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Studies on a $2n$th-order p-Laplacian differential equation with singularity

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Abstract

In this paper, we consider the $2n$th-order p-Laplacian differential equation with singularity

$$\left(\varphi_p\left(x(t)\right)^{(n)}\right)^{(n)} + f\left(x(t)\right)x'(t) + g\left(t, x(t-\sigma)\right) = e(t).$$

By applications of coincidence degree theory and some analysis techniques, sufficient conditions for the existence of positive periodic solutions are established.

MSC: 34C25; 34K13; 34K40

Keywords: positive periodic solution; p-Laplacian; $2n$th-order; singularity

1 Introduction

Generally speaking, differential equations with singularities have been considered from the very beginning of the discipline. The main reason is that singular forces are ubiquitous in applications, gravitational and electromagnetic forces being the most obvious examples. In 1979, Taliaferro [1] discussed the model equation with singularity

$$y'' + \frac{q(t)}{y^\alpha} = 0, \quad 0 < t < 1, \tag{1.1}$$

subject to

$$y(0) = 0 = y(1),$$

and obtained the existence of a solution for the problem. Here $\alpha > 0$, $q \in C(0,1)$ with $q > 0$ on $(0,1)$ and $\int_0^1 t(1-t)q(t)\,dt < \infty$. We call it the equation with the strong force condition if $\alpha \geq 1$ and we call it the equation with the weak force condition if $0 < \alpha < 1$.

Ding's work has attracted the attention of many specialists in differential equations. More recently, topological degree theory [2–4], the Schauder fixed point theorem [5, 6], the Krasnoselskii fixed point theorem in a cone [7–9], the Poincaré-Birkhoff twist theorem [10–12], and the Leray-Schauder alternative principle [13–15] have been employed to investigate the existence of positive periodic solutions of singular second-order, third-order and fourth-order differential equations. In 1996, using coincidence degree theory, Zhang

[2] considered the existence of T-periodic solutions for the scalar Liénard equation

$$x''(t) + f\big(x(t)\big)x'(t) + g\big(t, x(t)\big) = 0,$$

when g becomes unbounded as $x \to 0^+$. The main emphasis was on the repulsive case, *i.e.* when $g(t,x) \to +\infty$, as $x \to 0^+$. In 2007, Torres [5] studied singular forced semilinear differential equation

$$x'' + a(t)x' = f(t,x) + e(t). \tag{1.2}$$

By the Schauder fixed point theorem, the author has shown that the additional assumption of a weak singularity enabled new criteria for the existence of periodic solutions. Afterwards, Wang [3] investigated the existence and multiplicity of positive periodic solutions of the singular systems (1.2) by the Krasnoselskii fixed point theorem. The conditions he presented to guarantee the existence of positive periodic solutions are beautiful. Recently, Cheng and Ren [14] discussed a kind of fourth-order singular differential equation,

$$x^{(4)}(t) + ax'''(t) + bx''(t) + cx'(t) + dx(t) = f\big(t, x(t)\big) + e(t). \tag{1.3}$$

By application of Green's function and some fixed point theorems, *i.e.*, the Leray-Schauder alternative principle and Schauder's fixed point theorem, the authors established two existence results of positive periodic solutions for nonlinear fourth-order singular differential equation.

Motivated by [2, 3, 5, 14], in this paper, we consider the high-order p-Laplacian differential equation with singularity

$$\big(\varphi_p\big(x(t)\big)^{(n)}\big)^{(n)} + f\big(x(t)\big)x'(t) + g\big(t, x(t-\sigma)\big) = e(t), \tag{1.4}$$

where $p \geq 2$, $\varphi_p(x) = |x|^{p-2}x$ for $x \neq 0$, and $\varphi_p(0) = 0$; g is continuous function defined on $\mathbb{R}^2$ and periodic in t with $g(t,\cdot) = g(t+T,\cdot)$, g has a singularity at $x = 0$; σ is a constant and $0 \leq \sigma < T$; $e : \mathbb{R} \to \mathbb{R}$ are continuous periodic functions with $e(t+T) \equiv e(t)$ and $\int_0^T e(t)\,dt = 0$. T is a positive constant; n is positive integer.

The paper is organized as follows. In Section 2, we introduce some technical tools and present all the auxiliary results; in Section 3, by applying coincidence degree theory and some new inequalities, we obtain sufficient conditions for the existence of positive periodic solutions for (1.4), an example is also given to illustrate our results. Our new results generalize in several aspects some recent results contained in [2, 3, 5].

2 Lemmas

For the sake of convenience, throughout this paper we will adopt the following notation:

$$|u|_\infty = \max_{t\in[0,T]}\big|u(t)\big|, \qquad |u|_0 = \min_{t\in[0,T]}\big|u(t)\big|,$$

$$|u|_p = \left(\int_0^T |u|^p\,dt\right)^{\frac{1}{p}}, \qquad \bar{h} = \frac{1}{T}\int_0^T h(t)\,dt.$$

Let X and Y be real Banach spaces and $L : D(L) \subset X \to Y$ be a Fredholm operator with index zero, here $D(L)$ denotes the domain of L. This means that $\operatorname{Im} L$ is closed in Y and $\dim \operatorname{Ker} L = \dim(Y/\operatorname{Im} L) < +\infty$. Consider supplementary subspaces X_1, Y_1 of X, Y, respectively, such that $X = \operatorname{Ker} L \oplus X_1$, $Y = \operatorname{Im} L \oplus Y_1$. Let $P : X \to \operatorname{Ker} L$ and $Q : Y \to Y_1$ denote the natural projections. Clearly, $\operatorname{Ker} L \cap (D(L) \cap X_1) = \{0\}$ and so the restriction $L_P := L|_{D(L)\cap X_1}$ is invertible. Let K denote the inverse of L_P.

Let Ω be an open bounded subset of X with $D(L) \cap \Omega \neq \emptyset$. A map $N : \overline{\Omega} \to Y$ is said to be L-compact in $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and the operator $K(I - Q)N : \overline{\Omega} \to X$ is compact.

Lemma 2.1 (Gaines and Mawhin [16]) *Suppose that X and Y are two Banach spaces, and $L : D(L) \subset X \to Y$ is a Fredholm operator with index zero. Let $\Omega \subset X$ be an open bounded set and $N : \overline{\Omega} \to Y$ be L-compact on $\overline{\Omega}$. Assume that the following conditions hold*:

(1) $Lx \neq \lambda Nx$, $\forall x \in \partial\Omega \cap D(L)$, $\lambda \in (0, 1)$;

(2) $Nx \notin \operatorname{Im} L$, $\forall x \in \partial\Omega \cap \operatorname{Ker} L$;

(3) $\deg\{JQN, \Omega \cap \operatorname{Ker} L, 0\} \neq 0$, *where* $J : \operatorname{Im}\ Q \to \operatorname{Ker} L$ *is an isomorphism.*

Then the equation $Lx = Nx$ has a solution in $\overline{\Omega} \cap D(L)$.

Lemma 2.2 ([17]) *If $\omega \in C^1(\mathbb{R}, \mathbb{R})$ and $\omega(0) = \omega(T) = 0$, then*

$$\int_0^T \left|\omega(t)\right|^p dt \leq \left(\frac{T}{\pi_p}\right)^p \int_0^T \left|\omega'(t)\right|^p dt,$$

where $1 \leq p < \infty$, $\pi_p = 2\int_0^{(p-1)/p} \frac{ds}{(1-\frac{s^p}{p-1})^{1/p}} = \frac{2\pi(p-1)^{1/p}}{p\sin(\pi/p)}$.

Lemma 2.3 *If $x(t) \in C^n(\mathbb{R}, \mathbb{R})$ and $x^{(j)}(t + T) = x^{(j)}(t)$, $j = 0, 1, 2, \ldots, n - 1$, then*

$$\int_0^T \left|x^{(i)}(t)\right|^p dt \leq \left(\frac{T}{\pi_p}\right)^{p(n-i)} \int_0^T \left|x^{(n)}(t)\right|^p dt, \quad i = 1, 2, \ldots, n - 1,$$

where $\frac{1}{p} + \frac{1}{q} = 1$, $p \geq 2$.

Proof From $x^{(i-1)}(0) = x^{(i-1)}(T)$, there is a point $t_i \in [0, T]$ such that $x^{(i)}(t_i) = 0$. Let $\omega_i(t) = x^{(i)}(t + t_i)$, and then $\omega_i(0) = \omega_i(T) = 0$. From $x^{(i)}(0) = x^{(i)}(T)$, there is a point $t_{i+1} \in [0, T]$ such that $x^{(i+1)}(t_{i+1}) = 0$. Let $\omega_{i+1}(t) = x^{(i+1)}(t + t_{i+1})$, and then $\omega_{i+1}(0) = \omega_{i+1}(T) = 0$. Continuing this way we get from $x^{(n-i)}(0) = x^{(n-i)}(T)$ a point $t_{n-i+1} \in [0, T]$ such that $x^{(n)}(t_{n-i+1}) = 0$. Let $\omega_{n-i}(t) = x^{(n-i+1)}(t + t_{n-i+1})$, and then $\omega_{n-i}(0) = \omega_{n-i}(T) = 0$. From Lemma 2.2, we have

$$\begin{aligned}
\int_0^T \left|x^{(i)}(t)\right|^p dt &= \int_0^T \left|\omega_i(t)\right|^p dt \\
&\leq \left(\frac{T}{\pi_p}\right)^p \int_0^T \left|\omega_i'(t)\right|^p dt \\
&= \left(\frac{T}{\pi_p}\right)^p \int_0^T \left|x^{(i+1)}(t)\right|^p dt \\
&= \left(\frac{T}{\pi_p}\right)^p \int_0^T \left|\omega_{i+1}(t)\right|^p dt \\
&\leq \left(\frac{T}{\pi_p}\right)^{2p} \int_0^T \left|\omega_{i+1}'(t)\right|^p dt
\end{aligned}$$

$$\begin{aligned} &\cdots \\ &\le \left(\frac{T}{\pi_p}\right)^{p(n-i)} \int_0^T \left|\omega'_{n-i-1}(t)\right|^p \, dt \\ &= \left(\frac{T}{\pi_p}\right)^{p(n-i)} \int_0^T \left|x^{(n)}(t)\right|^p \, dt. \end{aligned} \tag{2.1}$$

□

In order to apply coincidence degree theorem, we rewrite (1.4) in the form

$$\begin{cases} x_1^{(n)}(t) = \varphi_q(x_2(t)), \\ x_2^{(n)}(t) = -f(x_1(t))x_1'(t) - g(t, x_1(t-\sigma)) + e(t), \end{cases} \tag{2.2}$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Clearly, if $x(t) = (x_1(t), x_2(t))^\top$ is a T-periodic solution to (2.2), then $x_1(t)$ must be a T-periodic solution to (1.4). Thus, the problem of finding a T-periodic solution for (1.4) reduces to finding one for (2.2).

Now, set $X = \{x = (x_1(t), x_2(t)) \in C(\mathbb{R}, \mathbb{R}^2) : x(t+T) \equiv x(t)\}$ with the norm $|x|_\infty = \max\{|x_1|_\infty, |x_2|_\infty\}$; $Y = \{x = (x_1(t), x_2(t)) \in C^1(\mathbb{R}, \mathbb{R}^2) : x(t+T) \equiv x(t)\}$ with the norm $\|x\| = \max\{|x|_\infty, |x'|_\infty\}$. Clearly, X and Y are both Banach spaces. Meanwhile, define

$$L : D(L) = \left\{x \in C^{2n}\left(\mathbb{R}, \mathbb{R}^2\right) : x(t+T) = x(t), t \in \mathbb{R}\right\} \subset X \to Y$$

by

$$(Lx)(t) = \begin{pmatrix} x_1^{(n)}(t) \\ x_2^{(n)}(t) \end{pmatrix}$$

and $N : X \to Y$ by

$$(Nx)(t) = \begin{pmatrix} \varphi_q(x_2(t)) \\ -f(x_1)x_1'(t) - g(t, x_1(t-\sigma)) + e(t) \end{pmatrix}. \tag{2.3}$$

Then (2.2) can be converted into the abstract equation $Lx = Nx$. From the definition of L, one can easily see that

$$\operatorname{Ker} L \cong \mathbb{R}^2, \qquad \operatorname{Im} L = \left\{ y \in Y : \int_0^T \begin{pmatrix} y_1(s) \\ y_2(s) \end{pmatrix} ds = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}.$$

So L is a Fredholm operator with index zero. Let $P : X \to \operatorname{Ker} L$ and $Q : Y \to \operatorname{Im} Q \subset \mathbb{R}^2$ be defined by

$$Px = \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix}; \qquad Qy = \frac{1}{T} \int_0^T \begin{pmatrix} y_1(s) \\ y_2(s) \end{pmatrix} ds,$$

then $\operatorname{Im} P = \operatorname{Ker} L$, $\operatorname{Ker} Q = \operatorname{Im} L$. Setting $L_P = L|_{D(L) \cap \operatorname{Ker} P}$ and L_P^{-1}: $\operatorname{Im} L \to D(L)$ denoting the inverse of L_P, then

$$\left[L_P^{-1} y\right](t) = \begin{pmatrix} (Gy_1)(t) \\ (Gy_2)(t) \end{pmatrix},$$

$$[Gy_1](t) = \sum_{i=1}^{n-1} \frac{1}{i!} x_1^{(i)}(0) t^i + \frac{1}{(n-1)!} \int_0^t (t-s)^{n-1} y_1(s)\,ds, \tag{2.4}$$

$$[Gy_2](t) = \sum_{i=1}^{n-1} \frac{1}{i!} x_2^{(i)}(0) t^i + \frac{1}{(n-1)!} \int_0^t (t-s)^{n-1} y_2(s)\,ds,$$

where $x_j^{(i)}(0)$, $i = 1, 2, \ldots, n-1$ and $j = 1, 2$, are defined by the following:

$$E_1 Z = B, \quad \text{where } E_1 = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ c_1 & 1 & 0 & \cdots & 0 & 0 \\ c_2 & c_1 & 1 & \cdots & 0 & 0 \\ \cdots & & & & & \\ c_{n-3} & c_{n-4} & c_{n-5} & \cdots & 1 & 0 \\ c_{n-2} & c_{n-3} & c_{n-4} & \cdots & c_1 & 0 \end{pmatrix}_{(n-1)\times(n-1)}.$$

$Z = (x_1^{(n-1)}(0), \ldots, x_1''(0), x_1'(0))^\top$, $B = (b_1, b_2, \ldots, b_{n-1})^\top$, $b_i = -\frac{1}{i!T}\int_0^T (T-s)^i y_1(s)\,ds$, and $c_k = \frac{T^k}{(k+1)!}$, $k = 1, 2, \ldots, n-2$.

From (2.3) and (2.4), it is clearly that QN and $K(I-Q)N$ are continuous, $QN(\overline{\Omega})$ is bounded and then $K(I-Q)N(\overline{\Omega})$ is compact for any open bounded $\Omega \subset X$, which means N is L-compact on $\bar{\Omega}$.

3 Existence of positive periodic solutions for (1.1)

Assume that

$$\psi(t) = \lim_{x\to+\infty} \sup \frac{g(t,x)}{x^{p-1}}, \tag{3.1}$$

exists uniformly a.e. $t \in [0, T]$, *i.e.*, for any $\varepsilon > 0$ there is $g_\varepsilon \in L^2(0, T)$ such that

$$g(t,x) \le \bigl(\psi(t) + \varepsilon\bigr) x^{p-1} + g_\varepsilon(t) \tag{3.2}$$

for all $x > 0$ and a.e. $t \in [0, T]$. Moreover, $\psi \in C(\mathbb{R}, \mathbb{R})$ and $\psi(t + T) = \psi(t)$.

For the sake of convenience, we list the following assumptions which will be used repeatedly in the sequel:

(H$_1$) There exist constants $0 < D_1 < D_2$ such that if x is a positive continuous T-periodic function satisfying

$$\int_0^T g\bigl(t, x(t)\bigr)\,dt = 0,$$

then

$$D_1 \le x(\tau) \le D_2$$

for some $\tau \in [0, T]$.

(H$_2$) $\bar{g}(x) < 0$ for all $x \in (0, D_1)$, and $\bar{g}(x) > 0$ for all $x > D_2$.

(H$_3$) $g(t,x) = g_0(x) + g_1(t,x)$, where $g_0 \in C((0,\infty); \mathbb{R})$ and $g_1 : [0,T] \times [0,\infty) \to \mathbb{R}$ is an L^2-Carathéodory function, *i.e.* it is measurable in the first variable and continuous in the second variable, and for any $b > 0$ there is $h_b \in L^2(0,T;\mathbb{R}_+)$ such that

$$\left|g_1(t,x)\right| \le h_b(t), \quad \text{a.e. } t \in [0,T], \forall 0 \le x \le b.$$

(H_4) $\int_0^1 g_0(x)\,dx = -\infty$.

Theorem 3.1 *Assume that conditions* (H_1)-(H_4) *hold. If* $|\psi|_\infty \frac{T^{\frac{p}{q}+1}}{2^{p-1}}(\frac{T}{\pi_p})^{p(n-1)} < 1$, *then* (1.4) *has at least a positive T-periodic solution.*

Proof Consider the equation

$$Lx = \lambda Nx, \quad \lambda \in (0,1).$$

Set $\Omega_1 = \{x : Lx = \lambda Nx, \lambda \in (0,1)\}$. If $x(t) = (x_1(t), x_2(t))^\top \in \Omega_1$, then

$$\begin{cases} x_1^{(n)}(t) = \lambda\varphi_q(x_2(t)), \\ x_2^{(n)}(t) = -\lambda f(x_1(t))x_1'(t) - \lambda g(t, x_1(t-\sigma)) + \lambda e(t). \end{cases} \tag{3.3}$$

Substituting $x_2(t) = \lambda^{1-p}\varphi_p[x_1^{(n)}(t)]$ into the second equation of (3.3)

$$\left(\varphi_p\left(x_1^{(n)}(t)\right)\right)^{(n)} + \lambda^p f\left(x_1(t)\right)x_1'(t) + \lambda^p g\left(t, x_1(t-\sigma)\right) = \lambda^p e(t). \tag{3.4}$$

Integrating both sides of (3.4) from 0 to T, we have

$$\int_0^T g\left(t, x_1(t-\sigma)\right)dt = 0. \tag{3.5}$$

In view of (H_1), there exist positive constants D_1, D_2, and $\xi \in [0,T]$ such that

$$D_1 \le \left|x_1(\xi)\right| \le D_2.$$

Then we have

$$\left|x_1(t)\right| = \left|x_1(\xi) + \int_\xi^t x_1'(s)\,ds\right| \le D_2 + \int_\xi^t \left|x_1'(s)\right|ds, \quad t \in [\xi, \xi + T],$$

and

$$\left|x_1(t)\right| = \left|x_1(t-T)\right| = \left|x_1(\xi) - \int_{t-T}^\xi x_1'(s)\,ds\right| \le D_2 + \int_{t-T}^\xi \left|x_1'(s)\right|ds, \quad t \in [\xi, \xi + T].$$

Combing the above two inequalities, we obtain

$$\begin{aligned} |x_1|_\infty &= \max_{t\in[0,T]}\left|x_1(t)\right| = \max_{t\in[\xi,\xi+T]}\left|x_1(t)\right| \\ &\le \max_{t\in[\xi,\xi+T]}\left\{D_2 + \frac{1}{2}\left(\int_\xi^t \left|x_1'(s)\right|ds + \int_{t-T}^\xi \left|x_1'(s)\right|ds\right)\right\} \\ &\le D_2 + \frac{1}{2}\int_0^T \left|x_1'(s)\right|ds. \end{aligned} \tag{3.6}$$

Multiplying both sides of (3.4) by $x_1(t)$ and integrating over interval $[0, T]$, we get

$$\begin{aligned} &\int_0^T \bigl(\varphi_p\bigl(x_1^{(n)}(t)\bigr)\bigr)^{(n)} x_1(t)\,dt + \lambda^p \int_0^T f\bigl(x_1(t)\bigr)x_1'(t)x_1(t)\,dt + \lambda^p \int_0^T g\bigl(t, x_1(t-\sigma)\bigr)x_1(t)\,dt \\ &\quad= \lambda^p \int_0^T e(t)x_1(t)\,dt. \end{aligned} \tag{3.7}$$

Substituting $\int_0^T (\varphi_p(x_1^{(n)}(t)))^{(n)} x_1(t)\,dt = (-1)^n \int_0^T |x_1^{(n)}(t)|^p\,dt$, $\int_0^T f(x_1(t))x_1'(t)x_1(t)\,dt = 0$ into (3.7), we have

$$(-1)^n \int_0^T \bigl|x_1^{(n)}(t)\bigr|^p\,dt = -\lambda^p \int_0^T g\bigl(t, x_1(t-\sigma)\bigr)x_1(t)\,dt + \lambda^p \int_0^T e(t)x_1(t)\,dt.$$

Namely,

$$\begin{aligned} \int_0^T \bigl|x_1^{(n)}(t)\bigr|^p\,dt &\le \int_0^T \bigl|g\bigl(t, x_1(t-\sigma)\bigr)\bigr|\bigl|x_1(t)\bigr|\,dt + \int_0^T \bigl|e(t)\bigr|\bigl|x_1(t)\bigr|\,dt \\ &\le |x_1|_\infty \int_0^T \bigl|g\bigl(t, x_1(t-\sigma)\bigr)\bigr|\,dt + |x_1|_\infty |e|_\infty T. \end{aligned} \tag{3.8}$$

Write

$$I_+ = \bigl\{t \in [0, T] : g\bigl(t, x_1(t-\sigma)\bigr) \ge 0\bigr\}; \qquad I_- = \bigl\{t \in [0, T] : g\bigl(t, x_1(t-\sigma)\bigr) \le 0\bigr\}.$$

Then we get from (3.2) and (3.5)

$$\begin{aligned} \int_0^T \bigl|g\bigl(t, x_1(t-\sigma)\bigr)\bigr|\,dt &= \int_{I_+} g\bigl(t, x_1(t-\sigma)\bigr)\,dt - \int_{I_-} g\bigl(t, x_1(t-\sigma)\bigr)\,dt \\ &= 2\int_{I_+} g\bigl(t, x_1(t-\sigma)\bigr)\,dt \\ &\le 2\int_{I_+} \bigl(\bigl(\psi(t) + \varepsilon\bigr)x_1^{p-1}(t-\sigma) + g_\varepsilon(t)\bigr)\,dt \\ &\le 2\bigl(|\psi|_\infty + \varepsilon\bigr) \int_0^T \bigl|x_1(t)\bigr|^{p-1}\,dt + 2\int_0^T \bigl|g_\varepsilon(t)\bigr|\,dt. \end{aligned} \tag{3.9}$$

Substituting (3.9) into (3.8), we have

$$\begin{aligned} \int_0^T \bigl|x_1^{(n)}(t)\bigr|^p\,dt &\le 2|x_1|_\infty\bigl(|\psi|_\infty + \varepsilon\bigr) \int_0^T \bigl|x_1(t)\bigr|^{p-1}\,dt \\ &\quad + |x_1|_\infty \left(2\int_0^T \bigl|g_\varepsilon(t)\bigr|\,dt + |e|_\infty T\right) \\ &\le 2\bigl(|\psi|_\infty + \varepsilon\bigr)T|x_1|_\infty^p + |x_1|_\infty \left(2T^{\frac{1}{2}}\left(\int_0^T \bigl|g_\varepsilon(t)\bigr|^2\,dt\right)^{\frac{1}{2}} + |e|_\infty T\right) \\ &= 2\bigl(|\psi|_\infty + \varepsilon\bigr)T|x_1|_\infty^p + |x_1|_\infty\bigl(2T^{\frac{1}{2}}|g_\varepsilon|_2 + |e|_\infty T\bigr). \end{aligned} \tag{3.10}$$

From (3.6) and Lemma 2.3, we have

$$\begin{aligned}|x_1|_\infty &\le D_2 + \frac{1}{2}\int_0^T \bigl|x_1'(t)\bigr|\,dt \le D_2 + \frac{T^{\frac{1}{q}}}{2}\left(\int_0^T \bigl|x_1'(t)\bigr|^p\,dt\right)^{\frac{1}{p}} \\ &\le D_2 + \frac{T^{\frac{1}{q}}}{2}\left(\frac{T}{\pi_p}\right)^{n-1}\left(\int_0^T \bigl|x_1^{(n)}(t)\bigr|^p\,dt\right)^{\frac{1}{p}}. \end{aligned} \tag{3.11}$$

Substituting (3.11) into (3.10), we have

$$\begin{aligned}&\int_0^T \bigl|x_1^{(n)}(t)\bigr|^p\,dt \\ &\quad\le 2\bigl(|\psi|_\infty+\varepsilon\bigr)T\left(D_2 + \frac{T^{\frac{1}{q}}}{2}\left(\frac{T}{\pi_p}\right)^{n-1}\left(\int_0^T \bigl|x_1^{(n)}(t)\bigr|^p\,dt\right)^{\frac{1}{p}}\right)^p \\ &\qquad+\left(D_2 + \frac{T^{\frac{1}{q}}}{2}\left(\frac{T}{\pi_p}\right)^{n-1}\left(\int_0^T \bigl|x_1^{(n)}(t)\bigr|^p\,dt\right)^{\frac{1}{p}}\right)\bigl(2T^{\frac{1}{2}}|g_\varepsilon|_2+|e|_\infty T\bigr) \\ &\quad= 2\bigl(|\psi|_\infty+\varepsilon\bigr)T\left(\frac{T^{\frac{p}{q}}}{2^p}\left(\frac{T}{\pi_p}\right)^{p(n-1)}\int_0^T \bigl|x_1^{(n)}(t)\bigr|^p\,dt + pD_2\frac{T^{\frac{p-1}{q}}}{2^{p-1}}\left(\frac{T}{\pi_p}\right)^{(p-1)(n-1)}\right. \\ &\qquad\left.\cdot\left(\int_0^T \bigl|x_1^{(n)}(t)\bigr|\,dt\right)^{\frac{p-1}{p}} + \cdots + pD_2^{p-1}\frac{T^{\frac{1}{q}}}{2}\left(\frac{T}{\pi_p}\right)^{n-1}\left(\int_0^T \bigl|x_1'(t)\bigr|^p\,dt\right)^{\frac{1}{p}} + D_2^p\right) \\ &\qquad+\left(D_2 + \frac{T^{\frac{1}{q}}}{2}\left(\frac{T}{\pi_p}\right)^{n-1}\left(\int_0^T \bigl|x_1^{(n)}(t)\bigr|^p\,dt\right)^{\frac{1}{p}}\right)\bigl(2T^{\frac{1}{2}}|g_\varepsilon|_2+|e|_\infty T\bigr) \\ &\quad= \bigl(|\psi|_\infty+\varepsilon\bigr)\frac{T^{\frac{p}{q}+1}}{2^{p-1}}\left(\frac{T}{\pi_p}\right)^{p(n-1)}\int_0^T \bigl|x_1^{(n)}\bigr|^p\,dt \\ &\qquad+\bigl(|\psi|_\infty+\varepsilon\bigr)pD_2\frac{T^{\frac{p-1}{q}+1}}{2^{p-2}}\left(\frac{T}{\pi_p}\right)^{(p-1)(n-1)}\left(\int_0^T \bigl|x_1^{(n)}(t)\bigr|^p\,dt\right)^{\frac{p-1}{p}} + \cdots \\ &\qquad+\bigl(2\bigl(|\psi|_\infty+\varepsilon\bigr)TpD_2^{p-1} + 2T^{\frac{1}{2}}|g_\varepsilon|_2+|e|_\infty T\bigr)\frac{T^{\frac{1}{q}}}{2}\left(\frac{T}{\pi_p}\right)^{n-1} \\ &\qquad\cdot\left(\int_0^T \bigl|x_1^{(n)}(t)\bigr|^p\,dt\right)^{\frac{1}{p}} + 2\bigl(|\psi|_\infty+\varepsilon\bigr)TD_2^p + D_2\bigl(2T^{\frac{1}{2}}|g_\varepsilon|_2+|e|_\infty T\bigr).\end{aligned}$$

Since ε sufficiently small, we know that $|\psi|_\infty\frac{T^{\frac{p}{q}+1}}{2^{p-1}}(\frac{T}{\pi_p})^{p(n-1)} < 1$. So, it is easy to see that there exists a positive constant M_1' such that

$$\int_0^T \bigl|x_1^{(n)}(t)\bigr|^p\,dt \le M_1'.$$

From (3.11), we have

$$\begin{aligned}|x_1|_\infty &\le D_2 + \frac{T^{\frac{1}{q}}}{2}\left(\frac{T}{\pi_p}\right)^{n-1}\left(\int_0^T \bigl|x_1^{(n)}(t)\bigr|^p\,dt\right)^{\frac{1}{p}} \\ &\le D_2 + \frac{T^{\frac{1}{q}}}{2}\left(\frac{T}{\pi_p}\right)^{n-1}\bigl(M_1'\bigr)^{\frac{1}{p}} := M_1.\end{aligned} \tag{3.12}$$

Since $x_1(0) = x_1(T)$, there exists a point $\eta_1 \in [0, T]$ such that $x_1'(\eta_1) = 0$. From Lemma 2.3, we can easily get

$$\begin{aligned} |x_1'|_\infty &\le \frac{1}{2}\int_0^T |x_1''(t)|\,dt \\ &\le \frac{T^{\frac{1}{q}}}{2}\left(\int_0^T |x_1''(t)|^p\,dt\right)^{\frac{1}{p}} \\ &\le \frac{T^{\frac{1}{q}}}{2}\left(\frac{T}{\pi_p}\right)^{(n-2)}\left(\int_0^T |x_1^{(n)}|^p\right)^{\frac{1}{p}} \\ &\le \frac{T^{\frac{1}{q}}}{2}\left(\frac{T}{\pi_p}\right)^{(n-2)}(M_1')^{\frac{1}{p}} := M_2. \end{aligned} \tag{3.13}$$

On the other hand, form $x_2^{(n-2)}(0) = x_2^{(n-2)}(T)$, there exists a point $\eta_2 \in [0, T]$ such that $x_2^{(n-1)}(\eta_2) = 0$, from the second equation of (3.3) and (3.9), we have

$$\begin{aligned} |x_2^{(n-1)}|_\infty &\le \frac{1}{2}\max\left|\int_0^T x_2^{(n)}(t)\,dt\right| \\ &\le \frac{\lambda}{2}\int_0^T \left|-f\big(x_1(t)\big)x_1'(t) - g\big(t, x_1\big(t, x_1(t-\sigma)\big)\big) + e(t)\right|dt \\ &\le \frac{\lambda}{2}\big(|f|_{M_1}TM_2 + 2\big(|\psi|_\infty + \varepsilon\big)TM_1^{p-1} + 2\sqrt{T}|g_\varepsilon|_2 + T|e|_\infty\big) := \lambda M_{n-1}, \end{aligned}$$

where $|f|_{M_1} = \max_{0<x_1(t)\le M_1} |f(x_1(t))|$. Since $x_2(0) = x_2(T)$, there exists a point $\eta_3 \in [0, T]$ such that $x_2'(\eta_3) = 0$. From the Wirtinger inequality (see [18], Lemma 2.4), we can easily get

$$\begin{aligned} |x_2'|_\infty &\le \frac{1}{2}\int_0^T |x_2''(t)|\,dt \le \frac{T^{\frac{1}{2}}}{2}\left(\int_0^T |x_2''(t)|^2\,dt\right)^{\frac{1}{2}} \\ &\le \frac{T}{2}\left(\frac{T}{2\pi}\right)^{(n-3)}|x_2^{(n-1)}|_\infty \\ &\le \frac{T}{2}\left(\frac{T}{2\pi}\right)^{(n-3)}(\lambda M_{n-1}) := \lambda M_3. \end{aligned} \tag{3.14}$$

By the first equation of (3.3), we have

$$\int_0^T |x_2(t)|^{q-2}x_2(t)\,dt = 0,$$

which implies that there is a constant $\eta_4 \in [0, T]$ such that $x_2(\eta_4) = 0$, so

$$|x_2|_\infty \le \frac{1}{2}\int_0^T |x_2'(t)|\,dt \le \frac{T}{2}|x_2'|_\infty \le \frac{\lambda T}{2}M_3 := \lambda M_4. \tag{3.15}$$

Next, it follows from (3.4) that

$$\big(\varphi_p\big(x_1^{(n)}(t+\sigma)\big)\big)^{(n)} + \lambda^p\big(f\big(x_1(t+\sigma)\big)x_1'(t+\sigma) + g\big(t+\sigma, x_1(t)\big)\big) = \lambda^p e(t+\sigma). \tag{3.16}$$

Namely,

$$\begin{aligned}&\left(\varphi_p\left(x_1^{(n)}(t+\sigma)\right)\right)^{(n)}+\lambda^p f\left(x_1(t+\sigma)\right)x_1'(t+\sigma)+\lambda^p\left(g_0\left(x_1(t)\right)+g_1\left(t+\sigma,x_1(t)\right)\right)\\&\quad=\lambda^p e(t+\sigma).\end{aligned}\tag{3.17}$$

Multiplying both sides of (3.17) by $x_1'(t)$, we get

$$\begin{aligned}&\left(\varphi_p\left(x_1^{(n)}(t+\sigma)\right)\right)^{(n)}x_1'(t)+\lambda^p f\left(x_1(t+\sigma)\right)x_1'(t+\sigma)x_1'(t)\\&\qquad+\lambda^p g_0\left(x_1(t)\right)x_1'(t)+\lambda^p g_1\left(t+\sigma,x_1(t)\right)x_1'(t)\\&\quad=\lambda^p e(t+\sigma)x_1'(t).\end{aligned}\tag{3.18}$$

Let $\tau\in[0,T]$, for any $\tau\le t\le T$, we integrate (3.18) on $[\tau,t]$ and get

$$\begin{aligned}&\lambda^p\int_{x_1(\tau)}^{x_1(t)}g_0(u)\,du\\&\quad=\lambda^p\int_\tau^t g_0\left(x_1(s)\right)x_1'(s)\,ds\\&\quad=-\int_\tau^t\left(\varphi_p\left(x_1^{(n)}(s+\sigma)\right)\right)^{(n)}x_1'(s)\,ds-\lambda^p\int_\tau^t f\left(x_1(s+\sigma)\right)x_1'(s+\sigma)x_1'(s)\,ds\\&\qquad-\lambda^p\int_\tau^t g_1\left(s+\sigma,x_1(s)\right)x_1'(s)\,ds+\lambda^p\int_\tau^t e(s+\sigma)x_1'(s)\,ds.\end{aligned}\tag{3.19}$$

By (3.12), (3.13), and (3.16), we have

$$\begin{aligned}&\left|\int_\tau^t\left(\varphi_p\left(x_1^{(n)}(s+\sigma)\right)\right)^{(n)}x_1'(s)\,ds\right|\\&\quad\le\int_\tau^t\left|\left(\varphi_p\left(x_1^{(n)}(s+\sigma)\right)\right)^{(n)}\right|\left|x_1'(s)\right|\,ds\\&\quad\le\left|x_1'\right|_\infty\int_0^T\left|\left(\varphi_p\left(x_1^{(n)}(t+\sigma)\right)\right)^{(n)}\right|\,dt\\&\quad\le\lambda^p\left|x_1'\right|_\infty\left(\int_0^T\left|f\left(x_1(t)\right)\right|\left|x_1'(t)\right|\,dt+\int_0^T\left|g\left(t,x_1(t-\sigma)\right)\right|\,dt+\int_0^T\left|e(t)\right|\,dt\right)\\&\quad\le\lambda^p M_2\left(|f|_{M_1}M_2+2\left(|\psi|_\infty+\varepsilon\right)TM_1^{p-1}+2T^{\frac12}\left|g_\varepsilon^+\right|_2+T|e|_\infty\right).\end{aligned}$$

Also we have

$$\begin{aligned}&\left|\int_\tau^t f\left(x_1(s+\sigma)\right)x_1'(s+\sigma)x_1'(s)\,ds\right|\le|f|_{M_1}M_2^2T,\\&\left|\int_\tau^t g\left(s+\sigma,x_1(s)\right)x_1'(s)\,ds\right|\le\left|x_1'\right|_\infty\int_0^T\left|g(t,x(t-\sigma)\right|\,dt\le M_2\sqrt{T}|g_{M_1}|_2,\end{aligned}$$

where $g_{M_1}=\max\limits_{0\le x\le M_1}|g_1(t,x)|\in L^2(0,T)$ is as in (H_3).

$$\left|\int_\tau^t e(t+\sigma)x_1'(t)\,dt\right|\le M_2T|e|_\infty.$$

From these inequalities we can derive form (3.19) that

$$\left|\int_{x_1(\tau)}^{x_1(t)} g_0(u)\,du\right| \le M_5', \tag{3.20}$$

for some constant M_5' which is independent on λ, x, and t. In view of the strong force condition (H_4), we know that there exists a constant $M_5 > 0$ such that

$$x_1(t) \ge M_5, \quad \forall t \in [\tau, T]. \tag{3.21}$$

The case $t \in [0,\tau]$ can be treated similarly.

From (3.12), (3.13), (3.14), (3.15), and (3.21), we get

$$\begin{aligned}\Omega = \bigl\{&x = (x_1, x_2)^\top : E_1 \le |x_1|_\infty \le E_2, \bigl|x_1'\bigr|_\infty \le E_3, |x_2|_\infty \le E_4 \text{ and} \\ &\bigl|x_2'\bigr|_\infty \le E_5, \forall t \in [0,T]\bigr\},\end{aligned}$$

where $0 < E_1 < \min(M_5, D_1)$, $E_2 > \max(M_1, D_2)$, $E_3 > M_2$, $E_4 > M_4$, and $E_5 > M_3$. $\Omega_2 = \{x : x \in \partial\Omega \cap \operatorname{Ker} L\}$, then $\forall x \in \partial\Omega \cap \operatorname{Ker} L$

$$QNx = \frac{1}{T}\int_0^T \begin{pmatrix} \varphi_q(x_2(t)) \\ -f(x_1(t))x_1'(t) - g(t, x_1(t-\sigma)) + e(t)\end{pmatrix} dt.$$

If $QNx = 0$, then $x_2(t) = 0$, $x_1 = E_2$ or $-E_2$. But if $x_1(t) = E_2$, we know

$$0 = \int_0^T \bigl\{g(t, E_2) - e(t)\bigr\}\,dt.$$

From assumption (H_2), we have $x_1(t) \le D_2 \le E_2$, which yields a contradiction. Similarly if $x_1 = -E_2$. We also have $QNx \ne 0$, *i.e.*, $\forall x \in \partial\Omega \cap \operatorname{Ker} L$, $x \notin \operatorname{Im} L$, so conditions (1) and (2) of Lemma 2.1 are both satisfied. Define the isomorphism $J : \operatorname{Im} Q \to \operatorname{Ker} L$ as follows:

$$J(x_1, x_2)^\top = (x_2, -x_1)^\top.$$

Let $H(\mu, x) = -\mu x + (1-\mu)JQNx$, $(\mu, x) \in [0,1] \times \Omega$, then $\forall(\mu, x) \in (0,1) \times (\partial\Omega \cap \operatorname{Ker} L)$,

$$H(\mu, x) = \begin{pmatrix} -\mu x_1 - \frac{1-\mu}{T}\int_0^T [g(t, x_1) - e(t)]\,dt \\ -\mu x_2 - (1-\mu)|x_2|^{q-2}x_2 \end{pmatrix}.$$

We have $\int_0^T e(t)\,dt = 0$. So, we can get

$$H(\mu, x) = \begin{pmatrix} -\mu x_1 - \frac{1-\mu}{T}\int_0^T g(t, x_1)\,dt \\ -\mu x_2 - (1-\mu)|x_2|^{q-2}x_2 \end{pmatrix}, \quad \forall(\mu, x) \in (0,1) \times (\partial\Omega \cap \operatorname{Ker} L).$$

From (H_2), it is obvious that $x^\top H(\mu, x) < 0$, $\forall(\mu, x) \in (0,1) \times (\partial\Omega \cap \operatorname{Ker} L)$. Hence

$$\begin{aligned}\deg\{JQN, \Omega \cap \operatorname{Ker} L, 0\} &= \deg\bigl\{H(0,x), \Omega \cap \operatorname{Ker} L, 0\bigr\} \\ &= \deg\bigl\{H(1,x), \Omega \cap \operatorname{Ker} L, 0\bigr\} \\ &= \deg\{I, \Omega \cap \operatorname{Ker} L, 0\} \ne 0.\end{aligned}$$

So condition (3) of Lemma 2.1 is satisfied. By applying Lemma 2.1, we conclude that the equation $Lx = Nx$ has a solution $x = (x_1, x_2)^\top$ on $\bar{\Omega} \cap D(L)$, *i.e.*, (1.4) has a positive T-periodic solution $x_1(t)$. □

Example 3.1 Consider the high-order p-Laplacian differential equation with singularity

$$\bigl(\varphi_p\bigl(x(t)'''\bigr)\bigr)'''' + f\bigl(x(t)\bigr)x'(t) + \frac{1}{6}(\sin 2t + 3)x^3(t-\sigma) - \frac{1}{x^\kappa(t-\sigma)} = \cos 2t, \tag{3.22}$$

where $\kappa \geq 1$ and $p = 4, f$ is continuous function, σ is a constant, and $0 \leq \sigma < T$.

It is clear that $T = \pi$, $n = 3$, $g(t,x) = \frac{1}{6}(\sin 2t + 3)x^3(t-\sigma) - \frac{1}{x^\kappa(t-\sigma)}$, $\psi(t) = \frac{1}{6}(\sin 2t + 3)$, $|\psi|_\infty = \frac{2}{3}$. It is obvious that (H_1)-(H_4) hold. Now we consider the assumption condition

$$\begin{aligned}
&|\psi|_\infty \frac{T^{\frac{p}{q}+1}}{2^{p-1}}\left(\frac{T}{\pi_p}\right)^{p(n-1)} \\
&= |\psi|_\infty \frac{T^{\frac{p}{q}+1}}{2^{p-1}}\left(\frac{T}{\frac{2\pi(p-1)^{1/p}}{p\sin(\pi/p)}}\right)^{p(n-1)} \\
&= \frac{2}{3}\cdot\frac{\pi^{\frac{4}{3}}}{2^3}\left(\frac{\pi}{\frac{2\pi(4-1)^{1/4}}{4\sin\pi/4}}\right)^{8} \\
&= \frac{4\pi^{\frac{4}{3}}}{27} < 1.
\end{aligned}$$

So by Theorem 3.1, we know (3.22) has at least one positive π-periodic solution.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
YX and SZ worked together in the derivation of the mathematical results. Both authors read and approved the final manuscript.

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