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Mock modular Mathieu moonshine modules

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Abstract

We construct super vertex operator algebras which lead to modules for moonshine relations connecting the four smaller sporadic simple Mathieu groups with distinguished mock modular forms. Starting with an orbifold of a free fermion theory, any subgroup of Co_0 that fixes a 3-dimensional subspace of its unique non-trivial 24-dimensional representation commutes with a certain $\mathcal{N}=4$ superconformal algebra. Similarly, any subgroup of Co_0 that fixes a 2-dimensional subspace of the 24-dimensional representation commutes with a certain $\mathcal{N}=2$ superconformal algebra. Through the decomposition of the corresponding twined partition functions into characters of the $\mathcal{N}=4$ (resp. $\mathcal{N}=2$) superconformal algebra, we arrive at mock modular forms which coincide with the graded characters of an infinite-dimensional $\mathbb{Z}$-graded module for the corresponding group. The Mathieu groups are singled out amongst various other possibilities by the moonshine property: requiring the corresponding weak Jacobi forms to have certain asymptotic behaviour near cusps. Our constructions constitute the first examples of explicitly realized modules underlying moonshine phenomena relating mock modular forms to sporadic simple groups. Modules for other groups, including the sporadic groups of McLaughlin and Higman–Sims, are also discussed.

1 Introduction

The investigation of moonshine connecting modular objects, sporadic groups, and 2d conformal field theories has been revitalized in recent years by the discovery of several new classes of examples. While monstrous moonshine [3, 15, 21, 30, 36, 37, 44] remains the best understood and prototypical case, a new class of umbral moonshines tying mock modular forms to automorphism groups of Niemeier lattices has recently been uncovered [13, 14] (cf. also [6]). The best studied example, and the first to be discovered, involves the group M_{24} and was discovered through the study of the elliptic genus of K3 [35]. The twining functions have been constructed in [4, 31, 38, 39] and were proved to be the graded characters of an infinite-dimensional M_{24}-module in [43]. Steps towards a better and deeper understanding of this mock modular moonshine can be found in [7, 8, 18, 40–42, 46, 56, 57, 59–61], and particularly in [9], where the importance of K3 surface geometry for all cases of umbral moonshine is elucidated. Possible connections to spacetime physics in string theory have been discussed in [5, 47–49, 63]. Evidence for a deep connection between monstrous and umbral moonshine has appeared in [25, 55].

In none of these cases, however, has a connection to an underlying conformal field theory[a] (whose Hilbert space furnishes the underlying module) been established. The goal of this paper is to provide first examples of mock modular moonshine for sporadic simple groups G, where the underlying G-module can be explicitly constructed in the state space of a simple and soluble conformal field theory.

Our starting point is the Conway module sketched in [36], studied in detail in [24], and revisited recently in [27]. The original construction was in terms of a supersymmetric theory of bosons on the E_8 root lattice, but this has the drawback of obscuring the true symmetries of the model. In [24], a different formulation of the same theory, as a $\mathbb{Z}_2$ orbifold of the theory of 24 free chiral fermions, was introduced. A priori, the theory has a Spin(24) symmetry. However, one can also view this theory as an $\mathcal{N} = 1$ superconformal field theory. The choice of $\mathcal{N} = 1$ structure breaks the Spin(24) symmetry to a subgroup. In [24] it was shown that the subgroup preserving the natural choice of $\mathcal{N} = 1$ structure is precisely the Conway group Co_0, a double cover of the sporadic group Co_1. In [24] it was shown that this action can be used to attach a normalized principal modulus (i.e. normalized Hauptmodul) for a genus zero group to every element of Co_0.

In this paper, we show that generalizations of the basic strategy of [24, 27] can be used to construct a wide variety of new examples of mock modular moonshine. Instead of choosing an $\mathcal{N} = 1$ superconformal structure, we choose larger extended chiral algebras $\mathcal{A}$. The subgroup of Spin(24) that commutes with a given choice can be determined by simple geometric considerations; in the cases of interest to us, it will be a subgroup that preserves point-wise a 2-plane or a 3-plane in the 24 dimensional representation of Co_0, or equivalently, a subgroup that acts trivially on two or three of the free fermions in some basis. In the rest of the paper, we will use **24** to denote the unique non-trivial 24-dimensional representation of Co_0, and use the term n-plane to refer to a n-dimensional subspace in **24**.

It is natural to ask about the role in moonshine, or geometry, of n-planes in **24** for other values of n. One of the inspirations for our analysis here is the recent result of Gaberdiel–Hohenegger–Volpato [40] which indicates the importance of 4-planes in **24** for non-linear K3 sigma models. The relationship between their results and the Co_0-module considered here is studied in [25], where connections to umbral moonshine for various higher n are also established. We refer the reader to Sect. 9, or the recent articles [1, 10], for a discussion of the interesting case that $n = 1$.

In this work we will focus on the cases where $\mathcal{A}$ is an $\mathcal{N} = 4$ or $\mathcal{N} = 2$ superconformal algebra, though other possibilities exist. In the first case, we demonstrate that any subgroup of Co_0 that preserves a 3-plane in the 24-dimensional representation can stabilise an $\mathcal{N} = 4$ structure. The groups that arise are discussed in e.g. Chapter 10 of the book by Conway and Sloane [16]. They include in particular the Mathieu groups M_{22} and M_{11}. In the second case, where the group need only fix a 2-plane in order to preserve an $\mathcal{N} = 2$ superconformal algebra, there are again many possibilities (again, see [16]), including the larger Mathieu groups M_{23} and M_{12}. Note that the larger the superconformal algebra we wish to preserve, the smaller the global symmetry group is. Corresponding to certain specific choices of the $\mathcal{N} = 0, 1, 2, 4$ algebras, we have the global symmetry groups $\mathrm{Spin}(24) \supset Co_0 \supset M_{23} \supset M_{22}$.

We should stress again that there are other Co_0 subgroups that preserve $\mathcal{N}=4$ resp. $\mathcal{N}=2$ superconformal algebras arising from 3-planes resp. 2-planes in **24**. Some examples are: the group $U_4(3)$ for the former case, and the McLaughlin (McL) and Higman–Sims (HS) sporadic groups, and also $U_6(2)$ for the latter case. However, only for the Mathieu groups do the twined partition functions of the module display uniformly a special property, which we regard as an essential feature of the moonshine phenomena. Namely, all the mock modular forms obtained via twining by elements of the Mathieu groups are encoded in Jacobi forms that are constant in the elliptic variable, in the limit as the modular variable tends to any cusp other than the infinite one. This property also holds for the Jacobi forms of the Mathieu moonshine mentioned above, and may be regarded as a counter-part to the genus zero property of monstrous moonshine, as we explain in more detail in Sect. 8.

The importance of this property is its predictive power: it allows us to write down trace functions for the actions of Mathieu group elements with little more information than a certain fixed multiplier system, and the levels of the functions we expect to find. A priori these are just guesses, but the constructions we present here verify their validity.

This may be compared to the predictive power of the genus zero property of monstrous moonshine: if $\Gamma < SL_2(\mathbb{R})$ determines a genus zero quotient of the upper-half plane, and if the stabilizer of $i\infty$ in Γ is generated by $\pm\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$, then there is a unique Γ-invariant holomorphic function satisfying $T_\Gamma(\tau) = q^{-1} + O(q)$ as $\tau \to i\infty$, for $q = e^{2\pi i\tau}$. The miracle of monstrous moonshine, and the content of the moonshine conjectures of Conway–Norton [15], is that for suitable choices of Γ, the function T_Γ is the trace of an element of the monster on some graded infinite-dimensional module (namely, the moonshine module of [36, 37]). The optimal growth property formulated in [14] plays the analogous predictive role in umbral moonshine, and is similar to the special property we formulate for the Mathieu moonshine considered here.

We mention here that although the moonshine conjectures have been proven in the monstrous case by Borcherds [3], and verified in [7, 28, 43] (see also [6]) for umbral moonshine, conceptual explanations of the genus zero property of monstrous moonshine, and of the analogous properties of umbral moonshine, and the Mathieu moonshine studied here, remain to be determined. An approach to establishing the genus zero property of monstrous moonshine via quantum gravity is discussed in [29].

The organization of the paper is as follows. We begin in Sect. 2 with a review of the module discussed in [27]. In Sect. 3, we describe methods for endowing this module with $\mathcal{N}=4$ and $\mathcal{N}=2$ structure. In Sect. 4, we discuss what this does to the manifest symmetry group of the model, reducing the symmetry from Co_0 to a variety of other possible groups which preserve a 3-plane (respectively 2-plane) in the **24** of Co_0. In Sects. 4 and 5, we discuss the action of these Co_0-subgroups on the modules and compute the corresponding twining functions. We identify M_{23}, M_{22}, McL, HS, $U_6(2)$ and $U_4(3)$ as some of the most interesting Co_0-subgroups preserving some extended superconformal algebra. In Sects. 6 and 7, we discuss in some detail the decomposition of the graded partition function of our chiral conformal field theory into characters of irreducible representations of the $\mathcal{N}=4$ and $\mathcal{N}=2$ superconformal algebras. In Sect. 8, we discuss the special property we require from a moonshine twining function, and show how this property singles out the Mathieu groups in our setup. We close with a discussion in Sect. 9. The appendices contain a number of tables: character tables for the various groups we discuss, tables of coefficients of the vector-valued mock modular forms that arise as our twining functions, and tables describing the decompositions of our modules into irreducible representations of the various groups.

2 The free field theory

The chiral 2d conformal field theory that will play a starring role in this paper has two different constructions. The first is described in [37] and starts with 8 free bosons X^i compactified on the 8-dimensional torus given by the E_8 root lattice, together with their Fermi superpartners ψ^i. One then orbifolds by the $\mathbb{Z}_2$ symmetry

$$(X^i, \psi^i) \to (-X^i, -\psi^i). \tag{2.1}$$

Note that, in more mathematical terms, a chiral 2d conformal field theory can be understood to mean a super vertex operator algebra (usually assumed to be simple, and of CFT-type in the sense of [23]), together with a (simple) canonically-twisted module. These two spaces are referred to as the Neveu–Schwarz (NS) and Ramond (R) sectors of the theory, respectively. The compactification of free bosons on the torus defined by a lattice L manifests, in the NS sector, as the usual lattice vertex algebra construction, and their Fermi superpartners are then realized by a Clifford module, or free fermion, super vertex algebra, where the underlying orthogonal space comes equipped with an isometric embedding of L. In the cases under consideration, there is a unique simple canonically-twisted module up to equivalence (cf. e.g. [24]), and hence a unique choice of R sector.

The orbifold procedure is described in detail in the language of vertex algebra in [24]. In what follows, a *field of dimension d* is a vertex operator or intertwining operator attached to a vector v in the NS or R sector, respectively, with $L(0)v = dv$. A *current* is a field of dimension 1. A field is called *primary* if its corresponding vector is a highest weight for the Virasoro (Lie) algebra. A *ground state* is an $L(0)$-eigenvector of minimal eigenvalue.

The E_8 construction just described has manifest $\mathcal{N} = 1$ supersymmetry, in the sense that the Neveu–Schwarz and Ramond algebras act naturally on the NS and R sectors, respectively. After orbifolding we obtain a $c = 12$ theory with no primary fields of dimension $\frac{1}{2}$. The partition functions (i.e., graded dimensions) of this free field theory can easily be computed. For example, the NS sector partition function is given by

$$Z_{\mathrm{NS},E8}(\tau) = \frac{1}{2}\left(\frac{E_4(\tau)\theta_3(\tau,0)^4}{\eta(\tau)^{12}} + 16\frac{\theta_4(\tau,0)^4}{\theta_2(\tau,0)^4} + 16\frac{\theta_2(\tau,0)^4}{\theta_4(\tau,0)^4}\right) \tag{2.2}$$

$$= q^{-1/2} + 0 + 276q^{1/2} + 2048q + 11202q^{3/2} + \cdots, \tag{2.3}$$

where E_4 is the weight 4 Eisenstein series, being the theta series of the E_8 lattice, $\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$ is the Dedekind eta function, and θ_i are the Jacobi theta functions recorded in "Appendix A". We have also set $q = \epsilon(\tau)$ and we use the shorthand notation $\mathrm{e}(x) = e^{2\pi ix}$ throughout this paper.

One recognizes representations of the Co_1 sporadic group appearing in the q-series (2.3): apart from 276, which is the minimal dimension of a faithful irreducible representation (cf. [31]), one can also observe

$$2048 = 1 + 276 + 1771, \tag{2.4}$$

$$11202 = 1 + 276 + 299 + 1771 + 8855, \tag{2.5}$$

$$\dots$$

In fact, this theory has a $\mathrm{Co}_0 \cong 2.\mathrm{Co}_1$ symmetry, which we call non-manifest since the action of Co_0 is not obvious from the given description. Note that we sometimes use n or $\mathbb{Z}_n$ to denote $\mathbb{Z}/n\mathbb{Z}$ depending on the context.

A better realization, for our purposes, was discussed in detail in [28] (cf. also [40]). The E_8 orbifold theory is equivalent to a theory of 24 free chiral fermions $\lambda_1, \lambda_2, \dots, \lambda_{24}$, also orbifolded by the $\mathbb{Z}_2$ symmetry $\lambda_\alpha \to -\lambda_\alpha$. This gives an alternative description of the Conway module above. The partition function from this "free fermion" point of view is more naturally written as

$$Z_{\mathrm{NS,fermion}}(\tau) = \frac{1}{2}\sum_{i=2}^{4} \frac{\theta_i^{12}(\tau,0)}{\eta^{12}(\tau)}\ . \tag{2.6}$$

This is equal to (2.2) according to non-trivial identities satisfied by theta functions. Note that $\theta_1(\tau,0) = 0$.

The free fermion theory has a manifest Spin(24) symmetry, but not a manifest $\mathcal{N} = 1$ supersymmetry. However, one can construct an $\mathcal{N} = 1$ supercurrent as follows. There is a unique (up to scale) NS ground state, but there are $2^{12} = 4096$ linearly independent Ramond sector ground states, which may be obtained by acting on a given fixed R sector ground state with the fermion zero modes $\lambda_i(0)$. It will be convenient to label the resulting 4096 Ramond sector ground states by vectors $\mathbf{s} \in \tilde{\mathbb{F}}_2^{12}$, where $\tilde{\mathbb{F}}_2 = \{-1/2, 1/2\}$.

We therefore have 4096 spin fields of dimension $\frac{3}{2}$ which implement the flow from the NS to the R sector. Denoting these fields as $\mathcal{W}_{\mathbf{s}}$, one can try to find a linear combination

$$W = \sum_{\mathbf{s}\in\tilde{\mathbb{F}}_2^{12}} c_{\mathbf{s}} \mathcal{W}_{\mathbf{s}} \tag{2.7}$$

which will serve as an $\mathcal{N} = 1$ supercurrent (i.e., field whose modes generate actions of the Neveu–Schwarz and Ramond super Lie algebras). As demonstrated in [28], and as we will review in the next section, there exists a set of values $c_{\mathbf{s}}$ such that the operator product expansion of W and the stress tensor T close properly, defining actions of the Neveu–Schwarz and Ramond algebras.

Any choice of W breaks the Spin(24) symmetry, since the Ramond sector ground states split into two 2048-dimensional irreducible representation of Spin(24). It is proven in [28] that the subgroup of Spin(24) that stabilizes a suitably chosen $\mathcal{N} = 1$ supercurrent is exactly the Conway group Co_0. In brief, the method of [28] is to identify a certain elementary abelian subgroup of order 2^{12} in Spin(24) (which should be regarded as a copy of the extended Golay code in Spin(24)). The action of this subgroup on the Ramond sector ground states singles out a particular choice of W, with the property that it is not annihilated by the zero mode of any dimension 1 field in the theory. It follows from this (cf. Proposition 4.8 of [28]) that the subgroup of Spin(24) that stabilizes W is a reductive algebraic group of dimension 0, and hence finite. On the other hand, one can show (cf. Proposition 4.7 of [28]) that this group contains Co_0, by virtue of the choice of subgroup 2^{12}. We obtain that the full stabilizer of W in Spin(24) is Co_0 by verifying (cf. Proposition 4.9 of [28]) that Co_0 is a maximal subgroup, subject to being finite.

In the rest of this paper, we extend this idea as follows. Instead of choosing an $\mathcal{N}=1$ supercurrent and viewing the theory as an $\mathcal{N}=1$ super conformal field theory, we choose various other super extensions of the Virasoro algebra. We will argue that $\mathcal{N}=4$ and $\mathcal{N}=2$ superconformal presentations of the theory are in one to one correspondence with choices of subgroups of Co_0 which fix a 3-plane (respectively, 2-plane) in the 24 dimensional representation. This leads us naturally to theories with various interesting symmetry groups, whose twining functions are easily computed in terms of the partition function (or elliptic genus) of the free fermion conformal field theory. These functions in turn are expressed nicely in terms of mock modular forms, and thus we establish mock modular moonshine relations for subgroups of Co_0 via this family of modules.

3 The superconformal algebras

We first discuss the largest superconformal algebra (SCA) we will consider, which gives rise to smaller global symmetry groups. We will construct an $\mathcal{N}=4$ SCA in the free fermion orbifold theory. Our strategy is to first construct the $SU(2)$ fields, and act with them on an $\mathcal{N}=1$ supercurrent to generate the full $\mathcal{N}=4$ SCA. We consequently obtain actions of the $\mathcal{N}=2$ SCA by virtue of its embeddings in the $\mathcal{N}=4$ SCA. In this process, we break the Co_0 symmetry group down to a proper subgroup as we will discuss in Sect. 4. We refer the reader to [32, 50, 52] for background on the $\mathcal{N}=4$ and $\mathcal{N}=2$ superconformal algebras.

We start with 24 real free fermions $\lambda_1, \lambda_2, \ldots, \lambda_{24}$. Picking out the first three fermions, we obtain the currents J_i:

$$J_i = -i\epsilon_{ijk}\lambda_j\lambda_k\,, \qquad i,j,k \in \{1,2,3\}\,. \tag{3.1}$$

They form an affine $SU(2)$ algebra with level 2 as may be seen from their operator product expansion (OPE),

$$J_i(z)J_j(0) \sim \frac{1}{z^2}\delta_{ij} + \frac{i}{z}\epsilon_{ijk}J_k(0)\,. \tag{3.2}$$

The next step is to pick an $\mathcal{N}=1$ supercurrent and act with J_i on it. As we reviewed in Sect. 2, an $\mathcal{N}=1$ supercurrent exists in this model and may be written as a linear combination of spin fields. Moreover, it may be chosen so that its stabilizer in Spin(24) is precisely Co_0. We will present a very general version of the construction now, and then extend it to find the $\mathcal{N}=4$ SCA.

To write the $\mathcal{N}=1$ supercurrent explicitly, we first group the 24 real fermions into 12 complex ones and bosonize them:

$$\begin{aligned}\psi_a &\equiv 2^{-1/2}(\lambda_{2a-1} + i\lambda_{2a}) \cong e^{iH_a},\\ \bar{\psi}_a &\equiv 2^{-1/2}(\lambda_{2a-1} - i\lambda_{2a}) \cong e^{-iH_a}, \quad a = 1, 2, \ldots, 12.\end{aligned} \tag{3.3}$$

In terms of the bosonic fields $\mathbf{H} = (H_1, \ldots, H_{12})$, an $\mathcal{N} = 1$ supercurrent W may be written as

$$W = \sum_{\mathbf{s}\in\tilde{\mathbb{F}}_2^{12}} w_{\mathbf{s}} e^{i\mathbf{s}\cdot\mathbf{H}} c_{\mathbf{s}}(\mathbf{p}), \tag{3.4}$$

where each component of $\mathbf{s} = (s_1, s_2, \ldots, s_{12})$ takes the values $\pm 1/2$, and the coefficients $w_{\mathbf{s}}$ belong to $\mathbb{C}$. We have introduced cocycle operators $c_{\mathbf{s}}(\mathbf{p})$ to ensure that the fields with integer spins (i.e., corresponding to even parity vectors) commute with all other operators, and the fields with half integral spins (corresponding to odd parity vectors) anticommute amongst themselves. The cocycle operators depend on the zero-mode operators $\mathbf{p}$ which are characterized by the commutation relation

$$\left[\mathbf{p}, e^{i\mathbf{k}\cdot\mathbf{H}}\right] = \mathbf{k}e^{i\mathbf{k}\cdot\mathbf{H}}, \tag{3.5}$$

where $\mathbf{k} = (k_1, \ldots, k_{12})$ is an arbitrary 12-tuple of complex numbers.

The associativity and closure of the OPE of the "dressed" vertex operators

$$V_{\mathbf{k}} = e^{i\mathbf{k}\cdot\mathbf{H}} c_{\mathbf{k}}(\mathbf{p}) \tag{3.6}$$

requires that

$$c_{\mathbf{k}}(\mathbf{p} + \mathbf{k}') c_{\mathbf{k}'}(\mathbf{p}) = e(\mathbf{k}, \mathbf{k}') c_{\mathbf{k}+\mathbf{k}'}(\mathbf{p}), \tag{3.7}$$

where the $\epsilon(\mathbf{k}, \mathbf{k}')$ satisfy the 2-cocycle condition

$$e(\mathbf{k}, \mathbf{k}')\, e(\mathbf{k} + \mathbf{k}', \mathbf{k}'') = e(\mathbf{k}', \mathbf{k}'')\, \epsilon(\mathbf{k}, \mathbf{k}' + \mathbf{k}''). \tag{3.8}$$

Moreover, in order for $V_{\mathbf{k}}$ to have the desired (anti)commutation relation, the condition

$$e(\mathbf{k}, \mathbf{k}') = (-1)^{\mathbf{k}\cdot\mathbf{k}' + \mathbf{k}^2\mathbf{k}'^2} e(\mathbf{k}', \mathbf{k}) \tag{3.9}$$

should be imposed. An explicit description of the cocycle for a general vertex operator $e^{i\mathbf{k}\cdot\mathbf{H}}$ may be chosen as

$$c_{\mathbf{k}}(\mathbf{p}) = e^{i\pi\mathbf{k}\cdot\mathbf{M}\cdot\mathbf{p}} \tag{3.10}$$

according to [33]. In our case, $\mathbf{M}$ is a 12×12 matrix that has the block form

$$\mathbf{M} = \begin{pmatrix} M_4 & 0 & 0 \\ 1_4 & M_4 & 0 \\ 1_4 & 1_4 & M_4 \end{pmatrix}, \qquad M_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -1 & 1 & -1 & 0 \end{pmatrix}, \tag{3.11}$$

where 1_4 is the 4×4 matrix with all entries set to 1.

Generically, the OPE of W with itself is

$$W(z)W(0) \sim \bar{w}w\left[\frac{1}{z^3} + \frac{T(0)}{4z}\right] + \frac{\bar{w}\Gamma^{\alpha\beta\gamma\delta}w}{96z}\lambda_\alpha\lambda_\beta\lambda_\gamma\lambda_\delta(0), \tag{3.12}$$

where we have defined

$$\bar{w}_{\mathbf{s}} = w_{-\mathbf{s}} c_{-\mathbf{s}}(\mathbf{s}), \tag{3.13}$$

$$\Gamma^{\alpha\beta\gamma\delta}_{\mathbf{s}\mathbf{s}'} = \left(\Gamma^\alpha\Gamma^\beta\Gamma^\gamma\Gamma^\delta\right)_{\mathbf{s}\mathbf{s}'} c_{-\mathbf{s}}(\mathbf{s}'-\mathbf{s})\left(-2s_{\lceil\frac{\alpha}{2}\rceil+1}\right)\cdots\left(-2s_{\lceil\frac{\beta}{2}\rceil}\right)\left(-2s_{\lceil\frac{\gamma}{2}\rceil+1}\right)\cdots\left(-2s_{\lceil\frac{\delta}{2}\rceil}\right), \tag{3.14}$$

for $\alpha < \beta < \gamma < \delta$. The other components of $\Gamma^{\alpha\beta\gamma\delta}$ are defined by the requirement that it is totally antisymmetrized. For W to be an $\mathcal{N} = 1$ supercurrent, the last terms in (3.12) must vanish,

$$\bar{w}\Gamma^{\alpha\beta\gamma\delta} w = 0\,, \qquad \forall \alpha,\beta,\gamma,\delta \in \{1,2,\ldots,12\}, \tag{3.15}$$

and the first two terms must have the correct normalization,

$$\bar{w}w = \sum_{\mathbf{s}\in\tilde{\mathbb{F}}_2^{12}} w_{-\mathbf{s}} w_{\mathbf{s}} c_{\mathbf{s}}(-\mathbf{s}) = 8. \tag{3.16}$$

From now on we presume to be chosen a solution $(w_{\mathbf{s}})$ such that W is an $N = 1$ supercurrent stabilized by Co_0, as described in Sect. 2.

We may now act with the $SU(2)$ currents J_i on our $\mathcal{N} = 1$ supercurrent W. In order to do this, write the $SU(2)$ currents in (3.1) in bosonized form,

$$J_1 = -\frac{1}{2}\left(e^{iH_1} - e^{-iH_1}\right)\left(e^{iH_2}e^{i\pi p_1} + e^{-iH_2}e^{-i\pi p_1}\right), \tag{3.17}$$

$$J_2 = \frac{i}{2}\left(e^{iH_1} + e^{-iH_1}\right)\left(e^{iH_2}e^{i\pi p_1} + e^{-iH_2}e^{-i\pi p_1}\right), \tag{3.18}$$

$$J_3 = i\partial H_1. \tag{3.19}$$

Here we have included the cocycles $e^{\pm i\pi p_1}$. We now extract the singular terms of the OPEs,

$$J_i(z)W(0) \sim -\frac{i}{2z} W_i(0), \tag{3.20}$$

where W_i are slightly modified combinations of spin fields,

$$W_1 = -\sum_{\mathbf{s}} 2s_2 w_{R\mathbf{s}} e^{i\mathbf{s}\cdot\mathbf{H}} c_{\mathbf{s}}(\mathbf{p}), \tag{3.21}$$

$$W_2 = i\sum_{\mathbf{s}} 4s_1 s_2 w_{R\mathbf{s}} e^{i\mathbf{s}\cdot\mathbf{H}} c_{\mathbf{s}}(\mathbf{p}), \tag{3.22}$$

$$W_3 = i\sum_{\mathbf{s}} 2s_1 w_{\mathbf{s}} e^{i\mathbf{s}\cdot\mathbf{H}} c_{\mathbf{s}}(\mathbf{p}), \tag{3.23}$$

and where $R\mathbf{s} \equiv (-s_1, -s_2, s_3, \ldots, s_{12})$.

We claim that all three W_i defined above are valid $\mathcal{N}=1$ supercurrents. This is because we may obtain, for instance, W_3 from W by rotating the 1–2 plane by π, and the conditions (3.15) and (3.16), for being an $\mathcal{N}=1$ supercurrent, are invariant under $SO(24)$ rotations. We may obtain W_2 and W_3 similarly. This shows that each of the W_i is an $\mathcal{N}=1$ supercurrent.

Furthermore, we can check using the identity (3.15) that the OPEs of the W_i are given by

$$W_i(z)W_j(0) \sim \delta_{ij}\left[\frac{8}{z^3}+\frac{2}{z}T(0)\right]+2i\epsilon_{ijk}\left[\frac{2}{z^2}J_k(0)+\frac{1}{z}\partial J_k(0)\right], \tag{3.24}$$

$$W(z)W_i(0) \sim -2i\left(\frac{2}{z^2}+\frac{\partial}{z}\right)J_i(0), \tag{3.25}$$

$$J_i(z)W_j(0) \sim \frac{i}{2z}(\delta_{ij}W+\epsilon_{ijk}W_k). \tag{3.26}$$

This shows that W, W_i, and J_i, together with the stress tensor T, defined as

$$T=-\frac{1}{2}\sum_\alpha \lambda_\alpha\partial\lambda_\alpha=-\frac{1}{2}\sum_a \partial H_a\partial H_a, \tag{3.27}$$

form an $\mathcal{N}=4$ SCA with central charge $c=12$. We may recombine the four $\mathcal{N}=1$ supercurrents W, W_i into the more conventional $\mathcal{N}=4$ supercurrents

$$W_1^\pm \equiv 2^{-1/2}(W\pm iW_3)\,, \qquad W_2^\pm \equiv \pm 2^{-1/2}i(W_1\pm iW_2), \tag{3.28}$$

which transform according to the representation $\mathbf{2}+\bar{\mathbf{2}}$ of $SU(2)$. In terms of these supercurrents we obtain the standard (small) $\mathcal{N}=4$ SCA with central charge $c=12$, characterized by the following set of OPEs:

$$T(z)T(0) \sim \frac{6}{z^4}+\frac{2}{z^2}T(0)+\frac{1}{z}\partial T(0), \tag{3.29}$$

$$T(z)W_a^\pm(0) \sim \frac{3}{2z^2}W_a^\pm(0)+\frac{1}{z}\partial W_a^\pm(0), \tag{3.30}$$

$$T(z)J_i(0) \sim \frac{1}{z^2}J_i(0)+\frac{1}{z}\partial J_i(0), \tag{3.31}$$

$$W_a^+(z)W_b^-(0) \sim \delta_{ab}\left[\frac{8}{z^3}+\frac{2}{z}T(0)\right]-2\sigma^i_{ab}\left[\frac{2}{z^2}J_i(0)+\frac{1}{z}\partial J_i(0)\right], \tag{3.32}$$

$$W_a^+(z)W_b^+(0) \sim W_a^-(z)W_b^-(0) \sim 0, \tag{3.33}$$

$$J_i(z)W_a^+(0) \sim -\frac{1}{2z}\sigma^i_{ab}W_b^+(0), \tag{3.34}$$

$$J_i(z)W_a^-(0) \sim \frac{1}{2z}\sigma^{i*}_{ab}W_b^-(0), \tag{3.35}$$

$$J_i(z)J_j(0) \sim \frac{1}{z^2}\delta_{ij} + \frac{i}{z}\epsilon_{ijk}J_k(0). \tag{3.36}$$

Here σ^i are the Pauli matrices.

Now we can generalize our formula for the partition function (2.6) to include a grading by the $U(1)$ charge under the Cartan generator of the $SU(2)$. The $U(1)$ charge operator J_0 is, by definition, twice the zero-mode of the J_3 current. From this and the definition $J_3 = -i\lambda_1\lambda_2 = \psi_1\bar{\psi}_1$, we see that under J_0 the complex fermion ψ_1 has charge 2 while the other 11 complex fermions are neutral. Therefore, the $U(1)$-graded NS sector partition function becomes

$$Z_{\rm NS}(\tau, z) = \frac{1}{2}\sum_{i=2}^{4} \frac{\theta_i(\tau, 2z)\,\theta_i(\tau, 0)^{11}}{\eta(\tau)^{12}}. \tag{3.37}$$

In the above discussion, we have chosen the first three fermions out of a total of 24 to generate a set of $SU(2)$ currents. Together with an $\mathcal{N}=1$ supercurrent they generate a full $\mathcal{N}=4$ SCA. It is clear that we are free to choose any three fermions for this purpose. In fact, we could choose an arbitrary three-dimensional subspace of the 24-dimensional vector space spanned by the fermions, and obtain an $\mathcal{N}=4$ SCA. For a given $\mathcal{N}=1$ supercurrent, not all choices of 3-plane are equivalent, as we will see in Sect. 4.

Observe that we could instead have chosen to single out only two real fermions, and construct a $U(1)$ current algebra instead of an $SU(2)$ current algebra. Completely analogous manipulations then show that each such choice provides an $\mathcal{N}=2$ superconformal algebra. As a result we can equip the Co_0 theory with $\mathcal{N}=2$ structure in such a way that the global symmetry group is broken to subgroups G of Co_0 which stabilize 2-planes in **24**.

To summarize the results of this section, we have shown how to construct an $\mathcal{N}=1$ supercurrent for the chiral conformal field theory described, in the previous section, as an orbifold of 24 free fermions. We have also shown how choices of 2- and 3-planes in the space spanned by the generating fermions give rise to actions of the $\mathcal{N}=2$ and $\mathcal{N}=4$ superconformal algebras (respectively) on the theory. As reviewed in Sect. 2, a suitable choice of $\mathcal{N}=1$ structure reduces the global symmetry of the theory to Co_0. In the next section we will discuss the finite simple groups that appear when we impose the richer, $\mathcal{N}=2$ and $\mathcal{N}=4$ superconformal structures.

4 Global symmetries

Enhancing the $\mathcal{N}=1$ structure of the theory to $\mathcal{N}=4$ breaks the Co_0 symmetry. We now show that for a specific choice of 3-plane in **24**, resulting in a specific copy of the $\mathcal{N}=4$ SCA, the stabilising subgroup of Co_0 is the sporadic group M_{22}. Similarly, for a specific choice of 2-plane, resulting in a specific copy of the $\mathcal{N}=2$ SCA, the stabilising subgroup of Co_0 is the sporadic group M_{23}. This amounts to a proof that the model described in Sect. 2 results in an infinite-dimensional M_{22} (resp. M_{23})-module underlying the mock modular forms described in Sect. 6 (resp. Sect. 7) arising from its interpretation as an $\mathcal{N}=4$ (resp. $\mathcal{N}=2$) module. More generally, we establish the modules for 3- (2-)plane-fixing subgroups of the largest Mathieu group M_{24} by fixing a specific copy of $\mathcal{N}=4$ ($\mathcal{N}=2$) SCA.

Recall that the theory regarded as an $\mathcal{N}=0$ theory (i.e., with no extension of the Virasoro action) has a Spin(24) symmetry resulting from the $SO(24)$ rotations on the 24-dimensional space, and a suitable choice of $\mathcal{N}=1$ supercurrent breaks the Spin(24) group down to its subgroup Co_0. The group Co_0 is the automorphism group of the Leech lattice Λ_{Leech}, and various interesting subgroups of Co_0 can be identified as stabilizers of suitably chosen lattice vectors in Λ_{Leech}. To study the automorphism group of the module when fixing more structure—more supersymmetries in this case—it will therefore be useful to describe the enhanced supersymmetries in terms of Leech lattice vectors.

In Chapter 10 of [16] it is shown that if we choose an appropriate tetrahedron in the Leech lattice, whose edges have lengths squared $16\times(2,2,2,2,3,3)$ in the normalisation described below, the subgroup of Co_0 that leaves all vertices of the tetrahedron invariant is M_{22}. To be more precise, let $\mathbf{e}_\gamma$, for $\gamma\in\{1,2,\ldots,24\}$, be an orthonormal basis of $\mathbb{R}^{24}$, and choose a copy $\mathcal{G}$ of the extended binary Golay code in $\mathcal{P}(\{1,\ldots,24\})$. Then we may realize Λ_{Leech} as the lattice generated by the vectors $2\sum_{\gamma\in C}\mathbf{e}_\gamma$ for $C\in\mathcal{G}$ together with $-4\mathbf{e}_1+\sum_{\gamma=1}^{24}\mathbf{e}_\gamma$. (One can show that all 24 vectors of the form $-4\mathbf{e}_\alpha+\sum_{\gamma=1}^{24}\mathbf{e}_\gamma$ are in Λ_{Leech}.) Define the tetrahedron $T_{\{\alpha,\beta\}}$ to be that whose four vertices are $O=0$, $X_\alpha=4\mathbf{e}_\alpha+\sum_{\gamma=1}^{24}\mathbf{e}_\gamma$, $X_\beta=4\mathbf{e}_\beta+\sum_{\gamma=1}^{24}\mathbf{e}_\gamma$ and $P_{\alpha\beta}=4\mathbf{e}_\alpha+4\mathbf{e}_\beta$, for any $\alpha,\beta\in\{1,2,\ldots,24\}$ with $\alpha\neq\beta$. For every such $T_{\{\alpha,\beta\}}$, the subgroup fixing every vertex is a copy of M_{22}, a sporadic simple group of order $2^7\cdot3^2\cdot5\cdot7\cdot11=443,520$ and the subgroup of M_{24} fixing $\mathbf{e}_\alpha$ and $\mathbf{e}_\beta$.

From the above discussion, it is clear that given $\{\alpha,\beta\}$, a copy of M_{22} stabilises the real span of $\mathbf{e}_\alpha$, $\mathbf{e}_\beta$ and $\sum_{\gamma=1}^{24}\mathbf{e}_\gamma$. Given a suitable choice of the $\mathcal{N}=1$ superconformal algebra, this copy of $\mathbb{R}^3$ in **24** then determines, up to rotations, the three fermions, denoted $\lambda_{1,2,3}$, from which the $SU(2)$ current algebra was built in Sect. 3. By definition then, a copy of M_{22} leaves the $\mathcal{N}=4$ superconformal algebra invariant.

A natural question is: what is the symmetry group G that fixes a given choice of $\mathcal{N}=2$ superconformal structure? Given the above description of the M_{22} action, we can choose the $\mathbb{R}^2\subset\mathbb{R}^3$ generated by $\mathbf{e}_\alpha$ and $\sum_{\gamma=1}^{24}\mathbf{e}_\gamma$ and use the two free fermions lying in the $\mathbb{R}^2$ to construct the $\mathcal{N}=2$ sub-algebra of the $\mathcal{N}=4$ SCA. Specifically, the $U(1)$ action is rotation of the $\mathbb{R}^2$. From the above discussion, it is not hard to see that there is a copy of M_{23} fixing $\mathbf{e}_\alpha$ and $\sum_{\gamma=1}^{24}\mathbf{e}_\gamma$ and hence stabilising the $\mathcal{N}=2$ structure. Recall that M_{23} is a sporadic simple group of order $2^7\cdot3^2\cdot5\cdot7\cdot11\cdot23=10,200,960$. In terms of the Leech lattice, it corresponds to the fact that the stabiliser of the triangle in Λ_{Leech} whose edges have lengths squared $16\times(6,3,2)$, with vertices chosen to be O, X_α and $2\sum_{\gamma=1}^{24}\mathbf{e}_\gamma$, is a copy of M_{23} inside the copy of Co_0 stabilising Λ_{Leech}.

This furnishes a proof that the theory described in Sect. 2 leads to modules for M_{22} and M_{23} which explicitly realize the mock modular forms to be defined in Sects. 6 and 7.

We should mention that by stabilizing different choices of geometric structure, other than the tetrahedron and triangle just discussed, leading to M_{22} and M_{23}, respectively, we can determine other global symmetry groups G. Indeed, our method constructs a G-module with $\mathcal{N}=4$ ($\mathcal{N}=2$) superconformal symmetry for any subgroup $G<Co_0$

which fixes a 3-plane (2-plane) in **24**. Since, as we will see in Sect. 6 (Sect. 7), such modules furnish assignments of mock modular forms to the elements of their global symmetry groups, it is an interesting question to classify the global symmetry groups $G < Co_0$ that can arise. We conclude this section with a discussion of some of these possibilities. Certainly a full classification is beyond the scope of this work, so we restrict our attention (mostly) to sporadic simple examples.

Indeed, the Conway group Co_0 is a rich source of sporadic simple groups, for no less than 12 of the 26 sporadic simple groups are involved in Co_0 (cf. [17]), in the sense that they may be obtained by taking quotients of subgroups of Co_0. Of these 12, all but 3 are actually realised as subgroups, and 6 of these 9 sporadic simple groups appear as subgroups of Co_0 fixing (at least) a 2-plane in **24**. These six 2-plane fixing groups are the smaller Mathieu groups, M_{23}, M_{22}, M_{12} and M_{11}, the Higman–Sims group HS, and the McLaughlin group McL. Some 2-planes they fix are described explicitly in Chapter 10 of [16].

$\mathcal{N} = 4$ **modules**

From the character tables (cf. [17]) of the six sporadic 2-plane fixing subgroups of Co_0 it is clear that M_{22} and M_{11} are the only examples that fix a 3-plane. Even though M_{11} is not a subgroup of M_{22}, it turns out that the mock modular forms attached to M_{11} by our $\mathcal{N} = 4$ construction (and the analysis of Sect. 6) are a proper subset of those attached to M_{22}, since the conjugacy classes of Co_0 appearing in a 3-plane-fixing subgroup M_{11} are a proper subset of those appearing in a subgroup M_{22}. For this reason we focus on M_{22} when discussing mock modular forms attached to sporadic simple groups via the $\mathcal{N} = 4$ construction in this work.

If we expand our attention to simple, not necessarily sporadic subgroups of Co_0, then there is one example which is larger than M_{22} (which has order 443,520). Namely, the group $U_4(3)$, with order 3,265,920, can arise as the stabilizer of a suitably chosen 3-plane in the **24** of Co_0 [16]. The $U_4(3)$ characters are presented in Appendix Table 20, the coefficients in the associated (twined) vector valued mock modular forms in Appendix Tables 3 and 4, and the decomposition of the module into irreducible representations of the group in Appendix Tables 26 and 27.

As we shall see in Sect. 8, the Jacobi forms attached to M_{22} (and therefore also those attached to M_{11}) by the $\mathcal{N} = 4$ construction are distinguished in that they satisfy a natural analogue of the genus zero condition of monstrous moonshine. By contrast, this property does not hold for all the Jacobi forms arising from $U_4(3)$. This is the main reason for our focus on M_{22} in the context of $\mathcal{N} = 4$ supersymmetry (Appendix Tables 1, 2, 5, 6, 15).

$\mathcal{N} = 2$ **modules**

We have focused on the example of M_{23}, with order 10,200,960, in this section. Since M_{22} and M_{11} are subgroups of M_{23} we do not consider them further in the context of $\mathcal{N} = 2$ structures. Of the remaining sporadic simple 2-plane-fixing subgroups of Co_0, the largest is the McLaughlin group McL, which is actually considerably larger than M_{23}, having order 898,128,000. Its characters are presented in Appendix Table 21, the coefficients of the (twined) mock modular forms in Appendix Tables 7 and 8, and the decomposition of the module into irreducible representations of the group in Appendix Tables 33, 34, 35, 36, 37, 38.

The next largest example, also larger than M_{23}, is the Higman–Sims group HS, with order 44,352,000. Its characters are presented in Appendix Table 22, the coefficients of the (twined) mock modular forms in Appendix Tables 9 and 10, and the decomposition of the module into irreducible representations of the group in Appendix Tables 39, 40, 41, 42, 43, 44.

If we expand our attention to simple groups fixing a 2-plane in **24** then there is one example larger than McL. Namely, the group $U_6(2)$, of order 9,196,830,720, fixes any triangle in **24** whose three sides are vectors of minimal length in the Leech lattice. The characters of $U_6(2)$ are given in Appendix Tables 17, 18, 19, the coefficients of the (twined) mock modular forms in Appendix Tables 11, 12, 13, 14, and the decomposition of the module into irreducible representations of the group in Appendix Tables 45, 46, 47, 48, 49, 50, 51, 52, 53, 54.

In direct analogy with the case of $\mathcal{N}=4$ structure, it will develop in Sect. 8 that the Jacobi forms attached to M_{12} and M_{23} satisfy a natural analogue of the genus zero condition of monstrous moonshine, and, contrastingly, this property fails in general for the modular forms arising from the other, non-Mathieu, 2-plane-fixing simple groups mentioned above. For these reasons, and since M_{12} is relatively small, we focus on M_{23} in our discussion of $\mathcal{N}=2$ supersymmetry.

5 Twining the module

In the last sections, we have described how to equip the orbifolded free fermion theory with $\mathcal{N}=4$ and $\mathcal{N}=2$ superconformal structures. In this section we will use the Ramond sector of our theory to attach two variable formal power series—the g-twined graded R sector partition function, cf. (5.10)—to each element $g \in Co_0$ that preserves at least a 2-plane in **24**.

Let us denote the Ramond sector by V, and let us choose a $U(1)$ charge operator J_0. This will be twice the Cartan generator of the $SU(2)$ in the $\mathcal{N}=4$ case, or the single $U(1)$ generator in the case of $\mathcal{N}=2$ SCA. Then it is natural to define the Ramond-sector $U(1)$-graded partition function, or *elliptic genus*,

$$Z(\tau,z) = \mathrm{T}\,r_V(-1)^F q^{L_0-c/24} y^{J_0} \tag{5.1}$$

$$= \frac{1}{2}\frac{1}{\eta^{12}(\tau)}\sum_{i=2}^{4}(-1)^{i+1}\theta_i(\tau,2z)\theta_i^{11}(\tau,0) \tag{5.2}$$

$$= \frac{1}{2}\frac{E_4(\tau)\theta_1^4(\tau,z)}{\eta^{12}(\tau)} + 8\sum_{i=2}^{4}\left(\frac{\theta_i(\tau,z)}{\theta_i(\tau,0)}\right)^4, \tag{5.3}$$

where we have introduced a chemical potential for the J_0 charges and set $y = \mathrm{e}(z)$ for $z \in \mathbb{C}$. Also, we define $(-1)^F$ as an operator on V by requiring that it act as Id on the untwisted free fermion contribution to V, and as $-$Id on the twisted fermion contribution.

As is expected for 2d conformal field theories with $\mathcal{N} \geq 2$ supersymmetry, the elliptic genus (5.1) transforms as a Jacobi form of weight 0, index $m = \frac{c}{6}$ and level 1.

Explicitly, and since $c = 12$ in our case, this means that $Z|_2(\lambda,\mu) = Z$ for all $\lambda,\mu \in \mathbb{Z}$, and $Z|_{0,2}\gamma = Z$ for all $\gamma \in SL_2(\mathbb{Z})$, where the elliptic and modular slash operators are defined by

$$\begin{aligned}(\phi|_m(\lambda,\mu))(\tau,z) &= e(m(\lambda^2\tau + 2\lambda z))\phi(\tau, z+\lambda\tau+\mu),\\ (\phi|_{k,m}\gamma)(\tau,z) &= e(-m\tfrac{cz^2}{c\tau+d})(c\tau+d)^{-k}\phi\left(\tfrac{a\tau+b}{c\tau+d},\tfrac{z}{c\tau+d}\right),\end{aligned} \tag{5.4}$$

respectively, for $\lambda,\mu \in \mathbb{Z}$ and $\gamma = \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in SL_2(\mathbb{Z})$. (A Jacobi form of level N is only required to satisfy $\phi|_{k,m}\gamma = \phi$ for γ in the congruence subgroup $\Gamma_0(N)$ (cf. (6.24)).)

As we have seen in Sect. 2, the two different ways of writing this function, (5.2) and (5.3), are intuitively connected more closely with the free fermion and E_8 root lattice descriptions of the theory, respectively. Of course, the $U(1)$-graded NS sector partition function (3.37) is related to the above, graded Ramond sector partition function by a spectral flow transformation

$$Z_{\rm NS}(\tau,z) = q^{1/2}y^{-2}Z(\tau, z - \tfrac{\tau+1}{2}). \tag{5.5}$$

There is a natural way in which one can twine the above function under certain subgroups of Co_0. From the previous discussions, we see that the representation **24** plays a central role in the way various subgroups of Co_0 act on the model. Let's denote by $\ell_{g,k}$ and $\bar{\ell}_{g,k}$, for $k = 1,\dots 12$, the 12 complex conjugate pairs of eigenvalues of $g \in Co_0$ when acting on **24**. This information is conveniently encoded in the so-called Frame shape of g, given by

$$\Pi_g = \prod_n {L_n}^{m_n}, \quad 1 \le L_1 < L_2 < L_3 \cdots, \quad \text{and} \quad m_n \in \mathbb{Z}, m_n \neq 0,$$

satisfying $\sum_n L_n m_n = 24$, through the fact that the 12 pairs $\{\ell_{g,k}, \bar{\ell}_{g,k}\}$ are precisely the 24 roots solving the equation

$$\prod_n (x^{L_n} - 1)^{m_n} = 0.$$

As discussed in Sects. 3 and 4, in order to preserve at least $\mathcal{N} = 2$ superconformal symmetry and hence be able to twine the graded R-sector partition function (5.1), the subgroup G must leave at least a 2-dimensional subspace in **24** pointwise invariant. In the graded partition function this corresponds to leaving the factor $\theta_i(\tau, 2z)$ in (5.2) invariant. As a result, for every conjugacy class $[g]$ of such a group G we can choose $\ell_{g,1} = \bar{\ell}_{g,1} = 1$. It is easy to see that when acting on the untwisted free fermions of the theory, contributing the terms involving θ_i with $i = 3,4$ in (5.2), the group element g simply replaces $\theta_i^{11}(\tau,0)$ with

$$\prod_{k=2}^{12} \theta_i(\tau, \rho_{g,k}) \tag{5.6}$$

where $\mathrm{e}(\rho_{g,k}) = \ell_{g,k}$.

When trying to do the same for the contribution from the twisted fermions, contributing the term involving θ_2 in (5.2), however, we see that the above simple consideration suffers from an ambiguity. This can be seen from the fact that $\theta_2(\tau,\rho) = -\theta_2(\tau,\rho+1)$, and hence the answer cannot be determined simply by looking at the g-eigenvalues on **24**. This of course is a reflection of the fact that the global symmetry group, with no superconformal structure imposed, is Spin(24), which is a 2-fold cover of $SO(24)$. As a result, to specify the twining of the twisted fermion contribution, we also need to know the action of G on the faithful 2^{12}-dimensional representation of Spin(24) spanned by Ramond sector ground states in the free fermion theory (cf. Sect. 2), henceforth denoted **4**096, which decomposes as $\mathbf{4096} = \mathbf{1} + \mathbf{276} + \mathbf{1771} + \mathbf{24} + \mathbf{2024}$ in terms of the irreducible representations of Co_0.

Note that, according to the orbifold construction, just "half" of the Ramond sector ground states in the free fermion theory will contribute to the Ramond sector V of the orbifold theory under consideration. In terms of the Co_0 action, the two "halves" are $\mathbf{24} + \mathbf{2024}$, where Co_0 acts faithfully, and $\mathbf{1} + \mathbf{276} + \mathbf{1771}$, where the action factors through $Co_1 = Co_0/2$. In practice, both choices give rise to equivalent theories (i.e., isomorphic super vertex operator algebras, cf. [24, 27]), but they are inequivalent as Co_0 -modules. For us, the ground states represented by $\mathbf{24} + \mathbf{2024}$ lie in the R sector, V, and the $\mathbf{1}$ in $\mathbf{1} + \mathbf{276} + \mathbf{1771}$ represents the Co_0-invariant $\mathcal{N} = 1$ supercurrent in the NS sector of our orbifold theory.

The above discussion serves to remind us that there is, really, a vanishing term

$$0 = \frac{1}{2}\frac{1}{\eta^{12}(\tau)}\theta_1(\tau,2z)\theta_1^{11}(\tau,0) \tag{5.7}$$

in (5.2), which, for certain $g \in Co_0$, will make a non-vanishing contribution to the g-twined version of (5.1). It vanishes when $g = e$ is the identity because the Ramond sector ground states in the free fermion theory come in pairs with opposite eigenvalues for $(-1)^F$. Moreover, exchanging the pair corresponds to complex conjugation $\psi_a \leftrightarrow \bar{\psi}_a$, for $a = 1, \ldots, 12$, of the complex fermions. Recall that one of the complex fermions, denoted ψ_1 in (3.19), was used to construct the $U(1)$ charge operator J_0, and we are interested in the graded partition function where we introduce a chemical potential z for this operator. Because exchanging $\psi_1 \leftrightarrow \bar{\psi}_1$ also induces a flip of $U(1)$ charges, captured by $z \leftrightarrow -z$, the contribution of the first complex fermion does not vanish, corresponding to the fact that the identity

$$\theta_1(\tau,z) = \theta_1(\tau,z+2) = -\theta_1(\tau,-z) \tag{5.8}$$

only forces $\theta_1(\tau,z)$ to vanish at $z \in \mathbb{Z}$. Consequently, the g-twining of (5.7) makes a non-zero contribution to the g-twining of (5.2) if and only if $\rho_{g,k} \notin \mathbb{Z}$ for all $k = 2, \ldots, 12$. In other words, it is non-zero only when the cyclic group generated by g fixes nothing but a 2-plane.

By inspection we find that, among the groups we consider, such group elements must be in the conjugacy classes $23\mathrm{AB} \subset M_{23}$, $6\mathrm{AB}, 12\mathrm{AB}, 12\mathrm{DE}, 18\mathrm{AB} \subset U_6(2)$, $15\mathrm{AB}, 30\mathrm{AB} \subset McL$, or $20\mathrm{AB} \subset HS$. The pairs of these conjugacy classes corresponding to the letters A and B (or D and E) are mutually inverse, and so their respective traces, on any representation, are related by complex conjugation. In terms of our construction, choosing one over the other is the same as choosing what one labels ψ_1 and $\bar{\psi}_1$, and the same as choosing an orientation on the 2-plane fixed by the group element in **24**. As a result, from (5.8) we see that the θ_1 term in the partition functions twined by these conjugate A (D) and B (E) classes come with an opposite sign.

Let us work with the principal branch of the logarithm, and choose $\rho_{g,k} \in [0, 1/2]$ in (5.6). Then, by direct computation—we must compute directly, for the choice of labels for mutually inverse conjugacy classes is not natural—we find that the signs in (5.10) are

$$l\epsilon_{g,1} = 1 \text{ for } \text{ g in } \begin{cases} 23A \subset M_{23}, \\ 20A \subset HS, \\ 15A \cup 30A \subset McL, \\ 12A \cup 12D \cup 6B \cup 18B \subset U_6(2), \end{cases} \tag{5.9}$$

and $\epsilon_{g,1} = -1$ for the inverse classes, $23B \subset M_{23}$, $20B \subset HS$,&c.

Putting these different contributions together, we conclude that for every $[g] \subset G$ where G is a subgroup of Co_0 preserving (at least) a 2-plane in **24**, the corresponding g-twined $U(1)$-graded R sector partition function reads

$$Z_g(\tau, z) = \mathrm{T}\, r_V g(-1)^F q^{L_0 - c/24} y^{J_0} \tag{5.10}$$

$$= \frac{1}{2} \frac{1}{\eta(\tau)^{12}} \sum_{i=1}^{4} (-1)^{i+1} \epsilon_{g,i}\, \theta_i(\tau, 2z) \prod_{k=2}^{12} \theta_i(\tau, \rho_{g,k}), \tag{5.11}$$

where

$$\epsilon_{g,2} = \begin{cases} \frac{\mathrm{T}\, r_{\mathbf{4096}} g}{2^{12} \prod_{k=1}^{12} \cos(\pi \rho_{g,k})} \in \{-1, 1\} & \text{when } \prod_{k=1}^{12} \cos(\pi \rho_{g,k}) \neq 0 \\ 0 & \text{when } \prod_{k=1}^{12} \cos(\pi \rho_{g,k}) = 0 \end{cases} \tag{5.12}$$

$$\epsilon_{g,3} = \epsilon_{g,4} = 1, \tag{5.13}$$

and where the $\epsilon_{g,1}$ are as determined in the preceding paragraph.

In this section we have introduced the g-twined $U(1)$-graded Ramond sector partition function, or g-twined elliptic genus of our theory, Z_g, for any $g \in Co_0$ fixing a 2-plane in **24**. We have also derived an explicit formula (5.10) for Z_g, in terms of the Frame shapes Π_g and values $\mathrm{T}\, r_{\mathbf{4096}} g$. This Frame shape and trace value data is collected, for $g \in G$, for various $G \subset \mathrm{Co}_0$, in "Appendix B". In Sects. 6 and 7 we will see how the above twining leads to the mock modular forms playing the role of the McKay–Thompson series in these new examples of mock modular moonshine.

6 The $\mathcal{N} = 4$ decompositions

From the discussion in Sect. 3 it is clear that the orbifold theory discussed in Sect. 2 can be equipped with $\mathcal{N} = 4$ superconformal structure. In this section we will study the decomposition of the Ramond sector V into irreducible representations of the $\mathcal{N} = 4$

SCA and see how the decomposition leads to mock modular forms relevant for the M_{22} moonshine which we will discuss in Sect. 8.

Recall (cf. [32]) that the $\mathcal{N}=4$ superconformal algebra contains subalgebras isomorphic to the affine $SU(2)$ and Virasoro Lie algebras. In a unitary representation the former of these acts with level $m-1$, for some integer $m>1$, and the latter with central charge $c=6(m-1)$.

The unitary irreducible highest weight representations $v^{\mathcal{N}=4}_{m;h,j}$ are labeled by the eigenvalues of L_0 and $\frac{1}{2}J^3_0$ acting on the highest weight state, which we denote by h and j, respectively. Cf. [33, 34]. The superconformal algebra has two types of highest weight Ramond sector representations: the *massless* (or *BPS*) representations with $h=\frac{c}{24}=\frac{m-1}{4}$ and $j\in\{0,\frac{1}{2},\dots,\frac{m-1}{2}\}$, and the *massive* (or *non-BPS*) representations with $h>\frac{m-1}{4}$ and $j\in\{\frac{1}{2},1,\dots,\frac{m-1}{2}\}$. Their graded characters, defined as

$$\mathrm{ch}^{\mathcal{N}=4}_{m;h,j}(\tau,z)=\mathrm{tr}_{v^{\mathcal{N}=4}_{m;h,j}}\Big((-1)^{J^3_0}y^{J^3_0}q^{L_0-c/24}\Big),\tag{6.1}$$

are given by

$$\mathrm{ch}^{\mathcal{N}=4}_{m;h,j}(\tau,z)=(\Psi_{1,1}(\tau,z))^{-1}\mu_{m;j}(\tau,z)\tag{6.2}$$

and

$$\mathrm{ch}^{\mathcal{N}=4}_{m;h,j}(\tau,z)=(\Psi_{1,1}(\tau,z))^{-1}\,q^{h-\frac{c}{24}-\frac{j^2}{m}}\,\big(\theta_{m,2j}(\tau,z)-\theta_{m,-2j}(\tau,z)\big)\tag{6.3}$$

in the massless and massive cases, respectively, [34]. In the above formulas, the function $\mu_{m;j}(\tau,z)$ is defined by setting

$$\mu_{m;j}(\tau,z)=(-1)^{1+2j}\sum_{k\in\mathbb{Z}}q^{mk^2}y^{2mk}\,\frac{(yq^k)^{-2j}+(yq^k)^{-2j+1}+\cdots+(yq^k)^{1+2j}}{1-yq^k},\tag{6.4}$$

and $\Psi_{1,1}$ is a meromorphic Jacobi form (cf. Sect. 8 of [19] for more on meromorphic Jacobi forms) of weight 1 and index 1 given by

$$\Psi_{1,1}(\tau,z)=-i\,\frac{\theta_1(\tau,2z)\,\eta(\tau)^3}{(\theta_1(\tau,z))^2}=\frac{y+1}{y-1}-(y^2-y^{-2})q+\cdots.\tag{6.5}$$

Finally, we have used the theta functions

$$\theta_{m,r}(\tau,z)=\sum_{k=r\pmod{2m}}\mathrm{e}(\tfrac{k}{2})\,q^{k^2/4m}y^k,\tag{6.6}$$

defined for all $2m\in\mathbb{Z}_{>0}$ and $r-m\in\mathbb{Z}$, and satisfying

$$\theta_{m,r}(\tau,z)=\theta_{m,r+2m}(\tau,z)=\mathrm{e}(m)\,\theta_{m,-r}(\tau,-z).$$

Note that the vector-valued theta function $\theta_m=(\theta_{m,r})$, $r-m\in\mathbb{Z}/2m\mathbb{Z}$, is a vector-valued Jacobi form of weight 1/2 and index m satisfying

$$
\begin{aligned}
\theta_m(\tau,z) &= \sqrt{\frac{1}{2m}}\sqrt{\frac{i}{\tau}}\,\mathrm{e}(-\tfrac{m}{\tau}z^2)\,\mathcal{S}_\theta.\theta_m(-\tfrac{1}{\tau},\tfrac{z}{\tau})\\
&= \mathcal{T}_\theta.\theta_m(\tau+1,z)\\
&= \theta_m(\tau,z+1) = \mathrm{e}(m(\tau+2z+1))\theta_m(\tau,z+\tau),
\end{aligned}
\tag{6.7}
$$

where the $\mathcal{S}_\theta$ and $\mathcal{T}_\theta$ matrices are $2m \times 2m$ matrices with entries

$$
(\mathcal{S}_\theta)_{r,r'} = \mathrm{e}(\tfrac{rr'}{2m})\mathrm{e}(\tfrac{-r+r'}{2}), \quad (\mathcal{T}_\theta)_{r,r'} = \mathrm{e}(-\tfrac{r^2}{4m})\,\delta_{r,r'}. \tag{6.8}
$$

We will take $m \in \mathbb{Z}$ for the rest of this section. When we consider $\mathcal{N}=2$ decompositions in the next section, we will use the theta function with half-integral indices.

From the above discussion, it is clear that the graded partition function of a module for the $c = 6(m-1)$ $\mathcal{N}=4$ SCA admits the following decomposition

$$
\mathcal{Z}^{\mathcal{N}=4,m} = \sum_{\substack{n\geq 0, 0\leq r\leq m-1\\ r\neq 0 \text{ when } n>0}} c'_r(n-\tfrac{r^2}{4m})\,\mathrm{ch}^{\mathcal{N}=4}_{m;\frac{m-1}{4}+n,\frac{r}{2}}(\tau,z). \tag{6.9}
$$

Furthermore, from the identity

$$
\mu_{m;\frac{r}{2}} = (-1)^r(r+1)\mu_{m;0} + (-1)^{n-1}\sum_{n=1}^{r} n\,q^{-\frac{(r-n+1)^2}{4m}}\left(\theta_{m,r-n+1}-\theta_{m,-(r-n+1)}\right)
$$

we arrive at

$$
\mathcal{Z}^{\mathcal{N}=4,m} = (\Psi_{1,1}(\tau,z))^{-1}\left(c_0\,\mu_{m;0}(\tau,z) + \sum_{r\in\mathbb{Z}/2m\mathbb{Z}} F^{(m)}_r(\tau)\,\theta_{m,r}(\tau,z)\right), \tag{6.10}
$$

where

$$
F^{(m)}_r(\tau) = \sum_{n=0}^{\infty} c_r\left(n-\tfrac{r^2}{4m}\right)q^{n-\frac{r^2}{4m}}, \quad 1\leq r\leq m-1, \tag{6.11}
$$

$$
c_0 = \sum_{r=0}^{m-1}(-1)^r\,(r+1)\,c'_r\left(-\tfrac{r^2}{4m}\right), \tag{6.12}
$$

$$
c_r\left(n-\tfrac{r^2}{4m}\right) = \begin{cases} \sum_{r'=r}^{m-1}(-1)^{r'-r}(r'+1-r)\,c'_{r'}\left(-\tfrac{r'^2}{4m}\right), & n=0\\ c'_r\left(n-\tfrac{r^2}{4m}\right), & n>0. \end{cases} \tag{6.13}
$$

The rest of the components of $F^{(m)} = (F^{(m)}_r)$, $r \in \mathbb{Z}/2m\mathbb{Z}$, are defined by setting

$$
F^{(m)}_r(\tau) = -F^{(m)}_{-r}(\tau) = F^{(m)}_{r+2m}(\tau). \tag{6.14}
$$

Recall that $\mu_{m;0}(\tau,z) = -f_0^{(m)}(\tau,z) + f_0^{(m)}(\tau,-z)$, a specialisation of the Appell–Lerch sum

$$f_u^{(m)}(\tau,z) = \sum_{k\in\mathbb{Z}} \frac{q^{mk^2} y^{2mk}}{1-yq^k \mathrm{e}(-u)} \tag{6.15}$$

studied in [64], has the following relation to the modular group $SL_2(\mathbb{Z})$: let the (non-holomorphic) completion of $\mu_{m;0}(\tau,z)$ be

$$\begin{aligned}\hat{\mu}_{m;0}(\tau,\bar{\tau},z) = \mu_{m;0}(\tau,z) - \mathrm{e}\left(-\tfrac{1}{8}\right)\frac{1}{\sqrt{2m}} \\ \times \sum_{r\in\mathbb{Z}/2m\mathbb{Z}} \theta_{m,r}(\tau,z)\int_{-\bar{\tau}}^{i\infty}(\tau'+\tau)^{-1/2}\overline{S_{m,r}(-\bar{\tau}')}\,\mathrm{d}\tau'\,.\end{aligned} \tag{6.16}$$

Then $\hat{\mu}_{m;0}$ transforms like a Jacobi form of weight 1 and index m for $SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$. Here $S_m = (S_{m,r})$ is the vector-valued cusp form for $SL_2(\mathbb{Z})$ whose components are given by the unary theta functions

$$S_{m,r}(\tau) = \sum_{k=r \pmod{2m}} \mathrm{e}\left(\tfrac{k}{2}\right) k\, q^{k^2/4m} = \frac{1}{2\pi i}\frac{\partial}{\partial z}\theta_{m,r}(\tau,z)|_{z=0}.$$

For later use, note that the theta series $S_{m,r}(\tau)$ is defined for all $2m \in \mathbb{Z}$ and $r - m \in \mathbb{Z}/2m\mathbb{Z}$.

The way in which the functions $\mathcal{Z}^{(m)}$ and $\hat{\mu}_{m;0}$ transform under the Jacobi group shows that the non-holomorphic function $\sum_{r\in\mathbb{Z}/2m\mathbb{Z}} \hat{F}_r^{(m)}(\tau)\,\theta_{m,r}(\tau,z)$ transforms as a Jacobi form of weight 1 and index m under $SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$, where

$$\hat{F}_r^{(m)}(\tau) = F_r^{(m)}(\tau) + c_0\mathrm{e}\left(-\tfrac{1}{8}\right)\frac{1}{\sqrt{2m}}\int_{-\bar{\tau}}^{i\infty}(\tau'+\tau)^{-1/2}\overline{S_{m,r}(-\bar{\tau}')}\,\mathrm{d}\tau'.$$

In other words, $F^{(m)} = (F_r^{(m)})$, $r \in \mathbb{Z}/2m\mathbb{Z}$ is a vector-valued mock modular form with a vector-valued shadow $c_0\, S_m$, whose r-th component is given by $S_{m,r}(\tau)$, with the multiplier for $SL_2(\mathbb{Z})$ given by the inverse of the multiplier system of S_m (cf. (6.8)).

Now we are ready to apply the above discussion to the $U(1)$-graded Ramond sector partition function of the theory, discussed in Sect. 5. Recall that in this case we have $c = 12$, so $m = 3$ in (6.2) and (6.3). The $\mathcal{N} = 4$ decomposition of (5.1) gives

$$\begin{aligned}Z(\tau,z) = 21\,\mathrm{ch}^{\mathcal{N}=4}_{3;\frac12,0} + \mathrm{ch}^{\mathcal{N}=4}_{3;\frac12,1} + \left(560\,\mathrm{ch}^{\mathcal{N}=4}_{3;\frac32,\frac12} + 8470\,\mathrm{ch}^{\mathcal{N}=4}_{3;\frac52,\frac12} + 70576\,\mathrm{ch}^{\mathcal{N}=4}_{3;\frac72,\frac12} + \cdots\right) \\ + \left(210\,\mathrm{ch}^{\mathcal{N}=4}_{3;\frac32,1} + 4444\,\mathrm{ch}^{\mathcal{N}=4}_{3;\frac52,1} + 42560\,\mathrm{ch}^{\mathcal{N}=4}_{3;\frac72,1} + \cdots\right)\end{aligned} \tag{6.17}$$

$$= (\Psi_{1,1}(\tau,z))^{-1}\left(24\,\mu_{3;0}(\tau,z) + \sum_{r\in\mathbb{Z}/6\mathbb{Z}} h_r(\tau)\theta_{3,r}(\tau,z)\right) \tag{6.18}$$

where $\cdots$ stand for terms with expansion $\Psi_{1,1}^{-1}q^{\alpha}y^{\beta}$ with $\alpha-\beta^2/12>3$. More Fourier coefficients of the functions $h_r(\tau)$ are recorded in "Appendix C", where $h=h_g$ for $[g]=1A$. Note that all the graded multiplicities $c_r'(n-\frac{r^2}{12})$ appear to be non-negative. Of course, this is guaranteed by the fact that V is a module for the $\mathcal{N}=4$ SCA as shown in Sect. 3. In particular, the Fourier coefficients of $h_r(\tau)$ appear to be all non-negative apart from that of the polar term $-2q^{-1/12}$ in h_1.

From the above discussion we see that $h=(h_r)$, for $r\in\mathbb{Z}/6\mathbb{Z}$, is a weight 1/2 vector-valued mock modular form for $SL_2(\mathbb{Z})$ with 6 components (but just 2 linearly independent components, since $h_0=h_3=0$, $h_{-1}=-h_1$, and $h_{-2}=-h_2$), with shadow given by $24\,S_3$, and multiplier system inverse to that of S_3.

This is to be contrasted with the elliptic genus of a generic non-chiral super conformal field theory. For example, the sigma model of a K3 surface has $c=6$, and the elliptic genus is given by

$$\begin{aligned}\mathbf{EG}(\tau,z;K3) &= \mathrm{Tr}_{\mathcal{H}_{RR}}(-1)^{F_L+F_R}y^{J_0}q^{L_0-c/24}\bar{q}^{\tilde{L}_0-\tilde{c}/24}\\ &= 20\,\mathrm{ch}^{\mathcal{N}=4}_{2;\frac14,0}-2\,\mathrm{ch}^{\mathcal{N}=4}_{2;\frac14,\frac12}+\big(90\,\mathrm{ch}^{\mathcal{N}=4}_{2;\frac54,\frac12}+462\,\mathrm{ch}^{\mathcal{N}=4}_{2;\frac94,\frac12}+1540\,\mathrm{ch}^{\mathcal{N}=4}_{2;\frac{13}{4},\frac12}+\cdots\big)\end{aligned} \tag{6.19}$$

$$\begin{aligned}&=(\Psi_{1,1}(\tau,z))^{-1}\Big\{24\,\mu_{2;0}(\tau,z)+(\theta_{2,1}(\tau,z)-\theta_{2,-1}(\tau,z))\\ &\quad\times(-2q^{-1/8}+90q^{7/8}+462q^{15/8}+1540q^{23/8}+\cdots)\Big\},\end{aligned} \tag{6.20}$$

where $\cdots$ stand for terms with expansion $\Psi_{1,1}^{-1}q^{\alpha}y^{\beta}$ with $\alpha-\beta^2/8>3$. In this case, the coefficient multiplying the massless character $\mathrm{ch}^{\mathcal{N}=4}_{2;\frac14,\frac12}$ is negative, arising from the Witten index of the right-moving massless multiplets paired with the representation $v^{\mathcal{N}=4}_{2;\frac14,\frac12}$ of the left-moving $\mathcal{N}=4$ SCA.

In Sect. 3 we have shown that the theory under consideration, as a module for the $\mathcal{N}=4$ SCA, admits a faithful action via automorphisms by a group G, as long as G is a subgroup of Co_0 fixing at least a 3-plane. For any such $g\in G$, the g-twined graded partition function $Z_g(\tau,z)$ is given by (5.10), and from the fact that the action of g commutes with the $\mathcal{N}=4$ SCA, we expect $Z_g(\tau,z)$ to admit a decomposition

$$Z_g(\tau,z)=(\Psi_{1,1}(\tau,z))^{-1}\left((\mathrm{Tr}_{\mathbf{24}}g)\,\mu_{3;0}(\tau,z)+\sum_{r\in\mathbb{Z}/6\mathbb{Z}}h_{g,r}(\tau)\theta_{3,r}(\tau,z)\right). \tag{6.21}$$

Moreover, the coefficients of

$$h_{g,r}(\tau)=a_rq^{-r^2/12}+\sum_{n=1}^{\infty}(\mathrm{Tr}_{V^G_{r,n}}g)\,q^{n-r^2/12} \tag{6.22}$$

must be characters of the G-module

$$V^G=\bigoplus_{r=1,2}\bigoplus_{n=1}^{\infty}V^G_{r,n} \tag{6.23}$$

arising from the orbifold theory discussed in Sect. 2.

Indeed, the multiplicities of the $\mathcal{N}=4$ multiplets in the decomposition (6.17) are suggestive of the following group theoretic interpretation[b]: the 21 $h=1/2$, $j=0$ massless representations transform as the 21-dimensional irreducible representation of M_{22}, and similarly, the 560 $h=3/2$, $j=1/2$ massive representations transform as $\chi_{10}+\chi_{11}$ (see "Appendix B"), or "$280+\overline{280}$", under M_{22}, etc.

We have explicitly computed the first 30 or so coefficients of each q-series $h_{g,r}(\tau)$ for all conjugacy classes $[g]$ of G, for $G=M_{22}$ and $G=U_4(3)$. These can be found in the tables in "Appendix C". Subsequently, we compute the first 30 or so G-modules $V^G_{r,n}$ in terms of their decompositions into irreducible representations. They can be found in the tables in "Appendix D".

Finally we would like to discuss the mock modular property of the functions $h_g=(h_{g,r})$. Recall that the *Hecke congruence subgroups* of $SL_2(\mathbb{Z})$ are defined as

$$\Gamma_0(N)=\left\{\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in SL_2(\mathbb{Z})\mid \text{c=0 mod N}\right\}. \tag{6.24}$$

We expect Z_g to be a weak Jacobi form of weight zero and index 2 (possibly with multiplier) for the group $\Gamma_0(\mathrm{o}_g)\ltimes\mathbb{Z}^2$, where o_g is the order of the group element $g\in G$. This can be verified explicitly from the expression (5.10). Repeating the similar arguments as above, we conclude that each vector-valued function h_g is a vector-valued mock modular form of weight 1/2 with shadow $(\mathrm{Tr}_{\mathbf{24}}g)S_3$ for the congruence subgroup $\Gamma_0(\mathrm{o}_g)$. Note that $(\mathrm{Tr}_{\mathbf{24}}g)\neq 0$ for all $g\in M_{22}$ which are the cases of our main interest. For these cases the multiplier of h_g is again given by the inverse of the multiplier system of S_3, now restricted to $\Gamma_0(o_g)$.

In this section we have analyzed the decomposition of the Ramond sector of our orbifold theory into irreducible modules for the $\mathcal{N}=4$ SCA, and we have demonstrated that the generating functions of irreducible $\mathcal{N}=4$ SCA module multiplicities furnish a vector-valued mock modular form. We have also demonstrated that these multiplicities are dimensions of modules for subgroups $G<Co_0$ that point-wise fix a 3-plane in **24**, and we have analyzed the modularity of the resulting, g-twined multiplicity generating functions, for $g\in G$. We have verified that each such g-twining results in a vector-valued mock modular form with a specified shadow function. In the next section we will present directly analogous considerations for $\mathcal{N}=2$ superconformal structures arising from 2-planes in **24**.

7 The $\mathcal{N}=2$ decompositions

As discussed in Sect. 4, the theory presented in Sect. 2 can be regarded as a module for an $\mathcal{N}=2$ SCA as well as for an $\mathcal{N}=4$ SCA. Moreover, for every subgroup $G<Co_0$ fixing a 2-plane there is an $\mathcal{N}=2$ SCA commuting with the action of G on the theory. As a result, and as we will now demonstrate, the decomposition of the partition function (5.10) twined by elements of G into $\mathcal{N}=2$ characters leads to sets of vector-valued mock modular forms, now of weight 1 / 2 and index 3 / 2, which are the graded characters of an infinite-dimensional G-module inherited from the Co_0-module structure on V (cf. Sect. 5).

To see what these (vector-valued) mock modular forms $\tilde{h}_g = (\tilde{h}_{g,j})$ are, let us start by recalling the characters of the irreducible representations of the $\mathcal{N}=2$ SCA. For the SCA with central charge $c = 3(2\ell+1) = 3\hat{c}$, the unitary irreducible highest weight representations $v^{\mathcal{N}=2}_{\ell;h,Q}$ are labeled by the two quantum numbers h and Q which are the eigenvalues of L_0 and J_0, respectively, when acting on the highest weight state [22, 51]. Just as in the $\mathcal{N}=4$ case, there are two types of Ramond sector highest weight representations: the *massless* (or *BPS*) representations with $h = \frac{c}{24} = \frac{\hat{c}}{8}$ and $Q \in \{-\frac{\hat{c}}{2}+1, -\frac{\hat{c}}{2}+2, \ldots, \frac{\hat{c}}{2}-1, \frac{\hat{c}}{2}\}$, and the *massive* (or *non-BPS*) representations with $h > \frac{\hat{c}}{8}$ and $Q \in \{-\frac{\hat{c}}{2}+1, -\frac{\hat{c}}{2}+2, \ldots, \frac{\hat{c}}{2}-2, \frac{\hat{c}}{2}-1, \frac{\hat{c}}{2}\}$, $Q \neq 0$. From now on we will concentrate on the case when ℓ is half-integral, and hence $\hat{c}$ and c are even.

The graded characters, defined as

$$\mathrm{ch}^{\mathcal{N}=2}_{\ell;h,Q}(\tau,z) = \mathrm{tr}_{v^{\mathcal{N}=2}_{\ell;h,Q}}\left((-1)^{J^3_0} y^{\tilde{J}^3_0} q^{L_0-c/24}\right), \tag{7.1}$$

are given by

$$\begin{aligned} &\mathrm{ch}^{\mathcal{N}=2}_{\ell;h,Q}(\tau,z) = \mathrm{e}(\tfrac{\ell}{2})(\Psi_{1,-\frac{1}{2}}(\tau,z))^{-1} q^{h-\frac{c}{24}-\frac{j^2}{4\ell}}\theta_{\ell,j}(\tau,z), \\ &j = \mathrm{sgn}(Q)\,(|Q|-1/2), \end{aligned} \tag{7.2}$$

for the massive representations, and

$$\begin{aligned} &\mathrm{ch}^{\mathcal{N}=2}_{\ell;c/24,Q}(\tau,z) = \mathrm{e}\left(\tfrac{\ell+Q+1/2}{2}\right)(\Psi_{1,-\frac{1}{2}}(\tau,z))^{-1}\, y^{Q+\frac{1}{2}} f^{(\ell)}_u(\tau,z+u), \\ &u = \tfrac{1}{2} + \tfrac{(1+2Q)\tau}{4\ell}, \end{aligned} \tag{7.3}$$

for the massless representations (with $Q \neq \frac{\hat{c}}{2}$). The character $\mathrm{ch}^{\mathcal{N}=2}_{\ell;c/24,Q}(\tau,z)$ for $Q = \frac{\hat{c}}{2}$ is given in (7.5). In the above formula, we have used the Appell–Lerch sum (6.15) and defined

$$\Psi_{1,-\frac{1}{2}} = -i\,\frac{\eta(\tau)^3}{\theta_1(\tau,z)} = \frac{1}{y^{1/2}-y^{-1/2}} + q\,(y^{1/2}-y^{-1/2}) + O(q^2).$$

Note that the above characters transform according to the rule

$$\mathrm{ch}^{\mathcal{N}=2}_{\ell;c/24,Q}(\tau,z) = \mathrm{ch}^{\mathcal{N}=2}_{\ell;c/24,-Q}(\tau,-z)$$

under charge conjugation.

From the relation between the massless and massive characters

$$\begin{aligned} \mathrm{ch}^{\mathcal{N}=2}_{\ell;c/24,Q} + \mathrm{ch}^{\mathcal{N}=2}_{\ell;c/24,-Q} = q^{-n}\sum_{k=0}^{|Q|-1}(-1)^k\left(\mathrm{ch}^{\mathcal{N}=2}_{\ell,n+c/24,Q-k} + \mathrm{ch}^{\mathcal{N}=2}_{\ell,n+c/24,k-Q}\right) \\ + 2(-1)^Q \mathrm{ch}^{\mathcal{N}=2}_{\ell;c/24,0}, \quad n>0, |Q| \le \ell, \end{aligned} \tag{7.4}$$

$$\begin{aligned} \mathrm{ch}^{\mathcal{N}=2}_{\ell;c/24,\frac{\hat{c}}{2}} = q^{-n}\left(\mathrm{ch}^{\mathcal{N}=2}_{\ell;n+c/24,\frac{\hat{c}}{2}} + \sum_{k=1}^{\frac{\hat{c}}{2}-1}(-1)^k\left(\mathrm{ch}^{\mathcal{N}=2}_{\ell,n+c/24,\frac{\hat{c}}{2}-k} + \mathrm{ch}^{\mathcal{N}=2}_{\ell,n+c/24,k-\frac{\hat{c}}{2}}\right)\right) \\ + (-1)^{\frac{\hat{c}}{2}}\mathrm{ch}^{\mathcal{N}=2}_{\ell;c/24,0}, \end{aligned} \tag{7.5}$$

as well as the charge conjugation symmetry of the theory, we expect the $U(1)$-graded Ramond sector partition function of a theory that is invariant under charge conjugation to admit a decomposition

$$\begin{aligned}\mathcal{Z}^{\mathcal{N}=2,\ell} &= C_0' \,\mathrm{ch}^{\mathcal{N}=2}_{\ell;\frac{c}{24},0}(\tau,z) + \sum_{n\geq 0} C_\ell'\left(n-\tfrac{\ell}{4}\right) \mathrm{ch}^{\mathcal{N}=2}_{\ell;\frac{c}{24}+n,\ell+\frac{1}{2}}(\tau,z) < \mathit{error}- \\ &\quad + \sum_{n\geq 0, j\in\{\frac{1}{2},\frac{3}{2},\ldots,\ell-1\}} C_j'\left(n-\tfrac{j^2}{4\ell}\right)\left(\mathrm{ch}^{\mathcal{N}=2}_{\ell;\frac{c}{24}+n,j+\frac{1}{2}}(\tau,z) + \mathrm{ch}^{\mathcal{N}=2}_{\ell;\frac{c}{24}+n,-\left(j+\frac{1}{2}\right)}(\tau,z)\right)\end{aligned} \tag{7.6}$$

$$= \mathrm{e}\left(\tfrac{\ell}{2}\right)(\Psi_{1,-\frac{1}{2}})^{-1}\left(C_0\,\tilde{\mu}_{\ell;0}(\tau,z) + \sum_{j-\ell\in\mathbb{Z}/2\ell\mathbb{Z}} \tilde{F}_j^{(\ell)}(\tau)\theta_{\ell,j}(\tau,z)\right) \tag{7.7}$$

when the $\mathcal{N}=2$ SCA has even central charge, $c = 3(2\ell+1)$. In the last equation, we have defined

$$\tilde{\mu}_{\ell;0} = \mathrm{e}\left(\tfrac{1}{4}\right) y^{1/2} f_u^{(\ell)}(\tau, u+z), \quad u = \frac{1}{2} + \frac{\tau}{4\ell},$$

and

$$\tilde{F}_j^{(\ell)}(\tau) = \tilde{F}_{-j}^{(\ell)}(\tau) = \tilde{F}_{j+2\ell}^{(\ell)}(\tau) = \sum_{n\geq 0} C_j\left(n-\tfrac{j^2}{4\ell}\right) q^{n-\frac{j^2}{4\ell}}, \tag{7.8}$$

$$C_0 = C_0' + 2\sum_{j\in\{\frac{1}{2},\frac{3}{2},\ldots,\ell\}} (-1)^{j+\frac{1}{2}} C_j'\left(-\tfrac{j^2}{4\ell}\right), \tag{7.9}$$

$$C_j\left(n-\tfrac{j^2}{4\ell}\right) = \begin{cases} \sum_{k=0}^{\ell-j}(-1)^k C_{j+k}'\left(-\frac{(j+k)^2}{4\ell}\right) & n=0 \\ C_j'\left(n-\frac{j^2}{4\ell}\right) & n>0. \end{cases} \tag{7.10}$$

Similar to the case of massless $\mathcal{N}=4$ characters, through its relation to the Appell–Lerch sum, $\tilde{\mu}_{\ell;0}$ admits a completion which transforms as a weight one, half-integral index Jacobi form under the Jacobi group. More precisely, define $\widehat{\tilde{\mu}_{\ell;0}}$ by replacing $\mu_{m;0}$ with $\tilde{\mu}_{\ell;0}$ and the integer m with the half-integral ℓ in (6.16). Then $\widehat{\tilde{\mu}_{\ell;0}}$ transforms like a Jacobi form of weight 1 and index ℓ under the group $SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$. Following the same computation as in the previous section, we hence conclude that $\tilde{F}^{(\ell)} = (\tilde{F}_j^{(\ell)})$, where $j-1/2 \in \mathbb{Z}/2\ell\mathbb{Z}$, is a vector-valued mock modular form with a vector-valued shadow $C_0\, S_\ell = C_0(S_{\ell,j}(\tau))$.

Now we are ready to apply the above discussion to the $U(1)$-graded R sector partition function of the orbifold theory of Sect. 2. The $\mathcal{N}=2$ decomposition gives

$$\begin{aligned}Z(\tau,z) &= 23\,\mathrm{ch}^{\mathcal{N}=2}_{\frac{3}{2};\frac{1}{2},0} + \mathrm{ch}^{\mathcal{N}=2}_{\frac{3}{2};\frac{1}{2},2} + \left(770\left(\mathrm{ch}^{\mathcal{N}=2}_{\frac{3}{2};\frac{3}{2},1} + \mathrm{ch}^{\mathcal{N}=2}_{\frac{3}{2};\frac{3}{2},-1}\right)\right. \\ &\quad \left. +13915\left(\mathrm{ch}^{\mathcal{N}=2}_{\frac{3}{2};\frac{5}{2},1} + \mathrm{ch}^{\mathcal{N}=2}_{\frac{3}{2};\frac{5}{2},-1}\right) + \cdots\right) \\ &\quad + \left(231\,\mathrm{ch}^{\mathcal{N}=2}_{\frac{3}{2};\frac{3}{2},2} + 5796\,\mathrm{ch}^{\mathcal{N}=2}_{\frac{3}{2};\frac{5}{2},2} + \ldots\right)\end{aligned} \tag{7.11}$$

$$= \mathrm{e}\left(\tfrac{3}{4}\right)\Psi^{-1}_{1,-\frac{1}{2}}\Big(24\,\tilde{\mu}_{\frac{3}{2};0} + \Big(-q^{-\frac{1}{24}} + 770\,q^{\frac{23}{24}} + 13915\,q^{\frac{47}{24}} + c\ldots\Big)\Big(\theta_{\frac{3}{2},\frac{1}{2}} + \theta_{\frac{3}{2},-\frac{1}{2}}\Big)$$
$$+ (q^{-\frac{3}{8}} + 231\,q^{\frac{5}{8}} + 5796q^{\frac{13}{8}} + \cdots)\,\theta_{\frac{3}{2},\frac{3}{2}}\Big) \tag{7.12}$$

where $\cdots$ denote the terms with expansion $\Psi^{-1}_{1,-\frac{1}{2}}q^{\alpha}y^{\beta}$ with $\alpha - \beta^2/6 > 2$. Again, we observe that all the multiplicities of the representations with characters $\mathrm{ch}^{\mathcal{N}=2}_{\frac{3}{2};h,Q}$ appear to be non-negative, consistent with our construction of V as an $\mathcal{N} = 2$ SCA module.

In general, from the previous sections we have seen that the graded partition function twined by any element g of a subgroup G of Co_0 should admit a decomposition into $\mathcal{N} = 2$ characters. We write

$$Z_g(\tau,z) = \mathrm{e}(\tfrac{3}{4})\Psi^{-1}_{1,-\frac{1}{2}}\left((\mathrm{Tr}_{\mathbf{24}}g)\,\tilde{\mu}_{\frac{3}{2};0}(\tau,z) + \sum_{j\in\{1/2,-1/2,3/2\}}\tilde{h}_{g,j}(\tau)\theta_{\frac{3}{2},j}\right). \tag{7.13}$$

Moreover, from the discussion in Sect. 2 we have seen that

$$\tilde{h}_{g,1/2}(\tau) = \overline{\tilde{h}_{g,-1/2}(-\bar{\tau})}, \tag{7.14}$$

and the coefficients of these functions

$$\tilde{h}_{g,j}(\tau) = a_j q^{-j^2/6} + \sum_{n=1}^{\infty}(\mathrm{Tr}_{\tilde{V}^G_{r,n}}g)\; q^{n-j^2/6} \tag{7.15}$$

are given by characters of a G-module

$$\tilde{V}^G = \bigoplus_{j=-1/2,1/2,3/2}\bigoplus_{n=1}^{\infty}\tilde{V}^G_{j,n} \tag{7.16}$$

which descends from the orbifold theory in Sect. 2. In particular, for any n, the G-module $\tilde{V}^G_{-1/2,n}$ is the dual of $\tilde{V}^G_{1/2,n}$. From the above discussion, we conclude that $\tilde{h}_g = (\tilde{h}_{g,j})$ is a vector-valued mock modular form for $\Gamma_0(o_g)$ with shadow $(\mathrm{Tr}_{\mathbf{24}}g)S_{3/2}$. Recall that $(\mathrm{Tr}_{\mathbf{24}}g) \neq 0$ for all $g \in M_{22}$ which are the cases of our main interest. For these case the multiplier of $\tilde{h}_g$ is given by the inverse of the multiplier system of $S_{3/2}$, restricted to $\Gamma_0(o_g)$.

To summarize, we have analyzed the decomposition of the Ramond sector of our orbifold theory into irreducible modules for the $\mathcal{N} = 2$ SCA in this section, and we have demonstrated that the generating functions of irreducible $\mathcal{N} = 2$ SCA module multiplicities also furnish a vector-valued mock modular form. We have demonstrated that these multiplicities are dimensions of modules for subgroups $G < Co_0$ that point-wise fix a 2-plane in **24**, and we have observed that the resulting, g-twined multiplicity generating functions, for $g \in G$, are vector-valued mock modular forms with a certain extra symmetry, relating the components labelled by $\pm 1/2$ by complex conjugation.

8 Mathieu moonshine

In the previous sections we have seen that the orbifold theory described in Sect. 2 leads to infinite-dimensional G-modules underlying a set of vector-valued mock modular forms from its $\mathcal{N}=4$ ($\mathcal{N}=2$) structures for any subgroup G of Co_0 fixing at least a 3-plane (2-plane) in **24**. In this section we will discuss a natural property of the vector-valued mock modular forms that distinguishes subgroups of M_{24} from other 3-plane (2-plane) fixing subgroups. These considerations lead to mock modular Mathieu moonshine involving distinguished vector-valued mock modular forms of weight 1 / 2.

Recall the celebrated genus zero condition of monstrous moonshine, which states that the monstrous McKay–Thompson series are Hauptmoduls with only a polar term q^{-1} at the cusp represented by $i\infty$, and no poles at any other cusps. To be more precise, denote by $T_g(\tau)=\sum_{n\geq -1} q^n \mathrm{tr}_{V^\natural_n} g$ the graded character of the moonshine module $V^\natural = \bigoplus_{n\geq -1} V^\natural_n$ of Frenkel–Lepowsky–Meurman [37]. Then $T_g(\tau)$ is a function invariant under the action of a particular $\Gamma_g < SL_2(\mathbb{R})$ (specified in [15]), such that

$$\begin{aligned} &\text{(i)} \quad qT_g(\tau)=O(1) \text{ as } \tau\to i\infty, \text{ and}\\ &\text{(ii)} \quad T_g(\gamma\tau)=O(1) \text{ as } \tau\to i\infty \text{ whenever } \gamma\in SL_2(\mathbb{Z}) \text{ and } \gamma\infty\notin\Gamma_g\infty. \end{aligned} \tag{8.1}$$

Similarly, in Mathieu moonshine [35], it follows from the results of [7] that if $g\in M_{24}$ and Z_g denotes the g-twined K3 elliptic genus then

$$\begin{aligned} &\text{(i)} \quad Z_g(\tau,z)=c_g+y+y^{-1} \text{ as } \tau\to i\infty, \text{ and}\\ &\text{(ii)} \quad Z_g|_{0,1}\gamma(\tau,z)=c_{g,\gamma} \text{ as } \tau\to i\infty \text{ whenever } \gamma\in SL_2(\mathbb{Z}) \text{ and } \gamma\infty\notin\Gamma_g\infty, \end{aligned} \tag{8.2}$$

(cf. (5.4)) for some $c_g, c_{g,\gamma}\in\mathbb{C}$. In other words, the $\tau\to i\infty$ limit of $Z_g|_{0,1}\gamma$ is a z-independent constant whenever γ is not in the invariance group Γ_g. (Note that Γ_g is always a subgroup of $SL_2(\mathbb{Z})$ for $g\in M_{24}$).

A natural question is therefore: among the subgroups of Co_0 fixing 2- or 3-planes for which we have constructed a module in this work, for which of these do the associated modular objects satisfy a condition analogous to the preceding cases of moonshine described above? We will see presently that the Mathieu groups M_{23}, M_{22}, M_{12} and M_{11} are distinguished in our setting, in that the graded characters of their respective modules yield weight zero weak Jacobi forms satisfying the conditions

$$\begin{aligned} &\text{(i)} \quad Z_g(\tau,z)=c_g+y^2+y^{-2} \text{ as } \tau\to i\infty, \text{ and}\\ &\text{(ii)} \quad Z_g|_{0,2}\gamma(\tau,z)=c_{g,\gamma} \text{ as } \tau\to i\infty \text{ whenever } \gamma\in SL_2(\mathbb{Z}) \text{ and } \gamma\infty\notin\Gamma_g\infty, \end{aligned} \tag{8.3}$$

for some $c_g, c_{g,\gamma}\in\mathbb{C}$. On the other hand, all the other groups mentioned in Sect. 4 contain elements g for which the condition (ii) in (8.3) is not satisfied. Thus the conditions (8.3) single out the Mathieu groups as the sporadic simple subgroups of Co_0 with this moonshine property. The constructions we have presented in this paper provide concrete realizations of the underlying mock modular Mathieu moonshine modules.

Note that the conditions (8.3) impose restrictions on the degrees of the poles (if any) of the mock modular forms h_g, $\tilde{h}_g$ (cf. Sects. 6, 7) at all cusps. In fact, for the case that G is a copy of M_{22} or M_{11} preserving $\mathcal{N}=4$ supersymmetry, the corresponding mock modular forms h_g only have poles at the infinite cusp. This property also holds for the mock modular forms attached to M_{24} via $\mathcal{N}=4$ decomposition of the twined K3 elliptic genera of Mathieu moonshine, satisfying (8.2), as was demonstrated in [7]. In more physical terms, (8.2) and (8.3) can be interpreted as the condition that the elliptic genus of any cyclic orbifold of the theory receives no contributions from $U(1)$-charged ground states in twisted sectors.

To investigate the behaviour of Z_g at cusps other than $i\infty$, we first note that for any positve integer N,

$$Z_g|_{0,2}\left(\begin{smallmatrix}1 & 0\\ N & 1\end{smallmatrix}\right)(\tau,z) = \mathrm{e}\left(N\sum_{\ell=1}^{12}\frac{\rho_\ell^2}{2}\right)\sum_{i=1}^{4}\epsilon_{i,N}(-1)^{i+1}\Theta_{i,N}(\tau,z),$$

where

$$\Theta_{i,N}(\tau,z) = \frac{\theta_i(\tau,2z)}{\eta^{12}(\tau)}\prod_{k=2}^{12}\mathrm{e}\left(\frac{\rho_k^2N^2}{2}\tau\right)\theta_i(\tau,\rho_k+N\rho_k\tau), \tag{8.4}$$

and $\epsilon_{i,N}=\epsilon_i$ when $2|N$, and

$$\begin{cases}\epsilon_{1,N} & = -\epsilon_1\\ \epsilon_{2,N} & = -\epsilon_3 = -1\\ \epsilon_{3,N} & = -\epsilon_2\\ \epsilon_{4,N} & = -\epsilon_4 = -1\end{cases} \tag{8.5}$$

otherwise. The above expressions can be derived from the transformation laws of Jacobi theta functions under $SL_2(\mathbb{Z})$.

Near the infinite cusp, $\tau\to i\infty$, the different contributions have the following leading behaviour:

$$\begin{aligned}
\Theta_{1,N}(\tau,z) &= \mathrm{e}\left(\tfrac{1}{2}+\sum_{k=1}^{12}(\tfrac{1}{2}-\rho_k)\lfloor N\rho_k\rfloor-\tfrac{\rho_k}{2}\right)q^{f_{N,1}(\Pi_g)/2}(y-y^{-1})\,[1+O(q^{1/o_g})]\\
\Theta_{2,N}(\tau,z) &= \mathrm{e}\left(\sum_{k=1}^{12}-\rho_k\lfloor N\rho_k\rfloor-\tfrac{\rho_k}{2}\right)q^{f_{N,1}(\Pi_g)/2}(y+y^{-1})\,[1+O(q^{1/o_g})]\\
\Theta_{3,N}(\tau,z) &= \mathrm{e}\left(-\sum_{k=1}^{12}\rho_k\lfloor\tfrac{1}{2}+N\rho_k\rfloor\right)q^{f_{N,2}(\Pi_g)/2}\left(1+q^{1/2}(y^2+y^{-2})\right)\\
&\quad\times\left(\prod_{k=2}^{12}\left(1+\mathrm{e}(-\rho_k)\,q^{\lfloor\frac{1}{2}+N\rho_k\rfloor+\frac{1}{2}-N\rho_k}\right)\prod_{n=1}^{\lfloor\frac{1}{2}+N\rho_k\rfloor}\left(1+\mathrm{e}(\rho_k)q^{\frac{1}{2}+N\rho_k-n}\right)\right)\\
&\quad\times[1+O(q^{1/o_g})]\\
\Theta_{4,N}(\tau,z) &= \mathrm{e}\left(-\sum_{k=1}^{12}(\tfrac{1}{2}+\rho_k)\lfloor\tfrac{1}{2}+N\rho_k\rfloor\right)q^{f_{N,2}(\Pi_g)/2}\left(1-q^{1/2}(y^2+y^{-2})\right)\\
&\quad\times\left(\prod_{k=2}^{12}\left(1-\mathrm{e}(-\rho_k)\,q^{\lfloor\frac{1}{2}+N\rho_k\rfloor+\frac{1}{2}-N\rho_k}\right)\prod_{n=1}^{\lfloor\frac{1}{2}+N\rho_k\rfloor}\left(1-\mathrm{e}(\rho_k)\,q^{\frac{1}{2}+N\rho_k-n}\right)\right)\\
&\quad\times[1+O(q^{1/o_g})]
\end{aligned} \tag{8.6}$$

with

$$f_{N,1}(\Pi_g) = -1+\sum_{k=1}^{12}(N\rho_k-1/2)^2+\lfloor N\rho_k\rfloor(1+\lfloor N\rho_k\rfloor-2N\rho_k)\,, \tag{8.7}$$

$$f_{N,2}(\Pi_g) = -1 + \sum_{k=1}^{12} N^2\rho_k^2 - \lfloor \tfrac{1}{2} + N\rho_k \rfloor (2N\rho_k - \lfloor \tfrac{1}{2} + N\rho_k \rfloor)\,, \tag{8.8}$$

where $\lfloor x \rfloor$ denotes the largest integer that is not greater than x.

Comparing with the second condition in (8.3), we see that it is satisfied for $\gamma = \left(\begin{smallmatrix} 1 & 0 \\ N & 1 \end{smallmatrix}\right)$ if and only if the Π_g satisfies

$$f_{N,1}(\Pi_g) > 0, \tag{8.9}$$

together with

$$f_{N,2}(\Pi_g) > -1,\ \ \frac{f_{N,2}(\Pi_g)}{2} + \left| \frac{1}{2} + \lfloor N\rho_k \rfloor - N\rho_k \right| \geq 0 \tag{8.10}$$

for the case $\epsilon_{3,N}\epsilon_{4,N}\epsilon(\frac{1}{2}\sum_{k=1}^{12}\lfloor \frac{1}{2}N\rho_k \rfloor) = 1$, and

$$f_{N,2}(\Pi_g) \geq 0 \tag{8.11}$$

for the case $\epsilon_{3,N}\epsilon_{4,N}\epsilon(\frac{1}{2}\sum_{k=1}^{12}\lfloor \frac{1}{2}N\rho_k \rfloor) = -1$.

Now we are left to check explicitly the condition (ii) in (8.3) for the various 2- and 3-plane preserving subgroups discussed in Sect. 4. First, recall that, for n a positive integer, each cusp of $\Gamma_0(n)$ is represented by a rational number of the form $u\ /\ v$ where u and v are coprime positive integers, v a divisor of n, and $u\ /\ v$ is equivalent u'/v' if and only if $v = v'$, and $u = u' \mod (v, n/v)$. (note that the infinite cusp is also represented by $1\ /\ n$.) Via direct computation using the data of the eigenvalues of $g \in Co_0$ acting on $\mathbf{24}$, we note that among all the groups we have considered in Sect. 4, the groups $U_4(3)$, $U_6(2)$ and McL all contain a conjugacy classe with Frame shape $\Pi_g = 3^9/1^3$, and HS has a conjugacy class with Frame shape $\Pi_g = 5^5/1^1$. One can explicitly check that $f_{1,1}(\Pi_g) = 0$ for these classes and hence the corresponding twining $Z_g|_{0,2}\left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)(\tau, z)$ has a non-vanishing coefficient for q^0y^1 as $\tau \to i\infty$. (Note that the cusp at $\tau = 1$ is not equivalent to the cusp at $i\infty$ in these cases.) This excludes the groups $U_4(3)$, $U_6(2)$, McL and HS as candidates for moonshine satisfying (8.3).

For the subgroups of M_{24}, a simple analysis of the cusp representatives of $\Gamma_g = \Gamma(o_g)$ shows that it is sufficient to verify (8.3) for $\gamma = \left(\begin{smallmatrix} 1 & 0 \\ N & 1 \end{smallmatrix}\right)$ for all $N|n$ and $N < n$. For these cases, we explicitly verify that (8.9)–(8.11) are satisfied and hence the moonshine condition (8.3) is met.

We therefore conclude that we have established mock modular moonshine for all but the largest of the sporadic simple Mathieu groups, together with explicit constructions of the corresponding modules. Moreoever, the corresponding twined graded characters are mock modular forms arising from Jacobi forms satisfying the distinguishing conditions (8.3), which we may recognise as furnishing a natural analogue of the powerful principal modulus property (a.k.a. genus zero property) of monstrous moonshine.

The reader will note that many of the numbers which occur as dimensions of irreducible representations of M_{23} also occur as dimensions of irreducible representations for M_{24}. Indeed, looking at the Tables in "Appendix C", one is tempted to guess that there is an alternative construction, or hidden symmetry in our model, which yields an M_{24} -module with the same graded dimensions. In fact, the procedure we have explained for computing twinings can be carried out for any element of M_{24}, regarded as a subgroup of Co_0, for any such element fixes a 2-space in **24**. However, there is no 2-space that is fixed by every element of a given copy of M_{24}, and explicit computations reveal that any M_{24}-module structure on the module we have constructed would have to involve virtual representations. This indicates that there is no direct extension to M_{24} of the Mathieu moonshine modules we have considered here, despite the prominent role M_{24} plays in incorporating the different groups of moonshine in the current setting. Nevertheless, there is a certain modification of our method for which M_{24} is now known to play a leading role. We refer the reader to the next, and final section for a description of this.

9 Discussion

In this paper we have demonstrated that, starting with the free field Co_0 module of [24], one can construct explicit examples of modules for various subgroups $G \subset \mathrm{Co}_0$ which underlie certain mock modular forms. In particular, subgroups which preserve a 3-plane (respectively 2-plane) in the **24** give rise to $\mathcal{N} = 4$ ($\mathcal{N} = 2$) modules with G symmetry. This gives completely explicit examples of mock modular moonshine for the smaller Mathieu groups, where the modules are known, and where the twining functions are distinguished in a manner directly similar to Mathieu moonshine. Other examples, including modules for the sporadic groups McL and HS, are also described.

There are several future directions. We considered here the $\mathcal{N} = 2$ and $\mathcal{N} = 4$ extended chiral algebras, and the subgroups of Co_0 that they preserve. Other extended chiral algebras may also yield interesting results. For instance, supersymmetric sigma models with target a Spin(7) manifold give rise to an extended chiral algebra [58], whose representations were studied in [45]. It is an extension of the $\mathcal{N} = 1$ superconformal algebra where instead of adding a $U(1)$ current (which extends the theory to an $\mathcal{N} = 2$ superconformal theory), one chooses an additional Ising factor. Conjectural characters for the unitary, irreducible representations of this algebra were worked out in [2], where it was shown that there is a suggestive relation between the decomposition of the elliptic genus of a Spin(7) manifold into these characters and irreducible representations of finite groups. This connection was made precise in [12], where it was shown that the same $c = 12$ theory can be viewed as an SCFT with extended $\mathcal{N} = 1$ symmetry, and thus yields theories with global symmetry groups M_{24}, Co_2, and Co_3. The partition function twined by these symmetries, when decomposed into characters of the Spin(7) algebra, gives rise to two-component vector-valued mock modular forms encoding infinite-dimensional modules for the corresponding sporadic groups.

The motivation that led, eventually, to the present study was actually to find connections between geometrical target manifolds associated to $c = 12$ conformal field theories, and sporadic groups. The elliptic genera of Calabi-Yau fourfolds were computed in [54], for instance; their structure is reminiscent of some of the modules we have seen here, and we intend to further explore and describe some of these connections in a future publication. Likewise, hyperkähler fourfolds, as well as the Spin(7) manifolds mentioned above, provide a wide class of geometries where an analogue of the connections between M_{24} and K3 may be sought.

Last but not least, there are suggestive connections between the trace functions in moonshine modules, and certain special properties of the underlying conformal field theory. Both the CFT appearing in monstrous moonshine and the Co_0 module that played a starring role in this paper appear to play special roles also in AdS_3 quantum gravity, where they are candidates for CFT duals to pure (super)gravity [62]. The genus zero property of the twining functions in monstrous moonshine can be reformulated as a condition that these class functions should be expressed as Rademacher sums based on a fixed polar part [7, 29]; this latter description then applies uniformly to monstrous moonshine and Mathieu moonshine.

In this paper we demonstrate that a similar criterion also applies to our mock modular Mathieu moonshines. In particular, we have shown that the Jacobi forms relevant for Mathieu moonshine display a specific asymptotic behaviour near non-infinite cusps, which, in physical terms, can be interpreted as a condition on orbifolds of the theory. Pursuing a deeper understanding of this property constitutes an enticing direction for the future.

10 Endnote

[a]See however the recent work [26] which constructs the super vertex operator algebra underlying the $X = E_8^3$ case of umbral moonshine.

[b]The observation that the decomposition into $\mathcal{N} = 4$ characters of (a multiple of) the function $Z(\tau, z)$ returns positive integers that are suggestive of representations of the Mathieu group M_{22} was first communicated privately by Jeff Harvey to J.D. in 2010.

Authors' contributions
Each author contributing equally to all aspects of the production of this article. All authors read and approved the final manuscript.

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Compliance with ethical guidelines

Competing interests
The authors declare that they have no competing interests.

Appendix A: Jacobi theta functions

We define the *Jacobi theta functions* $\theta_i(\tau, z)$ as follows for $q = e(\tau)$ and $y = e(z)$:

$$\theta_1(\tau, z) = -iq^{1/8}y^{1/2}\prod_{n=1}^{\infty}(1-q^n)(1-yq^n)(1-y^{-1}q^{n-1}), \tag{9.1}$$

$$\theta_2(\tau, z) = q^{1/8}y^{1/2}\prod_{n=1}^{\infty}(1-q^n)(1+yq^n)(1+y^{-1}q^{n-1}), \tag{9.2}$$

$$\theta_3(\tau, z) = \prod_{n=1}^{\infty} (1 - q^n)(1 + y q^{n-1/2})(1 + y^{-1} q^{n-1/2}), \tag{9.3}$$

$$\theta_4(\tau, z) = \prod_{n=1}^{\infty} (1 - q^n)(1 - y q^{n-1/2})(1 - y^{-1} q^{n-1/2}). \tag{9.4}$$

They transform in the following way under the group $SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$.

$$\begin{aligned}
\theta_1(\tau, z) &= i\,\alpha^{-1}(\tau, z)\theta_1\left(-\frac{1}{\tau}, \frac{z}{\tau}\right) = e(-1/8)\,\theta_1(\tau+1, z) \\
&= (-1)^{\lambda+\mu} e\Big(\tfrac{1}{2}(\lambda^2\tau + 2\lambda z)\Big)\theta_1(\tau, z + \lambda\tau + \mu), \\
\theta_2(\tau, z) &= \alpha^{-1}(\tau, z)\theta_4\left(-\frac{1}{\tau}, \frac{z}{\tau}\right) = e(-1/8)\,\theta_2(\tau+1, z) \\
&= (-1)^{\mu} e\Big(\tfrac{1}{2}(\lambda^2\tau + 2\lambda z)\Big)\theta_2(\tau, z + \lambda\tau + \mu), \\
\theta_3(\tau, z) &= \alpha^{-1}(\tau, z)\theta_3\left(-\frac{1}{\tau}, \frac{z}{\tau}\right) = \theta_4(\tau+1, z) \\
&= e\Big(\tfrac{1}{2}(\lambda^2\tau + 2\lambda z)\Big)\theta_3(\tau, z + \lambda\tau + \mu), \\
\theta_4(\tau, z) &= \alpha^{-1}(\tau, z)\theta_2\left(-\frac{1}{\tau}, \frac{z}{\tau}\right) = \theta_3(\tau+1, z) \\
&= (-1)^{\lambda} e\Big(\tfrac{1}{2}(\lambda^2\tau + 2\lambda z)\Big)\theta_4(\tau, z + \lambda\tau + \mu).
\end{aligned} \tag{9.5}$$

Here $\alpha(\tau, z) = \sqrt{-i\tau}\,\mathrm{e}(\frac{z^2}{2\tau})$, and $\lambda, \mu \in \mathbb{Z}$.

The weight four Eisenstein series E_4 can be written in terms of the Jacobi theta functions as

$$E_4(\tau) = \frac{1}{2}\Big(\theta_2(\tau, 0)^8 + \theta_3(\tau, 0)^8 + \theta_4(\tau, 0)^8\Big)\,. \tag{9.6}$$

Appendix B: Character tables

B1: Frame shapes and spinor representations

See Tables 1, 2, 3, 4, 5, 6.

Table 1 Frame shapes and spinor characters for M_{22}

$[g]$	1A	2A	3A	4A	4B	5A	6A	7A	7B	8A	11A	11B
Π_g	1^{24}	$1^8 2^8$	$1^6 3^6$	$1^4 2^4 4^4$	$1^4 2^4 4^4$	$1^4 5^4$	$1^2 2^2 3^3 6^2$	$1^3 7^3$	$1^3 7^3$	$1^2 2.4.8^2$	$1^2 11^2$	$1^2 11^2$
$\mathrm{Tr}_{\mathbf{4096}}\, g$	2,048	0	64	0	0	0	0	8	8	0	4	4

Table 2 Frame shapes and spinor characters for $U_4(3)$

$[g]$	1A	2A	3A	3BCD	4AB	5A	6A	6BC	7AB	8A	9ABCD	12A
Π_g	1^{24}	$1^8 2^8$	$\frac{3^9}{1^3}$	$1^6 3^6$	$1^4 2^2 4^4$	$1^4 5^4$	$\frac{1^5 3.6^4}{2^4}$	$1^2 2^2 3^3 6^2$	$1^3 7^3$	$1^2 2.4.8^2$	$\frac{1^3 9^3}{3^2}$	$\frac{1.2^2 3.12^2}{4^2}$
R_g	2,048	0	−8	64	0	0	72	0	8	0	4	0

Table 3 Frame shapes and spinor characters for M_{23}

$[g]$	1A	2A	3A	4A	5A	6A	7AB	8A	11AB	14AB	15AB	23AB
Π_g	1^{24}	$1^8 2^8$	$1^6 3^6$	$1^4 2^2 4^4$	$1^4 5^4$	$1^2 2^2 3^3 6^2$	$1^3 7^3$	$1^2 2.4.8^2$	$1^2 11^2$	$1.2.7.14$	$1.3.5.15$	1.23
$\mathrm{Tr}_{\mathbf{4096}}g$	2,048	0	64	0	0	0	8	0	4	0	4	2

Table 4 Frame shapes and spinor characters for *McL*

$[g]$	1A	2A	3A	3B	4A	5A	5B	6A	6B	7AB
Π_g	1^{24}	$1^8 2^8$	$\frac{3^9}{1^3}$	$1^6 3^6$	$1^4 2^2 4^4$	$\frac{5^5}{1}$	$1^4 5^4$	$\frac{1^5 3.6^4}{2^4}$	$1^2 2^2 3^3 6^2$	$1^3 7^3$
$\mathrm{Tr}_{\mathbf{4096}}g$	2,048	0	−8	64	0	−4	0	72	0	8

$[g]$	8A	9AB	10A	11AB	12A	14AB	15AB	30AB
Π_g	$1^2 2.4.8^2$	$\frac{1^3 9^3}{3^2}$	$\frac{1^3 5.10^2}{2^2}$	$1^2 11^2$	$\frac{1.2^2 3.12^2}{4^2}$	$1.2.7.14$	$\frac{1^2 15^2}{3.5}$	$\frac{2.3.5.30}{6.10}$
$\mathrm{Tr}_{\mathbf{4096}}g$	0	4	20	4	0	0	2	2

Table 5 Frame shapes and spinor characters for *HS*

$[g]$	1A	2A	2B	3A	4A	4BC	5A	5BC	6A	6B
Π_g	1^{24}	$1^8 2^8$	2^{12}	$1^6 3^6$	$\frac{2^6 4^4}{1^4}$	$1^4 2^2 4^4$	$\frac{5^5}{1}$	$1^4 5^4$	$2^3 6^3$	$1^2 2^2 3^3 6^2$
$\mathrm{Tr}_{\mathbf{4096}}g$	2,048	0	0	64	0	0	−4	0	0	0

$[g]$	7A	8ABC	10A	10B	11AB	12A	15A	20AB
Π_g	$1^3 7^3$	$1^2 2.4.8^2$	$\frac{1^3 5.10^2}{2^2}$	$2^2 10^2$	$1^2 11^2$	$\frac{1^2 4.6^2 12}{3^2}$	$1.3.5.15$	$\frac{1.2.10.20}{4.5}$
$\mathrm{Tr}_{\mathbf{4096}}g$	8	0	20	0	4	0	4	0

Table 6 Frame shapes and spinor characters for $U_6(2)$

$[g]$	1A	2A	2B	2C	3A	3B	3C	4A	4B	4CDE	4F	4G
Π_g	1^{24}	$\frac{2^{16}}{1^8}$	$1^8 2^8$	2^{12}	$1^6 3^6$	$\frac{3^9}{1^3}$	$1^6 3^6$	$\frac{1^8 4^8}{2^8}$	$\frac{4^8}{2^4}$	$1^4 2^2 4^4$	$2^4 4^4$	$1^4 2^2 4^4$
$\mathrm{Tr}_{\mathbf{4069}}g$	2048	0	0	0	64	–8	64	256	0	0	0	0

$[g]$	5A	6AB	6C	6D	6E	6F	6G	6H	7A	8A	8BCD
Π_g	$1^4 5^4$	$\frac{1.6^6}{2^2 3^3}$	$\frac{2^4 6^4}{1^2 3^2}$	$\frac{1^5 3.6^4}{2^4}$	$1^2 2^2 3^2 6^2$	$\frac{1^4 2.6^5}{3^4}$	$1^2 2^2 3^2 6^2$	$2^3 6^3$	$1^3 7^3$	$\frac{1^4 8^4}{2^2 4^2}$	$1^2 2.4.8^2$
$\mathrm{Tr}_{\mathbf{4096}}g$	0	0	0	72	0	0	0	0	8	32	0

$[g]$	9ABC	10A	11AB	12AB	12C	12DE	12FGH	12I	15A	18AB
Π_g	$\frac{1^3 9^3}{3^2}$	$\frac{1^2 2.10^3}{5^2}$	$1^2 11^2$	$\frac{2.3^3 12^3}{1.4.6^3}$	$\frac{1^2 3^2 4^2 12^2}{2^2 6^2}$	$\frac{1^3 12^3}{2.3.4.6}$	$\frac{1.2^2 3.12^2}{4^2}$	$2.4.6.12$	$1.3.5.15$	$\frac{1.2.18^2}{6.9}$
$\mathrm{Tr}_{\mathbf{4096}}g$	4	0	4	4	16	12	0	0	4	0

B2: Irreducible representations

See Tables 7, 8, 9, 10, 11, 12, 13, 14.

Table 7 Character table of M_{22}. $b_p = (-1 + i\sqrt{p})/2$

$[g]$	1A	2A	3A	4A	4B	5A	6A	7A	7B	8A	11A	11B
$[g^2]$	1A	1A	3A	2A	2A	5A	3A	7A	7B	4A	11B	11A
$[g^3]$	1A	2A	1A	4A	4B	5A	2A	7B	7A	8A	11A	11B
$[g^5]$	1A	2A	3A	4A	4B	1A	6A	7B	7A	8A	11A	11B
$[g^7]$	1A	2A	3A	4A	4B	5A	6A	1A	1A	8A	11B	11A
$[g^{11}]$	1A	2A	3A	4A	4B	5A	6A	7A	7B	8A	1A	1A
χ_1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	21	5	3	1	1	1	−1	0	0	−1	−1	−1
χ_3	45	−3	0	1	1	0	0	b_7	$\overline{b_7}$	−1	1	1
χ_4	45	−3	0	1	1	0	0	$\overline{b_7}$	b_7	−1	1	1
χ_5	55	7	1	3	−1	0	1	−1	−1	1	0	0
χ_6	99	3	0	3	−1	−1	0	1	1	−1	0	0
χ_7	154	10	1	−2	2	−1	1	0	0	0	0	0
χ_8	210	2	3	−2	−2	0	−1	0	0	0	1	1
χ_9	231	7	−3	−1	−1	1	1	0	0	−1	0	0
χ_{10}	280	−8	1	0	0	0	1	0	0	0	b_{11}	$\overline{b_{11}}$
χ_{11}	280	−8	1	0	0	0	1	0	0	0	$\overline{b_{11}}$	b_{11}
χ_{12}	385	1	−2	1	1	0	−2	0	0	1	0	0

Table 8 Character table of $U_4(3)$. $b_p = (-1 + i\sqrt{p})/2$

$[g]$	1A	2A	3A	3B	3C	3D	4A	4B	5A	6A	6B	6C	7A	7B	8A	9A	9B	9C	9D	12A
$[g^2]$	1A	1A	3A	3B	3C	3D	2A	2A	5A	3A	3B	3C	7A	7B	4A	9B	9A	9D	9C	6A
$[g^3]$	1A	2A	1A	1A	1A	1A	4A	4B	5A	2A	2A	2A	7B	7A	8A	3A	3A	3A	3A	4A
$[g^5]$	1A	2A	3A	3B	3C	3D	4A	4B	1A	6A	6B	6C	7B	7A	8A	9B	9A	9D	9C	12A
$[g^7]$	1A	2A	3A	3B	3C	3D	4A	4B	5A	6A	6B	6C	1A	1A	8A	9A	9B	9C	9D	12A
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	21	5	−6	3	3	3	1	1	1	2	−1	−1	0	0	−1	0	0	0	0	−2
χ_3	35	3	8	8	−1	−1	3	−1	0	0	0	3	0	0	−1	2	2	−1	−1	0
χ_4	35	3	8	−1	8	−1	3	−1	0	0	3	0	0	0	−1	−1	−1	2	2	0
χ_5	90	10	9	9	9	0	−2	2	0	1	1	1	−1	−1	0	0	0	0	0	1
χ_6	140	12	5	−4	−4	5	4	0	0	−3	0	0	0	0	0	−1	−1	−1	−1	1
χ_7	189	−3	27	0	0	0	5	1	−1	3	0	0	0	0	1	0	0	0	0	−1
χ_8	210	2	21	3	3	3	−2	−2	0	5	−1	−1	0	0	0	0	0	0	0	1
χ_9	280	−8	10	10	1	1	0	0	0	−2	−2	1	0	0	0	$1+3b_3$	$1+3\overline{b_3}$	1	1	0
χ_{10}	280	−8	10	10	1	1	0	0	0	−2	−2	1	0	0	0	$1+3\overline{b_3}$	$1+3b_3$	1	1	0
χ_{11}	280	−8	10	1	10	1	0	0	0	−2	1	−2	0	0	0	1	1	$1+3b_3$	$1+3\overline{b_3}$	0
χ_{12}	280	−8	10	1	10	1	0	0	0	−2	1	−2	0	0	0	1	1	$1+3\overline{b_3}$	$1+3b_3$	0
χ_{13}	315	11	−9	18	−9	0	−1	−1	0	−1	2	−1	0	0	1	0	0	0	0	−1
χ_{14}	315	11	−9	−9	18	0	−1	−1	0	−1	−1	2	0	0	1	0	0	0	0	−1
χ_{15}	420	4	−39	6	6	−3	4	0	0	1	−2	−2	0	0	0	0	0	0	0	1
χ_{16}	560	−16	−34	2	2	2	0	0	0	2	2	2	0	0	0	−1	−1	−1	−1	0
χ_{17}	640	0	−8	−8	−8	1	0	0	0	0	0	0	b_7	$\overline{b_7}$	0	1	1	1	1	0
χ_{18}	640	0	−8	−8	−8	1	0	0	0	0	0	0	$\overline{b_7}$	b_7	0	1	1	1	1	0
χ_{19}	729	9	0	0	0	0	−3	1	−1	0	0	0	1	1	−1	0	0	0	0	0
χ_{20}	896	0	32	−4	−4	−4	0	0	1	0	0	0	0	0	0	−1	−1	−1	−1	0

Table 9 Character table of M_{23}. $b_p = (-1 + i\sqrt{p})/2$

$[g]$	1A	2A	3A	4A	5A	6A	7A	7B	8A	11A	11B	14A	14B	15A	15B	23A	23B
$[g^2]$	1A	1A	3A	2A	5A	3A	7A	7B	4A	11B	11A	7A	7B	15A	15B	23A	23B
$[g^3]$	1A	2A	1A	4A	5A	2A	7B	7A	8A	11A	11B	14B	14A	5A	5A	23A	23B
$[g^5]$	1A	2A	3A	4A	1A	6A	7B	7A	8A	11A	11B	14B	14A	3A	3A	23B	23A
$[g^7]$	1A	2A	3A	4A	5A	6A	1A	1A	8A	11B	11A	2A	2A	15B	15A	23B	23A
$[g^{11}]$	1A	2A	3A	4A	5A	6A	7A	7B	8A	1A	1A	14A	14B	15B	15A	23B	23A
$[g^{23}]$	1A	2A	3A	4A	5A	6A	7A	7B	8A	11A	11B	14A	14B	15A	15B	1A	1A
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	22	6	4	2	2	0	1	1	0	0	0	-1	-1	-1	-1	-1	-1
χ_3	45	-3	0	1	0	0	b_7	$\overline{b_7}$	-1	1	1	$-b_7$	$-\overline{b_7}$	0	0	-1	-1
χ_4	45	-3	0	1	0	0	$\overline{b_7}$	b_7	-1	1	1	$-\overline{b_7}$	$-b_7$	0	0	-1	-1
χ_5	230	22	5	2	0	1	-1	-1	0	-1	-1	1	1	0	0	0	0
χ_6	231	7	6	-1	1	-2	0	0	-1	0	0	0	0	1	1	1	1
χ_7	231	7	-3	-1	1	1	0	0	-1	0	0	0	0	b_{15}	$\overline{b_{15}}$	1	1
χ_8	231	7	-3	-1	1	1	0	0	-1	0	0	0	0	$\overline{b_{15}}$	b_{15}	1	1
χ_9	253	13	1	1	-2	1	1	1	-1	0	0	-1	-1	1	1	0	0
χ_{10}	770	-14	5	-2	0	1	0	0	0	0	0	0	0	0	0	b_{23}	$\overline{b_{23}}$
χ_{11}	770	-14	5	-2	0	1	0	0	0	0	0	0	0	0	0	$\overline{b_{23}}$	b_{23}
χ_{12}	896	0	-4	0	1	0	0	0	0	b_{11}	$\overline{b_{11}}$	0	0	1	1	-1	-1
χ_{13}	896	0	-4	0	1	0	0	0	0	$\overline{b_{11}}$	b_{11}	0	0	1	1	-1	-1
χ_{14}	990	-18	0	2	0	0	b_7	$\overline{b_7}$	0	0	0	b_7	$\overline{b_7}$	0	0	1	1
χ_{15}	990	-18	0	2	0	0	$\overline{b_7}$	b_7	0	0	0	$\overline{b_7}$	b_7	0	0	1	1
χ_{16}	1035	27	0	-1	0	0	-1	-1	1	1	1	-1	-1	0	0	0	0
χ_{17}	2024	8	-1	0	-1	-1	1	1	0	0	0	1	1	-1	-1	0	0

Table 10 Character table of *McL*. $b_p = (-1 + i\sqrt{p})/2$, $a = 1 + 3b_3$

$[g]$	1A	2A	3A	3B	4A	5A	5B	6A	6B	7A	7B	8A	9A	9B	10A	11A	11B	12A	14A	14B	15A	15B	30A	30B
$[g^2]$	1A	1A	3A	3B	2A	5A	5B	3A	3B	7A	7B	4A	9B	9A	5A	11B	11A	6A	7A	7B	15A	15B	15A	15B
$[g^3]$	1A	2A	1A	1A	4A	5A	5B	2A	2A	7B	7A	8A	3A	3A	10A	11A	11B	4A	14B	14A	5A	5A	10A	10A
$[g^5]$	1A	2A	3A	3B	4A	1A	1A	6A	6B	7B	7A	8A	9B	9A	2A	11A	11B	12A	14B	14A	3A	3A	6A	6A
$[g^7]$	1A	2A	3A	3B	4A	5A	5B	6A	6B	1A	1A	8A	9A	9B	10A	11B	11A	12A	2A	2A	15B	15A	30B	30A
$[g^{11}]$	1A	2A	3A	3B	4A	5A	5B	6A	6B	7A	7B	8A	9B	9A	10A	1A	1A	12A	14A	14B	15B	15A	30B	30A
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	22	6	−5	4	2	−3	2	3	0	1	1	0	1	1	1	0	0	−1	−1	−1	0	0	−2	−2
χ_3	231	7	15	6	−1	6	1	7	−2	0	0	−1	0	0	2	0	0	−1	0	0	0	0	2	2
χ_4	252	28	9	9	4	2	2	1	1	0	0	0	0	0	−2	−1	−1	1	0	0	−1	−1	1	1
χ_5	770	−14	−13	5	−2	−5	0	7	1	0	0	0	−1	−1	1	0	0	1	0	0	b_{15}	$\overline{b_{15}}$	b_{15}	$\overline{b_{15}}$
χ_6	770	−14	−13	5	−2	−5	0	7	1	0	0	0	−1	−1	1	0	0	1	0	0	$\overline{b_{15}}$	b_{15}	$\overline{b_{15}}$	b_{15}
χ_7	896	0	32	−4	0	−4	1	0	0	0	0	0	−1	−1	0	b_{11}	$\overline{b_{11}}$	0	0	0	2	2	0	0
χ_8	896	0	32	−4	0	−4	1	0	0	0	0	0	−1	−1	0	$\overline{b_{11}}$	b_{11}	0	0	0	2	2	0	0
χ_9	1750	70	−5	13	2	0	0	−5	1	0	0	0	−2	−2	0	1	1	−1	0	0	0	0	0	0
χ_{10}	3520	64	−44	10	0	−5	0	4	−2	−1	−1	0	1	1	−1	0	0	0	1	1	1	1	−1	−1
χ_{11}	3520	−64	−44	10	0	−5	0	−4	2	−1	−1	0	1	1	1	0	0	0	−1	−1	1	1	1	1
χ_{12}	4500	20	45	−9	4	0	0	5	−1	−1	−1	0	0	0	0	1	1	1	−1	−1	0	0	0	0
χ_{13}	4752	−48	54	0	0	2	2	−6	0	−1	−1	0	0	0	2	0	0	0	1	1	−1	−1	−1	−1
χ_{14}	5103	63	0	0	3	3	−2	0	0	0	0	1	0	0	3	−1	−1	0	0	0	0	0	0	0
χ_{15}	5544	−56	36	9	0	19	−1	4	1	0	0	0	0	0	−1	0	0	0	0	0	1	1	−1	−1
χ_{16}	8019	−45	0	0	3	−6	−1	0	0	$-b_7$	$-\overline{b_7}$	−1	0	0	0	0	0	0	$-b_7$	$-\overline{b_7}$	0	0	0	0
χ_{17}	8019	−45	0	0	3	−6	−1	0	0	$-\overline{b_7}$	$-b_7$	−1	0	0	0	0	0	0	$-\overline{b_7}$	$-b_7$	0	0	0	0
χ_{18}	8250	10	15	6	−2	0	0	−5	−2	$-b_7$	$-\overline{b_7}$	0	0	0	0	0	0	1	b_7	$\overline{b_7}$	0	0	0	0
χ_{19}	8250	10	15	6	−2	0	0	−5	−2	$-\overline{b_7}$	$-b_7$	0	0	0	0	0	0	1	$\overline{b_7}$	b_7	0	0	0	0
χ_{20}	9625	105	40	−5	−3	0	0	0	3	0	0	−1	1	1	0	0	0	0	0	0	0	0	0	0

Table 10 continued

$[g]$	1A	2A	3A	3B	4A	5A	5B	6A	6B	7A	7B	8A	9A	9B	10A	11A	11B	12A	14A	14B	15A	15B	30A	30B
χ_{21}	9856	0	−80	−8	0	6	1	0	0	0	0	0	a	$\overline{a}$	0	0	0	0	0	0	0	0	0	0
χ_{22}	9856	0	−80	−8	0	6	1	0	0	0	0	0	$\overline{a}$	a	0	0	0	0	0	0	0	0	0	0
χ_{23}	10395	−21	27	0	−1	−5	0	3	0	0	0	1	0	0	−1	0	0	−1	0	0	b_{15}	$\overline{b_{15}}$	$-b_{15}$	$-\overline{b_{15}}$
χ_{24}	10395	−21	27	0	−1	−5	0	3	0	0	0	1	0	0	−1	0	0	−1	0	0	$\overline{b_{15}}$	b_{15}	$-\overline{b_{15}}$	$-b_{15}$

Table 11 Character table of *HS*. $b_p = (-1 + i\sqrt{p})/2, a_p = i\sqrt{p}$

$[g]$	1A	2A	2B	3A	4A	4B	4C	5A	5B	5C	6A	6B	7A	8A	8B	8C	10A	10B	11A	11B	12A	15A	20A	20B
$[g^2]$	1A	1A	1A	3A	2A	2A	2A	5A	5B	5C	3A	3A	7A	4B	4C	4C	5A	5B	11B	11A	6B	15A	10A	10A
$[g^3]$	1A	2A	2B	1A	4A	4B	4C	5A	5B	5C	2B	2A	7A	8A	8B	8C	10A	10B	11A	11B	4A	5B	20A	20B
$[g^5]$	1A	2A	2B	3A	4A	4B	4C	1A	1A	1A	6A	6B	7A	8A	8B	8C	2A	2B	11A	11B	12A	3A	4A	4A
$[g^7]$	1A	2A	2B	3A	4A	4B	4C	5A	5B	5C	6A	6B	1A	8A	8B	8C	10A	10B	11B	11A	12A	15A	20A	20B
$[g^{11}]$	1A	2A	2B	3A	4A	4B	4C	5A	5B	5C	6A	6B	7A	8A	8B	8C	10A	10B	1A	1A	12A	15A	20B	20A
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	22	6	−2	4	−6	2	2	−3	2	2	−2	0	1	0	0	0	1	−2	0	0	0	−1	−1	−1
χ_3	77	13	1	5	5	5	1	2	−3	2	1	1	0	1	−1	−1	−2	1	0	0	−1	0	0	0
χ_4	154	10	10	1	−2	6	−2	4	4	−1	1	1	0	0	0	0	0	0	0	0	1	1	−2	−2
χ_5	154	10	−10	1	−10	−2	2	4	4	−1	−1	1	0	0	2	−2	0	0	0	0	−1	1	0	0
χ_6	154	10	−10	1	−10	−2	2	4	4	−1	−1	1	0	0	−2	2	0	0	0	0	−1	1	0	0
χ_7	175	15	11	4	15	−1	3	0	5	0	2	0	0	−1	1	1	0	1	−1	−1	0	−1	0	0
χ_8	231	7	−9	6	15	−1	−1	6	1	1	0	−2	0	−1	−1	−1	2	1	0	0	0	1	0	0
χ_9	693	21	9	0	21	5	1	−7	3	−2	0	0	0	1	−1	−1	1	−1	0	0	0	0	1	1
χ_{10}	770	34	−10	5	−14	2	−2	−5	0	0	−1	1	0	−2	0	0	−1	0	0	0	1	0	1	1
χ_{11}	770	−14	10	5	−10	−2	−2	−5	0	0	1	1	0	0	0	0	1	0	0	0	−1	0	a_5	$\overline{a_5}$
χ_{12}	770	−14	10	5	−10	−2	−2	−5	0	0	1	1	0	0	0	0	1	0	0	0	−1	0	$\overline{a_5}$	a_5
χ_{13}	825	25	9	6	−15	1	1	0	−5	0	0	−2	−1	1	1	1	0	−1	0	0	0	1	0	0
χ_{14}	896	0	16	−4	0	0	0	−4	1	1	−2	0	0	0	0	0	0	1	b_{11}	$\overline{b_{11}}$	0	1	0	0
χ_{15}	896	0	16	−4	0	0	0	−4	1	1	−2	0	0	0	0	0	0	1	$\overline{b_{11}}$	b_{11}	0	1	0	0
χ_{16}	1056	32	0	−6	0	0	0	6	−4	1	0	2	−1	0	0	0	2	0	0	0	0	−1	0	0
χ_{17}	1386	−6	18	0	6	−2	−2	11	6	1	0	0	0	0	0	0	−1	−2	0	0	0	0	1	1
χ_{18}	1408	0	16	4	0	0	0	8	−7	−2	−2	0	1	0	0	0	0	1	0	0	0	−1	0	0
χ_{19}	1750	−10	10	−5	−10	6	2	0	0	0	1	−1	0	−2	0	0	0	0	1	1	−1	0	0	0
χ_{20}	1925	5	−19	−1	5	5	−3	0	5	0	−1	−1	0	1	1	1	0	1	0	0	−1	−1	0	0
χ_{21}	1925	5	1	−1	−35	−3	1	0	5	0	1	−1	0	1	−1	−1	0	1	0	0	1	−1	0	0

Table 11 continued

$[g]$	1A	2A	2B	3A	4A	4B	4C	5A	5B	5C	6A	6B	7A	8A	8B	8C	10A	10B	11A	11B	12A	15A	20A	20B
χ_{22}	2520	24	0	0	24	−8	0	−5	0	0	0	0	0	0	0	0	−1	0	1	1	0	0	−1	−1
χ_{23}	2750	−50	−10	5	10	2	2	0	0	0	−1	1	−1	0	0	0	0	0	0	0	1	0	0	0
χ_{24}	3200	0	−16	−4	0	0	0	0	−5	0	2	0	1	0	0	0	0	−1	−1	−1	0	1	0	0

Table 12 Character table of $U_6(2)$ - Part I

$[g]$	1A	2A	2B	2C	3A	3B	3C	4A	4B	4C	4D	4E	4F	4G	5A
$[g^2]$	1A	1A	1A	1A	3A	3B	3C	2A	2A	2B	2B	2B	2B	2B	5A
$[g^3]$	1A	2A	2B	2C	1A	1A	1A	4A	4B	4C	4D	4E	4F	4G	5A
$[g^5]$	1A	2A	2B	2C	3A	3B	3C	4A	4B	4C	4D	4E	4F	4G	1A
$[g^7]$	1A	2A	2B	2C	3A	3B	3C	4A	4B	4C	4D	4E	4F	4G	5A
$[g^{11}]$	1A	2A	2B	2C	3A	3B	3C	4A	4B	4C	4D	4E	4F	4G	5A
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	22	−10	6	−2	4	−5	4	6	−2	2	2	2	−2	2	2
χ_3	231	39	7	−9	6	15	6	23	7	−1	−1	−1	−1	−1	1
χ_4	252	60	28	12	9	9	9	12	−4	4	4	4	4	4	2
χ_5	385	−95	17	−7	25	7	−2	1	9	5	5	5	−7	−3	0
χ_6	440	120	24	8	35	8	−1	24	8	0	0	0	8	0	0
χ_7	560	−80	−16	16	20	20	2	16	−16	0	0	0	0	0	0
χ_8	616	−24	40	8	−14	−5	13	8	8	8	8	8	0	0	1
χ_9	770	−30	−14	10	5	−13	5	34	−6	−2	−2	−2	2	−2	0
χ_{10}	770	−30	−14	10	5	−13	5	34	−6	−2	−2	−2	2	−2	0
χ_{11}	1155	195	35	19	30	−6	3	−13	3	11	−5	−5	3	3	0
χ_{12}	1155	195	35	19	30	−6	3	−13	3	−5	11	−5	3	3	0
χ_{13}	1155	195	35	19	30	−6	3	−13	3	−5	−5	11	3	3	0
χ_{14}	1386	−246	58	−30	36	9	9	−6	2	−2	−2	−2	−6	6	1
χ_{15}	1540	260	4	−28	55	1	1	−12	−12	4	4	4	4	−4	0
χ_{16}	3080	−440	40	−8	65	2	−7	40	−8	0	0	0	−8	0	0
χ_{17}	3080	−440	40	−8	65	2	−7	40	−8	0	0	0	−8	0	0
χ_{18}	3520	−320	64	0	10	−44	10	64	0	0	0	0	0	0	0
χ_{19}	4620	−180	44	−36	−15	57	12	28	−20	4	4	4	−4	−4	0
χ_{20}	4928	320	64	64	−4	68	5	0	0	0	0	0	0	0	−2
χ_{21}	5544	−24	−56	24	9	36	9	72	24	0	0	0	−8	0	−1
χ_{22}	6160	400	−48	−16	40	−50	4	48	16	0	0	0	0	0	0
χ_{23}	6160	400	−48	−16	40	−50	4	48	16	0	0	0	0	0	0
χ_{24}	6930	690	98	42	45	−36	−9	18	−6	6	6	6	10	−2	0
χ_{25}	8064	384	128	0	−36	−36	18	0	0	0	0	0	0	0	−1
χ_{26}	9240	−360	88	−8	−30	33	−3	−8	−8	24	−8	−8	0	0	0
χ_{27}	9240	−360	88	−8	−30	33	−3	−8	−8	−8	24	−8	0	0	0
χ_{28}	9240	−360	88	−8	−30	33	−3	−8	−8	−8	−8	24	0	0	0
χ_{29}	10395	315	−21	−45	0	27	0	75	−13	−1	−1	−1	3	−1	0
χ_{30}	10395	315	−21	−45	0	27	0	75	−13	−1	−1	−1	3	−1	0
χ_{31}	10395	315	−21	−45	0	27	0	−21	19	15	−1	−1	3	−1	0
χ_{32}	10395	315	−21	−45	0	27	0	−21	19	−1	15	−1	3	−1	0
χ_{33}	10395	315	−21	−45	0	27	0	−21	19	−1	−1	15	3	−1	0
χ_{34}	11264	1024	0	0	104	32	−4	0	0	0	0	0	0	0	−1
χ_{35}	13860	420	100	36	−45	9	−18	84	20	4	4	4	−4	4	0
χ_{36}	14784	−1344	64	0	114	−12	6	−64	0	0	0	0	0	0	−1
χ_{37}	18711	−1161	−57	63	81	0	0	−9	15	3	3	3	−1	−5	1
χ_{38}	18711	1431	87	−9	81	0	0	−9	−9	−9	−9	−9	−1	−1	1
χ_{39}	20790	−810	6	−18	0	54	0	6	14	−6	−6	−6	6	2	0
χ_{40}	20790	−810	6	−18	0	54	0	6	14	−6	−6	−6	6	2	0
χ_{41}	24640	−960	64	−64	−20	−92	−11	0	0	0	0	0	0	0	0
χ_{42}	25515	−405	−117	27	0	0	0	27	−21	3	3	3	3	3	0
χ_{43}	25515	−405	−117	27	0	0	0	27	−21	3	3	3	3	3	0
χ_{44}	32768	0	0	0	−64	−64	8	0	0	0	0	0	0	0	−2
χ_{45}	37422	270	30	54	−81	0	0	−18	6	−6	−6	−6	−2	−6	2
χ_{46}	40095	1215	−81	−9	0	0	0	−81	−9	3	3	3	−9	3	0

Table 13 Character table of $U_6(2)$ - Part II. $a_q = q + 6i\sqrt{3}, c_{-3} = (1 - 3i\sqrt{3})/2$

$[g]$	6A	6B	6C	6D	6E	6F	6G	6H	7A	8A	8B	8C	8D	9A	9B	9C
$[g^2]$	3B	3B	3A	3B	3A	3C	3C	3C	7A	4B	4C	4D	4E	9B	9A	9C
$[g^3]$	2A	2A	2A	2B	2B	2A	2B	2C	7A	8A	8B	8C	8D	3B	3B	3B
$[g^5]$	6B	6A	6C	6D	6E	6F	6G	6H	7A	8A	8B	8C	8D	9B	9A	9C
$[g^7]$	6A	6B	6C	6D	6E	6F	6G	6H	1A	8A	8B	8C	8D	9A	9B	9C
$[g^{11}]$	6B	6A	6C	6D	6E	6F	6G	6H	7A	8A	8B	8C	8D	9B	9A	9C
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	-1	-1	-4	3	0	2	0	-2	1	2	0	0	0	1	1	1
χ_3	3	3	6	7	-2	0	-2	0	0	3	-1	-1	-1	0	0	0
χ_4	-3	-3	9	1	1	3	1	3	0	0	0	0	0	0	0	0
χ_5	-5	-5	1	-1	5	-2	2	2	0	1	-1	-1	-1	1	1	1
χ_6	12	12	3	0	3	3	3	-1	-1	0	0	0	0	2	2	-1
χ_7	-8	-8	4	-4	-4	-2	2	-2	0	0	0	0	0	-1	-1	2
χ_8	3	3	-6	-5	-2	3	1	-1	0	0	0	0	0	-2	-2	-2
χ_9	a_{-3}	$\bar{a}_{-3}$	-3	7	1	-3	1	1	0	2	0	0	0	-1	-1	-1
χ_{10}	$\bar{a}_{-3}$	a_{-3}	-3	7	1	-3	1	1	0	2	0	0	0	-1	-1	-1
χ_{11}	6	6	6	2	2	-3	-1	1	0	-1	3	-1	-1	0	0	0
χ_{12}	6	6	6	2	2	-3	-1	1	0	-1	-1	3	-1	0	0	0
χ_{13}	6	6	6	2	2	-3	-1	1	0	-1	-1	-1	3	0	0	0
χ_{14}	-3	-3	-12	1	4	-3	1	-3	0	-2	0	0	0	0	0	0
χ_{15}	17	17	-1	1	-5	-1	1	-1	0	0	0	0	0	1	1	1
χ_{16}	a_{-8}	$\bar{a}_{-8}$	1	-2	1	1	1	1	0	0	0	0	0	c_{-3}	$\bar{c}_{-3}$	-1
χ_{17}	$\bar{a}_{-8}$	a_{-8}	1	-2	1	1	1	1	0	0	0	0	0	$\bar{c}_{-3}$	c_{-3}	-1
χ_{18}	4	4	-14	4	-2	4	-2	0	-1	0	0	0	0	1	1	1
χ_{19}	9	9	9	-7	5	0	-4	0	0	0	0	0	0	0	0	0
χ_{20}	-4	-4	-4	4	4	5	1	1	0	0	0	0	0	-1	-1	2
χ_{21}	12	12	9	4	1	-3	1	-3	0	0	0	0	0	0	0	0
χ_{22}	a_4	$\bar{a}_4$	-8	-6	0	-2	0	2	0	0	0	0	0	$-c_{-3}$	$-\bar{c}_{-3}$	1
χ_{23}	$\bar{a}_4$	a_4	-8	-6	0	-2	0	2	0	0	0	0	0	$-\bar{c}_{-3}$	$-c_{-3}$	1
χ_{24}	-12	-12	-3	-4	5	-3	-1	-3	0	2	0	0	0	0	0	0
χ_{25}	-12	-12	12	-4	-4	0	2	0	0	0	0	0	0	0	0	0
χ_{26}	9	9	-6	1	-2	-3	1	1	0	0	0	0	0	0	0	0
χ_{27}	9	9	-6	1	-2	-3	1	1	0	0	0	0	0	0	0	0
χ_{28}	9	9	-6	1	-2	-3	1	1	0	0	0	0	0	0	0	0
χ_{29}	-9	-9	0	3	0	0	0	0	0	-1	1	1	1	0	0	0
χ_{30}	-9	-9	0	3	0	0	0	0	0	-1	1	1	1	0	0	0
χ_{31}	-9	-9	0	3	0	0	0	0	0	-1	-3	1	1	0	0	0
χ_{32}	-9	-9	0	3	0	0	0	0	0	-1	1	-3	1	0	0	0
χ_{33}	-9	-9	0	3	0	0	0	0	0	-1	1	1	-3	0	0	0
χ_{34}	16	16	-8	0	0	4	0	0	1	0	0	0	0	-1	-1	-1
χ_{35}	-3	-3	3	1	-5	0	-2	0	0	0	0	0	0	0	0	0
χ_{36}	-12	-12	-6	4	-2	0	-2	0	0	0	0	0	0	0	0	0
χ_{37}	0	0	9	0	-3	0	0	0	0	-1	1	1	1	0	0	0
χ_{38}	0	0	9	0	-3	0	0	0	0	-1	-1	-1	-1	0	0	0
χ_{39}	$18i\sqrt{3}$	$-18i\sqrt{3}$	0	-6	0	0	0	0	0	2	0	0	0	0	0	0
χ_{40}	$-18i\sqrt{3}$	$18i\sqrt{3}$	0	-6	0	0	0	0	0	2	0	0	0	0	0	0
χ_{41}	12	12	12	4	4	3	1	-1	0	0	0	0	0	-2	-2	1
χ_{42}	0	0	0	0	0	0	0	0	0	-1	-1	-1	-1	0	0	0
χ_{43}	0	0	0	0	0	0	0	0	0	-1	-1	-1	-1	0	0	0
χ_{44}	0	0	0	0	0	0	0	0	1	0	0	0	0	2	2	-1

Table 13 continued

χ_{45}	0	0	−9	0	3	0	0	0	0	−2	0	0	0	0	0	0
χ_{46}	0	0	0	0	0	0	0	0	−1	3	1	1	1	0	0	0

Table 14 Character table of $U_6(2)$ - Part III. $a_q = q + 6i\sqrt{3}$, $b_p = (-1 + i\sqrt{p})/2$, $d^n_m = m + in\sqrt{3}$

$[g]$	10A	11A	11B	12A	12B	12C	12D	12E	12F	12G	12H	12I	15A	18A	18B
$[g^2]$	5A	11B	11A	6B	6A	6C	6B	6A	6D	6D	6D	6E	15A	9B	9A
$[g^3]$	10A	11A	11B	4A	4A	4A	4B	4B	4C	4D	4E	4F	5A	6A	6B
$[g^5]$	2A	11A	11B	12B	12A	12C	12E	12D	12F	12G	12H	12I	3A	18B	18A
$[g^7]$	10A	11B	11A	12A	12B	12C	12D	12E	12F	12G	12H	12I	15A	18A	18B
$[g^{11}]$	10A	1A	1A	12B	12A	12C	12E	12D	12F	12G	12H	12I	15A	18B	18A
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	0	0	0	−3	−3	0	1	1	−1	−1	−1	−2	−1	−1	−1
χ_3	−1	0	0	5	5	2	1	1	−1	−1	−1	2	1	0	0
χ_4	0	−1	−1	3	3	−3	−1	−1	1	1	1	1	−1	0	0
χ_5	0	0	0	1	1	1	−3	−3	−1	−1	−1	−1	0	1	1
χ_6	0	0	0	6	6	3	2	2	0	0	0	−1	0	0	0
χ_7	0	−1	−1	−2	−2	4	2	2	0	0	0	0	0	1	1
χ_8	1	0	0	−1	−1	2	−1	−1	−1	−1	−1	0	1	0	0
χ_9	0	0	0	d^{-3}_{-2}	$\bar{d}^{-3}_{-2}$	1	d^{-1}_0	d^1_0	1	1	1	−1	0	d^1_0	d^{-1}_0
χ_{10}	0	0	0	$\bar{d}^{-3}_{-2}$	d^{-3}_{-2}	1	d^1_0	d^{-1}_0	1	1	1	−1	0	d^{-1}_0	d^1_0
χ_{11}	0	0	0	−4	−4	2	0	0	2	−2	−2	0	0	0	0
χ_{12}	0	0	0	−4	−4	2	0	0	−2	2	−2	0	0	0	0
χ_{13}	0	0	0	−4	−4	2	0	0	−2	−2	2	0	0	0	0
χ_{14}	−1	0	0	3	3	0	−1	−1	1	1	1	0	1	0	0
χ_{15}	0	0	0	−3	−3	3	−3	−3	1	1	1	1	0	−1	−1
χ_{16}	0	0	0	d^{-3}_{-5}	$\bar{d}^{-3}_{-5}$	1	d^{-1}_1	$\bar{d}^{-1}_1$	0	0	0	1	0	$d^{-1/2}_{-1/2}$	$\bar{d}^{-1/2}_{-1/2}$
χ_{17}	0	0	0	$\bar{d}^{-3}_{-5}$	d^{-3}_{-5}	1	$\bar{d}^{-1}_1$	d^{-1}_1	0	0	0	1	0	$\bar{d}^{-1/2}_{-1/2}$	$d^{-1/2}_{-1/2}$
χ_{18}	0	0	0	−8	−8	−2	0	0	0	0	0	0	0	1	1
χ_{19}	0	0	0	1	1	1	1	1	1	1	1	−1	0	0	0
χ_{20}	0	0	0	0	0	0	0	0	0	0	0	0	1	−1	−1
χ_{21}	1	0	0	0	0	−3	0	0	0	0	0	1	−1	0	0
χ_{22}	0	0	0	$-6b_3$	$-6\bar{b}_3$	0	d^{-1}_1	$\bar{d}^{-1}_1$	0	0	0	0	0	$d^{-1/2}_{-1/2}$	$\bar{d}^{-1/2}_{-1/2}$
χ_{23}	0	0	0	$-6\bar{b}_3$	$-6b_3$	0	$\bar{d}^{-1}_1$	d^{-1}_1	0	0	0	0	0	$\bar{d}^{-1/2}_{-1/2}$	$d^{-1/2}_{-1/2}$
χ_{24}	0	0	0	0	0	−3	0	0	0	0	0	1	0	0	0
χ_{25}	−1	1	1	0	0	0	0	0	0	0	0	0	−1	0	0
χ_{26}	0	0	0	1	1	−2	1	1	−3	1	1	0	0	0	0
χ_{27}	0	0	0	1	1	−2	1	1	1	−3	1	0	0	0	0
χ_{28}	0	0	0	1	1	−2	1	1	1	1	−3	0	0	0	0
χ_{29}	0	0	0	$-\bar{a}_{-3}$	$-a_{-3}$	0	d^{-2}_{-1}	$\bar{d}^{-2}_{-1}$	−1	−1	−1	0	0	0	0
χ_{30}	0	0	0	$-a_{-3}$	$-\bar{a}_{-3}$	0	$\bar{d}^{-2}_{-1}$	d^{-2}_{-1}	−1	−1	−1	0	0	0	0
χ_{31}	0	0	0	−3	−3	0	1	1	3	−1	−1	0	0	0	0
χ_{32}	0	0	0	−3	−3	0	1	1	−1	3	−1	0	0	0	0
χ_{33}	0	0	0	−3	−3	0	1	1	−1	−1	3	0	0	0	0
χ_{34}	−1	0	0	0	0	0	0	0	0	0	0	0	−1	1	1
χ_{35}	0	0	0	3	3	3	−1	−1	1	1	1	−1	0	0	0
χ_{36}	1	0	0	8	8	2	0	0	0	0	0	0	−1	0	0
χ_{37}	−1	0	0	0	0	−3	0	0	0	0	0	−1	1	0	0

Table 14 continued

χ_{38}	1	0	0	0	0	-3	0	0	0	0	0	-1	1	0	0
χ_{39}	0	0	0	$6b_3$	$6\bar{b}_3$	0	d^1_{-1}	$\bar{d}^1_{-1}$	0	0	0	0	0	0	0
χ_{40}	0	0	0	$6\bar{b}_3$	$6b_3$	0	$\bar{d}^1_{-1}$	d^1_{-1}	0	0	0	0	0	0	0
χ_{41}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{42}	0	$-b_{11}$	$-\bar{b}_{11}$	0	0	0	0	0	0	0	0	0	0	0	0
χ_{43}	0	$-\bar{b}_{11}$	$-b_{11}$	0	0	0	0	0	0	0	0	0	0	0	0
χ_{44}	0	-1	-1	0	0	0	0	0	0	0	0	0	1	0	0
χ_{45}	0	0	0	0	0	3	0	0	0	0	0	1	-1	0	0
χ_{46}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Appendix C: Coefficient tables

See Tables 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28.

Table 15 The twined series for M_{22}. The table shows the Fourier coefficients multiplying $q^{-D/12}$ in the q-expansion of the function $h_{g,1}(\tau)$

$-D[g]$	1A	2A	3A	4AB	5A	6A	7AB	8A	11AB
-1	-2	-2	-2	-2	-2	-2	-2	-2	-2
11	560	-16	2	0	0	2	0	0	-1
23	8470	54	-8	-2	0	0	0	2	0
35	70576	-144	16	0	-4	0	2	0	0
47	435820	332	4	4	0	-4	0	0	0
59	2187328	-704	-50	0	8	-2	-4	0	0
71	9493330	1394	58	-6	0	2	0	-2	0
83	36792560	-2640	38	0	0	6	0	0	2
95	130399766	4822	-172	6	-14	4	0	-2	2
107	429229920	-8480	174	0	0	-2	0	0	-2
119	1327987562	14506	104	-6	22	-8	6	2	0
131	3895785632	-24288	-502	0	-8	-6	4	0	2
143	10912966810	39770	466	10	0	2	-10	2	-2
155	29351354032	-63888	316	0	-28	12	0	0	0
167	76141761850	101018	-1232	-14	0	8	0	-2	0
179	191223891936	-157344	1098	0	56	-6	0	0	0
191	466389602756	241764	710	12	-14	-18	0	0	0
203	1107626293840	-366960	-2876	0	0	-12	10	0	0
215	2567229121428	550676	2472	-12	-62	8	6	4	4
227	5818567673360	-817840	1658	0	0	26	-20	0	2
239	12917927405852	1203068	-6148	20	112	20	0	0	-4
251	28135083779792	-1753840	5174	0	-48	-10	-4	0	0
263	60194978116000	2535488	3382	-24	0	-34	0	-4	2
275	126660856741328	-3637232	-12634	0	-112	-26	0	0	-3
287	262393981258310	5179526	10430	22	0	14	20	-2	0

Table 16 The twined series for M_{22}. The table shows the Fourier coefficients multiplying $q^{-D/12}$ in the q-expansion of the function $h_{g,2}(\tau)$

$-D[g]$	1A	2A	3A	4AB	5A	6A	7AB	8A	11AB
−4	1	1	1	1	1	1	1	1	1
8	210	2	3	−2	0	−1	0	0	1
20	4444	−4	7	4	−1	−1	−1	0	0
32	42560	16	−19	4	0	1	0	−2	1
44	281512	−24	19	−8	7	3	0	0	0
56	1481964	12	24	−12	−6	0	1	0	0
68	6649200	−16	−81	16	0	−1	5	0	−3
80	26455264	64	70	24	−6	−2	−4	4	0
92	95731405	−83	70	−27	0	−2	0	1	0
104	320626372	36	−248	−36	22	0	−4	0	0
116	1006567156	−44	217	44	−14	1	0	0	1
128	2990338680	168	183	52	0	3	0	−6	−1
140	8469129448	−216	−656	−72	−17	0	3	0	−1
152	23000871960	88	558	−88	0	−2	10	0	0
164	60186506768	−112	431	112	48	−1	−11	0	3
176	152335803872	416	−1582	144	−28	2	0	8	−1
188	374172530930	−494	1340	−166	0	4	−5	−2	0
200	894352929498	202	981	−202	−42	1	0	0	−5
212	2085157528300	−244	−3545	244	0	−1	0	0	0
224	4751675601024	864	2946	280	104	−6	9	−12	0
236	10602363945184	−1056	2077	−352	−51	−3	18	0	0
248	23199658816580	420	−7480	−420	0	0	−20	0	4
260	49851590654096	−496	6146	496	−84	2	0	0	0
272	105323108387200	1792	4228	608	0	4	−10	16	0
284	219021850730991	−2097	−15099	−697	196	−3	0	3	0

Table 17 The twined series for $U_4(3)$. The table shows the Fourier coefficients multiplying $q^{-D/12}$ in the q-expansion of the function $h_{g,1}(\tau)$

$-D[g]$	1A	2A	3A	3BCD	4AB	5A	6A	6BC	7AB	8A	9ABCD	12A
−1	−2	−2	−2	−2	−2	−2	−2	−2	−2	−2	−2	−2
11	560	−16	−34	2	0	0	2	2	0	0	−1	0
23	8470	54	−116	−8	−2	0	0	0	0	2	1	−2
35	70576	−144	−272	16	0	−4	0	0	2	0	1	0
47	435820	332	−662	4	4	0	2	−4	0	0	−2	−2
59	2187328	−704	−1454	−50	0	8	−2	−2	−4	0	4	0
71	9493330	1394	−2732	58	−6	0	−4	2	0	−2	1	0
83	36792560	−2640	−5254	38	0	0	6	6	0	0	−7	0
95	130399766	4822	−9802	−172	6	−14	10	4	0	−2	2	0
107	429229920	−8480	−16782	174	0	0	−2	−2	0	0	3	0
119	1327987562	14506	−28930	104	−6	22	−14	−8	6	2	−7	0
131	3895785632	−24288	−49066	−502	0	−8	−6	−6	4	0	5	0
143	10912966810	39770	−79058	466	10	0	14	2	−10	2	4	−2
155	29351354032	−63888	−127484	316	0	−28	12	12	0	0	−11	0
167	76141761850	101018	−203264	−1232	−14	0	−4	8	0	−2	13	−2
179	191223891936	−157344	−313578	1098	0	56	−6	−6	0	0	3	0
191	466389602756	241764	−482842	710	12	−14	−6	−18	0	0	−22	0
203	1107626293840	−366960	−736772	−2876	0	0	−12	−12	10	0	10	0
215	2567229121428	550676	−1098876	2472	−12	−62	−4	8	6	4	9	0
227	5818567673360	−817840	−1634074	1658	0	0	26	26	−20	0	−22	0
239	12917927405852	1203068	−2412226	−6148	20	112	38	20	0	0	17	2
251	28135083779792	−1753840	−3502486	5174	0	−48	−10	−10	−4	0	14	0
263	60194978116000	2535488	−5067686	3382	−24	0	−58	−34	0	−4	−32	0
275	126660856741328	−3637232	−7287046	−12634	0	−112	−26	−26	0	0	32	0
287	262393981258310	5179526	−10348570	10430	22	0	38	14	20	−2	11	−2

Table 18 The twined series for $U_4(3)$. The table shows the Fourier coefficients multiplying $q^{-D/12}$ in the q-expansion of the function $h_{g,2}(\tau)$

$-D[g]$	1A	2A	3A	3BCD	4AB	5A	6A	6BC	7AB	8A	9ABCD	12A
−4	1	1	1	1	1	1	1	1	1	1	1	1
8	210	2	21	3	−2	0	5	−1	0	0	0	1
20	4444	−4	97	7	4	−1	−7	−1	−1	0	1	1
32	42560	16	197	−19	4	0	13	1	0	−2	−1	1
44	281512	−24	577	19	−8	7	−15	3	0	0	−2	1
56	1481964	12	1176	24	−12	−6	24	0	1	0	3	0
68	6649200	−16	2313	−81	16	0	−31	−1	5	0	−3	1
80	26455264	64	4552	70	24	−6	40	−2	−4	4	−2	0
92	95731405	−83	8440	70	−27	0	−56	−2	0	1	7	0
104	320626372	36	14440	−248	−36	22	72	0	−4	0	−5	0
116	1006567156	−44	25759	217	44	−14	−89	1	0	0	1	−1
128	2990338680	168	42861	183	52	0	117	3	0	−6	9	1
140	8469129448	−216	69904	−656	−72	−17	−144	0	3	0	−5	0
152	23000871960	88	114066	558	−88	0	178	−2	10	0	−9	2
164	60186506768	−112	181097	431	112	48	−223	−1	−11	0	11	1
176	152335803872	416	280208	−1582	144	−28	272	2	0	8	−10	0
188	374172530930	−494	436490	1340	−166	0	−326	4	−5	−2	−4	2
200	894352929498	202	662499	981	−202	−42	403	1	0	0	24	−1
212	2085157528300	−244	993025	−3545	244	0	−487	−1	0	0	−17	1
224	4751675601024	864	1485462	2946	280	104	582	−6	9	−12	−6	−2
236	10602363945184	−1056	2189455	2077	−352	−51	−705	−3	18	0	25	−1
248	23199658816580	420	3186440	−7480	−420	0	840	0	−20	0	−16	0
260	49851590654096	−496	4635278	6146	496	−84	−994	2	0	0	−16	−2
272	105323108387200	1792	6654706	4228	608	0	1186	4	−10	16	34	2
284	219021850730991	−2097	9475383	−15099	−697	196	−1401	−3	0	3	−30	−1

Table 19 The twined series for M_{23}. The table shows the Fourier coefficients multiplying $q^{-D/24}$ in the q-expansion of the function $\tilde{h}_{g,\frac{1}{2}}(\tau)$.$b_p = (-1 + i\sqrt{p})/2$

$-D[g]$	1A	2A	3A	4A	5A	6A	7AB	8A	11AB	14AB	15AB	23A	23B
−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1
23	770	−14	5	−2	0	1	0	0	0	0	0	b_{23}	$\overline{b_{23}}$
47	13915	43	10	−1	0	−2	−1	1	0	1	0	0	0
71	132825	−119	21	5	−5	1	0	−1	0	0	1	0	0
95	915124	308	31	4	4	−1	0	0	1	0	1	0	0
119	5069867	−693	59	−13	2	3	−2	−1	0	0	−1	0	0
143	24053215	1407	85	−9	0	−3	4	−1	−1	0	0	−1	−1
167	101268540	−2772	135	24	0	3	2	2	−1	0	0	0	0
191	387746282	5306	200	14	−18	−4	0	0	0	0	0	0	0
215	1372935090	−9710	300	−38	15	4	−5	2	−1	−1	0	0	0
239	4552039296	17136	414	−20	11	−6	0	2	2	0	−1	0	0
263	14265412315	−29589	610	63	0	6	0	−3	0	0	0	1	1
287	42568680715	50155	835	35	0	−5	−6	−1	0	0	0	0	0
311	121665949240	−83160	1165	−108	−45	9	10	−2	0	0	0	0	0
335	334658246604	135148	1581	−60	34	−11	3	−4	0	−1	1	0	0
359	889413095662	−216482	2158	170	22	10	0	4	3	0	−2	0	0
383	2291482148835	342259	2865	87	0	−11	−9	1	0	1	0	0	0
407	5739333227670	−533610	3855	−250	0	15	0	2	−2	0	0	0	0
431	14008317423968	821296	5051	−124	−107	−17	0	6	−1	0	1	−1	−1
455	33388385201699	−1250717	6656	371	79	16	−11	−5	0	1	1	0	0
479	77853744768906	1886234	8649	190	56	−19	22	0	−4	0	−1	1	1
503	177881794535250	−2816798	11250	−554	0	22	8	−4	3	2	0	0	0
527	398808419854845	4167709	14430	−283	0	−26	0	−7	0	0	0	1	1
551	878461575586727	−6116665	18581	799	−218	29	−22	7	0	−2	1	0	0
575	1903241478167799	8909383	23631	395	154	−29	0	1	0	0	1	$\overline{b_{23}}$	b_{23}

Table 20 The twined series for M_{23}. The table shows the Fourier coefficients multiplying $q^{-D/24}$ in the q-expansion of the function $\tilde{h}_{g,\frac{3}{2}}(\tau)$

$-D[g]$	1A	2A	3A	4A	5A	6A	7AB	8A	11AB	14AB	15AB	23AB
−9	1	1	1	1	1	1	1	1	1	1	1	1
15	231	7	6	−1	1	−2	0	−1	0	0	1	1
39	5796	−28	18	4	1	2	0	0	−1	0	−2	0
63	65505	97	−15	1	0	1	−1	1	0	−1	0	1
87	494385	−239	60	−7	0	4	3	1	1	−1	0	0
111	2922381	525	90	−3	−9	−6	0	1	0	0	0	1
135	14525511	−1113	−75	15	6	−3	0	−1	0	0	0	−1
159	63447087	2255	228	7	7	−4	−3	−1	0	1	−2	0
183	250188435	−4333	360	−29	0	8	0	−1	2	0	0	1
207	907876585	7945	−260	−15	0	4	0	−3	0	0	0	0
231	3073155810	−14174	762	50	−25	10	−3	2	−1	1	2	0
255	9804660777	24809	1062	25	17	−10	9	1	−1	1	2	0
279	29717775186	−42286	−759	−78	16	−7	1	2	0	1	1	0
303	86116649220	70308	2070	−36	0	−18	0	4	−3	0	0	0
327	239806592730	−115046	2880	122	0	16	−9	−2	1	−1	0	−1
351	644418434331	185563	−1926	59	−64	10	0	−1	1	0	−1	0
375	1676994065901	−294483	5256	−195	46	24	0	−3	0	0	1	−1
399	4238788584987	460571	6948	−101	37	−28	−9	−5	0	−1	−2	0
423	10432762525295	−711985	−4645	295	0	−13	19	3	0	−1	0	0
447	25058448433770	1088842	12120	146	0	−32	3	2	4	−1	0	0
471	58848028309224	−1647128	15912	−424	−136	40	0	4	0	0	2	0
495	135349351964727	2466359	−10281	−201	92	23	−16	7	−1	0	−1	0
519	305326880332593	−3660335	26646	617	68	46	0	−7	−2	0	−4	0
543	676433083819185	5388145	34110	305	0	−50	0	−3	0	0	0	0
567	1473468429847035	−7867397	−21840	−901	0	−32	−15	−5	−5	1	0	1

Table 21 The twined series for *McL*. The table shows the Fourier coefficients multiplying $q^{-D/24}$ in the q-expansion of the function $\tilde{h}_{g,\frac{1}{2}}(\tau)$. $b_p = (-1 + i\sqrt{p})/2$

$-D[g]$	1A	2A	3A	3B	4A	5A	5B	6A	6B	7AB	8A	9AB	10A	11AB	12A	14AB	$15A, \overline{15B}$	$30A, \overline{30B}$
−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1
23	770	−14	−13	5	−2	−5	0	7	1	0	0	−1	1	0	1	0	b_{15}	b_{15}
47	13915	43	−17	10	−1	−10	0	7	−2	−1	1	1	−2	0	−1	1	$-b_{15}$	b_{15}
71	132825	−119	−42	21	5	−25	−5	22	1	0	−1	0	1	0	2	0	$-b_{15}$	b_{15}
95	915124	308	−68	31	4	−26	4	32	−1	0	0	−2	−2	1	−2	0	b_{15}	b_{15}
119	5069867	−693	−112	59	−13	−58	2	60	3	−2	−1	2	2	0	2	0	$-b_{15}$	$(b_{15}-2)$
143	24053215	1407	−167	85	−9	−85	0	81	−3	4	−1	1	−3	−1	−3	0	$-\overline{b_{15}}$	$(b_{15}-1)$
167	101268540	−2772	−279	135	24	−135	0	141	3	2	2	−3	3	−1	3	0	$(b_{15}-1)$	$(b_{15}-1)$
191	387746282	5306	−394	200	14	−218	−18	194	−4	0	0	2	−4	0	−4	0	$-2b_{15}$	$2b_{15}$
215	1372935090	−9710	−600	300	−38	−285	15	304	4	−5	2	0	5	−1	4	−1	0	$2b_{15}$
239	4552039296	17136	−837	414	−20	−404	11	411	−6	0	2	−3	−4	2	−5	0	$(2b_{15}-1)$	$(2b_{15}-3)$
263	14265412315	−29589	−1208	610	63	−610	0	612	6	0	−3	4	6	0	6	0	2	$(2b_{15}-2)$
287	42568680715	50155	−1667	835	35	−835	0	829	−5	−6	−1	1	−5	0	−7	0	$-b_{15}$	$(3b_{15}-2)$
311	121665949240	−83160	−2345	1165	−108	−1210	−45	1179	9	10	−2	−5	10	0	9	0	$(b_{15}-2)$	$(3b_{15}-2)$
335	334658246604	135148	−3153	1581	−60	−1546	34	1567	−11	3	−4	3	−12	0	−9	−1	$(1-2b_{15})$	$(4b_{15}-1)$
359	889413095662	−216482	−4313	2158	170	−2138	22	2167	10	0	4	1	8	3	11	0	$\overline{b_{15}}$	$(5b_{15}-3)$
383	2291482148835	342259	−5748	2865	87	−2865	0	2860	−11	−9	1	−6	−11	0	−12	1	$(4b_{15}-1)$	$(4b_{15}-3)$
407	5739333227670	−533610	−7692	3855	−250	−3855	0	3864	15	0	2	6	15	−2	14	0	$(2-2b_{15})$	$(4b_{15}-4)$
431	14008317423968	821296	−10096	5051	−124	−5157	−107	5032	−17	0	6	2	−19	−1	−16	0	$(1-b_{15})$	$(5b_{15}-3)$
455	33388385201699	−1250717	−13333	6656	371	−6576	79	6679	16	−11	−5	−7	18	0	17	1	$(4b_{15}-1)$	$(6b_{15}-3)$
479	77853744768906	1886234	−17280	8649	190	−8594	56	8624	−19	22	0	6	−16	−4	−20	0	$(1-3b_{15})$	$(7b_{15}-5)$
503	177881794535250	−2816798	−22500	11250	−554	−11250	0	11272	22	8	−4	0	22	3	22	2	0	$(8b_{15}-4)$
527	398808419854845	4167709	−28887	14430	−283	−14430	0	14413	−26	0	−7	−9	−26	0	−25	0	$(3-3b_{15})$	$(9b_{15}-5)$
551	878461575586727	−6116665	−37138	18581	799	−18798	−218	18602	29	−22	7	8	30	0	28	−2	$(2-5b_{15})$	$(9b_{15}-6)$
575	1903241478167799	8909383	−47253	23631	395	−23476	154	23599	−29	0	1	3	−32	0	−31	0	2	$(10b_{15}-6)$

Table 22 The twined series for *McL*. The table shows the Fourier coefficients multiplying $q^{-D/24}$ in the q-expansion of the function $\tilde{h}_{g,\frac{3}{2}}(\tau)$

$-D[g]$	1A	2A	3A	3B	4A	5A	5B	6A	6B	7AB	8A	9AB	10A	11AB	12A	14AB	15AB	30AB
−9	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
15	231	7	15	6	−1	6	1	7	−2	0	−1	0	2	0	−1	0	0	2
39	5796	−28	45	18	4	21	1	5	2	0	0	0	−3	−1	1	0	0	0
63	65505	97	30	−15	1	30	0	22	1	−1	1	0	2	0	−2	−1	0	2
87	494385	−239	150	60	−7	60	0	22	4	3	1	0	−4	1	2	−1	0	2
111	2922381	525	225	90	−3	81	−9	57	−6	0	1	0	5	0	−3	0	0	2
135	14525511	−1113	159	−75	15	161	6	63	−3	0	−1	3	−3	0	3	0	−1	3
159	63447087	2255	570	228	7	237	7	122	−4	−3	−1	0	5	0	−2	1	0	2
183	250188435	−4333	900	360	−29	360	0	164	8	0	−1	0	−8	2	4	0	0	4
207	907876585	7945	505	−260	−15	510	0	289	4	0	−3	−5	10	0	−3	0	0	4
231	3073155810	−14174	1905	762	50	735	−25	361	10	−3	2	0	−9	−1	5	1	0	6
255	9804660777	24809	2655	1062	25	1077	17	551	−10	9	1	0	9	−1	−5	1	0	6
279	29717775186	−42286	1518	−759	−78	1536	16	710	−7	1	2	0	−16	0	6	1	3	5
303	86116649220	70308	5175	2070	−36	2070	0	1071	−18	0	4	0	18	−3	−9	0	0	6
327	239806592730	−115046	7200	2880	122	2880	0	1408	16	−9	−2	0	−16	1	8	−1	0	8
351	644418434331	185563	3879	−1926	59	3806	−64	1999	10	0	−1	9	18	1	−13	0	−1	9
375	1676994065901	−294483	13140	5256	−195	5301	46	2580	24	0	−3	0	−23	0	12	0	0	10
399	4238788584987	460571	17370	6948	−101	6987	37	3530	−28	−9	−5	0	31	0	−14	−1	0	10
423	10432762525295	−711985	9260	−4645	295	9270	0	4532	−13	19	3	−10	−30	0	16	−1	0	12
447	25058448433770	1088842	30300	12120	146	12120	0	6124	−32	3	2	0	32	4	−16	−1	0	14
471	58848028309224	−1647128	39780	15912	−424	15774	−136	7876	40	0	4	0	−38	0	20	0	0	16
495	135349351964727	2466359	20562	−10281	−201	20652	92	10442	23	−16	7	0	44	−1	−18	0	−3	17
519	305326880332593	−3660335	66615	26646	617	26718	68	13231	46	0	−7	0	−50	−2	23	0	0	16
543	676433083819185	5388145	85275	34110	305	34110	0	17155	−50	0	−3	0	50	0	−25	0	0	20
567	1473468429847035	−7867397	43725	−21840	−901	43710	0	21637	−32	−15	−5	15	−62	−5	29	1	0	22

Table 23 The twined series for HS. The table shows the Fourier coefficients multiplying $q^{-D/24}$ in the q-expansion of the function $\tilde{h}_{g,\frac{1}{2}}(\tau)$. $a_p = i\sqrt{p}$

$-D[g]$	1A	2A	2B	3A	4A	4BC	5A	5BC	6A	6B	7A	8ABC	10A	10B	11AB	12A	15A	$20A, \overline{20B}$
−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1
23	770	−14	10	5	−10	−2	−5	0	1	1	0	0	1	0	0	−1	0	a_5
47	13915	43	−45	10	−25	−1	−10	0	0	−2	−1	1	−2	0	0	2	0	0
71	132825	−119	129	21	−35	5	−25	−5	3	1	0	−1	1	−1	0	1	1	$\overline{a_5}$
95	915124	308	−300	31	−60	4	−26	4	−3	−1	0	0	−2	0	1	−3	1	0
119	5069867	−693	667	59	−125	−13	−58	2	1	3	−2	−1	2	2	0	1	−1	0
143	24053215	1407	−1425	85	−185	−9	−85	0	−3	−3	4	−1	−3	0	−1	1	0	a_5
167	101268540	−2772	2820	135	−240	24	−135	0	3	3	2	2	3	0	−1	−3	0	$\overline{a_5}$
191	387746282	5306	−5278	200	−394	14	−218	−18	−4	−4	0	0	−4	2	0	2	0	$(\overline{a_5}+1)$
215	1372935090	−9710	9634	300	−630	−38	−285	15	4	4	−5	2	5	−1	−1	0	0	a_5
239	4552039296	17136	−17176	414	−860	−20	−404	11	−4	−6	0	2	−4	−1	2	−2	−1	0
263	14265412315	−29589	29715	610	−1145	63	−610	0	6	6	0	−3	6	0	0	4	0	0
287	42568680715	50155	−50085	835	−1645	35	−835	0	−9	−5	−6	−1	−5	0	0	−1	0	$\overline{a_5}$
311	121665949240	−83160	82944	1165	−2420	−108	−1210	−45	9	9	10	−2	10	−1	0	−5	0	0
335	334658246604	135148	−135268	1581	−3244	−60	−1546	34	−7	−11	3	−4	−12	2	0	5	1	(a_5+1)
359	889413095662	−216482	216822	2158	−4126	170	−2138	22	12	10	0	4	8	2	3	2	−2	$(\overline{a_5}-1)$
383	2291482148835	342259	−342085	2865	−5665	87	−2865	0	−13	−11	−9	1	−11	0	0	−7	0	a_5
407	5739333227670	−533610	533110	3855	−7930	−250	−3855	0	13	15	0	2	15	0	−2	5	0	a_5
431	14008317423968	821296	−821544	5051	−10260	−124	−5157	−107	−15	−17	0	6	−19	1	−1	3	1	$\overline{a_5}$
455	33388385201699	−1250717	1251459	6656	−12909	371	−6576	79	18	16	−11	−5	18	−1	0	−6	1	$(\overline{a_5}+1)$
479	77853744768906	1886234	−1885854	8649	−17146	190	−8594	56	−21	−19	22	0	−16	−4	−4	5	−1	$(\overline{a_5}-1)$
503	177881794535250	−2816798	2815690	11250	−23010	−554	−11250	0	22	22	8	−4	22	0	3	0	0	$2a_5$
527	398808419854845	4167709	−4168275	14430	−29195	−283	−14430	0	−24	−26	0	−7	−26	0	0	−8	0	0
551	878461575586727	−6116665	6118263	18581	−36305	799	−18798	−218	27	29	−22	7	30	−2	0	7	1	$2\overline{a_5}$
575	1903241478167799	8909383	−8908593	23631	−46925	395	−23476	154	−33	−29	0	1	−32	2	0	1	1	0

Table 24 The twined series for *HS*. The table shows the Fourier coefficients multiplying $q^{-D/24}$ in the q-expansion of the function $\tilde{h}_{g,\frac{3}{2}}(\tau)$

$-D[g]$	1A	2A	2B	3A	4A	4BC	5A	5BC	6A	6B	7A	8ABC	10A	10B	11AB	12A	15A	20AB
−9	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
15	231	7	−9	6	15	−1	6	1	0	−2	0	−1	2	1	0	0	1	0
39	5796	−28	36	18	36	4	21	1	0	2	0	0	−3	1	−1	0	−2	1
63	65505	97	−95	−15	65	1	30	0	−5	1	−1	1	2	0	0	−1	0	0
87	494385	−239	225	60	105	−7	60	0	0	4	3	1	−4	0	1	0	0	0
111	2922381	525	−531	90	189	−3	81	−9	0	−6	0	1	5	−1	0	0	0	−1
135	14525511	−1113	1143	−75	319	15	161	6	9	−3	0	−1	−3	−2	0	1	0	−1
159	63447087	2255	−2241	228	471	7	237	7	0	−4	−3	−1	5	−1	0	0	−2	1
183	250188435	−4333	4275	360	675	−29	360	0	0	8	0	−1	−8	0	2	0	0	0
207	907876585	7945	−7975	−260	1025	−15	510	0	−10	4	0	−3	10	0	0	2	0	0
231	3073155810	−14174	14274	762	1554	50	735	−25	0	10	−3	2	−9	−1	−1	0	2	−1
255	9804660777	24809	−24759	1062	2169	25	1077	17	0	−10	9	1	9	1	−1	0	2	−1
279	29717775186	−42286	42130	−759	2930	−78	1536	16	19	−7	1	2	−16	0	0	−1	1	0
303	86116649220	70308	−70380	2070	4140	−36	2070	0	0	−18	0	4	18	0	−3	0	0	0
327	239806592730	−115046	115290	2880	5850	122	2880	0	0	16	−9	−2	−16	0	1	0	0	0
351	644418434331	185563	−185445	−1926	7835	59	3806	−64	−36	10	0	−1	18	0	1	−4	−1	0
375	1676994065901	−294483	294093	5256	10269	−195	5301	46	0	24	0	−3	−23	−2	0	0	1	−1
399	4238788584987	460571	−460773	6948	13851	−101	6987	37	0	−28	−9	−5	31	−3	0	0	−2	1
423	10432762525295	−711985	712575	−4645	18775	295	9270	0	45	−13	19	3	−30	0	0	1	0	0
447	25058448433770	1088842	−1088550	12120	24450	146	12120	0	0	−32	3	2	32	0	4	0	0	0
471	58848028309224	−1647128	1646280	15912	31320	−424	15774	−136	0	40	0	4	−38	0	0	0	2	0
495	135349351964727	2466359	−2466761	−10281	41015	−201	20652	92	−59	23	−16	7	44	4	−1	5	−1	0
519	305326880332593	−3660335	3661569	26646	53817	617	26718	68	0	46	0	−7	−50	4	−2	0	−4	2
543	676433083819185	5388145	−5387535	34110	68625	305	34110	0	0	−50	0	−3	50	0	0	0	0	0
567	1473468429847035	−7867397	7865595	−21840	86395	−901	43710	0	90	−32	−15	−5	−62	0	−5	−2	0	0

Table 25 The twined series for $U_6(2)$. The table shows the Fourier coefficients multiplying $q^{-D/24}$ in the q-expansion of the function $\tilde{h}_{g,\frac{1}{2}}(\tau)$ - part I. $a_p = i\sqrt{p}$

$-D[g]$	1A	2A	2B	2C	3A	3B	3C	4A	4B	4CDE	4F	4G	5A	$6A, \overline{6B}$
−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1
23	770	−30	−14	10	5	−13	5	34	−6	−2	2	−2	0	$-3-6a_3$
47	13915	−5	43	−45	10	−17	10	139	−13	−1	3	−1	0	−5
71	132825	−199	−119	129	21	−42	21	505	−15	5	−7	5	−5	$-10-12a_3$
95	915124	180	308	−300	31	−68	31	1412	−28	4	−4	4	4	$-18-18a_3$
119	5069867	−917	−693	667	59	−112	59	3627	−69	−13	11	−13	2	$-26-30a_3$
143	24053215	1055	1407	−1425	85	−167	85	8559	−97	−9	7	−9	0	$-43-36a_3$
167	101268540	−3300	−2772	2820	135	−279	135	19068	−108	24	−20	24	0	$-69-78a_3$
191	387746282	4490	5306	−5278	200	−394	200	40154	−190	14	−14	14	−18	$-100-90a_3$
215	1372935090	−10894	−9710	9634	300	−600	300	81250	−334	−38	42	−38	15	$-148-156a_3$
239	4552039296	15456	17136	−17176	414	−837	414	158736	−440	−20	24	−20	11	$-213-204a_3$
263	14265412315	−32005	−29589	29715	610	−1208	610	300987	−541	63	−69	63	0	$-298-306a_3$
287	42568680715	46795	50155	−50085	835	−1667	835	555307	−805	35	−37	35	0	$-419-408a_3$
311	121665949240	−87784	−83160	82944	1165	−2345	1165	1001016	−1264	−108	104	−108	−45	$-583-606a_3$
335	334658246604	128780	135148	−135268	1581	−3153	1581	1767388	−1652	−60	52	−60	34	$-793-768a_3$
359	889413095662	−225074	−216482	216822	2158	−4313	2158	3062830	−1978	170	−162	170	22	$-1073-1092a_3$
383	2291482148835	330755	342259	−342085	2865	−5748	2865	5217427	−2789	87	−85	87	0	$-1444-1428a_3$
407	5739333227670	−548970	−533610	533110	3855	−7692	3855	8751462	−4090	−250	254	−250	0	$-1914-1938a_3$
431	14008317423968	801024	821296	−821544	5051	−10096	5051	14473168	−5192	−124	136	−124	−107	$-2532-2496a_3$
455	33388385201699	−1277277	−1250717	1251459	6656	−13333	6656	23625763	−6269	371	−381	371	79	$-3327-3366a_3$
479	77853744768906	1851562	1886234	−1885854	8649	−17280	8649	38101658	−8478	190	−190	190	56	$-4328-4284a_3$
503	177881794535250	−2861710	−2816798	2815690	11250	−22500	11250	60762514	−11782	−554	546	−554	0	$-5614-5658a_3$
527	398808419854845	4109885	4167709	−4168275	14430	−28887	14430	95895325	−14739	−283	269	−283	0	$-7237-7194a_3$
551	878461575586727	−6190873	−6116665	6118263	18581	−37138	18581	149872439	−17753	799	−785	799	−218	$-9268-9318a_3$
575	1903241478167799	8814743	8909383	−8908593	23631	−47253	23631	232094167	−23265	395	−393	395	154	$-11827-11766a_3$

Table 26 The twined series for $U_6(2)$. The table shows the Fourier coefficients multiplying $q^{-D/24}$ in the q-expansion of the function $\tilde{h}_{g,\frac{1}{2}}(\tau)$ - part II. $a_p = i\sqrt{p}$

$-D[g]$	6C	6D	6E	6F	6G	6H	7A	8A	8BCD	9ABC	10A	11AB	$12A, \overline{12B}$	12C	$12D, \overline{12E}$	12FGH	12I	15A	$18A, \overline{18B}$
−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1
23	−3	7	1	−3	1	1	0	2	0	−1	0	0	$-2+3a_3$	1	a_3	1	−1	0	$-a_3$
47	−14	7	−2	4	−2	0	−1	−1	1	1	0	0	$-5+6a_3$	−2	$-1-2a_3$	−1	0	0	1
71	−19	22	1	−1	1	3	0	1	−1	0	1	0	$-8+12a_3$	1	0	2	−1	1	$-1+a_3$
95	−33	32	−1	−3	−1	−3	0	0	0	−2	0	1	$-19+15a_3$	−1	$-1+a_3$	−2	−1	1	0
119	−53	60	3	1	3	1	−2	3	−1	2	−2	0	$-27+27a_3$	3	$3+a_3$	2	−1	−1	$1+a_3$
143	−91	81	−3	5	−3	−3	4	−5	−1	1	0	−1	$-45+42a_3$	-3	$-1-2a_3$	−3	1	0	−1
167	−129	141	3	−9	3	3	2	4	2	−3	0	−1	$-66+69a_3$	3	$3a_3$	3	1	0	$-a_3$
191	−208	194	−4	8	−4	−4	0	−2	0	2	0	0	$-103+99a_3$	-4	$-1-3a_3$	−4	−2	0	2
215	−292	304	4	−4	4	4	−5	6	2	0	1	−1	$-146+150a_3$	4	$2+2a_3$	4	0	0	$-1+a_3$
239	−426	411	−6	0	−6	−4	0	−4	2	−3	1	2	$-213+210a_3$	−6	$-5-2a_3$	−5	0	−1	$-a_3$
263	−598	612	6	2	6	6	0	3	−3	4	0	0	$-297+303a_3$	6	$5+a_3$	6	0	0	2
287	−845	829	−5	7	−5	−9	−6	−5	−1	1	0	0	$-425+414a_3$	−5	$-1-2a_3$	−7	−1	0	$-2+a_3$
311	−1147	1179	9	−19	9	9	10	8	−2	−5	1	0	$-576+585a_3$	9	$2+7a_3$	9	−1	0	$-1-2a_3$
335	−1603	1567	−11	17	−11	−7	3	−16	−4	3	0	0	$-797+792a_3$	−11	$-5-8a_3$	−9	1	1	$2+a_3$
359	−2138	2167	10	−8	10	12	0	14	4	1	−4	3	$-1067+1080a_3$	10	$5+4a_3$	11	0	−2	$-2+a_3$
383	−2887	2860	−11	−1	−11	−13	−9	−9	1	−6	0	0	$-1448+1434a_3$	−11	$-8-2a_3$	−12	−1	0	$-1-a_3$
407	−3825	3864	15	−3	15	13	0	18	2	6	0	−2	$-1911+1923a_3$	15	$11+5a_3$	14	−1	0	$3+a_3$
431	−5085	5032	−17	21	−17	−15	0	−12	6	2	−1	−1	$-2540+2526a_3$	−17	$-8-10a_3$	−16	1	1	$-3+a_3$
455	−6624	6679	16	−30	16	18	−11	11	−5	−7	3	0	$-3314+3333a_3$	16	$4+11a_3$	17	0	1	$-3a_3$
479	−8687	8624	−19	31	−19	−21	22	−18	0	6	2	−4	$-4342+4320a_3$	−19	$-6-12a_3$	−20	−1	−1	4
503	−11206	11272	22	−22	22	22	8	18	−4	0	0	3	$-5603+5625a_3$	22	$11+11a_3$	22	0	0	$-4+2a_3$
527	−14482	14413	−26	8	−26	−24	0	−35	−7	−9	0	0	$-7244+7221a_3$	−26	$-18-9a_3$	−25	2	0	$-1-2a_3$
551	−18523	18602	29	−13	29	27	−22	35	7	8	2	0	$-9259+9285a_3$	29	$19+11a_3$	28	1	1	$5+a_3$
575	−23689	23599	−29	35	−29	−33	0	−25	1	3	−2	0	$-11846+11811a_3$	−29	$-12-15a_3$	−31	−3	1	$-4+a_3$

Table 27 **The twined series for $U_6(2)$. The table shows the Fourier coefficients multiplying $q^{-D/24}$ in the q-expansion of the function $\tilde{h}_{g,\frac{3}{2}}(\tau)$ - part I**

$-D[g]$	1A	2A	2B	2C	3A	3B	3C	4A	4B	4CDE	4F	4G	5A	6AB	6C	6D
−9	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
15	231	39	7	−9	6	15	6	23	7	−1	−1	−1	1	3	6	7
39	5796	36	−28	36	18	45	18	84	20	4	−4	4	1	9	18	5
63	65505	225	97	−95	−15	30	−15	353	33	1	1	1	0	18	33	22
87	494385	−15	−239	225	60	150	60	993	49	−7	9	−7	0	30	60	22
111	2922381	909	525	−531	90	225	90	2685	93	−3	5	−3	−9	45	90	57
135	14525511	−505	−1113	1143	−75	159	−75	6487	167	15	−17	15	6	71	149	63
159	63447087	3183	2255	−2241	228	570	228	14719	239	7	−9	7	7	114	228	122
183	250188435	−2925	−4333	4275	360	900	360	31395	323	−29	27	−29	0	180	360	164
207	907876585	10025	7945	−7975	−260	505	−260	64553	505	−15	9	−15	0	269	524	289
231	3073155810	−11166	−14174	14274	762	1905	762	127490	802	50	−46	50	−25	381	762	361
255	9804660777	29097	24809	−24759	1062	2655	1062	243945	1097	25	−23	25	17	531	1062	551
279	29717775186	−36270	−42286	42130	−759	1518	−759	453842	1426	−78	82	−78	16	738	1497	710
303	86116649220	78660	70308	−70380	2070	5175	2070	824596	2052	−36	44	−36	0	1035	2070	1071
327	239806592730	−103590	−115046	115290	2880	7200	2880	1465578	2986	122	−126	122	0	1440	2880	1408
351	644418434331	201115	185563	−185445	−1926	3879	−1926	2555051	3947	59	−61	59	−64	1963	3898	1999
375	1676994065901	−273555	−294483	294093	5256	13140	5256	4376077	5037	−195	189	−195	46	2628	5256	2580
399	4238788584987	488475	460571	−460773	6948	17370	6948	7377627	6875	−101	91	−101	37	3474	6948	3530
423	10432762525295	−675025	−711985	712575	−4645	9260	−4645	12257551	9535	295	−289	295	0	4592	9227	4532
447	25058448433770	1137450	1088842	−1088550	12120	30300	12120	20093306	12298	146	−142	146	0	6060	12120	6124
471	58848028309224	−1583640	−1647128	1646280	15912	39780	15912	32531448	15448	−424	432	−424	−136	7956	15912	7876
495	135349351964727	2548791	2466359	−2466761	−10281	20562	−10281	52070711	20407	−201	215	−201	92	10350	20631	10442
519	305326880332593	−3553935	−3660335	3661569	26646	66615	26646	82458145	27217	617	−631	617	68	13323	26646	13231
543	676433083819185	5524785	5388145	−5387535	34110	85275	34110	129280929	34465	305	−311	305	0	17055	34110	17155
567	1473468429847035	−7692805	−7867397	7865595	−21840	43725	−21840	200804347	42747	−901	891	−901	0	21761	43616	21637

Table 28 The twined series for $U_6(2)$. The table shows the Fourier coefficients multiplying $q^{-D/24}$ in the q-expansion of the function $\tilde{h}_{g,\frac{3}{2}}(\tau)$ - part II

$-D[g]$	6E	6F	6G	6H	7A	8A	8BCD	9ABC	10A	11AB	12AB	12C	12DE	12FGH	12I	15A	18AB
−9	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
15	−2	0	−2	0	0	3	−1	0	−1	0	5	2	1	−1	2	1	0
39	2	0	2	0	0	0	0	0	1	−1	3	−6	−1	1	2	−2	0
63	1	3	1	−5	−1	1	1	0	0	0	20	5	0	−2	1	0	0
87	4	0	4	0	3	−3	1	0	0	1	30	0	−2	2	0	0	0
111	−6	0	−6	0	0	9	1	0	−1	0	39	−6	3	−3	2	0	0
135	−3	−7	−3	9	0	−5	−1	3	0	0	79	1	−1	3	1	0	−1
159	−4	0	−4	0	−3	3	−1	0	3	0	130	16	2	−2	0	−2	0
183	8	0	8	0	0	−9	−1	0	0	2	156	−24	−4	4	0	0	0
207	4	14	4	−10	0	9	−3	−5	0	0	275	20	7	−3	0	0	−1
231	10	0	10	0	−3	−6	2	0	−1	−1	383	2	−5	5	2	2	0
255	−10	0	−10	0	9	9	1	0	−3	−1	513	−18	5	−5	−2	2	0
279	−7	−21	−7	19	1	−14	2	0	0	0	764	5	−8	6	1	1	0
303	−18	0	−18	0	0	24	4	0	0	−3	1069	34	9	−9	2	0	0
327	16	0	16	0	−9	−18	−2	0	0	1	1368	−72	−8	8	0	0	0
351	10	28	10	−36	0	15	−1	9	0	1	1985	50	5	−13	2	−1	1
375	24	0	24	0	0	−27	−3	0	0	0	2644	16	−12	12	0	1	0
399	−28	0	−28	0	−9	27	−5	0	5	0	3414	−60	14	−14	4	−2	0
423	−13	−43	−13	45	19	−25	3	−10	0	0	4654	19	−14	16	−1	0	2
447	−32	0	−32	0	3	30	2	0	0	4	6152	92	16	−16	−4	0	0
471	40	0	40	0	0	−36	4	0	0	0	7788	−168	−20	20	0	2	0
495	23	69	23	−59	−16	55	7	0	−4	−1	10400	119	28	−18	−1	−1	0
519	46	0	46	0	0	−51	−7	0	0	−2	13357	34	−23	23	2	−4	0
543	−50	0	−50	0	0	45	−3	0	0	0	16917	−138	25	−25	−2	0	0
567	−32	−94	−32	90	−15	−69	−5	15	0	−5	21907	52	−33	29	0	0	−1

Appendix D: Decomposition tables

See Tables 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60.

Table 29 The table shows the decomposition of the Fourier coefficients multiplying $q^{-D/12}$ in the function $h_{g,1}(\tau)$ into irreducible representations χ_n of M_{22}

−D	χ_1	χ_2	χ_3	χ_4	χ_5	χ_6	χ_7	χ_8	χ_9	χ_{10}	χ_{11}	χ_{12}
−1	−2	0	0	0	0	0	0	0	0	0	0	0
11	0	0	0	0	0	0	0	0	0	1	1	0
23	0	0	0	0	2	2	4	4	6	4	4	8
35	0	2	8	8	6	16	22	34	32	48	48	60
47	2	26	42	42	60	100	160	208	232	268	268	380
59	2	92	228	228	258	480	738	1028	1132	1394	1394	1900
71	26	472	952	952	1204	2130	3334	4508	4966	5966	5966	8240
83	78	1710	3754	3754	4516	8192	12708	17410	19112	23284	23284	31928
95	300	6220	13194	13194	16254	29148	45402	61752	68016	82218	82218	113216
107	950	20228	43616	43616	53078	95744	148822	203204	223388	271160	271160	372564
119	3040	63080	134624	134624	164946	296536	461482	628868	691920	838076	838076	1152800
131	8706	184100	395460	395460	482650	869408	1352058	1844428	2028652	2459952	2459952	3381724
143	24720	517270	1106932	1106932	1354036	2436238	3790274	5167368	5684522	6888684	6888684	9473140
155	66016	1388930	2978518	2978518	3638646	6551148	10189794	13897114	15285968	18531236	18531236	25478420
167	171904	3606408	7724632	7724632	9443990	16996718	26440708	36052396	39659114	48067156	48067156	66095608
179	430782	9052216	19403004	19403004	23710442	42682664	66393106	90540888	99592828	120725462	120725462	165992492
191	1052210	22086048	47318486	47318486	57840438	104106714	161947152	220829728	242915594	294432862	294432862	404852680
203	2496320	52439410	112383780	112383780	137347678	247235148	384582826	524442118	576882244	699266588	699266588	961480364
215	5789798	121561770	260469430	260469430	318366892	573046530	891413422	1215547168	1337108322	1620714052	1620714052	2228498796
227	13116978	275489850	590364314	590364314	721533716	1298781032	2020314748	2754999510	3030488952	3673355264	3673355264	5050837768
239	29128906	611659466	1310656938	1310656938	1601947284	2883475316	4485422600	6116448648	6728109656	8155231968	8155231968	11213482124

Table 30 The table shows the decomposition of the Fourier coefficients multiplying $q^{-D/12}$ in the function $h_{g,2}(\tau)$ into irreducible representations χ_n of M_{22}

−D	χ_1	χ_2	χ_3	χ_4	χ_5	χ_6	χ_7	χ_8	χ_9	χ_{10}	χ_{11}	χ_{12}
−4	1	0	0	0	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	1	0	0	0	0
20	0	1	1	1	1	1	2	2	1	3	3	4
32	0	1	5	5	5	10	15	18	24	26	26	38
44	2	15	28	28	35	61	96	136	147	179	179	242
56	2	70	149	149	184	332	516	706	770	936	936	1284
68	15	310	675	675	821	1486	2307	3138	3468	4196	4196	5778
80	62	1260	2686	2686	3286	5906	9192	12528	13770	16702	16702	22964
92	215	4535	9711	9711	11871	21367	33238	45338	49855	60440	60440	83094
104	716	15162	32528	32528	39754	71562	111316	151798	167022	202408	202408	278332
116	2277	47678	102132	102132	124829	224684	349513	476604	524228	635466	635466	873748
128	6752	141611	303407	303407	370835	667490	1038325	1415886	1557452	1887842	1887842	2595774
140	19067	400933	859281	859281	1050214	1890431	2940645	4009962	4411059	5346659	5346659	7351704
152	51870	1089096	2333682	2333682	2852302	5134124	7986426	10890628	11979584	14520766	14520766	19965996
164	135731	2849794	6106599	6106599	7463621	13434478	20898099	28497414	31347100	37996546	37996546	52245218
176	343436	7212750	15456158	15456158	18890818	34003540	52894358	72128538	79341684	96171541	96171541	132236044
188	843662	17716592	37963923	37963923	46400388	83520644	129921032	177165164	194881422	236220080	236220080	324802452
200	2016488	42346314	90741956	90741956	110906886	199632354	310539240	423462681	465808750	564616772	564616772	776347956
212	4701307	98728774	211562274	211562274	258575987	465436954	724012941	987290154	1086019814	1316387235	1316387235	1810032796
224	10713686	224984946	482109963	482109963	589245588	1060641874	1649887462	2249846600	2474830808	2999795266	2999795266	4124718268
236	23905058	502005985	1075726834	1075726834	1314777297	2366599091	3681376388	5020058920	5522064385	6693411661	6693411661	9203440774
248	52307782	1098468036	2353861448	2353861448	2876941610	5178495256	8055436866	10984686398	12083156304	14646249036	14646249036	20138593100
260	112400070	2360397820	5057994232	5057994232	6181993066	11127587228	17309580294	23603973264	25964369548	31471963978	31471963978	43273949924

Table 31 The table shows the decomposition of the Fourier coefficients multiplying $q^{-D/12}$ in the function $h_{g,1}(\tau)$ into irreducible representations χ_n of $U_4(3)$ - part I

−D	χ_1	χ_2	χ_3	χ_4	χ_5	χ_6	χ_7	χ_8	χ_9	χ_{10}
−1	−2	0	0	0	0	0	0	0	0	0
11	0	0	0	0	0	0	0	0	0	0
23	0	0	0	0	0	0	0	0	0	0
35	0	0	0	0	0	2	4	4	7	7
47	0	6	4	4	14	22	22	26	34	34
59	0	12	20	20	52	82	120	132	189	189
71	4	72	102	102	270	422	532	606	801	801
83	8	232	380	380	984	1548	2112	2344	3164	3164
95	38	860	1398	1398	3618	5622	7492	8350	11125	11125
107	124	2748	4554	4554	11732	18306	24784	27532	36834	36834
119	420	8640	14228	14228	36676	57060	76674	85316	113706	113706
131	1156	24972	41622	41622	107060	166678	225288	250260	334074	334074
143	3368	70446	116944	116944	300968	468176	631070	701512	935210	935210
155	8908	188592	314208	314208	808100	1257446	1698156	1886748	2516637	2516637
167	23350	490188	815984	815984	2098802	3264774	4405146	4895338	6526850	6526850
179	58388	1229276	2048450	2048450	5267788	8195334	11065144	12294420	16394948	16394948
191	142936	3000480	4998134	4998134	12853814	19994842	26987278	29987754	39982901	39982901
203	338664	7121124	11868186	11868186	30518832	47475942	64096288	71217412	94962320	94962320
215	786382	16511002	27512230	27512230	70749080	110054184	148559990	165070996	220092869	220092869
227	1780628	37411832	62351670	62351670	160334544	249414368	336717232	374129064	498851309	498851309
239	3955942	83070266	138437866	138437866	355990060	553762164	747550924	830621184	1107491427	1107491427
251	8612692	180905952	301506846	301506846	775306952	1206043980	1628176080	1809082032	2412136251	2412136251
263	18432618	387072550	645093274	645093274	1658826474	2580396830	3483475608	3870548166	5160722937	5160722937
275	38777988	814425664	1357371466	1357371466	3490390608	5429518660	7329886448	8144312112	10859139417	10859139417
287	80345934	1687237678	2812006070	2812006070	7230903886	11248072954	15184775484	16872013154	22496001167	22496001167

Table 32 The table shows the decomposition of the Fourier coefficients multiplying $q^{-D/12}$ in the function $h_{g,1}(\tau)$ into irreducible representations χ_n of $U_4(3)$ - part II

$-D$	χ_{11}	χ_{12}	χ_{13}	χ_{14}	χ_{15}	χ_{16}	χ_{17}	χ_{18}	χ_{19}	χ_{20}
−1	0	0	0	0	0	0	0	0	0	0
11	0	0	0	0	0	1	0	0	0	0
23	0	0	2	2	2	1	2	2	2	2
35	7	7	6	6	10	16	14	14	16	16
47	34	34	46	46	62	74	86	86	100	116
59	189	189	206	206	290	391	432	432	480	596
71	801	801	934	934	1242	1626	1864	1864	2130	2586
83	3164	3164	3532	3532	4756	6379	7216	7216	8192	10064
95	11125	11125	12636	12636	16856	22345	25568	25568	29148	35728
107	36834	36834	41346	41346	55278	73819	84136	84136	95744	117656
119	113706	113706	128270	128270	171022	227677	260274	260274	296536	364172
131	334074	334074	375590	375590	501258	668609	763500	763500	869408	1068560
143	935210	935210	1053068	1053068	1404072	1871144	2138648	2138648	2436238	2993494
155	2516637	2516637	2830546	2830546	3775230	5034454	5751956	5751956	6551148	8051776
167	6526850	6526850	7345188	7345188	9793632	13055603	14921266	14921266	16996718	20888326
179	16394948	16394948	18442654	18442654	24593074	32792777	37473264	37473264	42682664	52460200
191	39982901	39982901	44986624	44986624	59982140	79970261	91395836	91395836	104106714	127950562
203	94962320	94962320	106828846	106828846	142445378	189931510	217054970	217054970	247235148	303871648
215	220092869	220092869	247617812	247617812	330156976	440195866	503083826	503083826	573046530	704309074
227	498851309	498851309	561199172	561199172	748280636	997717729	1140227136	1140227136	1298781032	1596305804
239	1107491427	1107491427	1245957224	1245957224	1661276556	2215005306	2531441198	2531441198	2883475316	3544000144
251	2412136251	2412136251	2713634904	2713634904	3618212136	4824304831	5513444680	5513444680	6280138936	7718796324
263	5160722937	5160722937	5805874860	5805874860	7741166340	10321492731	11796004996	11796004996	13436398852	16514369292
275	10859139417	10859139417	12216494216	12216494216	16288726840	21718346527	24820871652	24820871652	28272484272	34749166956
287	22496001167	22496001167	25308126986	25308126986	33744168882	44992097964	51419567602	51419567602	58570125574	71987317446

Table 33 The table shows the decomposition of the Fourier coefficients multiplying $q^{-D/12}$ in the function $h_{g,2}(\tau)$ into irreducible representations χ_n of $U_4(3)$ - part I

−D	χ_1	χ_2	χ_3	χ_4	χ_5	χ_6	χ_7	χ_8	χ_9	χ_{10}
−4	1	0	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	1	0	0
20	0	0	0	0	1	1	1	0	1	1
32	0	0	1	1	2	1	4	3	3	3
44	1	2	4	4	8	14	16	21	26	26
56	0	8	18	18	42	64	92	104	129	129
68	3	37	74	74	185	285	396	428	573	573
80	10	170	288	288	742	1142	1558	1720	2276	2276
92	30	606	1038	1038	2648	4114	5574	6189	8226	8226
104	98	2040	3458	3458	8852	13740	18616	20668	27505	27505
116	315	6453	10820	10820	27785	43187	58374	64812	86349	86349
128	932	19200	32105	32105	82484	128231	173260	192443	256447	256447
140	2585	54343	90862	90862	233470	363072	490425	544792	726201	726201
152	7068	147802	246650	246650	634020	986096	1331594	1479426	1972157	1972157
164	18479	386845	645246	645246	1658877	2580201	3483848	3870656	5160351	5160351
176	46678	979182	1632930	1632930	4198392	6530326	8817060	9796194	13060772	13060772
188	114644	2405530	4010504	4010504	10311866	16040080	21655492	24061078	32080056	32080056
200	273958	5750076	9585456	9585456	24646996	38338768	51759604	57509747	76677496	76677496
212	638587	13406477	22347452	22347452	57462835	89385017	120673444	134079840	178770283	178770283
224	1455274	30552132	50924466	50924466	130945830	203691226	274988136	305540174	407382190	407382190
236	3246731	68171414	113625730	113625730	292176084	454492898	613572808	681744339	908985651	908985651
248	7103972	149171154	248629024	248629024	639325160	994500692	1342587586	1491758880	1989002072	1989002072
260	15265066	320543266	534252608	534252608	1373783498	2136989558	2884951508	3205494608	4273978516	4273978516
272	32250408	677225340	1128728962	1128728962	2902432966	4514885396	6095118216	6772343354	9029770416	9029770416
284	67064204	1408309053	2347212278	2347212278	6035669290	9388804084	12674919317	14083228602	18777609455	18777609455

Table 34 The table shows the decomposition of the Fourier coefficients multiplying $q^{-D/12}$ in the function $h_{g,1}(\tau)$ into irreducible representations χ_n of $U_4(3)$ - part II

−D	χ_{11}	χ_{12}	χ_{13}	χ_{14}	χ_{15}	χ_{16}	χ_{17}	χ_{18}	χ_{19}	χ_{20}
−4	0	0	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	0	0	0
20	1	1	0	0	0	0	1	1	1	1
32	3	3	3	3	5	6	8	8	10	14
44	26	26	27	27	31	46	54	54	61	81
56	129	129	142	142	182	248	289	289	332	410
68	573	573	636	636	842	1124	1299	1299	1486	1842
80	2276	2276	2544	2544	3372	4512	5178	5178	5906	7278
92	8226	8226	9223	9223	12251	16366	18749	18749	21367	26305
104	27505	27505	30902	30902	41142	54888	62812	62812	71562	88062
116	86349	86349	97044	97044	129268	172448	197214	197214	224684	276276
128	256447	256447	288351	288351	384273	512502	585938	585938	667490	820618
140	726201	726201	816744	816744	1088680	1451756	1659543	1659543	1890431	2323911
152	1972157	1972157	2218282	2218282	2957154	3943264	4507162	4507162	5134124	6310848
164	5160351	5160351	5804744	5804744	7738814	10318996	11794091	11794091	13434478	16513050
176	13060772	13060772	14692426	14692426	19588686	26119004	29851826	29851826	34003540	41794712
188	32080056	32080056	36088522	36088522	48115930	64156020	73323427	73323427	83520644	102655962
200	76677496	76677496	86259890	86259890	115010066	153348868	175259316	175259316	199632354	245367882
212	178770283	178770283	201113152	201113152	268146424	357531400	408612706	408612706	465436954	572065334
224	407382190	407382190	458299758	458299758	611059410	814750620	931151309	931151309	1060641874	1303622708
236	908985651	908985651	1022601213	1022601213	1363458009	1817950926	2077669720	2077669720	2366599091	2908753703
248	1989002072	1989002072	2237616450	2237616450	2983474090	3977974852	4546274088	4546274088	5178495256	6364807716
260	4273978516	4273978516	4808209568	4808209568	6410924444	8547913920	9769068796	9769068796	11127587228	13676730320
272	9029770416	9029770416	10158468546	10158468546	13544593866	18059479228	20639439888	20639439888	23509622436	28895264904
284	18777609455	18777609455	21124777990	21124777990	28166327222	37555131344	42920200803	42920200803	48888805889	60088352145

Table 35 The table shows the decomposition of the Fourier coefficients multiplying $q^{-D/24}$ in the function $\tilde{h}_{g,\frac{1}{2}}(\tau)$ into irreducible representations χ_n of M_{23} - part I

−D	χ_1	χ_2	χ_3	χ_4	χ_5	χ_6	χ_7	χ_8	χ_9
−1	−1	0	0	0	0	0	0	0	0
23	0	0	0	0	0	0	0	0	0
47	0	0	0	0	1	1	0	0	0
71	0	0	1	1	3	3	2	2	4
95	1	4	4	4	24	23	21	21	24
119	0	10	23	23	110	115	113	113	122
143	3	57	104	104	555	552	547	547	604
167	11	217	450	450	2266	2289	2283	2283	2500
191	40	851	1705	1705	8792	8800	8789	8789	9646
215	132	2945	6066	6066	30883	31076	31062	31062	34002
239	455	9865	20061	20061	102784	103141	103119	103119	112980
263	1393	30717	62965	62965	321420	322980	322951	322951	353668
287	4196	91938	187731	187731	960228	964123	964080	964080	1056018
311	11898	262221	536800	536800	2742534	2754938	2754882	2754882	3017118
335	32865	722082	1476142	1476142	7546650	7578723	7578641	7578641	8300712
359	87129	1917740	3923759	3923759	20051805	20140197	20140092	20140092	22057824
383	224778	4942779	10108150	10108150	51668700	51891433	51891287	51891287	56834066
407	562443	12376668	25318792	25318792	129399884	129965529	129965340	129965340	142342008
431	1373557	30213098	61794663	61794663	315850951	317219639	317219382	317219382	347432516
455	3272651	72004775	147289246	147289246	752794475	756074520	756074191	756074191	828078940
479	7632762	167908464	343437978	343437978	1755376093	1762997610	1762997173	1762997173	1930905618
503	17436751	383624438	784701896	784701896	4010659821	4028113401	4028112844	4028112844	4411737282
527	39096804	860103679	1759278672	1759278672	8991927075	9030999117	9030998389	9030998389	9891102068
551	86113417	1894529513	3875207902	3875207902	19806533653	19892683368	19892682446	19892682446	21787212032
575	186578208	4104664777	8395853450	8395853450	42912264213	43098789349	43098788160	43098788160	47203452886

Table 36 The table shows the decomposition of the Fourier coefficients multiplying $q^{-D/24}$ in the function $\tilde{h}_{g,\frac{1}{2}}(\tau)$ into irreducible representations χ_n of M_{23} - part II

−D	χ_{10}	χ_{11}	χ_{12}	χ_{13}	χ_{14}	χ_{15}	χ_{16}	χ_{17}
−1	0	0	0	0	0	0	0	0
23	1	0	0	0	0	0	0	0
47	1	1	1	1	1	1	2	3
71	11	11	11	11	14	14	12	26
95	68	68	80	80	87	87	96	182
119	389	389	444	444	496	496	508	1003
143	1811	1811	2111	2111	2324	2324	2454	4777
167	7661	7661	8892	8892	9848	9848	10246	20084
191	29245	29245	34052	34052	37596	37596	39394	76950
215	103695	103695	120586	120586	133306	133306	139204	272375
239	343525	343525	399819	399819	441658	441658	462028	903231
263	1076965	1076965	1252987	1252987	1384656	1384656	1447084	2830347
287	3212975	3212975	3738996	3738996	4130944	4130944	4319566	8446311
311	9184195	9184195	10686484	10686484	11808192	11808192	12343520	24139819
335	25260382	25260382	29394631	29394631	32477570	32477570	33956132	66400838
359	67136828	67136828	78121437	78121437	86318725	86318725	90238598	176470194
383	172966448	172966448	201271974	201271974	222385345	222385345	232499596	454660159
407	433225520	433225520	504113505	504113505	557004065	557004065	582313360	1138754982
431	1057386979	1057386979	1230418627	1230418627	1359497348	1359497348	1421306694	2779430485
455	2520265222	2520265222	2932664346	2932664346	3240340814	3240340814	3387607678	6624675836
479	5876631765	5876631765	6838273397	6838273397	7555669131	7555669131	7899140768	15447177148
503	13427082844	13427082844	15624224127	15624224127	17263391744	17263391744	18048043408	35293998406
527	30103271271	30103271271	35029285569	35029285569	38704205367	38704205367	40463558496	79128667059
551	66309028068	66309028068	77159558254	77159558254	85254464108	85254464108	89129562686	174297913365
575	143662505416	143662505417	167170968143	167170968143	184708934751	184708934751	193104947248	377627303791

Table 37 The table shows the decomposition of the Fourier coefficients multiplying $q^{-D/24}$ in the function $\tilde{h}_{g,\frac{3}{2}}(\tau)$ into irreducible representations χ_n of M_{23} - part I

−D	χ_1	χ_2	χ_3	χ_4	χ_5	χ_6	χ_7	χ_8	χ_9
−9	1	0	0	0	0	0	0	0	0
15	0	0	0	0	0	1	0	0	0
39	0	1	0	0	1	0	0	0	0
63	0	0	0	0	2	1	2	2	2
87	1	2	2	2	10	12	10	10	12
111	0	8	12	12	72	71	65	65	76
135	1	29	66	66	317	324	327	327	354
159	8	148	278	278	1456	1451	1439	1439	1584
183	25	536	1108	1108	5614	5666	5650	5650	6186
207	90	1969	3996	3996	20527	20571	20585	20585	22554
231	301	6612	13574	13574	69200	69575	69539	69539	76160
255	979	21229	43224	43224	221297	222128	222072	222072	243296
279	2892	63977	131140	131140	669671	672826	672862	672862	736834
303	8477	185925	379810	379810	1942293	1950361	1950253	1950253	2136178
327	23485	516995	1058004	1058004	5406043	5430193	5430053	5430053	5947048
351	63229	1390158	2842550	2842550	14531106	14593220	14593319	14593319	15983498
375	164314	3616159	7398130	7398130	37808739	37974823	37974566	37974566	41590710
399	415733	9142805	18698262	18698262	95575487	95988502	95988148	95988148	105130940
423	1022444	22498248	46023364	46023364	235220486	236247135	236247364	236247364	258745612
447	2456955	54045257	110540364	110540364	564999523	567449989	567449375	567449375	621494632
471	5768329	126911807	259601046	259601046	1326827403	1332605575	1332604789	1332604789	1459516642
495	13269156	291907776	597070540	597070540	3051727864	3064982355	3064982875	3064982875	3356890620
519	29930003	658478668	1346907692	1346907692	6884144714	6914096414	6914095094	6914095094	7572573738
543	66312927	1458848820	2983976756	2983976756	15251512500	15317793334	15317791616	15317791616	16776640436
567	144441012	3177751973	6499993004	6499993004	33222076673	33366564639	33366565723	33366565723	36544317696

Table 38 The table shows the decomposition of the Fourier coefficients multiplying $q^{-D/24}$ in the function $\tilde{h}_{g,\frac{3}{2}}(\tau)$ into irreducible representations χ_n of M_{23} - part II

−D	χ_{10}	χ_{11}	χ_{12}	χ_{13}	χ_{14}	χ_{15}	χ_{16}	χ_{17}
−9	0	0	0	0	0	0	0	0
15	0	0	0	0	0	0	0	0
39	1	1	0	0	1	1	0	1
63	4	4	6	6	6	6	8	13
87	41	41	42	42	49	49	48	97
111	220	220	254	254	280	280	302	582
135	1099	1099	1278	1278	1418	1418	1462	2879
159	4783	4783	5568	5568	6143	6143	6460	12594
183	18920	18920	21967	21967	24308	24308	25342	49625
207	68482	68482	79749	79749	88055	88055	92194	180159
231	232064	232064	269912	269912	298347	298347	311662	609707
255	739984	739984	861169	861169	951374	951374	995038	1945438
279	2243393	2243393	2610275	2610275	2884379	2884379	3014774	5896262
303	6500043	6500043	7563999	7563999	8357121	8357121	8738192	17086836
327	18102016	18102016	21063317	21063317	23273934	23273934	24329868	47580312
351	48641673	48641673	56602448	56602448	62539370	62539370	65385226	127861367
375	126586391	126586391	147298443	147298443	162753713	162753713	170146608	332736007
399	319954666	319954666	372313294	372313294	411370026	411370026	430076524	841030812
423	787500731	787500731	916360448	916360448	1012501148	1012501148	1058511762	2069990475
447	1891483817	1891483817	2201005302	2201005302	2431907356	2431907356	2542467146	4971917548
471	4442040173	4442040173	5168908580	5168908580	5711193875	5711193875	5970765562	11676191108
495	10216574441	10216574441	11888393029	11888393029	13135596047	13135596047	13732710656	26855037539
519	23047036703	23047036703	26818346389	26818346389	29631903464	29631903464	30978745714	60580719175
543	51059233415	51059233415	59414411523	59414411523	65647584650	65647584650	68631657584	134212929307
567	111221993159	111221993159	129421909124	129421909124	142999706146	142999706146	149499558772	292354823899

Table 39 The table shows the decomposition of the Fourier coefficients multiplying $q^{-D/24}$ in the function $\tilde{h}_{g,\frac{1}{2}}(\tau)$ into irreducible representations χ_n of *McL* - part I

−D	χ_1	χ_2	χ_3	χ_4	χ_5	χ_6	χ_7	χ_8
−1	−1	0	0	0	0	0	0	0
23	0	0	0	0	1	0	0	0
47	0	0	0	0	0	0	0	0
71	0	0	0	0	1	1	0	0
95	0	1	1	1	2	1	1	1
119	0	1	2	1	7	7	5	5
143	0	3	8	8	23	22	24	24
167	1	5	27	29	93	92	101	101
191	0	13	103	113	336	336	386	386
215	2	37	357	381	1192	1191	1370	1370
239	6	122	1181	1292	3911	3909	4541	4541
263	18	357	3673	3990	12260	12259	14231	14231
287	49	1065	10971	11987	36504	36503	42467	42467
311	135	2983	31296	34082	104381	104379	121375	121375
335	379	8247	86127	94007	286919	286918	333865	333865
359	995	21796	228754	249432	762672	762670	887304	887304
383	2564	56231	589479	643218	1964549	1964545	2286050	2286050
407	6386	140568	1476138	1610016	4920870	4920869	5725731	5725731
431	15622	343337	3603184	3931104	12009753	12009751	13975118	13975118
455	37167	817799	8587440	9367446	28625805	28625800	33309272	33309272
479	86756	1907498	20024583	21845882	66746654	66746652	77669274	77669274
503	198014	4357044	45751192	49908836	152506325	152506321	177460315	177460315
527	444173	9769828	102575149	111902115	341912961	341912958	397863480	397863480
551	977997	21517663	225940987	246477854	753141922	753141920	876380147	876380147
575	2119404	46622397	489518783	534024793	1631720458	1631720453	1898731972	1898731972

Table 40 **The table shows the decomposition of the Fourier coefficients multiplying $q^{-D/24}$ in the function $\tilde{h}_{g,\frac{1}{2}}(\tau)$ into irreducible representations χ_n of McL - part II**

−D	χ_9	χ_{10}	χ_{11}	χ_{12}	χ_{13}	χ_{14}	χ_{15}	χ_{16}
−1	0	0	0	0	0	0	0	0
23	0	0	0	0	0	0	0	0
47	0	1	0	0	0	0	0	0
71	0	1	1	1	0	1	1	2
95	3	5	3	5	4	5	5	8
119	8	21	22	25	27	27	32	46
143	49	99	92	120	123	138	146	214
167	193	396	402	507	536	572	628	909
191	765	1535	1513	1946	2039	2212	2385	3459
215	2657	5375	5400	6874	7271	7783	8486	12269
239	8900	17881	17816	22816	24056	25887	28072	40626
263	27745	55882	55963	71465	75500	81007	88095	127410
287	83033	166945	166766	213315	225153	241943	262694	380029
311	236919	476743	476982	609553	643803	691150	751132	1086405
335	652314	1311878	1311412	1676845	1770483	1901666	2065592	2987879
359	1732646	3485567	3486207	4456239	4706100	5053141	5490489	7941451
383	4465544	8981556	8980403	11481470	12123772	13020310	14144455	20459305
407	11182141	22493230	22494840	28756211	30367390	32608993	35428696	51244665
431	27296592	54903746	54901023	70187986	74116978	79593979	86469919	125073387
455	65055014	130856090	130859914	167289297	176659468	189704760	206102830	298112039
479	151701090	305132577	305126393	390081042	411922132	442353736	480575987	695120690
503	346597295	697161444	697170138	891261397	941176611	1010687552	1098039598	1588233526
527	777084367	1563042221	1563028664	1998200428	2110092165	2265963373	2461774474	3560784919
551	1711669416	3442912979	3442931988	4401459954	4647951685	4991249366	5422610680	7843413933
575	3708476391	7459319124	7459290309	9536046829	10070049592	10813885859	11748391635	16993217734

Table 41 The table shows the decomposition of the Fourier coefficients multiplying $q^{-D/24}$ in the function $\tilde{h}_{g,\frac{1}{2}}(\tau)$ into irreducible representations χ_n of McL - part III

−D	χ_{17}	χ_{18}	χ_{19}	χ_{20}	χ_{21}	χ_{22}	χ_{23}	χ_{24}
−1	0	0	0	0	0	0	0	0
23	0	0	0	0	0	0	0	0
47	0	0	0	0	0	0	0	1
71	2	1	1	1	1	1	1	2
95	8	8	8	10	10	10	11	11
119	46	46	46	53	53	55	59	60
143	214	221	221	261	261	263	279	279
167	909	928	928	1076	1076	1110	1175	1175
191	3459	3561	3561	4167	4167	4252	4487	4489
215	12269	12607	12607	14687	14687	15064	15899	15900
239	40626	41815	41815	48824	48824	49950	52683	52683
263	127410	131025	131025	152796	152796	156541	165130	165131
287	380029	391030	391030	456320	456320	467137	492677	492679
311	1086405	1117568	1117568	1303640	1303640	1335140	1408229	1408230
335	2987879	3074116	3074116	3586785	3586785	3672503	3873309	3873312
359	7941451	8169871	8169871	9531019	9531019	9760341	10294270	10294273
383	20459305	21049092	21049092	24558072	24558072	25146550	26521644	26521644
407	51244665	52720052	52720052	61505504	61505504	62983039	66427796	66427799
431	125073387	128677387	128677387	150125530	150125530	153726332	162132945	162132948
455	298112039	306697753	306697753	357811165	357811165	366401965	386440375	386440376
479	695120690	715147222	715147222	834342808	834342808	854361992	901084137	901084142
503	1588233526	1633981014	1633981014	1906304694	1906304694	1952063468	2058818682	2058818686
527	3560784919	3663364558	3663364558	4273934999	4273934999	4376498280	4615836213	4615836219
551	7843413933	8069346991	8069346991	9414224038	9414224038	9640181689	10167382764	10167382771
575	16993217734	17482746319	17482746319	20396558046	20396558046	20886051640	22028253515	22028253520

Table 42 The table shows the decomposition of the Fourier coefficients multiplying $q^{-D/24}$ in the function $\tilde{h}_{g,\frac{3}{2}}(\tau)$ into irreducible representations χ_n of *McL* - part I

−D	χ_1	χ_2	χ_3	χ_4	χ_5	χ_6	χ_7	χ_8
−9	1	0	0	0	0	0	0	0
15	0	0	1	0	0	0	0	0
39	0	0	0	1	0	0	0	0
63	0	0	1	0	0	0	0	0
87	1	0	1	1	1	1	0	0
111	0	0	4	1	3	3	2	2
135	1	0	6	5	12	12	14	14
159	1	3	24	23	55	55	62	62
183	2	5	72	72	219	219	247	247
207	2	22	245	259	777	777	905	905
231	6	73	804	865	2644	2644	3060	3060
255	18	251	2556	2785	8404	8404	9775	9775
279	36	716	7659	8308	25493	25493	29645	29645
303	105	2128	22218	24238	73822	73822	85899	85899
327	276	5867	61728	67254	205653	205653	239219	239219
351	730	15803	165837	180937	552420	552420	642880	642880
375	1881	41051	431408	470406	1437895	1437895	1672987	1672987
399	4756	103927	1090489	1189748	3633969	3633969	4228700	4228700
423	11625	255433	2683334	2926781	8944652	8944652	10408024	10408024
447	27977	614030	6445575	7031919	21483285	21483285	24998994	24998994
471	65536	1441297	15135916	16510850	50453393	50453393	58708484	58708484
495	150811	3315763	34812830	37978515	116039405	116039405	135028617	135028617
519	339981	7478672	78530602	85667592	261770033	261770033	304603271	304603271
543	753426	16570437	173981507	189800269	579930706	579930706	674830133	674830133
567	1640491	36091901	378977806	413425592	1263264431	1263264431	1469977138	1469977138

Table 43 The table shows the decomposition of the Fourier coefficients multiplying $q^{-D/24}$ in the function $\tilde{h}_{g,\frac{3}{2}}(\tau)$ into irreducible representations χ_n of McL - part II

−D	χ_9	χ_{10}	χ_{11}	χ_{12}	χ_{13}	χ_{14}	χ_{15}	χ_{16}
−9	0	0	0	0	0	0	0	0
15	0	0	0	0	0	0	0	0
39	0	0	0	0	0	0	1	0
63	0	0	0	1	0	1	1	0
87	1	1	2	2	2	2	6	4
111	7	13	10	15	14	19	21	25
135	24	54	56	75	78	81	95	130
159	129	254	244	320	334	365	398	562
183	482	975	986	1251	1328	1414	1566	2235
207	1776	3566	3536	4559	4792	5173	5607	8093
231	5969	12026	12064	15394	16274	17445	19023	27451
255	19154	38471	38380	49139	51846	55752	60533	87506
279	57809	116393	116510	148894	157282	168786	183543	265369
303	167931	337637	337390	431517	455558	489419	531575	768802
327	467083	939698	940034	1201486	1268952	1362371	1480572	2141239
351	1255917	2525910	2525280	3228953	3409380	3661782	3977741	5753503
375	3267135	6572140	6573020	8402301	8873332	9527924	10352451	14973440
399	8260107	16613698	16612158	21238355	22426902	24084770	26165020	37845752
423	20326853	40887536	40889694	52272318	55200610	59275991	64401011	93150396
447	48828286	98212448	98208856	125553959	132583074	142379305	154680790	223734839
471	114662453	230638380	230643438	294852626	311367160	334361281	363262412	525430528
495	263731843	530473486	530465432	678158814	716131132	769034337	835486965	1208473427
519	594922426	1196650730	1196662060	1529814670	1615490472	1734806216	1884740065	2726136749
543	1318037615	2651128016	2651110530	3389217725	3579004576	3843378364	4175506851	6039574829
567	2871035001	5774897738	5774922228	7382694796	7796138458	8371967563	9095496261	13155976526

Table 44 The table shows the decomposition of the Fourier coefficients multiplying $q^{-D/24}$ in the function $\tilde{h}_{g,\frac{3}{2}}(\tau)$ into irreducible representations χ_n of *McL* - part III

−D	χ_{17}	χ_{18}	χ_{19}	χ_{20}	χ_{21}	χ_{22}	χ_{23}	χ_{24}
−9	0	0	0	0	0	0	0	0
15	0	0	0	0	0	0	0	0
39	0	0	0	0	0	0	0	0
63	0	0	0	1	1	1	1	1
87	4	5	5	5	5	5	6	6
111	25	27	27	32	32	31	34	34
135	130	132	132	153	153	161	168	168
159	562	583	583	685	685	695	733	733
183	2235	2298	2298	2671	2671	2743	2898	2898
207	8093	8336	8336	9753	9753	9968	10503	10503
231	27451	28225	28225	32895	32895	33718	35576	35576
255	87506	90070	90070	105135	105135	107589	113467	113467
279	265369	272959	272959	318375	318375	326136	343976	343976
303	768802	791067	791067	923068	923068	945024	996685	996685
327	2141239	2202779	2202779	2569637	2569637	2631602	2775597	2775597
351	5753503	5919487	5919487	6906560	6906560	7071840	7458445	7458445
375	14973440	15404421	15404421	17971134	17971134	18403192	19409807	19409807
399	37845752	38936669	38936669	45427164	45427164	46516151	49059813	49059813
423	93150396	95832723	95832723	111803325	111803325	114488565	120749919	120749919
447	223734839	230181494	230181494	268547565	268547565	274989746	290027769	290027769
471	525430528	540564243	540564243	630654439	630654439	645794430	681112299	681112299
495	1208473427	1243288847	1243288847	1450509683	1450509683	1485315445	1566542113	1566542113
519	2726136749	2804662530	2804662530	3272097750	3272097750	3350637733	3533877823	3533877823
543	6039574829	6213562155	6213562155	7249168236	7249168236	7423133737	7829083804	7829083804
567	13155976526	13534944287	13534944287	15790750779	15790750779	16169749982	17054036789	17054036789

Table 45 The table shows the decomposition of the Fourier coefficients multiplying $q^{-D/24}$ in the function $\tilde{h}_{g,\frac{1}{2}}(\tau)$ into irreducible representations χ_n of HS - part I

−D	χ_1	χ_2	χ_3	χ_4	χ_5	χ_6	χ_7	χ_8
−1	−1	0	0	0	0	0	0	0
23	0	0	0	0	0	0	0	0
47	0	0	0	0	0	0	0	0
71	0	0	0	1	0	0	1	0
95	0	2	3	2	5	5	3	6
119	0	2	8	19	14	14	21	24
143	0	16	44	80	90	90	92	131
167	4	50	174	357	339	339	407	515
191	7	203	681	1334	1372	1372	1521	2039
215	33	672	2372	4787	4720	4720	5435	7112
239	99	2292	7929	15766	15887	15887	17929	23777
263	330	7047	24733	49595	49392	49392	56348	74177
287	948	21207	73979	147695	148048	148048	167874	221911
311	2759	60245	211115	422628	422054	422054	480211	633338
335	7516	166233	581198	1161707	1162651	1162651	1320213	1743556
359	20108	440909	1543852	3088701	3087208	3087208	3509785	4631476
383	51591	1137213	3978753	7955779	7958171	7958171	9040892	11936173
407	129518	2846173	9963420	19929384	19925689	19925689	22646724	29890200
431	315659	6949871	24321125	48638189	48643912	48643912	55271192	72963281
455	753088	16560006	57964332	115934578	115925926	115925926	131743166	173892768
479	1754960	38620903	135165363	270321371	270334509	270334509	307184590	405495839
503	4011289	88231069	308818831	617651188	617631664	617631664	701874719	926456288
527	8990981	197828069	692381433	1384742351	1384771352	1384771352	1573573454	2077143971
551	19807924	435736187	1525098819	3050227006	3050184616	3050184616	3466163569	4575295928
575	42910275	944081846	3304250864	6608457916	6608519893	6608519893	7509616830	9912751919

Table 46 The table shows the decomposition of the Fourier coefficients multiplying $q^{-D/24}$ in the function $\tilde{h}_{g,\frac{1}{2}}(\tau)$ into irreducible representations χ_n of *HS* - part II

−D	χ_9	χ_{10}	χ_{11}	χ_{12}	χ_{13}	χ_{14}	χ_{15}	χ_{16}
−1	0	0	0	0	0	0	0	0
23	0	0	1	0	0	0	0	0
47	0	1	0	0	1	0	0	0
71	3	2	3	4	3	3	3	2
95	14	19	15	15	18	17	17	22
119	79	85	94	94	95	106	106	117
143	375	432	413	412	449	478	478	575
167	1585	1739	1776	1777	1887	2061	2061	2396
191	6059	6780	6708	6709	7219	7803	7803	9247
215	21454	23769	23898	23897	25544	27789	27789	32642
239	71118	79177	78950	78950	84686	91864	91864	108441
263	222914	247447	247836	247836	265365	288353	288353	339513
287	665120	739461	738796	738797	791857	859693	859693	1013725
311	1901066	2111639	2112736	2112736	2263145	2458353	2458353	2896427
335	5228981	5811159	5809370	5809369	6225096	6760018	6760018	7968570
359	13897177	15439550	15442403	15442404	16544170	17969139	17969139	21175541
383	35804287	39785466	39780944	39780943	42624415	46290660	46290660	54560442
407	89677291	99637115	99644150	99644149	106758449	115949075	115949075	136648460
431	218879656	243206606	243195756	243195757	260571613	282991729	282991729	333534637
455	521694040	579649779	579666270	579666271	621063719	674519766	674519766	794956158
479	1216464055	1351642677	1351617767	1351617768	1448172795	1572792424	1572792424	1853668197
503	2779404150	3088203718	3088240865	3088240863	3308813035	3593587213	3593587213	4235268766
527	6231379962	6923790672	6923735648	6923735648	7418312272	8056712554	8056712554	9495455515
551	13725964622	15251021256	15251101926	15251101928	16340430775	17746732420	17746732420	20915725781
575	29738144692	33042457875	33042340266	33042340266	35402559005	38449273233	38449273233	45315309735

Table 47 The table shows the decomposition of the Fourier coefficients multiplying $q^{-D/24}$ in the function $\tilde{h}_{g,\frac{1}{2}}(\tau)$ into irreducible representations χ_n of *HS* - part III

−D	χ_{17}	χ_{18}	χ_{19}	χ_{20}	χ_{21}	χ_{22}	χ_{23}	χ_{24}
−1	0	0	0	0	0	0	0	0
23	0	0	0	0	0	0	0	0
47	0	0	0	1	1	1	1	1
71	4	5	6	4	6	7	9	9
95	26	27	35	42	40	53	56	67
119	162	164	202	215	221	286	317	361
143	740	756	941	1054	1046	1371	1487	1743
167	3180	3230	4009	4374	4396	5744	6289	7290
191	12073	12281	15273	16866	16834	22047	24028	28003
215	42967	43635	54213	59519	59591	77977	85159	99000
239	142122	144411	179524	197695	197583	258692	282195	328520
263	445985	453032	563011	618937	619159	810439	884610	1029079
287	1329897	1351104	1679391	1847957	1847627	2418829	2639284	3071601
311	3802615	3862871	4800963	5280034	5280632	6912578	7544029	8777736
335	10457085	10623310	13203972	14526071	14525165	19015093	20749740	24146350
359	27795624	28236528	35094660	38601412	38602966	50534149	55147824	64169990
383	71606337	72743550	90413520	99459154	99456838	130198910	142079975	165332469
407	179357824	182203997	226459775	249099125	249102903	326096807	355863082	414090303
431	437754013	444703911	552723198	608005819	608000215	795929685	868568813	1010705622
455	1043395661	1059955647	1317413812	1449139547	1449148385	1897063405	2070216248	2408967373
479	2432916045	2471536933	3071871642	3379082403	3379069491	4423514093	4827235467	5617163685
503	5558825615	5647056684	7018707863	7720543541	7720563319	10106911327	11029385777	12834168544
527	12462733462	12660561531	15735791983	17309423366	17309394714	22659582306	24727692056	28774079262
551	27451966739	27887702980	34661549588	38127628080	38127670972	49912570342	54468122422	63381031865
575	59476232959	60420314767	75096290859	82606031374	82605970008	108138748123	118008497342	137319059194

Table 48 The table shows the decomposition of the Fourier coefficients multiplying $q^{-D/24}$ in the function $\tilde{h}_{g,\frac{3}{2}}(\tau)$ into irreducible representations χ_n of HS - part I

−D	χ_1	χ_2	χ_3	χ_4	χ_5	χ_6	χ_7	χ_8
−9	1	0	0	0	0	0	0	0
15	0	0	0	0	0	0	0	1
39	0	0	1	0	0	0	1	0
63	0	0	0	0	1	1	0	1
87	1	0	2	3	1	1	3	3
111	0	2	6	10	13	13	12	21
135	1	4	25	54	45	45	61	73
159	2	36	118	217	232	232	251	349
183	8	117	436	882	850	850	1000	1298
207	19	455	1585	3138	3191	3191	3567	4768
231	76	1504	5333	10710	10609	10609	12169	15972
255	224	4906	17078	34000	34168	34168	38662	51207
279	680	14653	51536	103287	102985	102985	117337	154628
303	1937	42824	149641	298883	299362	299362	339698	448887
327	5448	118790	416242	832940	832131	832131	946482	1248628
351	14492	319869	1119026	2237183	2238459	2238459	2542349	3357136
375	37900	831414	2911159	5823587	5821517	5821517	6617566	8733350
399	95506	2103267	7359765	14717082	14720250	14720250	16724285	22079147
423	235400	5173804	18111488	36226467	36221486	36221486	41165993	54334528
447	564823	12431424	43505965	87006255	87013768	87013768	98871502	130517592
471	1327268	29188063	102164805	204337166	204325655	204325655	232200431	306494073
495	3051197	67140916	234984819	469957788	469974829	469974829	534044327	704954744
519	6885123	151446502	530077046	1060170981	1060145461	1060145461	1204737801	1590230348
543	15250485	335540316	1174370946	2348714622	2348751902	2348751902	2668997408	3523111975
567	33223874	730874824	2558094287	5116227145	5116172310	5116172310	5813889540	7674283501

Table 49 The table shows the decomposition of the Fourier coefficients multiplying $q^{-D/24}$ in the function $\tilde{h}_{g,\frac{3}{2}}(\tau)$ into irreducible representations χ_n of *HS* - part II

−D	χ_9	χ_{10}	χ_{11}	χ_{12}	χ_{13}	χ_{14}	χ_{15}	χ_{16}
−9	0	0	0	0	0	0	0	0
15	0	0	0	0	0	0	0	0
39	0	0	0	0	0	0	0	0
63	1	1	0	0	1	1	1	3
87	7	7	10	10	9	10	10	10
111	46	54	48	48	56	54	54	71
135	227	239	254	254	268	299	299	344
159	990	1118	1089	1089	1183	1265	1265	1520
183	3908	4310	4367	4367	4655	5071	5071	5937
207	14181	15813	15709	15709	16879	18296	18296	21661
231	48025	53238	53426	53426	57171	62148	62148	73107
255	153185	170410	170086	170086	182391	197917	197917	233542
279	464346	515565	516126	516126	552755	600589	600589	707419
303	1345538	1495628	1494707	1494707	1601901	1739300	1739300	2050680
327	3747019	4162388	4163910	4163910	4460701	4845163	4845163	5709189
351	10068963	11189191	11186751	11186751	11986913	13017514	13017514	15344148
375	26203117	29112145	29116039	29116039	31194087	33880203	33880203	39927180
399	66230865	73593695	73587635	73587635	78846635	85629401	85629401	100925428
423	163012176	181118490	181127903	181127903	194061566	210766813	210766813	248396323
447	391537803	435051059	435036721	435036721	466117227	506225002	506225002	596634205
471	919501026	1021654112	1021675875	1021675875	1094643301	1188857894	1188857894	1401136588
495	2114832583	2349834128	2349801630	2349801630	2517659061	2734316595	2734316595	3222614325
519	4770733874	5300784874	5300833219	5300833219	5679443225	6168239640	6168239640	7269672253
543	10569264743	11743672359	11743601341	11743601341	12582461650	13665284242	13665284242	16105571907
567	23022947110	25580985954	25581089850	25581089850	27408265179	29767082573	29767082573	35082549764

Table 50 The table shows the decomposition of the Fourier coefficients multiplying $q^{-D/24}$ in the function $\tilde{h}_{g,\frac{3}{2}}(\tau)$ into irreducible representations χ_n of HS - part III

−D	χ_{17}	χ_{18}	χ_{19}	χ_{20}	χ_{21}	χ_{22}	χ_{23}	χ_{24}
−9	0	0	0	0	0	0	0	0
15	0	0	0	0	0	0	0	0
39	1	1	0	0	0	0	1	0
63	2	2	2	4	2	4	4	5
87	19	19	19	20	20	28	32	34
111	89	93	111	131	125	168	181	213
135	466	468	580	622	628	822	904	1043
159	1973	2007	2489	2770	2750	3612	3932	4587
183	7859	7976	9884	10828	10850	14204	15532	18023
207	28328	28783	35784	39464	39398	51610	56267	65550
231	96152	97662	121313	133278	133362	174568	190608	221641
255	306248	311150	386726	425729	425535	557162	607872	707534
279	929020	943669	1172777	1289537	1289797	1688384	1842742	2143915
303	2690703	2733527	3397533	3738214	3737680	4893216	5339398	6213691
327	7494835	7613625	9462567	10407432	10408176	13625020	14869371	17301389
351	20136861	20456746	25425987	27970908	27969538	36615272	39955933	46495887
375	52408277	53239679	66170567	72784072	72786002	95282852	103981276	120993550
399	132459084	134562341	167247845	183978541	183975195	240841718	262820443	305831383
423	326029062	331202866	411649683	452805738	452810502	592768390	646874829	752720667
447	783069157	795500581	988727552	1087614155	1087606345	1423778908	1553720168	1807974775
471	1839012867	1868200964	2321978347	2554155967	2554167059	3343632878	3648816574	4245880168
495	4229650434	4296791328	5340477195	5874555816	5874538248	7690311850	8392189392	9765479321
519	9541491409	9692937895	12047321308	13252008175	13252033063	17348106814	18931489848	22029335806
543	21138496831	21474037147	26690041661	29359113857	29359075739	38433714858	41941519745	48804725354
567	46045944874	46776819698	58138785609	63952566006	63952619706	83719772698	91360913541	106310810073

Table 51 The table shows the decomposition of the Fourier coefficients multiplying $q^{-D/24}$ in the function $\tilde{h}_{g,\frac{1}{2}}(\tau)$ into irreducible representations χ_n of $U_6(2)$ - part I

−D	χ_1	χ_2	χ_3	χ_4	χ_5	χ_6	χ_7	χ_8	χ_9	χ_{10}
−1	−1	0	0	0	0	0	0	0	0	0
23	0	0	0	0	0	0	0	0	0	1
47	0	0	0	0	0	0	0	0	0	0
71	0	0	0	0	0	0	0	0	1	1
95	0	1	1	0	0	0	0	1	1	2
119	0	1	2	1	0	2	2	0	4	5
143	0	3	5	2	1	3	3	3	7	9
167	1	4	10	8	3	11	13	10	22	27
191	0	8	28	17	18	33	32	32	56	61
215	2	15	67	57	54	99	112	103	172	180
239	3	36	187	156	196	280	312	330	475	487
263	9	73	477	461	592	807	971	998	1394	1411
287	12	179	1318	1279	1803	2265	2723	2936	3892	3914
311	33	404	3451	3556	5080	6244	7741	8287	10843	10878
335	66	1029	9188	9534	14074	16751	20811	22680	29067	29109
359	160	2473	23558	25066	37214	43863	55156	60008	76443	76503
383	334	6120	59837	63872	96109	111842	140844	154261	194953	195033
407	795	14678	147655	159223	240240	278362	352289	385653	486133	486240
431	1763	35220	358129	386854	586899	676374	856699	940408	1181472	1181610
455	4088	82414	848112	920162	1397738	1607611	2040543	2239707	2810456	2810645
479	9121	190679	1971916	2141266	3260343	3741059	4750527	5220237	6541054	6541292
503	20500	431992	4492407	4887625	7446731	8536684	10850420	11923091	14931553	14931866
527	45009	964995	10058179	10947884	16697919	19121267	24309087	26726129	33447564	33447965
551	98354	2117328	22125220	24103804	36775178	42092946	53536679	58860381	73643280	73643797
575	210968	4579167	47903632	52199413	79680703	91155986	115951616	127512423	159487514	159488167

Table 52 The table shows the decomposition of the Fourier coefficients multiplying $q^{-D/24}$ in the function $\tilde{h}_{g,\frac{1}{2}}(\tau)$ into irreducible representations χ_n of $U_6(2)$ - part II

−D	χ_{11}	χ_{12}	χ_{13}	χ_{14}	χ_{15}	χ_{16}	χ_{17}	χ_{18}	χ_{19}
−1	0	0	0	0	0	0	0	0	0
23	0	0	0	0	0	0	0	0	0
47	0	0	0	0	0	0	0	1	0
71	0	0	0	0	0	0	1	1	0
95	0	0	0	1	0	1	2	3	1
119	0	0	0	0	0	3	5	8	3
143	1	1	1	5	4	14	16	22	17
167	10	10	10	12	11	45	49	63	55
191	39	39	39	62	61	159	164	204	219
215	157	157	157	189	207	510	519	625	715
239	532	532	532	693	744	1642	1653	1946	2383
263	1732	1732	1732	2090	2302	4975	4992	5815	7273
287	5207	5207	5207	6424	7052	14671	14694	16983	21717
311	15071	15071	15071	18136	20097	41417	41450	47724	61506
335	41585	41585	41585	50419	55773	113397	113440	130238	169159
359	111048	111048	111048	133476	148102	299955	300016	343936	448042
383	286479	286479	286479	345208	382864	771309	771388	883312	1154153
407	718884	718884	718884	863390	958742	1928130	1928238	2206704	2886846
431	1755665	1755665	1755665	2110561	2343230	4702139	4702278	5378796	7045333
455	4187951	4187951	4187951	5027797	5584747	11198147	11198333	12806125	16782848
479	9767990	9767990	9767990	11731080	13029827	26101380	26101618	29842923	39131147
503	22326103	22326103	22326103	26797674	29770742	59614785	59615100	68152090	89385449
527	50061479	50061479	50061479	60096749	66762700	133631223	133631622	152753296	200393822
551	110289605	110289605	110289605	132364423	147060137	294300418	294300936	336394392	441360427
575	238965109	238965109	238965109	286811937	318652800	637563387	637564041	728720690	956216022

Table 53 The table shows the decomposition of the Fourier coefficients multiplying $q^{-D/24}$ in the function $\tilde{h}_{g,\frac{1}{2}}(\tau)$ into irreducible representations χ_n of $U_6(2)$ - part III

−D	χ_{20}	χ_{21}	χ_{22}	χ_{23}	χ_{24}	χ_{25}	χ_{26}	χ_{27}	χ_{28}
−1	0	0	0	0	0	0	0	0	0
23	0	0	0	0	0	0	0	0	0
47	0	0	0	0	0	0	0	0	0
71	1	1	0	1	0	0	0	0	0
95	0	2	1	2	1	1	1	1	1
119	4	8	5	7	5	4	4	4	4
143	12	24	22	24	20	22	24	24	24
167	59	89	81	85	83	86	97	97	97
191	204	281	291	296	302	345	388	388	388
215	750	942	982	991	1064	1194	1360	1360	1360
239	2420	2932	3180	3191	3473	4006	4564	4564	4564
263	7688	9022	9799	9816	10856	12481	14266	14266	14266
287	22754	26332	28983	29006	32232	37369	42728	42728	42728
311	65313	74730	82323	82356	92034	106605	122027	122027	122027
335	179162	203895	225671	225714	252683	293554	336080	336080	336080
359	476877	540363	598307	598368	671268	779662	892976	892976	892976
383	1227450	1387781	1539339	1539418	1728219	2009529	2301754	2301754	2301754
407	3076064	3471701	3851620	3851728	4327763	5031900	5764588	5764588	5764588
431	7505177	8462435	9395281	9395420	10559896	12283559	14072625	14072625	14072625
455	17892382	20159114	22383595	22383781	25167016	29274597	33540710	33540710	33540710
479	41714567	46979165	52179270	52179508	58675953	68265693	78215160	78215160	78215160
503	95319311	107312059	119196613	119196928	134058491	155968449	178705755	178705755	178705755
527	213691022	240528777	267203769	267204168	300539689	349688430	400669808	400669808	400669808
551	470720019	529752466	588518852	588519370	661990039	770250507	882558528	882558528	882558528
575	1019815583	1147598073	1274985629	1274986283	1434203044	1668815367	1912148601	1912148601	1912148601

Table 54 The table shows the decomposition of the Fourier coefficients multiplying $q^{-D/24}$ in the function $\tilde{h}_{g,\frac{1}{2}}(\tau)$ into irreducible representations χ_n of $U_6(2)$ - part IV

−D	χ_{29}	χ_{30}	χ_{31}	χ_{32}	χ_{33}	χ_{34}	χ_{35}	χ_{36}	χ_{37}
−1	0	0	0	0	0	0	0	0	0
23	0	0	0	0	0	0	0	0	0
47	1	0	0	0	0	0	0	0	0
71	1	0	0	0	0	0	1	0	0
95	4	3	1	1	1	1	3	0	1
119	11	9	4	4	4	6	12	4	11
143	43	39	26	26	26	30	47	30	44
167	140	135	105	105	105	124	181	141	208
191	504	495	430	430	430	476	641	581	769
215	1659	1647	1512	1512	1512	1679	2189	2112	2800
239	5391	5373	5106	5106	5106	5576	7089	7143	9196
263	16513	16488	15981	15981	15981	17469	21956	22579	29034
287	48944	48909	47964	47964	47964	52139	64970	67811	86401
311	138812	138765	137052	137052	137052	149008	184876	194405	247540
335	380846	380778	377745	377745	377745	409888	506964	535979	680273
359	1009253	1009164	1003898	1003898	1003898	1089307	1345063	1426142	1809476
383	2597544	2597424	2588431	2588431	2588431	2806561	3461147	3677682	4660418
407	6498438	6498279	6483169	6483169	6483169	7029317	8662775	9215765	11676421
431	15853998	15853785	15828769	15828769	15828769	17157026	21132786	22502085	28495813
455	37769072	37768797	37727937	37727937	37727937	40892977	50353789	53644469	67927686
479	88050558	88050195	87984244	87984244	87984244	95353051	117386017	125107300	158383477
503	201135952	201135486	201030242	201030242	201030242	217864013	268167991	285875651	361897629
527	450900478	450899874	450733710	450733710	450733710	488448796	601164912	640979164	811353013
551	993105129	993104358	992844595	992844595	992844595	1075912674	1324106543	1411960695	1787223481
575	2151522265	2151521277	2151118874	2151118874	2151118874	2331033239	2868612560	3059214800	3872097331

Table 55 The table shows the decomposition of the Fourier coefficients multiplying $q^{-D/24}$ in the function $\tilde{h}_{g,\frac{1}{2}}(\tau)$ into irreducible representations χ_n of $U_6(2)$ - part V

−D	χ_{38}	χ_{39}	χ_{40}	χ_{41}	χ_{42}	χ_{43}	χ_{44}	χ_{45}	χ_{46}
−1	0	0	0	0	0	0	0	0	0
23	0	0	0	0	0	0	0	0	0
47	0	0	0	0	0	0	0	0	0
71	0	0	0	0	1	1	1	0	0
95	2	2	2	3	3	3	3	3	2
119	9	11	11	12	17	17	18	21	17
143	49	55	55	67	69	69	87	94	91
167	199	228	228	265	295	295	362	409	415
191	786	881	881	1049	1088	1088	1384	1557	1631
215	2771	3105	3105	3657	3864	3864	4892	5575	5869
239	9251	10311	10311	12230	12681	12681	16220	18452	19609
263	28943	32259	32259	38155	39773	39773	50831	57985	61760
287	86560	96303	96303	114151	118296	118296	151675	172971	184763
311	247286	275091	275091	325787	338162	338162	433503	494841	528970
335	680702	756753	756753	896888	929109	929109	1192384	1360994	1456375
359	1808802	2010778	2010778	2382438	2469363	2469363	3168960	3618306	3873101
383	4661495	5180729	5180729	6140008	6359343	6359343	8164490	9321948	9982360
407	11674769	12974774	12974774	15375595	15927891	15927891	20449076	23351239	25008800
431	28498396	31668560	31668560	37532560	38869434	38869434	49911204	56994271	61050043
455	67923794	75478392	75478392	89450970	92643752	92643752	118961732	135851564	145527486
479	158389396	175998083	175998083	208588414	216007314	216007314	277390339	316772980	339359186
503	361888871	402117862	402117862	476572073	493536047	493536047	633786948	723786642	775414671
527	811366088	901543092	901543092	1068489749	1106464669	1106464669	1420941137	1622719280	1738525374
551	1787204411	1985829461	1985829461	2353547126	2437220934	2437220934	3129929312	3574428126	3829570352
575	3872125240	4302423193	4302423193	5099152355	5280310504	5280310504	6781185810	7744222864	8297132508

Table 56 The table shows the decomposition of the Fourier coefficients multiplying $q^{-D/24}$ in the function $\tilde{h}_{g,\frac{3}{2}}(\tau)$ into irreducible representations χ_n of $U_6(2)$ - part I

−D	χ_1	χ_2	χ_3	χ_4	χ_5	χ_6	χ_7	χ_8	χ_9	χ_{10}
−9	1	0	0	0	0	0	0	0	0	0
15	0	0	1	0	0	0	0	0	0	0
39	0	0	0	1	0	0	0	0	0	0
63	0	0	1	0	0	1	0	0	0	0
87	1	0	1	1	0	1	0	0	1	1
111	0	0	4	1	0	2	0	0	2	2
135	1	0	5	4	1	5	2	0	5	5
159	1	2	14	8	4	13	6	7	14	14
183	2	2	24	19	10	30	24	20	43	43
207	2	8	62	40	42	79	66	66	113	113
231	5	16	140	124	128	213	228	223	342	342
255	10	54	379	334	423	585	648	693	964	964
279	17	107	946	928	1242	1639	1952	2037	2782	2782
303	30	299	2574	2556	3642	4502	5434	5889	7697	7697
327	64	716	6655	6947	10035	12149	15072	16267	21026	21026
351	122	1817	17366	18206	27087	31983	39848	43501	55450	55450
375	287	4447	43965	47011	70199	82214	103514	112936	143220	143220
399	608	10973	109822	117781	177721	206063	259946	284997	359275	359275
423	1400	26176	267074	288683	436768	504591	639120	700427	881263	881263
447	3111	62208	638341	691001	1049704	1207687	1530976	1681362	2109987	2109987
471	7070	144100	1491474	1619959	2463620	2829836	3593518	3946210	4947675	4947675
495	15666	329562	3422339	3719775	5667740	6498220	8255052	9073094	11363145	11363145
519	34864	738943	7702776	8384938	12782104	14643912	18617246	20462399	25615443	25615443
543	75899	1632596	17046286	18562586	28320987	32418588	41222644	45326175	56711601	56711601
567	164118	3545538	37091514	40418996	61684063	70582578	89782368	98720482	123491729	123491729

Table 57 The table shows the decomposition of the Fourier coefficients multiplying $q^{-D/24}$ in the function $\tilde{h}_{g,\frac{3}{2}}(\tau)$ into irreducible representations χ_n of $U_6(2)$ - part II

−D	χ_{11}	χ_{12}	χ_{13}	χ_{14}	χ_{15}	χ_{16}	χ_{17}	χ_{18}	χ_{19}
−9	0	0	0	0	0	0	0	0	0
15	0	0	0	0	0	0	0	0	0
39	0	0	0	0	0	0	0	0	0
63	0	0	0	0	0	0	0	0	0
87	0	0	0	0	0	0	0	0	1
111	0	0	0	0	0	2	2	4	4
135	1	1	1	0	0	8	8	10	9
159	5	5	5	11	9	30	30	40	44
183	26	26	26	30	32	100	100	127	139
207	97	97	97	136	143	350	350	416	497
231	363	363	363	436	475	1106	1106	1310	1596
255	1172	1172	1172	1484	1607	3458	3458	4034	5088
279	3638	3638	3638	4370	4842	10248	10248	11856	15102
303	10611	10611	10611	12975	14297	29442	29442	33922	43783
327	29814	29814	29814	35856	39741	81284	81284	93406	121084
351	80286	80286	80286	97039	107493	217714	217714	249613	325236
375	209675	209675	209675	251918	279626	564586	564586	646742	844320
399	530506	530506	530506	638563	708547	1425020	1425020	1631017	2133713
423	1307537	1307537	1307537	1570043	1743695	3502378	3502378	4006776	5246147
447	3142048	3142048	3142048	3775558	4192520	8406902	8406902	9614460	12599677
471	7383374	7383374	7383374	8863180	9845654	19730666	19730666	22560343	29576649
495	16985215	16985215	16985215	20394871	22654755	45366836	45366836	51864617	68021751
519	38326695	38326695	38326695	46000805	51105811	102311008	102311008	116954636	153417371
543	84919350	84919350	84919350	101933701	113244297	226631616	226631616	259049862	339876627
567	185003006	185003006	185003006	222026326	246680793	493602894	493602894	564184563	740284029

Table 58 The table shows the decomposition of the Fourier coefficients multiplying $q^{-D/24}$ in the function $\tilde{h}_{g,\frac{3}{2}}(\tau)$ into irreducible representations χ_n of $U_6(2)$ - part III

−D	χ_{20}	χ_{21}	χ_{22}	χ_{23}	χ_{24}	χ_{25}	χ_{26}	χ_{27}	χ_{28}
−9	0	0	0	0	0	0	0	0	0
15	0	0	0	0	0	0	0	0	0
39	0	1	0	0	0	0	0	0	0
63	0	1	0	0	0	0	0	0	0
87	1	3	1	1	0	1	0	0	0
111	2	7	4	4	2	4	3	3	3
135	9	20	15	15	12	11	13	13	13
159	32	60	55	55	48	59	63	63	63
183	142	204	193	193	196	219	243	243	243
207	477	630	662	662	698	801	909	909	909
231	1672	2051	2165	2165	2352	2689	3057	3057	3057
255	5228	6225	6779	6779	7442	8627	9834	9834	9834
279	15977	18566	20286	20286	22548	26013	29759	29759	29759
303	46068	52957	58400	58400	65100	75588	86454	86454	86454
327	128664	146652	161869	161869	181184	210189	240627	240627	240627
351	345074	391657	433851	433851	486308	565195	647221	647221	647221
375	899003	1017030	1126986	1126986	1265124	1470206	1683993	1683993	1683993
399	2270753	2564491	2845570	2845570	3196124	3717138	4257972	4257972	4257972
423	5591173	6305860	6998318	6998318	7865522	9147020	10479406	10479406	10479406
447	13425927	15130943	16801605	16801605	18887958	21972919	25174005	25174005	25174005
471	31535156	35518958	39444133	39444133	44354258	51598009	59118234	59118234	59118234
495	72522016	81656325	90701880	90701880	102004268	118679661	135979088	135979088	135979088
519	163610220	184167767	204577702	204577702	230098050	267714806	306745221	306745221	306745221
543	362451055	407928442	453185010	453185010	509745480	593117732	679593708	679593708	679593708
567	789548632	888501021	987096186	987096186	1110356978	1291964951	1480349568	1480349568	1480349568

Table 59 The table shows the decomposition of the Fourier coefficients multiplying $q^{-D/24}$ in the function $\tilde{h}_{g,\frac{3}{2}}(\tau)$ into irreducible representations χ_n of $U_6(2)$ - part IV

−D	χ_{29}	χ_{30}	χ_{31}	χ_{32}	χ_{33}	χ_{34}	χ_{35}	χ_{36}	χ_{37}
−9	0	0	0	0	0	0	0	0	0
15	0	0	0	0	0	0	0	0	0
39	0	0	0	0	0	0	0	0	0
63	1	1	0	0	0	0	1	0	0
87	2	2	0	0	0	1	2	0	2
111	8	8	3	3	3	4	8	3	5
135	24	24	13	13	13	17	35	16	32
159	95	95	69	69	69	77	118	89	123
183	322	322	267	267	267	306	424	367	517
207	1124	1124	1011	1011	1011	1113	1471	1390	1822
231	3633	3633	3411	3411	3411	3763	4830	4796	6268
255	11444	11444	11019	11019	11019	12011	15140	15497	19864
279	34164	34164	33372	33372	33372	36393	45506	47238	60491
303	98541	98541	97104	97104	97104	105478	131014	137526	174934
327	272918	272918	270372	270372	270372	293697	363656	383784	487925
351	732055	732055	727612	727612	727612	789280	975053	1033033	1310274
375	1901107	1901107	1893498	1893498	1893498	2053906	2534016	2690675	3411830
399	4801562	4801562	4788741	4788741	4788741	5191571	6399104	6805582	8621691
423	11807813	11807813	11786521	11786521	11786521	12777678	15741552	16756387	21225130
447	28351578	28351578	28316673	28316673	28316673	30690911	37794342	40258951	50975995
471	66557162	66557162	66500652	66500652	66500652	72075158	88736216	94560774	119725056
495	153056164	153056164	152965722	152965722	152965722	165771974	204055733	217516112	275355244
519	345212969	345212969	345069777	345069777	345069777	373954834	460266342	490719721	621183917
543	764740650	764740650	764516151	764516151	764516151	828475182	1019607264	1087225658	1376174113
567	1665696713	1665696713	1665348005	1665348005	1665348005	1804658930	2220884907	2368380807	2997763988

Table 60 The table shows the decomposition of the Fourier coefficients multiplying $q^{-D/24}$ in the function $\tilde{h}_{g,\frac{3}{2}}(\tau)$ into irreducible representations χ_n of $U_6(2)$ - part V

−D	χ_{38}	χ_{39}	χ_{40}	χ_{41}	χ_{42}	χ_{43}	χ_{44}	χ_{45}	χ_{46}
−9	0	0	0	0	0	0	0	0	0
15	0	0	0	0	0	0	0	0	0
39	0	0	0	0	0	0	0	0	0
63	0	0	0	1	0	0	0	0	0
87	1	1	1	1	2	2	2	2	0
111	6	7	7	9	8	8	11	9	9
135	28	32	32	38	45	45	51	58	54
159	130	145	145	175	179	179	225	250	255
183	503	566	566	662	715	715	891	1012	1044
207	1846	2060	2060	2457	2536	2536	3233	3662	3861
231	6220	6951	6951	8205	8608	8608	10951	12474	13215
255	19937	22198	22198	26322	27278	27278	34934	39779	42381
279	60355	67201	67201	79550	82728	82728	105878	120825	128899
303	175147	194787	194787	230873	239200	239200	306831	350044	374214
327	487555	542191	542191	642249	666188	666188	854421	975422	1043352
351	1310841	1457089	1457089	1726958	1788772	1788772	2296034	2621064	2805657
375	3410897	3791267	3791267	4492367	4655102	4655102	5975070	6822630	7304754
399	8623112	9583033	9583033	11357471	11762638	11762638	15102657	17244669	18468738
423	21222871	23584843	23584843	27949916	28950941	28950941	37171558	42447872	45465426
447	50979357	56648802	56648802	67138492	69528362	69528362	89282411	101955129	109216491
471	119719865	133032283	133032283	157661257	163281960	163281960	209673562	239444613	256509711
495	275362888	305972205	305972205	362631248	375524639	375524639	482245140	550717850	589999521
519	621172387	690217270	690217270	818019198	847121958	847121958	1087869446	1242355809	1330999539
543	1376190840	1529134899	1529134899	1812299925	1876700029	1876700029	2410108430	2752364303	2948823657
567	2997739281	3330883649	3330883649	3947676846	4087990850	4087990850	5249918560	5995502658	6423520020

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