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Solutions for a fractional difference boundary value problem

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Abstract

Using a variational approach and critical point theory, we investigate the existence of solutions for a fractional difference boundary value problem.
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1 Introduction

In this work, using variational methods and critical point theory, we study the fractional difference boundary value problem

$$\begin{cases} {}_T\triangle_{t-1}^\nu({}_t\triangle_{\nu-1}^\nu x(t)) = f(x(t+\nu-1)), & t \in [0,T]_{\mathbb{N}_0}, \\ x(\nu-2) = [{}_t\triangle_{\nu-1}^\nu x(t)]_{t=T} = 0, \end{cases} \tag{1.1}$$

where $\nu \in (0,1)$, ${}_t\triangle_{\nu-1}^\nu$ and ${}_T\triangle_t^\nu$ are, respectively, the left fractional difference and the right fractional difference operators, $t \in [0,T]_{\mathbb{N}_0} := \{0,1,2,\dots,T\}$, and $f : \mathbb{R} \to \mathbb{R}$ is continuous.

Fractional calculus has a long history, and there is renewed interest in the study of both fractional calculus and fractional difference equations. In [1, 2], the authors discussed properties of the generalized falling function, a corresponding power rule for fractional delta-operators and the commutativity of fractional sums. A number of papers have appeared which build the theoretical foundations of discrete fractional calculus (for more details, we refer the reader to [3–8] and the references therein).

Atici and Eloe [3] considered the existence of positive solutions for the following two-point boundary value problem for a nonlinear finite fractional difference equation:

$$\begin{cases} -\triangle^\nu y(t) = f(t+\nu-1, y(t+\nu-1)), & t = 1,2,\dots,b+1, \\ y(\nu-2) = 0, & y(\nu+b+1) = 0. \end{cases} \tag{1.2}$$

In [4], the authors used the mountain pass theorem, a linking theorem, and Clark's theorem to establish the existence of multiple solutions for a fractional difference boundary value problem with a parameter. Under some suitable assumptions, they obtained some results which ensure the existence of a precise interval of parameters for which the problem admits multiple solutions. We note that there are many papers in the literature [9–18] which discuss discrete problems via variational and critical point theory.

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In [9], Tian and Henderson studied the $2n$th order nonlinear difference equation

$$\Delta^n\left(r(t-n)\Delta^n x(t-n)\right) + f\left(t, x(t)\right) = 0, \quad t \in \mathbb{Z}, \tag{1.3}$$

and established some existence results for anti-periodic solutions under various assumptions on the nonlinearity. In [10], Ye and Tang considered the second-order discrete Hamiltonian system

$$\Delta^2 u(t-1) + d(t)\left|u(t)\right|^{\mu-2} u(t) + \nabla H\left(t, u(t)\right) = 0, \quad \forall t \in \mathbb{Z},$$

and obtained an existence theorem for a nonzero T-periodic solution.

In the literature on discrete problem via critical point theory, the authors are interested in the existence of at least one solution or infinitely many solutions. The existence of a unique solution is not usually studied. In this paper, using Browder's theorem, first we present a uniqueness result in Section 3. Then a linking theorem is used to establish existence. Finally, assuming an Ambrosetti-Rabinowitz type condition, we show that problem (1.1) has many solutions if the nonlinearity is odd.

2 Preliminaries

For convenience, throughout this paper, we arrange $\sum_{i=j}^{m} x(i) = 0$ for $m < j$. We present some definitions and lemmas for discrete fractional operators.

For any integer β, let $\mathbb{N}_\beta := \{\beta, \beta + 1, \beta + 2, \ldots\}$ and $t^{(\nu)} := \Gamma(t+1)/\Gamma(t+1-\nu)$, where t and ν are determined by (1.1). We also appeal to the convention that if $t + 1 - \nu$ is a pole of the gamma function and $t + 1$ is not a pole, then $t^{(\nu)} = 0$.

Definition 2.1 (see [2, 3]) The νth fractional sum of f for $\nu > 0$ is defined by

$$\triangle_a^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \sum_{s=a}^{t-\nu} (t - s - 1)^{(\nu-1)} f(s) \quad \text{for } t \in \mathbb{N}_{a-\nu}. \tag{2.1}$$

We also define the νth fractional difference for $\nu > 0$ by $\triangle^\nu f(t) := \Delta^N \triangle^{\nu-N} f(t)$, where $t \in \mathbb{N}_{a+N-\nu}$ and $N \in \mathbb{N}$ is chosen so that $0 \le N - 1 < \nu \le N$.

Definition 2.2 (see [2, 3]) Let f be any real-valued function and $\nu \in (0, 1)$. The left discrete fractional difference and the right discrete fractional difference operators are, respectively, defined as

$$
\begin{aligned}
{}_t\triangle_a^\nu f(t) &= \triangle_t \triangle_a^{-(1-\nu)} f(t) \\
&= \frac{1}{\Gamma(1-\nu)} \triangle \sum_{s=a}^{t+\nu-1} (t-s-1)^{(-\nu)} f(s), \quad t \equiv a - \nu + 1 \,(\mathrm{mod}\,1), \\
{}_b\triangle_t^\nu f(t) &= -\triangle_b \triangle_t^{-(1-\nu)} f(t) \\
&= \frac{1}{\Gamma(1-\nu)} (-\triangle) \sum_{s=t+1-\nu}^{b} (s-t-1)^{(-\nu)} f(s), \quad t \equiv b + \nu - 1 \,(\mathrm{mod}\,1).
\end{aligned}
\tag{2.2}
$$

Definition 2.3 Suppose that X is a Banach space and $I : X \to \mathbb{R}$ is a functional defined on X. For given $x, y \in X$, assume that

$$\lim_{\varepsilon \to 0} \frac{I(x + \varepsilon y) - I(x)}{\varepsilon} \tag{2.3}$$

exists. Then I is Gateaux differentiable at x, the limit in (2.3) is called the Gateaux differential of I at x in direction y, and is denoted by $(I'(x), y)$, *i.e.*,

$$\big(I'(x), y\big) = \lim_{\varepsilon \to 0} \frac{I(x + \varepsilon y) - I(x)}{\varepsilon}. \tag{2.4}$$

Definition 2.4 (see [19, p.303]) Let X be a reflexive real Banach space and X^* its dual. The operator $L : X \to X^*$ is said to be demicontinuous if L maps strongly convergent sequences in X to weakly convergent sequences in X^*.

Lemma 2.5 (Browder theorem, see [19, Theorem 5.3.22]) *Let X be a reflexive real Banach space. Moreover, let $L : X \to X^*$ be an operator satisfying the conditions*

(i) *L is bounded and demicontinuous,*

(ii) *L is coercive, i.e., $\lim_{\|x\| \to \infty} \frac{(L(x),x)}{\|x\|} = +\infty$,*

(iii) *L is monotone on the space X, i.e., for all $x, y \in X$, we have*

$$\big(L(x) - L(y), x - y\big) \geq 0. \tag{2.5}$$

Then the equation $L(x) = f^*$ has at least one solution $x \in X$ for every $f^* \in X^*$. If, moreover, the inequality (2.5) is strict for all $x, y \in X$, $x \neq y$, then the equation $L(x) = f^*$ has precisely one solution $x \in X$ for all $f^* \in X^*$.

Definition 2.6 (see [20–22]) Let X be a real Banach space, $I \in C^1(X, \mathbb{R})$ and $c \in \mathbb{R}$. We say that I satisfies the $(PS)_c$ condition if any sequence $\{x_n\} \subset X$ such that $I(x_n) \to c$ and $I'(x_n) \to 0$ as $n \to \infty$ has a convergent subsequence.

Lemma 2.7 (Linking theorem, Rabinowitz, see [20–22]) *Let $X = Y \oplus Z$ be a Banach space with Z closed in X and $\dim Y < \infty$. Let $\rho > r > 0$, and let $z \in Z$ be such that $\|z\| = r$. Define*

$$\mathcal{M} := \big\{ u = y + \lambda z : \|u\| \leq \rho, \lambda \geq 0, y \in Y \big\}, \qquad \mathcal{N} := \big\{ u \in Z : \|u\| = r \big\},$$

$$\mathcal{M}_0 := \big\{ u = y + \lambda z : y \in Y, \|u\| = \rho \text{ and } \lambda \geq 0, \text{ or } \|u\| \leq \rho \text{ and } \lambda = 0 \big\}.$$

Let $I \in C^1(X, \mathbb{R})$ be such that

$$b := \inf_{u \in \mathcal{N}} I(u) > a := \max_{u \in \mathcal{M}_0} I(u).$$

If I satisfies the $(PS)_c$ condition with

$$c := \inf_{\gamma \in \Gamma} \max_{u \in \mathcal{M}} I\big(\gamma(u)\big), \quad \text{where } \Gamma := \big\{ \gamma \in C(\mathcal{M}, X) : \gamma|_{\mathcal{M}_0} = \text{id} \big\},$$

then c is a critical point of I.

Definition 2.8 (see [20–22]) Let X be a real Banach space and $I \in C^1(X, \mathbb{R})$. We say that I satisfies the Cerami condition ((C) condition for short) if any sequence $\{x_n\} \subset X$ such that $I(x_n)$ is bounded and $(1 + \|x_n\|)\|I'(x_n)\| \to 0$ as $n \to \infty$, there exists a subsequence of $\{x_n\}$ which is convergent in X.

Lemma 2.9 (Mountain pass theorem, see [20–22]) *Let X be a real Banach space, and let $I \in C^1(X, \mathbb{R})$ satisfy the (C) condition. If $I(\theta) = 0$ and the following conditions hold:*

(i) *there are two positive constants ρ, η and a closed linear subspace X_1 of X such that $\operatorname{codim} X_1 = l$ and $I|_{X_1 \cap \partial B_\rho} \geq \eta$, where B_ρ is an open ball of radius ρ with center θ;*

(ii) *there is a subspace X_2 with $\dim X_2 = m$, $m > l$, such that*

$$I(x) \to -\infty \quad as \ \|x\| \to \infty, x \in X_2.$$

Then I possesses at least $m - l$ distinct pairs of nontrivial critical points.

In what follows, we establish the variational framework for (1.1). Let

$$X := \left\{ x = \left(x(\nu - 1), x(\nu), \ldots, x(\nu + T - 1) \right)^+ : x(\nu + i - 1) \in \mathbb{R}, i = 0, 1, \ldots, T \right\}. \tag{2.6}$$

Then X is the $T + 1$-dimensional Hilbert space with the usual inner product and the usual norm

$$(x, z) = \sum_{t=\nu-1}^{T+\nu-1} x(t)z(t), \qquad \|x\| = \left(\sum_{t=\nu-1}^{T+\nu-1} |x(t)|^2 \right)^{\frac{1}{2}}, \quad x, z \in X. \tag{2.7}$$

For $\alpha > 1$, we define the α-norm on X: $\|x\|_\alpha = (\sum_{t=\nu-1}^{T+\nu-1} |x(t)|^\alpha)^{\frac{1}{\alpha}}$. Since $\dim X < \infty$, we see that there exist $c_{1\alpha} > 0$, $c_{2\alpha} > 0$ such that

$$c_{1\alpha}\|x\| \leq \|x\|_\alpha \leq c_{2\alpha}\|x\| \tag{2.8}$$

for all x belonging to X (or its subspace).

In view of [4, (3.4)], we can define an energy functional on X by

$$I(x) = \frac{1}{2} \sum_{t=-1}^{T} \left({}_t\triangle_{\nu-1}^\nu x(t) \right)^2 - \sum_{t=-1}^{T} F\left(x(t + \nu - 1) \right), \quad x \in X, \tag{2.9}$$

where

$$F\left(x(t + \nu - 1) \right) = \int_0^{x(t+\nu-1)} f(s)\,ds,$$

$$x(\nu - 2) = 0, \qquad \left[{}_t\triangle_{\nu-1}^\nu x(t) \right]_{t=T} = \frac{-\nu}{\Gamma(1 - \nu)} \sum_{s=\nu-1}^{T+\nu} (T + s - 1)^{(-\nu-1)} x(s) = 0.$$

Clearly, $I(\theta) = 0$. Let

$$E := \left\{ \chi = \left(x(\nu - 2), x(\nu - 1), \ldots, x(\nu + T) \right)^+ \right.$$

$$\left. \in \mathbb{R}^{T+3} : x(\nu - 2) = 0, \left[{}_t\triangle_{\nu-1}^\nu x(t) \right]_{t=T} = 0 \right\}. \tag{2.10}$$

Then, from the boundary conditions of (1.1), it is easy to see that E is isomorphic to X. In the following, when we say $x \in X$, we always imply that x can be extended to $\chi \in E$ if it is necessary. Now we claim that if $x = (x(v-1), x(v), \dots, x(v+T-1))^+ \in X$ is a critical point of I, then $\chi = (x(v-2), x(v-1), \dots, x(v+T))^+ \in E$ is precisely a solution of (1.1). Indeed, since I can be viewed as a continuously differentiable functional defined on the finite dimensional Hilbert space X, the Fréchet derivative $I'(x)$ is zero if and only if $\partial I(x)/\partial x(i) = 0$ for all $i = v-1, v, \dots, v+T-1$. From the relation between the Fréchet derivative and the Gateaux derivative, we obtain

$$
\begin{aligned}
\big(I'(x), y\big) &= \lim_{\varepsilon \to 0} \frac{I(x+\varepsilon y) - I(x)}{\varepsilon} = \frac{1}{\varepsilon}\left[\frac{1}{2} \sum_{t=-1}^{T} \big[\big({}_t\triangle_{v-1}^v\big(x(t) + \varepsilon y(t)\big)\big)^2 - \big({}_t\triangle_{v-1}^v x(t)\big)^2 \big] \right. \\
&\quad \left. - \sum_{t=-1}^{T} \big[F\big((x+\varepsilon y)(t+v-1)\big) - F\big(x(t+v-1)\big) \big] \right] \\
&= \sum_{t=-1}^{T} {}_t\triangle_{v-1}^v x(t)\, {}_t\triangle_{v-1}^v y(t) - \sum_{t=-1}^{T} f\big(x(t+v-1)\big) y(t+v-1).
\end{aligned}
\tag{2.11}
$$

Therefore, in order to obtain the existence of solutions for (1.1), we only need to study the existence of critical points of the energy functional I on X.

Next, noting Definition 2.2, for $t \in [-1, T]_{\mathbb{N}_{-1}}$, we let

$$
{}_t\triangle_{v-1}^v x(t) = \triangle \frac{1}{\Gamma(1-v)} \sum_{s=v-1}^{t-(1-v)} (t-s-1)^{(-v)} x(s) := \triangle z(t+v-1).
\tag{2.12}
$$

Then we have

$$z(v-2) = 0,$$

$$z(v-1) = \frac{1}{\Gamma(1-v)} \sum_{s=v-1}^{0-(1-v)} (-s-1)^{(-v)} x(s) = x(v-1),$$

$$z(v) = \frac{1}{\Gamma(1-v)} \sum_{s=v-1}^{1-(1-v)} (1-s-1)^{(-v)} x(s) = (1-v)x(v-1) + x(v),$$

$$
\begin{aligned}
z(v+1) &= \frac{1}{\Gamma(1-v)} \sum_{s=v-1}^{2-(1-v)} (2-s-1)^{(-v)} x(s) \\
&= \frac{(2-v)(1-v)}{2!} x(v-1) + (1-v)x(v) + x(v+1),
\end{aligned}
$$

$$\vdots$$

$$
\begin{aligned}
z(v+T-1) &= \frac{1}{\Gamma(1-v)} \sum_{s=v-1}^{T-(1-v)} (T-s-1)^{(-v)} x(s) \\
&= \frac{(T-v)(T-1-v)\cdots(1-v)}{T!} x(v-1) \\
&\quad + \frac{(T-1-v)(T-2-v)\cdots(1-v)}{(T-1)!} x(v) + \cdots \\
&\quad + (1-v)x(v+T-2) + x(v+T-1),
\end{aligned}
$$

i.e., $z = Bx$, where $z = (z(\nu - 1), z(\nu), \dots, z(\nu + T - 1))^+$, $x = (x(\nu - 1), x(\nu), \dots, x(\nu + T - 1))^+$,

$$
B = \begin{pmatrix}
1 & 0 & 0 & \cdots & 0 \\
1 - \nu & 1 & 0 & \cdots & 0 \\
\frac{(2-\nu)(1-\nu)}{2!} & 1 - \nu & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\frac{(T-\nu)(T-1-\nu)\cdots(1-\nu)}{T!} & \frac{(T-1-\nu)(T-2-\nu)\cdots(1-\nu)}{(T-1)!} & \cdots & \cdots & 1
\end{pmatrix}_{(T+1)\times(T+1)}.
$$

Clearly, $(B^{-1})^+ B^{-1}$ is a positive definite matrix. All the eigenvalues of $(B^{-1})^+ B^{-1}$ are positive. Let $\lambda_{\min}$ and $\lambda_{\max}$ denote respectively the minimum and the maximum eigenvalues of $(B^{-1})^+ B^{-1}$. Since $x = B^{-1}z$, we have

$$
\lambda_{\min} \|z\|^2 \leq \|x\|^2 = \left(z^+ \left(B^{-1} \right)^+, B^{-1} z \right) \leq \lambda_{\max} \|z\|^2. \tag{2.13}
$$

Next, let

$$
A = \begin{pmatrix}
2 & -1 & 0 & \cdots & 0 & 0 \\
-1 & 2 & -1 & \cdots & 0 & 0 \\
0 & -1 & 2 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 2 & -1 \\
0 & 0 & 0 & \cdots & -1 & 1
\end{pmatrix}_{(T+1)\times(T+1)}.
$$

By direct verification, we see that A is a positive definite matrix. Let $\eta_1, \dots, \eta_{T+1}$ be the orthonormal eigenvectors corresponding to the eigenvalues $\lambda_1, \dots, \lambda_{T+1}$ of A, where $0 < \lambda_1 < \cdots < \lambda_{T+1}$. Clearly, $X := \text{span}\{\eta_1, \dots, \eta_{T+1}\}$. Let $Y := \text{span}\{\eta_1, \dots, \eta_l\}$, $Z := \text{span}\{\eta_{l+1}, \dots, \eta_{T+1}\}$, for $l \in [1, T+1]_{\mathbb{N}_0}$. Then $X = Y \oplus Z$.

3 Main results

Now we state our main results and give their proof. For convenience, we list assumptions on f and F:

(H1) $f(\theta) \neq 0$, and there is a constant c such that $|f(x) - f(y)| \leq c|x - y|$, $\forall x, y \in X$.

(H2) There exist a constant $d > 0$ and $\mu \in (1, 2)$ such that $\limsup_{|x| \to 0} \frac{F(x)}{|x|^\mu} < d$.

(H3) $\frac{\lambda_l}{2\lambda_{\min}} x^2 \leq \int_0^x f(s)\, ds$ for all $x \in \mathbb{R}$.

(H4) There exist $\alpha > 2$ and $R > 0$ such that

$$
0 < \alpha \int_0^x f(s)\, ds \leq x f(x) \quad \text{for } |x| > R.
$$

(H5) There is a constant $\alpha > 2$ such that $\liminf_{|x| \to \infty} \frac{F(x)}{|x|^\alpha} > 0$.

(H6) There is $\gamma > 1$ such that

$$
\liminf_{|x| \to \infty} \frac{f(x)x - 2F(x)}{|x|^\gamma} > 0.
$$

(H7) $f(-x) + f(x) = 0$, $\forall x \in \mathbb{R}$.

Theorem 3.1 *Let* (H1) *hold. Then* (1.1) *has precisely one solution for* $c \in (0, \frac{\lambda_1}{\lambda_{\max}})$.

Proof We shall apply Lemma 2.5 to prove the result. From (2.11), we define the operator

$$(L(x), y) = \sum_{t=-1}^{T} {}_t \triangle_{\nu-1}^{\nu} x(t) {}_t \triangle_{\nu-1}^{\nu} y(t) - \sum_{t=-1}^{T} f(x(t+\nu-1)) y(t+\nu-1), \quad \forall x, y \in X. \qquad (3.1)$$

Clearly, if for all $y \in X$, there exists $x_0 \in X$ such that $(L(x_0), y) = 0$, then x_0 is a solution of (1.1). Let

$$(L_1(x), y) = \sum_{t=-1}^{T} {}_t \triangle_{\nu-1}^{\nu} x(t) {}_t \triangle_{\nu-1}^{\nu} y(t),$$

$$(L_2(x), y) = \sum_{t=-1}^{T} f(x(t+\nu-1)) y(t+\nu-1), \quad \forall x, y \in X.$$

We sketch the properties of L_1 and L_2. It is clear that L_1 is a linear operator, and furthermore, L_1 is bounded. Indeed, the Cauchy-Schwarz inequality enables us to obtain, notice (2.12) and (2.13),

$$\begin{aligned}
\left| (L_1(x), y) \right| &\le \sum_{t=-1}^{T} \left| {}_t \triangle_{\nu-1}^{\nu} x(t) \right| \left| {}_t \triangle_{\nu-1}^{\nu} y(t) \right| \\
&\le \left(\sum_{t=-1}^{T} \left| {}_t \triangle_{\nu-1}^{\nu} x(t) \right|^2 \right)^{\frac{1}{2}} \left(\sum_{t=-1}^{T} \left| {}_t \triangle_{\nu-1}^{\nu} y(t) \right|^2 \right)^{\frac{1}{2}} \\
&\le \frac{\lambda_{T+1}}{\lambda_{\min}} \|x\| \|y\| < \infty, \quad x, y \in X.
\end{aligned} \qquad (3.2)$$

Consequently, L_1 is continuous on X. Next, we show that L_2 is bounded and continuous. Let $y = \theta$ in (H1) and $|f(\theta)| = c_1 > 0$. Then we have from (H1)

$$\left| f(x) \right| \le c|x| + c_1, \quad \forall x \in X. \qquad (3.3)$$

From (3.3), the definition of L_2, and the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
\left| (L_2(x), y) \right| &\le \sum_{t=0}^{T} \left| f(x(t+\nu-1)) \right| \left| y(t+\nu-1) \right| \\
&\le \sum_{t=0}^{T} \left(c \left| x(t+\nu-1) \right| + c_1 \right) \left| y(t+\nu-1) \right| \\
&\le c\|x\|\|y\| + c_1\sqrt{T+1}\|y\| < \infty, \quad \forall x, y \in X,
\end{aligned} \qquad (3.4)$$

$$\begin{aligned}
\left| (L_2(x_1) - L_2(x_2), y) \right| &\le \sum_{t=0}^{T} \left| f(x_1(t+\nu-1)) - f(x_2(t+\nu-1)) \right| \left| y(t+\nu-1) \right| \\
&\le \sum_{t=0}^{T} c \left| x_1(t+\nu-1) - x_2(t+\nu-1) \right| \left| y(t+\nu-1) \right| \\
&\le c\|x_1 - x_2\|\|y\|, \quad \forall x, y \in X.
\end{aligned} \qquad (3.5)$$

Therefore, L_2 is bounded and continuous, as required. Hence, L is bounded and continuous, so demicontinuous.

From (3.3), notice (2.12) and (2.13), we see

$$
\begin{aligned}
\big(L(x), x\big) &= \sum_{t=-1}^{T}\big({}_t\triangle_{\nu-1}^{\nu} x(t)\big)^2 - \sum_{t=-1}^{T} f\big(x(t+\nu-1)\big)x(t+\nu-1)\\
&= \sum_{t=-1}^{T-1}\big(\triangle z(t+\nu-1)\big)^2 - \sum_{t=0}^{T} f\big(x(t+\nu-1)\big)x(t+\nu-1)\\
&\geq \frac{\lambda_1}{\lambda_{\max}}\|x\|^2 - \sum_{t=0}^{T}\big(c\big|x(t+\nu-1)\big| + c_1\big)\big|x(t+\nu-1)\big|\\
&\geq \left(\frac{\lambda_1}{\lambda_{\max}} - c\right)\|x\|^2 - c_1\sqrt{T+1}\|x\|.
\end{aligned}
\tag{3.6}
$$

Therefore, $\lim_{\|x\|\to\infty}\frac{(L(x),x)}{\|x\|} = +\infty$, *i.e.*, L is coercive on X.

Finally, we prove that L is strictly monotone. Indeed, from (H1), we have

$$
\begin{aligned}
\big(L(x) - L(y), x - y\big) &\geq \sum_{t=-1}^{T}\big({}_t\triangle_{\nu-1}^{\nu} x(t) - {}_t\triangle_{\nu-1}^{\nu} y(t)\big)^2\\
&\quad - \sum_{t=0}^{T}\big|f\big(x(t+\nu-1)\big) - f\big(y(t+\nu-1)\big)\big|\big|x(t+\nu-1) - y(t+\nu-1)\big|\\
&\geq \frac{\lambda_1}{\lambda_{\max}}\|x-y\|^2 - c\|x-y\|^2 > 0, \quad \text{for } x, y \in X \text{ and } x \neq y.
\end{aligned}
\tag{3.7}
$$

All the conditions of Lemma 2.5 are satisfied, as claimed. Hence, (1.1) has precisely one solution. This completes the proof. $\qquad\square$

Theorem 3.2 *Let* (H2)-(H4) *hold. Then* (1.1) *has at least one solution.*

Proof From (H2), there exists $\delta > 0$ with

$$
F(x) \leq d|x|^{\mu} \quad \text{for } |x| \leq \delta.
\tag{3.8}
$$

Thus, for $x \in Z$ with $\|x\| \leq \delta$, it follows from the Hölder inequality that

$$
\begin{aligned}
I(x) &= \frac{1}{2}\sum_{t=-1}^{T}\big({}_t\triangle_{\nu-1}^{\nu} x(t)\big)^2 - \sum_{t=-1}^{T} F\big(x(t+\nu-1)\big)\\
&= \frac{1}{2}\sum_{t=-1}^{T-1}\big(\triangle z(t+\nu-1)\big)^2 - \sum_{t=0}^{T} F\big(x(t+\nu-1)\big)\\
&\geq \frac{\lambda_{l+1}}{2}\|z\|^2 - \sum_{t=0}^{T} d\big|x(t+\nu-1)\big|^{\mu}\\
&\geq \frac{\lambda_{l+1}}{2\lambda_{\max}}\|x\|^2 - d\left(\sum_{t=0}^{T}\big|x(t+\nu-1)\big|^2\right)^{\frac{\mu}{2}}\left(\sum_{t=0}^{T} 1\right)^{1-\frac{\mu}{2}}\\
&= \frac{\lambda_{l+1}}{2\lambda_{\max}}\|x\|^2 - d(T+1)^{1-\frac{\mu}{2}}\|x\|^{\mu}.
\end{aligned}
\tag{3.9}
$$

By virtue of the inequality $1 < \mu < 2$, there exists $r > 0$ such that

$$b := \inf_{\|x\|=r, x \in Z} I(x) > 0. \tag{3.10}$$

From (H3), for $x \in Y$, we see

$$
\begin{aligned}
I(x) &= \frac{1}{2} \sum_{t=-1}^{T} \left({}_t\triangle_{\nu-1}^{\nu} x(t) \right)^2 - \sum_{t=-1}^{T} F\big(x(t+\nu-1)\big) \\
&= \frac{1}{2} \sum_{t=-1}^{T-1} \left(\Delta z(t+\nu-1) \right)^2 - \sum_{t=0}^{T} F\big(x(t+\nu-1)\big) \\
&\leq \frac{\lambda_l}{2\lambda_{\min}} \|x\|^2 - \sum_{t=0}^{T} F\big(x(t+\nu-1)\big) \\
&= \sum_{t=0}^{T} \left[\frac{\lambda_l}{2\lambda_{\min}} x^2(t+\nu-1) - F\big(x(t+\nu-1)\big) \right] \leq 0.
\end{aligned} \tag{3.11}
$$

From (H4), we see that there exist $c_2, c_3 > 0$ such that

$$\int_0^x f(s)\, ds \geq c_2 |x|^\alpha - c_3, \quad \forall x \in \mathbb{R}. \tag{3.12}$$

Hence, for $x \in X$, we find

$$
\begin{aligned}
I(x) &= \frac{1}{2} \sum_{t=-1}^{T} \left({}_t\triangle_{\nu-1}^{\nu} x(t) \right)^2 - \sum_{t=-1}^{T} F\big(x(t+\nu-1)\big) \\
&= \frac{1}{2} \sum_{t=-1}^{T-1} \left(\Delta z(t+\nu-1) \right)^2 - \sum_{t=0}^{T} F\big(x(t+\nu-1)\big) \\
&\leq \frac{\lambda_{T+1}}{2\lambda_{\min}} \|x\|^2 - c_2 \sum_{t=0}^{T} \big| x(t+\nu-1) \big|^\alpha + c_3(T+1).
\end{aligned} \tag{3.13}
$$

Set $z := r\frac{\eta_{l+1}}{\|\eta_{l+1}\|}$ with $r > 0$ is given in (3.10). For $Y \oplus \mathbb{R}_z \subset X$, (2.8) holds true. This, together with (3.13), implies

$$I(x) \leq \frac{\lambda_{T+1}}{2\lambda_{\min}} \|x\|^2 - c_2 c_{1\alpha}^\alpha \|x\|^\alpha + c_3(T+1).$$

Since $\alpha > 2$, we obtain

$$\lim_{\|x\| \to \infty, x \in Y \oplus \mathbb{R}_z} I(x) = -\infty. \tag{3.14}$$

Define

$$\mathcal{M} := \big\{ x = y + \lambda z : \|x\| \leq \rho, \lambda \geq 0, y \in Y \big\},$$

$$\mathcal{M}_0 := \big\{ x = y + \lambda z : y \in Y, \|x\| = \rho \text{ and } \lambda \geq 0, \text{ or } \|x\| \leq \rho \text{ and } \lambda = 0 \big\}.$$

Since $z \in Z$ and then $I(z) \geq b > 0$, (3.11) and (3.14) guarantee that there is $\rho > r$ such that

$$a := \max_{x \in \mathcal{M}_0} I(x) \leq 0.$$

It remains to prove that I satisfies the $(PS)_c$ condition. This will be the case if we show that any sequence $\{x_n\}_{n=1}^{\infty} \subset X$ such that

$$d := \sup_n I(x_n) < \infty, \qquad I'(x_n) \to 0,$$

contains a convergent subsequence. Note that $\dim X < \infty$, so we only need to show the boundedness of $\{x_n\}_{n=1}^{\infty}$. Take $\beta > 0$ such that $\beta^{-1} \in (2, \alpha)$ for n large enough, and (H4), (3.12) and (2.8) enable us to obtain

$$d + \|x_n\| \geq I(x_n) - \frac{1}{\beta}\left(I'(x_n), x_n\right)$$

$$= \left(\frac{1}{2} - \frac{1}{\beta}\right) \sum_{t=-1}^{T} \left({}_t\triangle_{\nu-1}^{\nu} x_n(t)\right)^2$$

$$+ \sum_{t=-1}^{T} \left(\frac{1}{\beta} f\big(x_n(t + \nu - 1)\big)x_n(t + \nu - 1) - F\big(x_n(t + \nu - 1)\big)\right)$$

$$\geq \left(\frac{1}{2} - \frac{1}{\beta}\right)\frac{\lambda_1}{\lambda_{\max}}\|x_n\|^2 + \sum_{t=0}^{T}\left(\frac{\alpha}{\beta} - 1\right)F\big(x_n(t + \nu - 1)\big)$$

$$\geq \left(\frac{1}{2} - \frac{1}{\beta}\right)\frac{\lambda_1}{\lambda_{\max}}\|x_n\|^2 + \sum_{t=0}^{T}\left(\frac{\alpha}{\beta} - 1\right)\big(c_2\big|x_n(t + \nu - 1)\big|^{\alpha} - c_3\big)$$

$$\geq \left(\frac{1}{2} - \frac{1}{\beta}\right)\frac{\lambda_1}{\lambda_{\max}}\|x_n\|^2 + \left(\frac{\alpha}{\beta} - 1\right)c_2 c_{1\alpha}^{\alpha}\|x_n\|^{\alpha} - \left(\frac{\alpha}{\beta} - 1\right)(T+1)c_3.$$

Since $\alpha > 2$ and $(\frac{\alpha}{\beta} - 1) > 0$, we see that $\{x_n\}_{n=1}^{\infty}$ is bounded.

Thus the functional I satisfies all the conditions of Lemma 2.7, and then I has a critical point, and (1.1) has at least one solution. This completes the proof. □

Theorem 3.3 *Let* (H2), (H5)-(H7) *hold. Then* (1.1) *has at least $m - l$ solutions.*

Proof We shall utilize Lemma 2.9 to prove the result. If $X_1 = Z = \text{span}\{\eta_{l+1}, \ldots, \eta_{T+1}\}$, we see $\text{codim } X_1 = l$. From (H2), noting (3.9), we can take $\rho = (dd_1\lambda_{\max}\lambda_{l+1}^{-1}(T+1)^{1-\frac{\mu}{2}})^{\frac{1}{2-\mu}}$ so that $\rho \leq \delta$, where $d_1 > 2$. Therefore,

$$I_{X_1 \cap \partial B_\rho} \geq \big(dd_1\lambda_{\max}\lambda_{l+1}^{-1}(T+1)^{1-\frac{\mu}{2}}\big)^{\frac{\mu}{2-\mu}}\left[\frac{\lambda_{l+1}}{2\lambda_{\max}}dd_1\lambda_{\max}\lambda_{l+1}^{-1}(T+1)^{1-\frac{\mu}{2}} - d(T+1)^{1-\frac{\mu}{2}}\right]$$

$$= \left(\frac{d_1}{2} - 1\right)d\big(dd_1\lambda_{\max}\lambda_{l+1}^{-1}(T+1)^{1-\frac{\mu}{2}}\big)^{\frac{\mu}{2-\mu}}(T+1)^{1-\frac{\mu}{2}}.$$

Thus (i) of Lemma 2.9 holds true.

Choose $X_2 := \text{span}\{\eta_1, \ldots, \eta_m\}$, where $m > l$, and $\dim X_2 = m$. From (H5), we see that there exist $c_4, c_5 > 0$ such that

$$F(x) \geq c_4|x|^{\alpha} - c_5, \quad \forall x \in \mathbb{R}.$$

Therefore, from (3.13) and (2.8), we arrive at

$$-I(x) \geq -\frac{\lambda_m}{2\lambda_{\min}}\|x\|^2 + c_4 \sum_{t=0}^{T}\left|x(t+\nu-1)\right|^\alpha - c_5(T+1)$$

$$\geq -\frac{\lambda_m}{2\lambda_{\min}}\|x\|^2 + c_4 c_{1\alpha}^\alpha \|x\|^\alpha - c_5(T+1).$$

Since $\alpha > 2$, $I(x) \to -\infty$ as $\|x\| \to \infty$, $x \in X_2$. Thus (ii) of Lemma 2.9 holds true.

Finally, we prove that I satisfies the (C) condition. Let $\{x_n\} \subset X$ be such that for some $M_1 > 0$,

$$\left|I(x_n)\right| \leq M_1, \qquad \left(1+\|x_n\|\right)\left\|I'(x_n)\right\| \to 0 \quad \text{as } n \to \infty.$$

We claim that $\|x_n\|$ is bounded. Otherwise, suppose that $\|x_n\| \to \infty$ as $n \to \infty$. It is easy to see that for any $n \in \mathbb{N}$, there exists M_2 such that

$$2I(x_n) - \left(I'(x_n), x_n\right) \leq M_2.$$

On the other hand, from (H6), there exist $c_6, c_7 > 0$ such that

$$f(x)x - 2F(x) \geq c_6|x|^\gamma - c_7, \quad \forall x \in \mathbb{R}.$$

Consequently, from (2.8),

$$2I(x_n) - \left(I'(x_n), x_n\right) = \sum_{t=0}^{T}\left[f\left(x_n(t+\nu-1)\right)x_n(t+\nu-1) - 2F\left(x_n(t+\nu-1)\right)\right]$$

$$\geq \sum_{t=0}^{T}\left(c_6\left|x_n(t+\nu-1)\right|^\gamma - c_7\right) \geq c_6 c_{1\gamma}^\gamma \|x_n\|^\gamma - c_7(T+1).$$

Let $n \to \infty$, and we get a contradiction.

It is easy to see that I is even and $I(\theta) = 0$. Thus all the conditions of Lemma 2.9 are satisfied, and (1.1) has at least $m - l$ solutions. The proof is complete. □

Examples

1. Let $f(x) = \eta_1 x + \eta_2$, where $\eta_1 \in (0, \frac{\lambda_1}{\lambda_{\max}})$ and $\eta_2 \neq 0$. Clearly, (H1) holds.
2. Let $f(x) = \frac{\lambda_l}{\lambda_{\min}}(x + x^3)e^{x^2} + 4x^3$. Then $F(x) = \frac{\lambda_l}{2\lambda_{\min}}x^2 e^{x^2} + x^4$. Thus, (H2) and (H3) hold automatically. For $R \geq 1$, $\alpha = 4$, we see

$$0 < \frac{2\lambda_l}{\lambda_{\min}}x^2 e^{x^2} + 4x^4 \leq xf(x) = \frac{\lambda_l}{\lambda_{\min}}\left(x^2 + x^4\right)e^{x^2} + 4x^4 \quad \text{for all } |x| > 1.$$

Therefore, (H4) holds.

3. Let $f(x) = 2x + 4x^3$. Then $F(x) = x^2 + x^4$ and (H2), (H7) hold. Choose $\alpha = 4$, $\gamma = 4$, and we see

$$\liminf_{|x| \to \infty}\frac{x^2 + x^4}{|x|^4} = 1 > 0, \qquad \liminf_{|x| \to \infty}\frac{f(x)x - 2F(x)}{|x|^4} = \liminf_{|x| \to \infty}\frac{2x^4}{|x|^4} = 2 > 0.$$

Hence (H5) and (H6) hold.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

WD and JX carried out the main results of this article and drafted the manuscript. DO directed the study and helped with the inspection. All the authors read and approved the final manuscript.

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References

1. Atici, FM, Eloe, PW: A transform method in discrete fractional calculus. Int. J. Differ. Equ. **2**, 165-176 (2007)
2. Atici, FM, Şengül, S: Modeling with fractional difference equations. J. Math. Anal. Appl. **369**, 1-9 (2010)
3. Atici, FM, Eloe, PW: Two point boundary value problems for finite fractional difference equations. J. Differ. Equ. Appl. **17**, 445-456 (2011)
4. Xie, ZS, Jin, YF, Hou, CM: Multiple solutions for a fractional difference boundary value problem via variational approach. Abstr. Appl. Anal. **2012**, 143914 (2012)
5. Goodrich, CS: On discrete sequential fractional boundary value problems. J. Math. Anal. Appl. **385**, 111-124 (2012)
6. Goodrich, CS: On a fractional boundary value problem with fractional boundary conditions. Appl. Math. Lett. **25**, 1101-1105 (2012)
7. Lv, WD: Existence of solutions for discrete fractional boundary value problems with a *p*-Laplacian operator. Adv. Differ. Equ. **2012**, 163 (2012)
8. Pan, YY, Han, ZL, Sun, SR, Huang, ZQ: The existence and uniqueness of solutions to boundary value problems of fractional difference equations. Math. Sci. **2012**, 6 (2012)
9. Tian, Y, Henderson, J: Anti-periodic solutions of higher order nonlinear difference equations: a variational approach. J. Differ. Equ. Appl. **19**, 1380-1392 (2013)
10. Ye, YW, Tang, CL: Periodic solutions for second-order discrete Hamiltonian system with a change of sign in potential. Appl. Math. Comput. **219**, 6548-6555 (2013)
11. Deng, XQ, Liu, X, Zhang, YB, Shi, HP: Periodic and subharmonic solutions for a 2*n*th-order difference equation involving *p*-Laplacian. Indag. Math. **24**, 613-625 (2013)
12. Zhang, GD, Sun, HR: Multiple solutions for a fourth-order difference boundary value problem with parameter via variational approach. Appl. Math. Model. **36**, 4385-4392 (2012)
13. Mawhin, J: Periodic solutions of second order nonlinear difference systems with ϕ-Laplacian: a variational approach. Nonlinear Anal. **75**, 4672-4687 (2012)
14. Zhang, X, Shi, YM: Homoclinic orbits of a class of second-order difference equations. J. Math. Anal. Appl. **396**, 810-828 (2012)
15. Wang, SL, Liu, JS: Nontrivial solutions of a second order difference systems with multiple resonance. Appl. Math. Comput. **218**, 9342-9352 (2012)
16. Iannizzotto, A, Tersian, SA: Multiple homoclinic solutions for the discrete *p*-Laplacian via critical point theory. J. Math. Anal. Appl. **403**, 173-182 (2013)
17. Bereanu, C, Jebelean, P, Şerban, C: Periodic and Neumann problems for discrete *p*(·)-Laplacian. J. Math. Anal. Appl. **399**, 75-87 (2013)
18. Galewski, M, Smejda, J: On the dependence on parameters for mountain pass solutions of second order discrete BVP's. Appl. Math. Comput. **219**, 5963-5971 (2013)
19. Drábek, P, Milota, J: Methods of Nonlinear Analysis: Applications to Differential Equations. Birkhäuser, Basel (2007)
20. Willem, M: Minimax Theorems. Birkhäuser, Boston (1996)
21. Struwe, M: Variational Methods: Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems. Springer, Berlin (2008)
22. Rabinowitz, PH: Minimax Methods in Critical Point Theory with Applications to Differential Equations. Am. Math. Soc., Providence (1986)