{Fix}(T_i)\}\neq\emptyset$. Take $\mu\in\mathbb{R}$ as follows:*

$$0<\mu<\frac{2\bar{\gamma}}{L^2}.$$

Let $\{x_n\}\subset H$ be defined by

$$\begin{cases} x_1\in C \quad \text{chosen arbitrarily}, \\ y_n=[I-bA^*(I-T_{n(\mathrm{mod}\,N)})A]x_n, \\ s_n=a_{n,0}y_n+\sum_{i=1}^{\infty}b_{n,i}S_{i,\gamma}y_n+\sum_{i=1}^{\infty}a_{n,i}J_{r_i}^{A_{f_i}}y_n, \\ x_{n+1}=\alpha_n x_n+(1-\alpha_n)(\beta_n\theta_n+(1-\beta_n V)s_n), \end{cases} \tag{3.11}$$

where $S_{i,\gamma}=\gamma I+(1-\gamma)S_i$, $\gamma\in[\delta,1)$, $f_i(x,y)=\langle y-x, M_i x\rangle$ for all $x,y\in C$, $i\in\mathbb{N}$, for each $n\in\mathbb{N}$, $\{\alpha_n,\beta_n\}\subset(0,1)$, $\{a_{n,i}: n,i\in\mathbb{N}\}\subset[0,1]$, $\{b_{n,i}: n,i\in\mathbb{N}\}\subset[0,1]$ and $\{r_i\}\subset(0,\infty)$. Assume the following:

(i) $0<\liminf_{n\to\infty}\alpha_n\leq\limsup_{n\to\infty}\alpha_n<1$;
(ii) $\lim_{n\to\infty}\beta_n=0$ *and* $\sum_{n=1}^{\infty}\beta_n=\infty$;

(iii) $a_{n,0} + \sum_{i=1}^{\infty} b_{n,i} + \sum_{i=1}^{\infty} a_{n,i} = 1$, $\liminf_{n\to\infty} a_{n,0}a_{n,i} > 0$, $\liminf_{n\to\infty} a_{n,0}b_{n,i} > 0$, $\forall i \in \mathbb{N}$ *and* $0 < b < \frac{1-\rho}{\|A\|^2}$;

(iv) $\lim_{n\to\infty} \theta_n = \theta$ *for some* $\theta \in H$.

Then $\lim_{n\to\infty} x_n = \bar{x}$, *where* $\bar{x} = P_{\Pi_{11}}(\bar{x} - V\bar{x} + \theta)$. *This point* $\bar{x}$ *is also a unique solution of the following hierarchical variational inequality*:

$$\langle V\bar{x} - \theta, q - \bar{x}\rangle \geq 0, \quad \forall q \in \Pi_{11}.$$

(VIII) For each $i \in \mathbb{N}$, let $\Psi_i : C \to C$ be a hemicontinuous pseudo-contractive mapping. We study split common fixed points for a countable family of hemicontinuous pseudo-contractive mappings, a countable family of quasi-nonexpansive mappings and a finite family of ρ_i-strictly pseudo-nonspreading mappings:

$$\begin{cases} \text{Find } \bar{x} \in C \text{ such that} \\ \bar{x} \in \bigcap_{i=1}^{\infty} \operatorname{Fix}(\Psi_i) \cap \bigcap_{i=1}^{\infty} \operatorname{Fix}(W_i) \end{cases}$$

and

$$\begin{cases} \bar{u} = A\bar{x} \in H_2 \text{ such that} \\ \bar{u} \in \bigcap_{i=1}^{N} \operatorname{Fix}(T_i). \end{cases}$$

For $i \in \mathbb{N}$, let $f_i(x,y) = \langle y - x, (I - \Psi_i)x\rangle$ for all $x, y \in C$ in Theorem 3.11, we have the following theorem.

Theorem 3.12 *For each* $i \in \mathbb{N}$, *let* A_{f_i} *be as in Lemma* 2.7 *and suppose that* Ψ_i *is bounded on any line segment of* C. *Suppose that* $\Pi_{12} := \{x \in C : x \in \bigcap_{i=1}^{\infty} \operatorname{Fix}(\Psi_i) \bigcap_{i=1}^{\infty} \operatorname{Fix}(W_i)$ *and* $Ax \in \bigcap_{i=1}^{N} \operatorname{Fix}(T_i)\} \neq \emptyset$. *Take* $\mu \in \mathbb{R}$ *as follows*:

$$0 < \mu < \frac{2\bar{\gamma}}{L^2}.$$

Let $\{x_n\} \subset H$ *be defined by*

$$\begin{cases} x_1 \in C \quad \textit{chosen arbitrarily}, \\ y_n = [I - bA^*(I - T_{n(\mathrm{mod}\, N)})A]x_n, \\ s_n = a_{n,0}y_n + \sum_{i=1}^{\infty} b_{n,i}W_i y_n + \sum_{i=1}^{\infty} a_{n,i}J_{r_i}^{A_{f_i}}, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)(\beta_n\theta_n + (1-\beta_n V)s_n), \end{cases} \tag{3.12}$$

where $f_i(x,y) = \langle y - x, x - \Psi_i x\rangle$ *for all* $x, y \in C$, *for each* $n \in \mathbb{N}$, $\{\alpha_n, \beta_n\} \subset (0,1)$, $\{a_{n,i} : n, i \in \mathbb{N}\} \subset [0,1]$, $\{b_{n,i} : n, i \in \mathbb{N}\} \subset [0,1]$ *and* $\{r_i\} \subset (0,\infty)$. *Assume the following*:

(i) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$;

(ii) $\lim_{n\to\infty} \beta_n = 0$ *and* $\sum_{n=1}^{\infty} \beta_n = \infty$;

(iii) $a_{n,0} + \sum_{i=1}^{\infty} b_{n,i} + \sum_{i=1}^{\infty} a_{n,i} = 1$, $\liminf_{n\to\infty} a_{n,0}a_{n,i} > 0$, $\liminf_{n\to\infty} a_{n,0}b_{n,i} > 0$, $\forall i \in \mathbb{N}$ *and* $0 < b < \frac{1-\rho}{\|A\|^2}$;

(iv) $\lim_{n\to\infty} \theta_n = \theta$ *for some* $\theta \in H$.

Then $\lim_{n\to\infty} x_n = \bar{x}$, *where* $\bar{x} = P_{\Pi_{12}}(\bar{x} - V\bar{x} + \theta)$. *This point* $\bar{x}$ *is also a unique solution of the following hierarchical variational inequality*:

$$\langle V\bar{x} - \theta, q - \bar{x}\rangle \geq 0, \quad \forall q \in \Pi_{12}.$$

Proof Let $M_i = I - \Psi_i$. Since, for each $i \in \mathbb{N}$, Ψ_i is a pseudo-contraction, it is easy to see that M_i is monotone. Then Theorem 3.12 follows from Theorem 3.10. □

Remark 3.6 Theorem 3.12 improves Theorem 3.1 of Cheng *et al.* in [13], Theorem 3.1 of Deng [14], Theorem 3.1 of Zegeye and Shahzad in [16] in the sense that our convergence is for the split feasibility problem and for the common fixed points of a countable family of quasi-nonexpansive mappings and a countable family hemicontinuous pseudo-contractive mappings without any closedness or Lipschitz continuity assumption on these pseudo-contractive mappings.

(IX) For each $i \in \mathbb{N}$, let $\Psi_i : C \to C$ be a hemicontinuous pseudo-contractive mapping. We study split common fixed points for a countable family of hemicontinuous pseudo-contractive mappings, a countable family of δ_i-strictly pseudo-nonspreading mappings and a finite family of ρ_i-strictly pseudo-nonspreading mappings:

$$\begin{cases} \text{Find } \bar{x} \in C \text{ such that} \\ \bar{x} \in \bigcap_{i=1}^{\infty} \operatorname{Fix}(\Psi_i) \cap \bigcap_{i=1}^{\infty} \operatorname{Fix}(S_i) \end{cases}$$

and

$$\begin{cases} \bar{u} = A\bar{x} \in H_2 \text{ such that} \\ \bar{u} \in \bigcap_{i=1}^{N} \operatorname{Fix}(T_i). \end{cases}$$

For each $i \in \mathbb{N}$, let $W_i = S_i$ in Theorem 3.12, we have the following theorem.

Theorem 3.13 *Suppose that for each $i \in \mathbb{N}$, M_i is bounded on any line segment of C and $\Pi_{13} := \{x \in C : x \in \bigcap_{i=1}^{\infty} \operatorname{Fix}(\Psi_i) \bigcap_{i=1}^{\infty} \operatorname{Fix}(S_i)$ and $Ax \in \bigcap_{i=1}^{N} \operatorname{Fix}(T_i)\} \neq \emptyset$. Take $\mu \in \mathbb{R}$ as follows*:

$$0 < \mu < \frac{2\bar{\gamma}}{L^2}.$$

Let $\{x_n\} \subset H$ be defined by

$$(3.13)\quad \begin{cases} x_1 \in C \quad \textit{chosen arbitrarily}, \\ y_n = [I - bA^*(I - T_{n(\bmod N)})A]x_n, \\ s_n = a_{n,0}y_n + \sum_{i=1}^{\infty} b_{n,i}S_{i,\gamma}y_n + \sum_{i=1}^{\infty} a_{n,i}J_{r_i}^{A_{f_i}}, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n)(\beta_n\theta_n + (1 - \beta_n V)s_n), \end{cases}$$

where $S_{i,\gamma} = \gamma I + (1-\gamma)S_i$, $\gamma \in [\delta, 1)$, $f_i(x,y) = \langle y - x, x - \Psi_i x\rangle$ for all $x, y \in C$, for each $n \in \mathbb{N}$, $\{\alpha_n, \beta_n\} \subset (0,1)$, $\{a_{n,i} : n, i \in \mathbb{N}\} \subset [0,1]$, $\{b_{n,i} : n, i \in \mathbb{N}\} \subset [0,1]$ and $\{r_i\} \subset (0,\infty)$. Assume the following:

(i) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$;
(ii) $\lim_{n\to\infty} \beta_n = 0$ *and* $\sum_{n=1}^{\infty} \beta_n = \infty$;
(iii) $a_{n,0} + \sum_{i=1}^{\infty} b_{n,i} + \sum_{i=1}^{\infty} a_{n,i} = 1$, $\liminf_{n\to\infty} a_{n,0}a_{n,i} > 0$, $\liminf_{n\to\infty} a_{n,0}b_{n,i} > 0$, $\forall i \in \mathbb{N}$ *and* $0 < b < \frac{1-\rho}{\|A\|^2}$;
(iv) $\lim_{n\to\infty} \theta_n = \theta$ *for some* $\theta \in H$.

Then $\lim_{n\to\infty} x_n = \bar{x}$, *where* $\bar{x} = P_{\Pi_{13}}(\bar{x} - V\bar{x} + \theta)$. *This point* $\bar{x}$ *is also a unique solution of the following hierarchical variational inequality*:

$$\langle V\bar{x} - \theta, q - \bar{x}\rangle \geq 0, \quad \forall q \in \Pi_{13}.$$

(X) For each $i \in \mathbb{N}$, let $\Psi_i : C \to C$ be a hemicontinuous pseudo-contractive mapping. We study a split feasibility problem for common fixed points of a countable family of hemicontinuous pseudo-contractive mappings, zeros for a countable family of mappings which are the sum of two monotone mappings and common fixed points of a finite family of ρ_i-strictly pseudo-nonspreading mappings.

That is,

$$\begin{cases} \text{Find } \bar{x} \in C \text{ such that} \\ \bar{x} \in \bigcap_{i=1}^{\infty}(G_i + B_i)^{-1}0 \cap \bigcap_{i=1}^{\infty} \operatorname{Fix}(\Psi_i) \end{cases}$$

and

$$\begin{cases} \bar{u} = A\bar{x} \in H_2 \text{ such that} \\ \bar{u} \in \bigcap_{i=1}^{N} \operatorname{Fix}(T_i). \end{cases}$$

For each $i \in \mathbb{N}$, let $W_i = J_{r_i}^{G_i}(I - r_iB_i)$ in Theorem 3.12, we have the following theorem.

Theorem 3.14 *Suppose that for each* $i \in \mathbb{N}$, Ψ_i *is bounded on any line segment of* C *and* $\Pi_{14} := \{x \in C : x \in \bigcap_{i=1}^{\infty} \operatorname{Fix}(\Psi_i) \bigcap_{i=1}^{\infty} \operatorname{Fix}(J_{r_i}^{G_i}(I - r_iB_i))$ *and* $Ax \in \bigcap_{i=1}^{N} \operatorname{Fix}(T_i)\} \neq \emptyset$. *Take* $\mu \in \mathbb{R}$ *as follows*:

$$0 < \mu < \frac{2\bar{\gamma}}{L^2}.$$

Let $\{x_n\} \subset H$ *be defined by*

$$\begin{cases} x_1 \in C \quad \textit{chosen arbitrarily}, \\ y_n = [I - bA^*(I - T_{n(\bmod N)})A]x_n, \\ s_n = a_{n,0}y_n + \sum_{i=1}^{\infty} b_{n,i}J_{\lambda_i}^{A_{f_i}}y_n + \sum_{i=1}^{\infty} a_{n,i}J_{r_i}^{G_i}(I - r_iB_i)y_n, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n)(\beta_n\theta_n + (1 - \beta_n V)s_n), \end{cases} \tag{3.14}$$

where $f_i(x,y) = \langle y - x, (I - \Psi_i)x\rangle$ *for all* $x, y \in C$, *for each* $n \in \mathbb{N}$, $\{\alpha_n, \beta_n\} \subset (0,1)$, $\{a_{n,i} : n, i \in \mathbb{N}\} \subset [0,1]$, $\{b_{n,i} : n, i \in \mathbb{N}\} \subset [0,1]$ *and* $\{\lambda_i, r_i\} \subset (0,\infty)$. *Assume the following*:

(i) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$;

(ii) $\lim_{n\to\infty} \beta_n = 0$ *and* $\sum_{n=1}^{\infty} \beta_n = \infty$;

(iii) $a_{n,0} + \sum_{i=1}^{\infty} b_{n,i} + \sum_{i=1}^{\infty} a_{n,i} = 1$, $\liminf_{n\to\infty} a_{n,0}a_{n,i} > 0$, $\liminf_{n\to\infty} a_{n,0}b_{n,i} > 0$, $\forall i \in \mathbb{N}$ *and* $0 < b < \frac{1-\rho}{\|A\|^2}$;

(iv) $0 < r_i < 2\kappa_i$ *for all* $i \in \mathbb{N}$ *and* $\lim_{n\to\infty} \theta_n = \theta$ *for some* $\theta \in H$.

Then $\lim_{n\to\infty} x_n = \bar{x}$, *where* $\bar{x} = P_{\Pi_{14}}(\bar{x} - V\bar{x} + \theta)$. *This point* $\bar{x}$ *is also a unique solution of the following hierarchical variational inequality*:

$$\langle V\bar{x} - \theta, q - \bar{x}\rangle \geq 0, \quad \forall q \in \Pi_{14}.$$

Remark 3.7 Theorem 3.14 improves Theorem 3.1 of Zegeye and Shahzad [10], Theorem 3.1 of Yao *et al.* [11], Theorem 4.1 of Takahashi *et al.* [28], Theorem 2.1 of Cho *et al.* [36] and Theorem 3.1 of Tang [9] in the sense that our convergence is for the split feasibility problem and for the common solutions of a countable family of variational inclusion problems and fixed points of a countable family hemicontinuous pseudo-contractive mappings without any closedness or Lipschitz continuity assumption on these pseudo-contractive mappings.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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References
1. Censor, Y, Elfving, T: A multiprojection algorithm using Bregman projection in a product space. Numer. Algorithms **8**, 221-239 (1994)
2. Byrne, C: Iterative oblique projection onto convex sets and the split feasibility problem. Inverse Probl. **18**, 441-453 (2002)
3. Byrne, C: A unified treatment of some iterative algorithms in signal processing and image reconstruction. Inverse Probl. **20**, 103-120 (2004)
4. Xu, HK: A variable Krasnosel'skii-Mann algorithm and the multiple-set split feasibility problem. Inverse Probl. **22**, 2021-2034 (2006)
5. Xu, HK: Iterative methods for the split feasibility problem in infinite-dimensional Hilbert spaces. Inverse Probl. **26**, 105018 (2010)
6. Xu, HK: Viscosity approximation methods for nonexpansive mappings. J. Math. Anal. Appl. **298**, 279-291 (2004)
7. Xu, HK: An iterative approach to quadratic optimization. J. Optim. Theory Appl. **116**, 659-678 (2003)
8. Chang, SS, Kim, JK, Cho, YJ, Sim, JY: Weak- and strong-convergence theorems of solutions to split feasibility problem for nonspreading type mappings in Hilbert spaces. Fixed Point Theory Appl. **2014**, 11 (2014)
9. Tang, Y: Strong convergence of viscosity approximation methods for the fixed point of pseudo-contractive and monotone mappings. Fixed Point Theory Appl. **2013**, 273 (2013)
10. Zegeye, H, Shahzad, N: Approximating a common point of fixed point of pseudocontractive and zero of sum of monotone mappings. Fixed Point Theory Appl. **2014**, 85 (2014)
11. Yao, Y, Agarwal, RP, Liou, YC: Iterative algorithms for quasi-variational inclusions and fixed point problems of pseudo-contractions. Fixed Point Theory Appl. **2014**, 82 (2014)
12. Wangkeree, R, Nammanee, K: New iterative methods for a common solution of fixed points for pseudo-contractive mappings and variational inequalities. Fixed Point Theory Appl. **2013**, 233 (2013)
13. Cheng, Q, Su, Y, Zhang, J: Convergence theorems of a three-step iteration method for a countable family of pseudocontractive mappings. Fixed Point Theory Appl. **2013**, 100 (2013)
14. Deng, WQ: Strong convergence of a relaxed three-step iterative algorithm for countable families of pseudocontractions. Fixed Point Theory Appl. **2013**, 217 (2013)
15. Ungchittrakool, K, Jarernsuk, A: Strong convergence by a hybrid algorithm for solving generalized mixed equilibrium problems and fixed point problems of a Lipschitz pseudo-contraction in Hilbert spaces. Fixed Point Theory Appl. **2012**, 147 (2012)
16. Zegeye, H, Shahzad, N: An algorithm for a common fixed point of a family of pseudocontractive mappings. Fixed Point Theory Appl. **2013**, 234 (2013)
17. Kurokawa, Y, Takahashi, W: Weak and strong convergence theorems for nonspreading mappings in Hilbert spaces. Nonlinear Anal. **73**, 1562-1568 (2010)
18. Osilike, MO, Isiogugu, FO: Weak and strong convergence theorems for nonspreading-type mappings in Hilbert spaces. Nonlinear Anal. **74**, 1814-1822 (2011)
19. Takahashi, W: Nonlinear Functional Analysis - Fixed Point Theory and Its Applications. Yokohama Publishers, Yokohama (2000)
20. Reich, S: A weak convergence theorem for the alternative method with Bregman distance. In: Kartsatos, AG (ed.) Theory and Applications of Nonlinear Operators of Accretive and Monotone Type, pp. 313-318. Dekker, New York (1996)
21. Combettes, PL: Solving monotone inclusions via compositions of nonexpansive averaged operators. Optimization **53**(5-6), 475-504 (2004)
22. Yu, ZT, Lin, LJ: Hierarchical problems with applications to mathematical programming with multiple sets split feasibility constraints. Fixed Point Theory Appl. **2013**, 283 (2013)

23. Maingé, PE: Strong convergence of projected subgradient methods for nonsmooth and nonstrictly convex minimization. Set-Valued Anal. **16**, 899-912 (2008)
24. Aoyama, K, Kimura, Y, Takahashi, W, Toyoda, M: Approximation of common fixed points of a countable family of nonexpansive mappings in a Banach space. Nonlinear Anal. **67**, 2350-2360 (2007)
25. Blum, E, Oettli, W: From optimization and variational inequalities to equilibrium problems. Math. Stud. **63**, 123-146 (1994)
26. Combettes, PL, Hirstoaga, SA: Equilibrium programming in Hilbert spaces. J. Nonlinear Convex Anal. **6**, 117-136 (2005)
27. Chang, SS, Kim, JK, Wang, XR: Modified block iterative algorithm for solving convex feasibility problems in Banach spaces. J. Inequal. Appl. **2010**, Article ID 869684 (2010)
28. Takahashi, S, Takahashi, W, Toyoda, M: Strong convergence theorems for maximal monotone operators with nonlinear mappings in Hilbert spaces. J. Optim. Theory Appl. **147**, 27-41 (2010)
29. Zhang, C, He, S: A general iterative algorithm for an infinite family of nonexpansive mappings. Fixed Point Theory Appl. **2013**, 138 (2013)
30. Ceng, LC, Wong, NC, Yao, JC: Strong and weak convergence theorems for an infinite family of nonexpansive mappings and applications. Fixed Point Theory Appl. **2012**, 117 (2012)
31. Tian, M, Jing, X: Strong convergence for quasi-nonexpansive mappings in Hilbert space. Fixed Point Theory Appl. **2012**, 88 (2012)
32. Yu, ZT, Lin, LJ, Chuang, CS: Mathematical programming with multiple sets split monotone variational inclusion constraints. Fixed Point Theory Appl. **2014**, 20 (2014)
33. Takahashi, W, Wong, NC, Yao, JC: Iterative common solution for monotone inclusion problems,fixed point problems and equilibrium problems. Fixed Point Theory Appl. **2012**, 181 (2012)
34. Ceng, LC, Gu, SM, Yao, JC: Hybrid iterative method for find common solutions of generalized mixed equilibrium and fixed point problems. Fixed Point Theory Appl. **2012**, 92 (2012)
35. Qing, Y, Shang, M: Convergence of an extragradient-like iterative algorithm for monotone mappings. Fixed Point Theory Appl. **2013**, 67 (2013)
36. Cho, SY, Qin, XL, Wang, L: Strong convergence of a splitting algorithm for treating monotone operators. Fixed Point Theory Appl. **2014**, 94 (2014)

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