S-duality and modular transformation as a non-perturbative deformation of the ordinary pq-duality

D. Galakhov,[a,b] **A. Mironov**[b,c] **and A. Morozov**[a]

[a] *ITEP,*
  Bol. Cheremushkinskaya, Moscow, Russia
[b] *NHETC and Department of Physics and Astronomy, Rutgers University,*
  Piscataway, NJ 08855-0849, U.S.A.
[c] *Theory Department, Lebedev Physics Institute,*
  Leninsky prospekt, Moscow, Russia

  E-mail: galakhov@physics.rutgers.edu, mironov@lpi.ru, morozov@itep.ru

ABSTRACT: A recent claim that the S-duality between $4d$ SUSY gauge theories, which is AGT related to the modular transformations of $2d$ conformal blocks, is no more than an ordinary Fourier transform at the perturbative level, is further traced down to the commutation relation $[\check{P}, \check{Q}] = -i\hbar$ between the check-operator monodromies of the exponential resolvent operator in the underlying Dotsenko-Fateev matrix models and β-ensembles. To this end, we treat the conformal blocks as eigenfunctions of the monodromy check operators, what is especially simple in the case of one-point toric block. The kernel of the modular transformation is then defined as the intertwiner of the two monodromies, and can be obtained straightforwardly, even when the eigenfunction interpretation of the blocks themselves is technically tedious. In this way, we provide an elementary derivation of the old expression for the modular kernel for the one-point toric conformal block.

Contents

1 Introduction

S-duality is one of the most interesting discoveries of modern string theory [1–7]. It is a far-going generalization of the $E - B$ duality of Maxwell electrodynamics with magnetic charges which states that the non-perturbative partition functions of different field theories can coincide after a non-linear transformation of coupling constants. S-dual theories can have different numbers of perturbative degrees of freedom and different gauge groups. A significant class of S-dual models can be described by the $M5$-brane construction of [8–10], where $6d$ theory on the brane is compactified on a $2d$ Riemann surface, which therefore controls the structure of emerging perturbative $4d$ gauge theory, and thus provides a natural explanation of the hidden integrable structure [11–14], the spectral surface of an integrable system being just the covering of the original Riemann surface. In this picture S-dualities get related to modular transformations of the Riemann surface. A quantitative realization of this idea [10] led to the AGT conjecture [15–18], which identifies the LMNS instanton

sums [19–22], expressed via Nekrasov functions [23, 24], with the conformal blocks of $2d$ conformal theories [25, 26]. This identification opens a way to a quantitative study of S-dualities, because constructing modular transformations of conformal blocks is a hard but still solvable problem.

The problem is that the original definition provides conformal blocks in a form of perturbative series in the variable x; in the case of the spherical 4-point conformal block, AGT related to the ordinary SU(2) SYM theory with 4 hypermultiples, x is just a double ratio of the four punctures, and the modular transformation relates the conformal blocks at points x and $1-x$. Normally one needs some non-perturbative completion of this definition to even pose the problem.

There is a variety of such definitions: exploiting an $SL_q(2)$ counterpart of conformal blocks in specific representations [27–29] or various equations that they can satisfy, from Ward identities for extended blocks with additional insertions of degenerate fields [30–35] to wonderful, still badly understood, relations to the Painlevé IV equation [36].

Remarkably, study of the AGT relations provides as a byproduct a kind of a more direct approach. The conformal blocks possess a matrix model (β-ensemble) realization [37–45], which is an advanced version of the old Dotsenko-Fateev trick [46] and the Felder construction [47] (in particular, the integration contours for screening charges are actually open, not always closed, but instead one suffices to use only one screening charge of two). Then one can study the genus expansion in this theory, which is actually preserved by S-duality. This can be also considered as studying S-duality for expansions at the point $g_s = 0$, where all terms are explicit "non-perturbative" functions in x.

In fact, one should be careful with the word "perturbative" in the present context. In the above mentioned standard definition, the conformal block is a perturbative series not only in x, but also in the dimensions of operators, i.e. in the string coupling constant $g_s^2 = \epsilon_1 \epsilon_2$. Non-perturbative corrections exist both in x and in g_s, and they are actually very different. In what follows we reserve the words "perturbative" and "non-perturbative" for the g_s^2-corrections, assuming that the x-behavior is completely fixed by switching to the matrix model description. Thus, "the perturbative conformal block" refers to the genus expansion of the β-ensemble controlled by the "topological recursion" formalism of [48–52] and [55–59], while "the non-perturbative conformal block" refers to a more obscure quantity, which still does not have a unique commonly accepted definition. Hopefully, all the existing suggestions, [27, 28, 30, 31, 36] and the one described in the present paper, would finally lead to the same outcome, but this still remains to be demonstrated and understood.

Anyhow, in what follows we concentrate on one particular definition, that of the Dotsenko-Fateev β-ensemble of [42–45] and use it to study the S-dualities (modular transformations) of both perturbative and non-perturbative conformal blocks.

Making use of this idea and calculational advances in AGT studies, we conjectured recently [60, 61] that *the S-duality is actually reduced to an ordinary Fourier transform in all orders of perturbation expansion in string coupling constant.* In [60] this was shown for the central charge $c = 1 - 6(\beta - 1/\beta)^2 = 1$ ($\beta = 1$), and, after a more accurate analysis of normalization factors in [61], this result was extended to an arbitrary β. This claim was reconsidered and confirmed from a slightly different viewpoint in [62, 63]. These results

are perturbative, and their exact relation to non-perturbative suggestions of [27, 28, 36] still remained obscure, despite the latter formulas are also consistent with the pure Fourier transform at the perturbative level.

Calculations of [60, 61] are quite tedious, what seems strange for such a simple outcome. Clearly some very simple explanation should exist, which does not require long calculations. It is the purpose of the present paper to provide such an explanation in precise and quantitative form. This calls for begins an investigation of the far-reaching corollaries of emerging formalism, which so far seemed to be just a funny technical tool in advanced matrix model theory [53, 54].

An intuitive idea has already been formulated in [30]: to describe the duality, one can treat dual conformal blocks as eigenfunctions of canonically conjugated quantum operators. The question is what are the operators and how they act on the correlation functions, and it is where the matrix model theory is of a great use. Namely, it puts the story into the context of Seiberg-Witten (SW) theory, where the partition function is defined as a function of flat moduli $\vec{a}$ by the equations

$$\left(\oint_{\vec{A}} \lambda\right) Z(\vec{a}) = \vec{a} Z(\vec{a}),$$
$$\left(\oint_{\vec{B}} \lambda\right) Z(\vec{a}) = \frac{\partial}{\partial \vec{a}} Z(\vec{a}) \tag{1.1}$$

which allows one to treat the periods of the SW differential as operators acting on functions on the moduli space. In the genus expansion of matrix models the role of SW differential is played by the one-point resolvent, which can be defined either in the usual style of [48–52] or alternatively reformulated as produced by the so-called check-operators [53, 54] which act on ramification points of the spectral curve. Accordingly,

i) the period integrals of the resolvent generating operators turn out to establish a set of canonically conjugated observables [53, 54], and

ii) partition functions are their eigenfunctions.

In fact, there are delicate points in this story. The genus expansion is the typical quasiclassical expansion, thus, it actually suffers from the Stokes like phenomena, which requires a careful interplay between different branches of the Seiberg-Witten differentials. Taking this into account provides a natural non-perturbative completion of the genus expansion, and can be used as yet another definition of the non-perturbative conformal blocks and non-perturbative modular transformations. We demonstrate that in the simplest examples the results seem consistent with the ansatz of [27–29]. An additional advantage of such an approach is a clear relation to the theory of wall crossing *a la* [64–68], to the cluster algebras [69–73] and to the Kontsevich-Soibelman formulas [74, 75]. We elaborate more on these relations in a separate text.

2 Duality and eigenfunctions of dual operators

An archetypical example of duality is provided by the switch between coordinate and momentum operators. Namely, consider the two operators $\hat{A} = e^{i\hat{\mathcal{P}}}$ and $\hat{B} = e^{i\hat{\mathcal{Q}}}$, with the

commutation relation

$$\hat{A}\hat{B} = e^{i\hbar}\hat{B}\hat{A} \tag{2.1}$$

Then, their eigenfunctions are related by the Fourier transform in the eigenvalue space:

$$\hat{A}Z_a(\mathcal{Q}) = e^{ia}Z_a(\mathcal{Q})$$
$$\overset{(2.1)}{\Longrightarrow} \qquad Z_a(\mathcal{Q}) = \int e^{\frac{iaa'}{\hbar}}\tilde{Z}_{a'}(\mathcal{Q})da' \tag{2.2}$$
$$\hat{B}\tilde{Z}_{a'}(\mathcal{Q}) = e^{ia'}\tilde{Z}_{a'}(\mathcal{Q})$$

This can be easily checked in this case by calculating the eigenfunctions explicitly:

$$Z_a(\mathcal{Q}) = e^{\frac{ia\mathcal{Q}}{\hbar}}, \qquad \tilde{Z}_{a'}(\mathcal{Q}) = \delta(\mathcal{Q} - a') \tag{2.3}$$

but this is *not necessary* in order to define what is the transformation kernel. Instead, one can substitute the two operators by their representatives in the eigenvalue space, which reproduce the right commutation relations:

$$\check{A} = e^{ia}, \qquad \check{B} = e^{\hbar\frac{\partial}{\partial a}} \tag{2.4}$$

which we call *check-operators*, following [53, 54]. Then the transformation kernel $M(a, a') = e^{\frac{iaa'}{\hbar}}$ is simply defined from the relation

$$\check{A}(a)M(a, a') = \check{B}(a')M(a, a') \tag{2.5}$$

Here a delicate point is the possibility to multiply the operator B in (2.4) by an arbitrary function of a, which commutes with A, or conjugate it by a periodic function of a, which does not change B. Both these ambiguities change the normalization of eigenfunctions, this normalization factors require an attention, see more sophisticated examples below.

This is the approach to duality transformations, which we are going to apply in general. That is, we will construct a pair of peculiar operators $\hat{A}$ and $\hat{B}$ such that the conformal block, i.e. the matrix model (β-ensemble) partition function is an eigenfunction of $\hat{A}$, while the modular transformed conformal block is an eigenfunction of $\hat{B}$. Then, the AGT correspondence guarantees the same relation between the S-dual $N = 2$ supersymmetric gauge theories.

As we shall see, the only difference of the modular S-duality from the above ordinary pq-duality is that the relevant operators $\hat{A}$ and $\hat{B}$ in the case of *non-perturbative* conformal blocks commute in a little less trivial way, but *perturbatively* they satisfy exactly (2.1). This explains the perturbative result of [60, 61] for the *properly normalized* conformal blocks and straightforwardly provides its non-perturbative generalization, which is in accordance with [27, 28].

3 Modular transformations: conformal blocks and β-ensembles

The infinite conformal symmetry in two dimensions allows one to expand any correlator in CFT into the conformal blocks [25, 26]. We consider here the one-point correlator on torus.

The corresponding conformal block is usually represented by a series in the torus modular parameter $q = e^{\pi i \tau}$ with coefficients depending on the external dimension Δ_{ext} and on the intermediate dimension Δ. Hereafter, we use the following useful parametrization of the CFT quantities

$$\Delta = \frac{Q^2}{4} - a^2, \quad \Delta_{\text{ext}} = \mu(Q - \mu), \quad c = 1 + 6Q^2, \quad Q = b + b^{-1} \tag{3.1}$$

We assume the conformal block to be normalized as follows

$$B_a(\tau|\mu) = 1 + q \left(\frac{\Delta_{\text{ext}}(1 - \Delta_{\text{ext}})}{2\Delta} + 1 \right) + O(q^2) \tag{3.2}$$

Throughout the paper we rescale the conformal dimensions to include g_s^2 so that the perturbative series correspond to the large a expansions.

The $6j$-symbols (the Racah coefficients) for the Virasoro algebra can be realized as the fusion relation connecting the conformal blocks at modular transformed moduli of the torus:

$$B_a(\tau|\mu) = \int db M(a, b) B_b \left(-\frac{1}{\tau} \Big| \mu \right) \tag{3.3}$$

The conformal block can be related to the elliptic β-ensemble partition function [76, 77]

$$Z_a(\tau|\mu) = q^{-a^2} \int_0^\pi dz_1 \ldots \int_0^\pi dz_N \prod_{i<j} \theta(z_i - z_j)^{-2b^2} \prod_i \theta(z_i)^{-2b\mu} e^{-4ia\left(\sum_i bz_i + \mu w\right)}, \tag{3.4}$$

with the number of integrals constrained by the condition $\mu + bN = 0$. Here w is the point of the toric conformal block, where the operator is inserted, and, because of this condition, the integral does not depend on it (it can be put just zero). The toric heat kernel here reads

$$\theta(z) = 2q^{\frac{1}{8}} \sin z \prod_{n=1}^\infty (1 - q^n)(1 - 2q^n \cos 2z + q^{2n}) = \sum_{n=0}^\infty (-1)^n q^{n(n+1)/2} \sin(2n + 1)z \tag{3.5}$$

The concrete relation of this partition function and the conformal block is described by the formula[1]

$$Z_a(\tau|\mu) = \frac{Z_a(i\infty|\mu)}{\eta(q)^\nu} B_a(\tau|\mu), \quad \nu = 3\Delta_{\text{ext}} + 3N - 1 \tag{3.6}$$

The claim of [60, 61] was that for any set of parameters

$$Z_a(\tau|\mu) = \int db e^{2\pi i ab} Z_b \left(-\tau^{-1} | \mu \right) \tag{3.7}$$

at any perturbative order in μ/a. Here the conformal block $Z_a(\tau|\mu)$ and its modular transformed $Z_b \left(-\tau^{-1} | \mu \right)$ play the role of $Z_a(\mathcal{Q})$ and $\tilde{Z}_b(\mathcal{Q})$ of s.2 correspondingly.

---

[1]We choose the Dedekind function to be $\eta(q) = q^{\frac{1}{24}} \prod_n (1 - q^n)$.

4 Modular transformation of β-ensemble: perturbative level

4.1 Loop equations and their symmetries

The key role in our consideration is played by the resolvent operator. A net definition for the n-point resolvent for the β-ensemble on some generic Riemann surface can be given as an average

$$R_n(\xi_1,\ldots,\xi_n) = \left\langle \left(\sum_{i_1} \frac{E'(\xi_1,z_{i_1})}{E(\xi_1,z_{i_1})} \right) \ldots \left(\sum_{i_n} \frac{E'(\xi_n,z_{i_n})}{E(\xi_n,z_{i_n})} \right) \right\rangle \tag{4.1}$$

over a β-ensemble like (3.4), with x_i being integration variables in the β-ensemble and the prime means differentiating with respect to the first argument. Here $E(z,w)$ is the prime form [78], its logarithm plays a role of Green function for the scalars; in our particular toric case it is given by expression (3.5) (up to inessential constant which cancels out in the ratio).

One can introduce (infinitely many) additional time variables, in order to generate the multi-point disconnected resolvents by an operator acting on these times, [48–52] so that

$$R_n(\xi_1,\ldots,\xi_n) = Z^{-1}\hat{\nabla}(\xi_1)\ldots\hat{\nabla}(\xi_n)Z \tag{4.2}$$

Similarly, one introduces a set of connected resolvents

$$\rho_n(\xi_1,\ldots,\xi_n) = \hat{\nabla}(\xi_1)\ldots\hat{\nabla}(\xi_n)\log Z \tag{4.3}$$

Following [79] an infinite set of Ward identities for the β-ensemble partition function can be derived in a simple way by the shift of the integration variables

$$z_i \to z_i + \epsilon\partial_\xi \log\theta(\xi - z_i) \tag{4.4}$$

Thus at the first ϵ-order one derives an identity

$$\left\langle \left(\sum_i \frac{\theta'(\xi-z_i)}{\theta(\xi-z_i)} \right)^2 - \sum_i \frac{\theta''(\xi-z_i)}{\theta(\xi-z_i)} \right.$$
$$+ (-2b^2\mu\partial_\xi\log\theta(\xi-w) + 4ib^2a)\sum_i \frac{\theta'(\xi-z_i)}{\theta(\xi-z_i)}$$
$$+ 2\mu b\sum_i \frac{\theta'(\xi-z_i)}{\theta(\xi-z_i)} \left(\sum_i \frac{\theta'(z_i-w)}{\theta(z_i-w)} - \sum_i \frac{\theta'(\xi-w)}{\theta(\xi-w)} \right)$$
$$\left. - 2b^2\sum_{i<j} \frac{\theta'(z_i-z_j)}{\theta(z_i-z_j)} \left(\sum_i \frac{\theta'(\xi-z_i)}{\theta(\xi-z_i)} - \sum_i \frac{\theta'(\xi-z_j)}{\theta(\xi-z_j)} \right) \right\rangle = 0 \tag{4.5}$$

Using the relation

$$\frac{\theta'(x-y)\theta'(x-z)}{\theta(x-y)\theta(x-z)} + \frac{\theta'(y-x)\theta'(y-z)}{\theta(y-x)\theta(y-z)} + \frac{\theta'(z-x)\theta'(z-y)}{\theta(z-x)\theta(z-y)}$$
$$= \frac{1}{2}\left(\frac{\theta''(x-y)}{\theta(x-y)} + \frac{\theta''(y-z)}{\theta(y-z)} + \frac{\theta''(y-z)}{\theta(y-z)} \right) + 3\eta_1$$

where $\eta_1 = 4\frac{\partial \log \eta}{\partial \log q}$, and after a little algebra one can derive the following loop equation

$$-b^2\langle R(\xi,\xi)\rangle - Qb\langle R'(\xi)\rangle + (-2\mu\partial_\xi \log\theta(z-w) - 4ia)b\langle R(\xi)\rangle + 3b\mu(N+1)\eta_1$$
$$-\frac{\theta'(\xi-w)}{\theta(\xi-w)}\partial_w\log Z - bN\mu\frac{\theta''(\xi-w)}{\theta(\xi-w)} + 4\frac{\partial\log Z}{\partial\log q} + 4a^2 - 4ia\mu\frac{\theta'(\xi-w)}{\theta(\xi-w)} = 0 \quad (4.6)$$

It is much more useful to apply slightly shifted definition of the resolvent operator

$$\hat{\nabla}(\xi)Z = \left\langle b\sum_i \partial_\xi \log\theta(\xi-z_i) + \mu\partial_\xi\log\theta(\xi-w) + 2ia \right\rangle Z \quad (4.7)$$

Then obviously, the partition function is an eigenfunction of the resolvent integral (we take into account that $\mu + bN = 0$)

$$\int_0^\pi d\xi\,\hat{\nabla}(\xi)Z = 2\pi i a Z \quad (4.8)$$

Reformulating the loop equation in these terms, one derives

$$\Big[\hat{\nabla}^2(z) + Q\partial_z(\hat{\nabla}(z) - \mu\partial_z\log\theta(z-w))$$
$$- (\zeta(z-w)\partial_w - \mu^2\wp(z)) + 4q\partial_q - 3\mu(b-\mu)\eta_1\Big]Z = 0 \quad (4.9)$$

where we used the standard elliptic functions [80]

$$\zeta(z) = -\partial_z\log\theta(z)$$
$$\wp(z) = \partial_z^2\log\theta(z) \quad (4.10)$$

The loop equation possesses a symmetry

$$\hat{\nabla}(\xi) \longrightarrow -\hat{\nabla}(\xi) - Q\partial_\xi\log\hat{\nabla}(\xi) - \frac{Q^2}{2}\partial_\xi\left(\frac{\hat{\nabla}'(\xi)}{\hat{\nabla}^2(\xi)}\right)$$
$$- \frac{Q^3}{4}\partial_\xi\left(-\frac{5}{2}\frac{(\hat{\nabla}'(\xi))^2}{\hat{\nabla}^4(\xi)} + \frac{\hat{\nabla}''(\xi)}{\hat{\nabla}^3(\xi)}\right) + O(Q^4) \quad (4.11)$$

Thus, there are two solution branches[2]

$$\oint_A dz\,\hat{\nabla}^{(+)}(z)Z_a^{(+)} = aZ_a^{(+)}, \quad \oint_A dz\,\hat{\nabla}^{(-)}(z)Z_a^{(-)} = -aZ_a^{(-)} \quad (4.13)$$

where the integrals run over the A-period of the spectral surface and the gauge-invariant quantities like the conformal block may depend only on the invariant $\Delta(a) = \frac{Q^2}{4} - a^2$.

---

[2]For the toric block all the higher terms are exact so do not contribute thus giving the symmetry $\oint_\gamma dz\,\hat{\nabla}(z) \leftrightarrow -\oint_\gamma dz\,\hat{\nabla}(z)$.

Notice that for the 4-punctured sphere the second term in the expansion gives a non-vanishing contribution so the symmetry is $\oint_\gamma dz\,\hat{\nabla}(z) \leftrightarrow Q - \oint_\gamma dz\,\hat{\nabla}(z)$ and the symmetric function (invariant) in terms of eigenvalues is the conformal dimension $\Delta(\alpha) = \alpha(Q-\alpha)$, since in this case

$$\oint_A dz\,\hat{\nabla}(z)Z = \alpha Z \quad (4.12)$$

As usual we switch to a symmetric notation assuming $\alpha = Q/2 + a$.
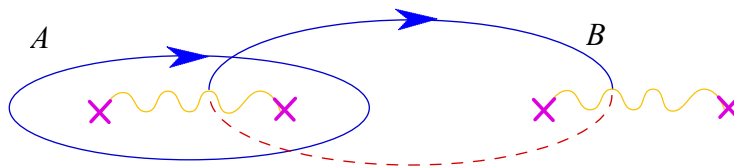
Figure 1. Cycles on the spectral curve

4.2 Resolvents via check-operators [53, 54]

The operator $\hat{\nabla}(z)$ acting on the matrix model partition function on sphere inserts $\sum_i \frac{1}{z-z_i}$ inside the correlators (similarly, it inserts $\sum_i \frac{\theta'(\xi-z_i)}{\theta(\xi-z_i)}$ when acting on the partition function on torus), and therefore is realized as acting on the infinite set of time variables in the partition function entering exponentially the measure, $e^{\sum_{i,k} t_k z_i^k}$:

$$\hat{\nabla}(z) = \sum_k \frac{1}{z^{k+1}} \frac{\partial}{\partial t_k} \tag{4.14}$$

These operators are very convenient in study of the Ward identities in the form of Virasoro constraints *a la* [48–52]. However, in the formalism of loop equations, one usually considers the partition function with most time variables vanishing, and only a few left, $t_k = T_k \neq 0$ for $k \leq N$ (N is an arbitrary integer parameterizing the class of solutions), the solution to the Ward identities being parameterized by an arbitrary function of these remaining variables. Hence, in the formalism of [53, 54] the operator $\check{\nabla}(z)$ can be interpreted as acting in the moduli space of solutions. Remarkably, the *result* of the action of $\hat{\nabla}(z)$ (the average of the resolvent) can be represented as an action of the *other* operator $\check{\nabla}(z)$, acting only on the moduli space. This is a somewhat difficult formalism, but it was developed rather far in [53, 54] and we can now use the results. For our purposes the main point is that while

$$\hat{\nabla}(z|t)Z(t)\Big|_{t=T} = \check{\nabla}(z|T)Z(T) \tag{4.15}$$

this is not true for repeated action of the resolvent operators:

$$\hat{\nabla}(z_1|t)\hat{\nabla}(z_2|t)Z(t)\Big|_{t=T} \neq \check{\nabla}(z_1|T)\check{\nabla}(z_2|T)Z(T) \tag{4.16}$$

Moreover, while $\hat{\nabla}(z)$ operators at different points z commute,

$$[\hat{\nabla}(z_1),\ \hat{\nabla}(z_2)] = 0 \tag{4.17}$$

this is not true for the check-operators:

$$[\check{\nabla}(z_1),\ \check{\nabla}(z_2)] \neq 0 \tag{4.18}$$

The most spectacular result of [53, 54] is that

$$\left[\oint_{A_I} \check{\nabla}(z),\ \oint_{B^J} \check{\nabla}(z)\right] = 2\pi i \delta_I^J \tag{4.19}$$

– 8 –

Actually, [53, 54] presented some evidence in favor of this conjecture by study of the first terms of the genus expansion only, however, hereafter we assume that this is true.

To clarify this relation, consider the 4-point conformal block on sphere (i.e. the matrix model on sphere, see e.g. [60, 35]) with operators inserted at points $(0, 1, q, \infty)$ and notice first that in the course of calculating the B-cycle integral one has to change the branch. Thus, it is more safe to consider "half-cycles" instead. According to [53, 54] the commutation relation for the check-resolvents reads

$$[\check{\nabla}(x), \check{\nabla}(y)] = -\frac{1}{\check{\nabla}(x)\check{\nabla}(y)}(\partial_x - \partial_y)\frac{\check{\nabla}(x)^2 - \check{\nabla}(y)^2}{x - y} + \text{higher orders} \qquad (4.20)$$

In the case of zero external dimensions in the conformal block, the expectation value of the check-resolvents gives a good spherical approximation

$$\langle \check{\nabla}(z) \rangle = \frac{(q(q-1)\partial_q \log Z)^{\frac{1}{2}}}{\sqrt{z(z-q)(z-1)}} \qquad (4.21)$$

where the average is taken in the matrix model on the sphere. Then, the commutator reads

$$[\check{\nabla}(x), \check{\nabla}(y)] = -\frac{1}{g(x)g(y)}(\partial_x - \partial_y)\frac{g(x)^2 - g(y)^2}{x - y} + \text{higher orders} \qquad (4.22)$$

where $g(z) = [z(z-q)(z-1)]^{-\frac{1}{2}}$. We introduce notations $A_{1/2}$ and $B_{1/2}$ for the half-cycles

$$\int_{A_{1/2}} dz = \int_0^q dz, \quad \int_{B_{1/2}} dz = \int_q^1 dz \qquad (4.23)$$

Thus the corresponding integral reads

$$-\int_0^q dx \int_q^1 \frac{1}{g(x)g(y)}(\partial_x - \partial_y)\frac{g(x)^2 - g(y)^2}{x - y} = \frac{\pi i}{2} \qquad (4.24)$$

This simple calculation appeals only to the spherical limit in a simple model, nevertheless, we assume on general grounds (see also [53, 54]) a *non-perturabative* relation

$$\left[\int_{A_{1/2}} dz \, \check{\nabla}^{(\pm)}(z), \int_{B_{1/2}} dz \, \check{\nabla}^{(\pm)}(z)\right] = \frac{\pi i}{2} \qquad (4.25)$$

A discussion of the relation between integrals over half-cycles and full cycles we postpone until the consideration of gauge invariant operators in s. 5.4.

Thus, we have constructed the operators manifestly realizing the *pq*-duality. They allow us to construct a dual pair of the operators A and B, the conformal block and the modular transformation like it was done in section 2.

4.3 The pair of dual check-operators

In order to construct the dual pair note that the action of the resolvent (or check-resolvent) operator on the partition function (conformal block) mimics inserting to the conformal

block the field degenerate at the second level, which can be described in terms of the β-ensemble partition function as

$$Z_{r.op.}(\xi) = \eta^{3\mu(b-\mu)} e^{b \int^{\xi} du \check{\nabla}(u)} Z = \eta^{3\mu(b-\mu)} \left\langle \prod_i \theta(\xi - z_i)^{b^2} \theta(\xi - w)^{\mu b} e^{2iab\xi} \right\rangle Z . \quad (4.26)$$

Indeed, the corresponding loop equation can be presented in the form of elliptic Calogero Schrödinger equation and coincides with the two-point conformal block on torus with one field degenerated at the second level, [35, eq. (34)]

$$\left[4q\partial_q + b^2\partial_z^2 - \left(\zeta(z-w)\partial_w + \Delta_\mu \wp(z-w) - \left(\frac{3}{2b^2} + 1 \right) \eta_1 \right) \right] \tilde{Z}_{r.op.}(z) = 0 \quad (4.27)$$

In other words the insertion of an external degenerate field into the conformal block can be, indeed, mimicked literally by the proper exponential of the check-resolvent. This gives us an operator expressing the monodromy of a degenerate field along the closed contour γ:

$$\mathcal{L}_\gamma \sim e^{b \oint_\gamma dz \, \check{\nabla}(z)} \quad (4.28)$$

It is supposed to represent a "quantum" version of the abelianization map discussed in [64–68].

Now, (4.19) implies that $\mathcal{L}_A$ and $\mathcal{L}_B$ form a pair of dual operators explicitly realized in pq-variables, much similar to the example of s.2. Hence, the corresponding modular transformation is nothing but the Fourier transform, in accordance with [60, 61].

5   Non-perturbative modular transformation

5.1   Phase ambiguity

As we mentioned in our basic example in s.2, to restore the integral kernel, one has to fix the normalization, which is otherwise is not essential. Therefore, it is important to specify the normalization constant $N(a)$ relating the partition functions and the conformal blocks

$$Z_a(\tau|\mu) = N(a|\mu)B_a(\tau|\mu) \quad (5.1)$$

is essential to determine the modular kernel. Now we consider these normalization constants in details.

Toric normalization constant. The normalization constant in the toric case can be simply determined from the partition function integral at $q = 0$

$$N(a) = \int dz_1 \dots \int dz_N \prod_{i<j} (\sin z_{ij})^{-2b^2} \prod_i (\sin z_i)^{-2b\mu} \prod_i e^{-4ibaz_i} \quad (5.2)$$

After changing the variables $z_i = -\frac{i}{2} \log t_i$, the integral reduces (up to an inessential factor) to

$$N(a) = \int dt_1 \dots \int dt_N \prod_{i<j} t_{ij}^{-2b^2} \prod_i t_i^{-2b\left(a+\frac{Q}{2}\right)} (t_i - 1)^{-2b\mu} \quad (5.3)$$

This expression is the Selberg integral which we discuss in appendix B

$$N(a) = \tilde{N}\left(\frac{Q}{2} + a, \mu, \frac{Q}{2} + a\right)$$

$$= \frac{\left(b^{2b^2 - b\mu + 1}/\Gamma(-b^2)\right)^{-\frac{\mu}{b}}}{\Gamma_b(0)\Gamma_b(Q - 2\mu)} \frac{\Gamma_b(-2a + \mu)\Gamma_b(-2a + Q - \mu)}{\Gamma_b(-2a)\Gamma_b(-2a + Q)} \quad (5.4)$$

where $\Gamma_b(x)$ is the Barnes double gamma function (see appendix A.1).

4-punctured sphere normalization constant. Here we have *an ambiguity* in the definition. Indeed consider *two* β-ensembles

$$Z_a^{(1)} = \int_0^x dz_1 \ldots \int_0^x dz_{N_1} \times$$

$$\times \int_0^1 dz_1 \ldots \int_0^1 dz_{N_1} \prod_{i<j} z_{ij}^{-2b^2} \prod_i z_i^{-2b\alpha_0}(z_i - x)^{-2b\alpha_x}(z_i - 1)^{-2b\alpha_1}$$

$$Z_a^{(2)} = \int_0^x dz_1 \ldots \int_0^x dz_{N_1} \times$$

$$\times \int_1^\infty dz_1 \ldots \int_1^\infty dz_{N_1} \prod_{i<j} z_{ij}^{-2b^2} \prod_i z_i^{-2b\alpha_0}(z_i - x)^{-2b\alpha_x}(z_i - 1)^{-2b\alpha_1} \quad (5.5)$$

They both give the same expressions for the conformal blocks, though the normalization constants are different

$$Z_a^{(1)} = \tilde{N}(\alpha, \alpha_x, \alpha_0)\tilde{N}(Q - \alpha_\infty, \alpha_1, \alpha)B_a \quad (5.6)$$
$$Z_a^{(2)} = \tilde{N}(\alpha, \alpha_x, \alpha_0)\tilde{N}(Q - \alpha, \alpha_\infty, \alpha_1)B_a$$

Modular kernels are going to be *different* for these two choices. In fact, there is even a larger ambiguity due to the possibility of using various combinations of the two screening charges (so far we used only one of them, see [42–45]) and there is no *a priori* way to choose between them. One can just say that the modular kernel is defined up to conjugation with these normalization factors.

5.2 Non-perturbative dual monodromies from the check-resolvent: toric example

In fact, the problem with expression (4.28) is that it is not gauge-invariant. One could make it gauge-invariant by taking a trace: a sum of the both branches

$$\mathcal{L}_\gamma \sim e^{b \oint_\gamma dz \, \check{\nabla}^{(+)}(z)} + e^{b \oint_\gamma dz \, \check{\nabla}^{(-)}(z)} \quad (5.7)$$

However, the partition functions are different at different branches, thus, one has to switch to the conformal block which is a gauge-invariant object

$$B_a(\tau|\mu) = \frac{Z_a(\tau|\mu)}{N(a)} \quad (5.8)$$

This means one has to twist the exponentials of the check operator by the corresponding normalization constants

$$e^{b \oint dz \, \check{\nabla}(z)} \longrightarrow \frac{1}{N(a)} e^{b \oint dz \, \check{\nabla}(z)} N(a) \tag{5.9}$$

The branches differ by the sign of the check operator, thus, ultimately the relation between the monodromy operator and the exponential of the check-resolvent reads

$$\mathcal{L}_\gamma^{\text{tor}} = \left[\frac{1}{N(a)} e^{b \oint_\gamma dz \, \check{\nabla}(z)} N(a) + \frac{1}{N(-a)} e^{-b \oint_\gamma dz \, \check{\nabla}(z)} N(-a) \right] \tag{5.10}$$

This operator is well-defined on the whole moduli space and the conformal blocks are eigenvectors of its A-periods.

We need only the a-dependent part of the normalization constant:

$$N(a) = \frac{\Gamma_b(2a + \mu) \Gamma_b(2a + Q - \mu)}{\Gamma_b(2a) \Gamma_b(2a + Q)} \tag{5.11}$$

Substituting the a-representation for the check operator,

$$\oint_A dz \, \check{\nabla}(z) = 2\pi i a, \quad \oint_B dz \, \check{\nabla}(z) = \frac{1}{2} \partial_a \tag{5.12}$$

one derives

$$\mathcal{L}_A = 2 \cos 2\pi b a \tag{5.13}$$

$$\mathcal{L}_B = \frac{\Gamma(2ab) \Gamma(bQ + 2ab)}{\Gamma(b\mu + 2ab) \Gamma(b(Q - \mu) + 2ab)} e^{\frac{b}{2} \partial_a} + \frac{\Gamma(-2ab) \Gamma(bQ - 2ab)}{\Gamma(b\mu - 2ab) \Gamma(b(Q - \mu) - 2ab)} e^{-\frac{b}{2} \partial_a}$$

Thus, we have constructed the two operators, $\mathcal{L}_A$ and $\mathcal{L}_B$ from the check operators with the canonical commutation relations. They provide the exchange relation in the CFT, that is, the modular transformation. Hence, the modular invariance is *a transformation induced by the pq-duality.* Moreover, in the perturbative regime (i.e. at large a) these operators contain only one of the two exponentials associated with one of the two branches, i.e. the modular transformation in this regime, indeed, reduces to the Fourier transformation [60, 61] as we discussed in the previous section (the pre-exponential factor in (5.10) in this case, when only one of the exponentials survives is absorbed into the normalization of the conformal block).

5.3 Check and surface operators

The dual operators $\mathcal{L}_{A,B}$ possess also an interpretation as line operators [30, 31]. The explicit expressions for them were already obtained in [30, 31] by some heuristic arguments, and they coincide with the result of our straightforward calculation in the previous subsection.

More concretely, the two fields degenerate at the second level of the Virasoro algebra have the following OPE:

$$\Phi_{(2,1)} \otimes \Phi_{(2,1)} = \Phi_{(1,1)} \oplus \Phi_{(3,1)} \tag{5.14}$$

And the field $\Phi_{(1,1)}$ has dimension 0 and can be thought as an operator acting in the space of conformal blocks. In other words one can perform the following operation $\mathcal{C}$ mapping n-point conformal blocks CB_n to the degenerate $n+2$-point blocks constructing a solution to the equation

$$\mathcal{C} : CB_n \longrightarrow CB_{n|2} \tag{5.15}$$

$$(b^2 L_{-1}^2 - L_{-2}) \langle V_{b/2}(z) V_{b/2}(w) \mathcal{O} \rangle = 0, \quad \langle V_{b/2}(z) V_{b/2}(w) \mathcal{O} \rangle \sim (z-w)^{\frac{b^2}{2}} \langle \mathcal{O} \rangle \tag{5.16}$$

Using the same differential equation, one can generate a monodromy transformation making a parallel transport of one of the degenerate fields along some contour γ:

$$\mathcal{M}_\gamma : CB_{n|2} \longrightarrow CB_{n|2} \tag{5.17}$$

In this way, one constructs the Verlinde (surface) [30, 31, 81, 82] operator

$$\mathcal{L}_\gamma = \mathcal{C}^{-1} \mathcal{M}_\gamma \mathcal{C} : CB_n \longrightarrow CB_n \tag{5.18}$$

It is important to show that this operator can be formulated as a differential operator acting on the conformal block, at least in some abstract form. This makes the Verlinde operator quite similar to the check operator constructed within the matrix model framework and means that the Verlinde operator is a kind of exponential of the check operator. This should be compared with what we did in [35] considering a slightly different operator $\mathcal{L}_\gamma = (C'_x)^{-1} \mathcal{M}_\gamma C'_x$

$$C'_x : CB_n \longrightarrow CB_{n|1} \tag{5.19}$$

and constructing it as a solution to the following equation

$$\left[b^2 z(z-1)\partial_z^2 - (2z-1)\partial_z - \frac{x(x-1)}{z-x}\partial_x + \Delta_{1/2b} \right.$$
$$\left. + \frac{\Delta_0}{z} - \frac{\Delta_1}{z-1} - \Delta_\infty + \frac{x^2 - (2x-1)z}{(z-x)^2}\Delta_x \right] B_{4|1}(z|x) = 0,$$
$$B_{4|1}(z|x) = B_4(x)(z-x)^{\frac{Q}{2} - \sqrt{\frac{Q^2}{4} - \Delta}}(1 + O((z-x))) \tag{5.20}$$

This is the equation for the 5-point conformal block with one field degenerate at the second level and with the corresponding intermediate dimension fixed, see [35, eqs. (26) and (30)] for details.

An explicit expression for those operators can be found in terms of CFT [30, 31], not only for the one-point toric but also for the four-point spherical conformal blocks:

- Toric:

$$\mathcal{L}_A = 2\cos 2\pi ba \tag{5.21}$$

$$\mathcal{L}_B = \frac{\Gamma(2ab)\Gamma(bQ + 2ab)}{\Gamma(b\mu + 2ab)\Gamma(b(Q-\mu) + 2ab)} e^{\frac{b}{2}\partial_a} + \frac{\Gamma(-2ab)\Gamma(bQ - 2ab)}{\Gamma(b\mu - 2ab)\Gamma(b(Q-\mu) - 2ab)} e^{-\frac{b}{2}\partial_a}$$

- 4-punctured sphere:

$$\mathcal{L}_A = \cos 2\pi ba, \tag{5.22}$$

$$\mathcal{L}_B = H_+(a)e^{b\partial_a} + H_0(a) + H_-(a)e^{-b\partial_a},$$

where

$$H_\pm(a) = 4\pi^2 \frac{\Gamma\left(b(Q/2 \pm 2a + b)\right)\Gamma\left(b(Q/2 \pm 2a)\Gamma\left(b(\pm 2a + b)\right)\Gamma\left(b(\pm 2a)\right)\right)}{\prod_{s_i=\pm}\Gamma\left(b(Q/2 \pm a + s_1\mu_1 + s_2\mu_2)\right)\Gamma\left(b(Q/2 \pm a + s_3\mu_3 + s_4\mu_4)\right)} \tag{5.23}$$

$$H_0(a) = \frac{\cos \pi b^2}{\cos 4\pi ba - \cos 2\pi b^2}(\cos 2\pi b\mu_2 \cos 2\pi b\mu_3 + \cos 2\pi b\mu_1 \cos 2\pi b\mu_4)$$

$$+ \frac{\cos 2\pi ba}{\cos 4\pi ba - \cos 2\pi b^2}(\cos 2\pi b\mu_1 \cos 2\pi b\mu_3 + \cos 2\pi b\mu_2 \cos 2\pi b\mu_4) \tag{5.24}$$

and the variables μ_i are related to the conformal dimensions in the 4-point spherical conformal block case as $\Delta_i = \mu_i(Q - \mu_i)$. Notice that our normalization for $H_\pm$ differs from [30] by 2π.

The result for the toric case coincides with formula (5.13) obtained in the previous subsection.

5.4 Towards the four-punctured sphere example

In the case of a punctured sphere our approach of s.5.2 becomes more subtle. The problem is that one has to switch branches while going along the B-cycle (see figure 1). However, having constructed the gauge-invariant operator, we expect a natural relation $\mathcal{L}_\gamma \sim \mathcal{L}_{\gamma_{1/2}}\mathcal{L}_{\gamma_{1/2}}$, where $\gamma_{1/2}$ denotes a "half" of the contour going just along one branch (either solid or dashed line on figure 1), though one can not exclude appearance of trace terms in this expression. Hence, generally this operator expansion reads

$$\mathcal{L}_\gamma = c_1\mathcal{L}_{\gamma_{1/2}}\mathcal{L}_{\gamma_{1/2}} + c_2 \tag{5.25}$$

The unknown coefficients c_1 and c_2 can be easily read off from the relation for monodromies along the A-cycle (5.13). Indeed, both the operator and the "half-operator" are well-defined

$$\mathcal{L}_A = 2\cos 2\pi ba, \quad \mathcal{L}_{(0,x)} = 2\cos \pi ba \tag{5.26}$$

Implementing a simple trigonometric identity $\cos 2x = 2\cos^2 x - 1$, one states the realization of the monodromy operator as a check operator

$$\mathcal{L}_\gamma^{4-\text{pun}} = \left[\frac{1}{N(a)}e^{b\int_{\gamma_{1/2}} dz\, \check{\nabla}(z)}N(a) + \frac{1}{N(-a)}e^{-b\int_{\gamma_{1/2}} dz\, \check{\nabla}(z)}N(-a)\right]^2 - 2 \tag{5.27}$$

This expression allows one to compute the shifting coefficients explicitly

$$\mathcal{L}_\gamma = H_+(a)e^{b\partial_a} + H_0(a) + H_-(a)e^{-b\partial_a}, \tag{5.28}$$

$$H_\pm(a) = \frac{N(\pm a + b)}{N(\pm a)},$$

$$H_0(a) = \frac{N(a + b/2)}{N(a)}\frac{N(-a)}{N(-a - b/2)} + \frac{N(-a + b/2)}{N(-a)}\frac{N(a)}{N(a - b/2)} - 2 \tag{5.29}$$

Now one suffices to substitute explicit expressions for $N(a)$ in order to obtain the final answer. However, as we already emphasized in s.5.1, there is an ambiguity in the normalization factor in this case. Hence, at the moment we just read off $N(a)$ from the known surface operator (5.23), leaving a discussion of this subtle point for a separate publication.

If one chooses

$$N(a) = \frac{\prod_{s_i=\pm} \Gamma_b(Q/2 + a + s_1\mu_1 + s_2\mu_2)\Gamma_b(Q/2 + a + s_3\mu_3 + s_4\mu_4)}{\Gamma_b(2a+Q)\Gamma_b(2a)} \tag{5.30}$$

this gives the value of $H_\pm$ coinciding with (5.23). Then, the real challenge is to reproduce the magnetic term contribution H_0. Formula (5.30) can not be appropriate for this purpose, since it is symmetric under permutation of μ_1 and μ_2, and formula (5.24) is not. Note, however, that (5.30) can be multiplied by any periodic function of a with period b, which does not effect $H_\pm$, while changing H_0.

Note also that at particular values $\mu_1 = \mu_4 = \frac{b}{4}$, the correct answer for H_0 is obtained directly from (5.30). Indeed, in this case

$$\frac{\Gamma_b(x + \mu_1 + b/2)\Gamma_b(x - \mu_1 + b/2)}{\Gamma_b(x + \mu_1)\Gamma_b(x - \mu_1)} = \frac{\sqrt{2\pi}b^{b(x-b/4-1/2)}}{\Gamma(bx - b^2/4)} \tag{5.31}$$

Applying this relation one finds the ratio

$$\begin{aligned}
F(a) &= \frac{N(a+b/2)}{N(a)}\frac{N(-a)}{N(-a-b/2)} \\
&= -4\frac{\prod_{s_i=\pm}\cos\left(\pi b\left(a + \frac{b}{4} + s_2\mu_2\right)\right)\cos\left(\pi b\left(a + \frac{b}{4} + s_3\mu_3\right)\right)}{\sin(2\pi ab)\sin(\pi b(2a+b))}
\end{aligned} \tag{5.32}$$

and, after a simple algebra, one indeed obtains

$$F(a) + F(-a) - 2 = H_0\left(a, \mu_1 = \frac{b}{4}, \mu_2, \mu_3, \mu_4 = \frac{b}{4}\right) \tag{5.33}$$

6 Modular kernel non-perturbatively

In this section we demonstrate that the modular kernel can be straightforwardly read off from the equation

$$\mathcal{L}_B(a)M(a,a') = \mathcal{L}_A(a')M(a,a') \tag{6.1}$$

much similar to eq. (2.5) of section 2. Let us consider the toric one-point conformal block, i.e. formulas (5.13). Note that the source of complexity of the modular kernel is a complicated structure of the conformal block asymptotic series $N(a) \neq N(-a)$. Let us be more specific in this place: divide the normalization factor in symmetric and non-symmetric parts $N(a) = N_n(a)N_s(a)$, where $N_s(-a) = N_s(a)$. Then, the monodromy operators can be simplified

$$\begin{aligned}
\mathcal{L}_\gamma &= \frac{1}{N_n(a)N_s(a)}e^{b\oint_\gamma dz\check{\nabla}(z)}N_n(a)N_s(a) \\
&\quad + \frac{1}{N_n(-a)N_s(-a)}e^{-b\oint_\gamma dz\check{\nabla}(z)}N_n(-a)N_s(-a) = N_s(a)^{-1}\mathcal{L}'_\gamma N_s(a), \\
\mathcal{L}'_\gamma &= N_n(a)^{-1}e^{b\oint_\gamma dz\check{\nabla}(z)}N_n(a) + N_n(-a)^{-1}e^{-b\oint_\gamma dz\check{\nabla}(z)}N_n(-a)
\end{aligned} \tag{6.2}$$

since $N_s(a)$ being Weyl symmetric coincides on different branches. Let us split the normalization factor

$$
\begin{aligned}
N(a) &= \frac{\Gamma_b(2a + \mu)\Gamma_b(2a + Q - \mu)}{\Gamma_b(2a)\Gamma_b(2a + Q)} \\
&= \frac{\Gamma_b(2a + \mu)\Gamma_b(Q - 2a)}{\Gamma_b(2a)\Gamma_b(Q - 2a - \mu)} \frac{\Gamma_b(2a + Q - \mu)\Gamma_b(-2a + Q - \mu)}{\Gamma_b(2a + Q)\Gamma_b(-2a + Q)} \\
&= \frac{S_b(2a + \mu)}{S_b(2a)} \left[\frac{\Gamma_b(2a + Q - \mu)\Gamma_b(-2a + Q - \mu)}{\Gamma_b(2a + Q)\Gamma_b(-2a + Q)} \right]
\end{aligned}
\tag{6.3}
$$

where $S_b(x)$ is the double sine function (see appendix A.2), and we throw away the last symmetric multiplier in the brackets.[3] Then, we obtain (cf. with [31, eq. (5.25)])

$$
N_a(a) = \frac{S_b(2a + \mu)}{S_b(2a)},
\tag{6.4}
$$

$$
\mathcal{L}'_A = \cos 2\pi ba, \quad \mathcal{L}'_B = \frac{1}{2}\left(\frac{\sin 2\pi b(a - \mu/2)}{\sin 2\pi ba} e^{-\frac{1}{2}b\partial_a} + \frac{\sin 2\pi b(a + \mu/2)}{\sin 2\pi ba} e^{\frac{1}{2}b\partial_a} \right)
\tag{6.5}
$$

Now one can solve the eigenvalue problem using expressions (6.4) in (6.1):

$$
\frac{1}{2}\left(\frac{\sin 2\pi b(a - \mu/2)}{\sin 2\pi ba} e^{-\frac{b}{2}\partial_a} + \frac{\sin 2\pi b(a + \mu/2)}{\sin 2\pi ba} e^{\frac{b}{2}\partial_a} \right) M(a, a') = \cos 2\pi ba'\, M(a, a')
\tag{6.6}
$$

It is simpler to solve this equation after performing the Fourier transform

$$
M(a, a') = \int_{-\infty}^{\infty} d\xi\, e^{4\pi i a\xi} f_{a'}(\xi)
\tag{6.7}
$$

This leads to the substitution

$$
e^{2\pi iba} \longrightarrow e^{-\frac{b}{2}\partial_\xi}, \quad e^{\frac{b}{2}\partial_a} \longrightarrow e^{2\pi ib\xi}
\tag{6.8}
$$

and we use the following variables

$$
\eta = e^{\pi ib^2}, \quad y = e^{\pi ib\mu}, \quad z = e^{2\pi ib\xi}, \quad s = e^{2\pi iba'}, \quad \hat{X}f(\xi) = e^{\frac{b}{2}\partial_\xi} f(\xi)
\tag{6.9}
$$

Then, the eigenvalue problem reduces to the following algebraic equation

$$
\left[\left(\hat{X}y - \hat{X}^{-1}y^{-1} \right)z^{-1} + \left(\hat{X}y^{-1} - \hat{X}^{-1}y \right)z \right] f_{a'}(\xi) = \left(\hat{X} - \hat{X}^{-1} \right)\left(s + \frac{1}{s} \right) f_{a'}(\xi) \Rightarrow
$$

$$
\Rightarrow \quad \hat{X}^2 f_{a'}(\xi) = \eta^2 \frac{(s - yz)(1 - zys)}{(sy - \eta^2 z)(y - \eta^2 sz)} f_{a'}(\xi)
\tag{6.10}
$$

or, equivalently, to

$$
f_{a'}(\xi + b) = \frac{\sin \pi b\left(\xi + \frac{\mu}{2} - a' \right) \sin \pi b\left(\xi + \frac{\mu}{2} + a' \right)}{\sin \pi b\left(\xi + b - \frac{\mu}{2} - a' \right) \sin \pi b\left(\xi + b - \frac{\mu}{2} + a' \right)} f_{a'}(\xi)
\tag{6.11}
$$

---

[3]In fact, the expressions for the surface operators in [31] and [30] differ exactly by this symmetric factor.

The solution reads

$$f_{a'}(\xi) = \tilde{C}_1(\xi) C_2(a') \frac{S_b\left(\xi + \frac{\mu}{2} - a'\right) S_b\left(\xi + \frac{\mu}{2} + a'\right)}{S_b\left(\xi + b - \frac{\mu}{2} - a'\right) S_b\left(\xi + b - \frac{\mu}{2} + a'\right)}$$

$$= C_1(\xi) C_2(a') \frac{S_b\left(\xi + \frac{\mu}{2} - a'\right) S_b\left(\xi + \frac{\mu}{2} + a'\right)}{S_b\left(\xi + Q - \frac{\mu}{2} - a'\right) S_b\left(\xi + Q - \frac{\mu}{2} + a'\right)}$$

where $C_1(\xi)$ is an arbitrary periodic function with period b, $C_2(a')$ is an arbitrary function and we used the fact that the function $G(x) = e^{\pi i x/b} \frac{S_b(x + 1/b)}{S_b(x)}$ is periodic with period b. Thus, finally,

$$M(a, a') = \int d\xi \, C_1(\xi) C_2(a') \frac{S_b\left(\xi + \frac{\mu}{2} - a'\right) S_b\left(\xi + \frac{\mu}{2} + a'\right)}{S_b\left(\xi + Q - \frac{\mu}{2} - a'\right) S_b\left(\xi + Q - \frac{\mu}{2} + a'\right)} e^{4\pi i a \xi}$$

and there is a freedom in this answer related with the choice of normalization of the conformal block. This result is consistent[4] with [29, (4.41)], and here it is obtained by solving directly the simple and explicit equation (6.1).

Acknowledgments

A   Useful quantum functions

In this appendix we list some useful definitions of the Barnes functions. We follow conventions of [31].

A.1   The double gamma function $\Gamma_b(x)$

This function satisfies the functional equation

$$\Gamma_b(x + b) = \frac{\sqrt{2\pi} b^{bx - \frac{1}{2}}}{\Gamma(bx)} \Gamma_b(x) \tag{A.1}$$

with the ordinary Γ-function in the denominator, and possesses the integral representation

$$\log \Gamma_b(x) = \int_0^\infty \frac{dt}{t} \left(\frac{e^{-xt} - e^{-Qt/2}}{\left(1 - e^{bt}\right)\left(1 - e^{t/b}\right)} - \frac{(Q - 2x)^2}{8e^t} - \frac{Q - 2x}{t} \right) \tag{A.2}$$

which immediately implies

$$\Gamma_b(x) = \Gamma_{1/b}(x) \tag{A.3}$$

---

[4]In order to compare the two answers, one has to use property (A.6).

A.2 The double sine function $S_b(x)$

This function is defined as

$$S_b(x) = \frac{\Gamma_b(x)}{\Gamma_b(Q-x)} \tag{A.4}$$

satisfies the difference equation

$$S_b(x+b) = 2\sin\pi bx \; S_b(x) \tag{A.5}$$

and enjoys the evident property

$$S_b(x)S_b(Q-x) = 1 \tag{A.6}$$

B Normalization of the matrix model partition function

The "holomorphic" three-point correlation function $\tilde{N}$ is defined through the Selberg integral

$$\tilde{N}(\alpha_3,\alpha_2,\alpha_1) = \frac{1}{(2\pi)^N N!} \prod_{i=1}^{N} \int_0^1 dz_i \prod_{i<j} z_{ij}^{-2b^2} \prod_{i=1}^{N} z_i^{-2b\alpha_1}(1-z_i)^{-2b\alpha_2},$$

$$\alpha_1 + \alpha_2 + bN = \alpha_3 \tag{B.1}$$

For any integer N this integral can be calculated explicitly

$$\tilde{N}(\alpha_3,\alpha_2,\alpha_1) = \prod_{j=1}^{N} \frac{\Gamma\left(-2b\alpha_1+1-b^2(j-1)\right)\Gamma\left(-2b\alpha_2+1-b^2(j-1)\right)\Gamma(1-b^2 j)}{\Gamma\left(-2b\alpha_1-2b\alpha_2+2-b^2(N+j-2)\right)\Gamma(1-b^2)} \tag{B.2}$$

Using the functional relation

$$\Gamma(bx) = \sqrt{2\pi}b^{bx-\frac{1}{2}}\frac{\Gamma_b(x)}{\Gamma_b(x+b)} \tag{B.3}$$

one can derive its analytic continuation to arbitrary N [41]

$$\tilde{N}(\alpha_1,\alpha_2,\alpha_3) = \left(\frac{b^{(N+2)b^2+1}}{\Gamma(-b^2)}\right)^N$$

$$\times \frac{\Gamma_b(2Q-\alpha_1-\alpha_2-\alpha_3)\Gamma_b(Q-\alpha_1+\alpha_2-\alpha_3)\Gamma_b(Q-\alpha_1-\alpha_2+\alpha_3)\Gamma_b(-\alpha_1+\alpha_2+\alpha_3)}{\Gamma_b(2Q-2\alpha_1)\Gamma_b(Q-2\alpha_2)\Gamma_b(Q-2\alpha_3)\Gamma_b(0)}$$

References

[1] C. Montonen and D.I. Olive, *Magnetic monopoles as gauge particles?*, *Phys. Lett.* **B 72** (1977) 117 [ɪɴSPIRE].

[2] P. Goddard, J. Nuyts and D.I. Olive, *Gauge theories and magnetic charge*, *Nucl. Phys.* **B 125** (1977) 1 [ɪɴSPIRE].

[3] E. Witten and D.I. Olive, *Supersymmetry algebras that include topological charges*, *Phys. Lett.* **B 78** (1978) 97 [ɪɴSPIRE].

[4] H. Osborn, *Topological charges for $N = 4$ supersymmetric gauge theories and monopoles of spin 1*, *Phys. Lett.* **B 83** (1979) 321 [ɪɴSPIRE].

[5] A. Sen, *Dyon-monopole bound states, selfdual harmonic forms on the multi-monopole moduli space and* $SL(2, \mathbb{Z})$ *invariance in string theory*, *Phys. Lett.* **B 329** (1994) 217 [hep-th/9402032] [ɪɴSPIRE].

[6] N. Seiberg and E. Witten, *Electric-magnetic duality, monopole condensation and confinement in $N = 2$ supersymmetric Yang-Mills theory*, *Nucl. Phys.* **B 426** (1994) 19 [*Erratum ibid.* **B 430** (1994) 485-486] [hep-th/9407087] [ɪɴSPIRE].

[7] N. Seiberg and E. Witten, *Monopoles, duality and chiral symmetry breaking in $N = 2$ supersymmetric QCD*, *Nucl. Phys.* **B 431** (1994) 484 [hep-th/9408099] [ɪɴSPIRE].

[8] E. Witten, *Solutions of four-dimensional field theories via M-theory*, *Nucl. Phys.* **B 500** (1997) 3 [hep-th/9703166] [ɪɴSPIRE].

[9] A. Mikhailov, *BPS states and minimal surfaces*, *Nucl. Phys.* **B 533** (1998) 243 [hep-th/9708068] [ɪɴSPIRE].

[10] D. Gaiotto, $N = 2$ *dualities*, *JHEP* **08** (2012) 034 [arXiv:0904.2715] [ɪɴSPIRE].

[11] A. Gorsky, I. Krichever, A. Marshakov, A. Mironov and A. Morozov, *Integrability and Seiberg-Witten exact solution*, *Phys. Lett.* **B 355** (1995) 466 [hep-th/9505035] [ɪɴSPIRE].

[12] A. Marshakov, M. Martellini and A. Morozov, *Insights and puzzles from branes: 4D SUSY Yang-Mills from 6D models*, *Phys. Lett.* **B 418** (1998) 294 [hep-th/9706050] [ɪɴSPIRE].

[13] A. Gorsky, S. Gukov and A. Mironov, *Multiscale $N = 2$ SUSY field theories, integrable systems and their stringy/brane origin. 1*, *Nucl. Phys.* **B 517** (1998) 409 [hep-th/9707120] [ɪɴSPIRE].

[14] A. Gorsky, S. Gukov and A. Mironov, *SUSY field theories, integrable systems and their stringy/brane origin. 2*, *Nucl. Phys.* **B 518** (1998) 689 [hep-th/9710239] [ɪɴSPIRE].

[15] L.F. Alday, D. Gaiotto and Y. Tachikawa, *Liouville correlation functions from four-dimensional gauge theories*, *Lett. Math. Phys.* **91** (2010) 167 [arXiv:0906.3219] [ɪɴSPIRE].

[16] N. Wyllard, A_{N-1} *conformal Toda field theory correlation functions from conformal $N = 2$* SU(N) *quiver gauge theories*, *JHEP* **11** (2009) 002 [arXiv:0907.2189] [ɪɴSPIRE].

[17] A. Mironov and A. Morozov, *The power of Nekrasov functions*, *Phys. Lett.* **B 680** (2009) 188 [arXiv:0908.2190] [ɪɴSPIRE].

[18] A. Mironov and A. Morozov, *On AGT relation in the case of U(3)*, *Nucl. Phys.* **B 825** (2010) 1 [arXiv:0908.2569] [ɪɴSPIRE].

[19] A. Losev, N. Nekrasov and S.L. Shatashvili, *Issues in topological gauge theory*, *Nucl. Phys.* **B 534** (1998) 549 [`hep-th/9711108`] [INSPIRE].

[20] A. Lossev, N. Nekrasov and S.L. Shatashvili, *Testing Seiberg-Witten solution*, `hep-th/9801061` [INSPIRE].

[21] G.W. Moore, N. Nekrasov and S. Shatashvili, *Integrating over Higgs branches*, *Commun. Math. Phys.* **209** (2000) 97 [`hep-th/9712241`] [INSPIRE].

[22] G.W. Moore, N. Nekrasov and S. Shatashvili, *D particle bound states and generalized instantons*, *Commun. Math. Phys.* **209** (2000) 77 [`hep-th/9803265`] [INSPIRE].

[23] N.A. Nekrasov, *Seiberg-Witten prepotential from instanton counting*, *Adv. Theor. Math. Phys.* **7** (2004) 831 [`hep-th/0206161`] [INSPIRE].

[24] N. Nekrasov and A. Okounkov, *Seiberg-Witten theory and random partitions*, `hep-th/0306238` [INSPIRE].

[25] A.Belavin, A.Polyakov and A.Zamolodchikov, *Infinite conformal symmetry in two dimensional quantum field theory*, *Nucl. Phys.* **B 241** (1984) 333.

[26] A. Zamolodchikov and Al. Zamolodchikov, *Conformal field theory and critical phenomena in 2d systems* (in Russian) (2009).

[27] B. Ponsot and J. Teschner, *Liouville bootstrap via harmonic analysis on a noncompact quantum group*, `hep-th/9911110` [INSPIRE].

[28] B. Ponsot and J. Teschner, *Clebsch-Gordan and Racah-Wigner coefficients for a continuous series of representations of $U_q(sl(2,\mathbb{R}))$*, *Commun. Math. Phys.* **224** (2001) 613 [`math/0007097`] [INSPIRE].

[29] J. Teschner, *From Liouville theory to the quantum geometry of Riemann surfaces*, `hep-th/0308031` [INSPIRE].

[30] L.F. Alday, D. Gaiotto, S. Gukov, Y. Tachikawa and H. Verlinde, *Loop and surface operators in $N = 2$ gauge theory and Liouville modular geometry*, *JHEP* **01** (2010) 113 [`arXiv:0909.0945`] [INSPIRE].

[31] N. Drukker, J. Gomis, T. Okuda and J. Teschner, *Gauge theory loop operators and Liouville theory*, *JHEP* **02** (2010) 057 [`arXiv:0909.1105`] [INSPIRE].

[32] C. Kozcaz, S. Pasquetti and N. Wyllard, *A & B model approaches to surface operators and Toda theories*, *JHEP* **08** (2010) 042 [`arXiv:1004.2025`] [INSPIRE].

[33] K. Maruyoshi and M. Taki, *Deformed prepotential, quantum integrable system and Liouville field theory*, *Nucl. Phys.* **B 841** (2010) 388 [`arXiv:1006.4505`] [INSPIRE].

[34] H. Awata, H. Fuji, H. Kanno, M. Manabe and Y. Yamada, *Localization with a surface operator, irregular conformal blocks and open topological string*, *Adv. Theor. Math. Phys.* **16** (2012) 725 [`arXiv:1008.0574`] [INSPIRE].

[35] A. Marshakov, A. Mironov and A. Morozov, *On AGT relations with surface operator insertion and stationary limit of β-ensembles*, *J. Geom. Phys.* **61** (2011) 1203 [`arXiv:1011.4491`] [INSPIRE].

[36] N. Iorgov, O. Lisovyy and Y. Tykhyy, *Painlevé VI connection problem and monodromy of $c = 1$ conformal blocks*, *JHEP* **12** (2013) 029 [`arXiv:1308.4092`] [INSPIRE].

[37] R. Dijkgraaf and C. Vafa, *Toda theories, matrix models, topological strings and $N = 2$ gauge systems*, `arXiv:0909.2453` [INSPIRE].

[38] H. Itoyama, K. Maruyoshi and T. Oota, *The quiver matrix model and* 2*d-*4*d conformal connection*, *Prog. Theor. Phys.* **123** (2010) 957 [arXiv:0911.4244] [INSPIRE].

[39] T. Eguchi and K. Maruyoshi, *Penner type matrix model and Seiberg-Witten theory*, *JHEP* **02** (2010) 022 [arXiv:0911.4797] [INSPIRE].

[40] T. Eguchi and K. Maruyoshi, *Seiberg-Witten theory, matrix model and AGT relation*, *JHEP* **07** (2010) 081 [arXiv:1006.0828] [INSPIRE].

[41] R. Schiappa and N. Wyllard, *An A_r threesome: matrix models,* 2*d CFTs and* 4*D $N = 2$ gauge theories*, *J. Math. Phys.* **51** (2010) 082304 [arXiv:0911.5337] [INSPIRE].

[42] A. Mironov, A. Morozov and S. Shakirov, *Matrix model conjecture for exact BS periods and Nekrasov functions*, *JHEP* **02** (2010) 030 [arXiv:0911.5721] [INSPIRE].

[43] A. Mironov, A. Morozov and S. Shakirov, *Conformal blocks as Dotsenko-Fateev integral discriminants*, *Int. J. Mod. Phys.* **A 25** (2010) 3173 [arXiv:1001.0563] [INSPIRE].

[44] A. Mironov, A. Morozov and S. Shakirov, *Brezin-Gross-Witten model as 'pure gauge' limit of Selberg integrals*, *JHEP* **03** (2011) 102 [arXiv:1011.3481] [INSPIRE].

[45] A. Mironov, A. Morozov and A. Morozov, *Conformal blocks and generalized Selberg integrals*, *Nucl. Phys.* **B 843** (2011) 534 [arXiv:1003.5752] [INSPIRE].

[46] Vl. Dotsenko and V.Fateev, *Conformal algebra and multipoint correlation functions in 2D statistical models*, *Nucl. Phys.* **B 240** (1984) 312.

[47] G. Felder, *BRST approach to minimal models*, *Nucl. Phys.* **B 317** (1989) 215 [*Erratum ibid.* **B 324** (1989) 548] [INSPIRE].

[48] A.S. Alexandrov, A. Mironov and A. Morozov, *Partition functions of matrix models as the first special functions of string theory. 1. Finite size Hermitean one matrix model*, *Int. J. Mod. Phys.* **A 19** (2004) 4127 [hep-th/0310113] [INSPIRE].

[49] A.S. Alexandrov, A. Mironov and A. Morozov, *M-theory of matrix models*, *Theor. Math. Phys.* **150** (2007) 153 [hep-th/0605171] [INSPIRE].

[50] A.S. Alexandrov, A. Mironov and A. Morozov, *Instantons and merons in matrix models*, *Physica* **D 235** (2007) 126 [hep-th/0608228] [INSPIRE].

[51] A. Alexandrov, A. Mironov and A. Morozov, *BGWM as second constituent of complex matrix model*, *JHEP* **12** (2009) 053 [arXiv:0906.3305] [INSPIRE].

[52] A.S. Alexandrov, A. Mironov, A. Morozov and P. Putrov, *Partition functions of matrix models as the first special functions of string theory. II. Kontsevich model*, *Int. J. Mod. Phys.* **A 24** (2009) 4939 [arXiv:0811.2825] [INSPIRE].

[53] A.S. Alexandrov, A. Mironov and A. Morozov, *Unified description of correlators in non-Gaussian phases of Hermitean matrix model*, *Int. J. Mod. Phys.* **A 21** (2006) 2481 [hep-th/0412099] [INSPIRE].

[54] A.S. Alexandrov, A. Mironov and A. Morozov, *Solving Virasoro constraints in matrix models*, *Fortsch. Phys.* **53** (2005) 512 [hep-th/0412205] [INSPIRE].

[55] B. Eynard, *Topological expansion for the 1-Hermitian matrix model correlation functions*, *JHEP* **11** (2004) 031 [hep-th/0407261] [INSPIRE].

[56] L. Chekhov and B. Eynard, *Hermitean matrix model free energy: Feynman graph technique for all genera*, *JHEP* **03** (2006) 014 [hep-th/0504116] [INSPIRE].

[57] L. Chekhov and B. Eynard, *Matrix eigenvalue model: Feynman graph technique for all genera*, *JHEP* **12** (2006) 026 [math-ph/0604014] [InSPIRE].

[58] B. Eynard and N. Orantin, *Invariants of algebraic curves and topological expansion*, *Commun. Num. Theor. Phys.* **1** (2007) 347 [math-ph/0702045] [InSPIRE].

[59] N. Orantin, *Symplectic invariants, Virasoro constraints and Givental decomposition*, arXiv:0808.0635 [InSPIRE].

[60] D. Galakhov, A. Mironov and A. Morozov, *S-duality as a beta-deformed Fourier transform*, *JHEP* **08** (2012) 067 [arXiv:1205.4998] [InSPIRE].

[61] N. Nemkov, *S-duality as Fourier transform for arbitrary ϵ_1, ϵ_2*, arXiv:1307.0773 [InSPIRE].

[62] M. Billó, M. Frau, L. Gallot, A. Lerda and I. Pesando, *Deformed $N = 2$ theories, generalized recursion relations and S-duality*, *JHEP* **04** (2013) 039 [arXiv:1302.0686] [InSPIRE].

[63] M. Billó, M. Frau, L. Gallot, A. Lerda and I. Pesando, *Modular anomaly equation, heat kernel and S-duality in $N = 2$ theories*, *JHEP* **11** (2013) 123 [arXiv:1307.6648] [InSPIRE].

[64] D. Gaiotto, G.W. Moore and A. Neitzke, *Four-dimensional wall-crossing via three-dimensional field theory*, *Commun. Math. Phys.* **299** (2010) 163 [arXiv:0807.4723] [InSPIRE].

[65] D. Gaiotto, G.W. Moore and A. Neitzke, *Wall-crossing, Hitchin systems and the WKB approximation*, arXiv:0907.3987 [InSPIRE].

[66] D. Gaiotto, G.W. Moore and A. Neitzke, *Framed BPS states*, arXiv:1006.0146 [InSPIRE].

[67] D. Gaiotto, G.W. Moore and A. Neitzke, *Wall-crossing in coupled 2d-4d systems*, arXiv:1103.2598 [InSPIRE].

[68] D. Gaiotto, G.W. Moore and A. Neitzke, *Spectral networks*, *Annales Henri Poincaré* **14** (2013) 1643 [arXiv:1204.4824] [InSPIRE].

[69] S. Fomin and A. Zelevinsky, *Cluster algebras I: foundations*, *Amer. Math. Soc.* **15** (2002) 497 [math/0104151].

[70] S. Fomin and A. Zelevinsky, *Cluster algebras IV: coefficients*, *Composito Math.* **143** (2007) 112 [math/0602259].

[71] V.V. Fock and A.B. Goncharov, *Moduli spaces of local systems and higher Teichmuller theory*, *Publ. Math. Inst. Hautes Études Sci.* **103** (2006) 1 [math/0311149].

[72] S. Fomin, M. Shapiro and D. Thurston, *Cluster algebras and triangulated surfaces. Part I: cluster complexes*, *Acta Math.* **201** (2008) 83 [math/0608367].

[73] S.Fomin and D.Thurston, *Cluster algebras and triangulated surfaces. Part II: Lambda lengths*, arXiv:1210.5569.

[74] M. Kontsevich and Y. Soibelman, *Stability structures, motivic Donaldson-Thomas invariants and cluster transformations*, arXiv:0811.2435 [InSPIRE].

[75] B. Pioline, *Four ways across the wall*, *J. Phys. Conf. Ser.* **346** (2012) 012017 [arXiv:1103.0261] [InSPIRE].

[76] K. Maruyoshi and F. Yagi, *Seiberg-Witten curve via generalized matrix model*, *JHEP* **01** (2011) 042 [arXiv:1009.5553] [InSPIRE].

[77] A. Mironov, A. Morozov and S. Shakirov, *On 'Dotsenko-Fateev' representation of the toric conformal blocks*, *J. Phys.* **A 44** (2011) 085401 [arXiv:1010.1734] [InSPIRE].

[78] J. Fay, *Theta functions on Riemann surfaces*, Lecture Notes in Mathematics volume 352, Springer, Germany (1973).

[79] G. Bonelli, K. Maruyoshi and A. Tanzini, *Quantum Hitchin systems via beta-deformed matrix models*, arXiv:1104.4016 [INSPIRE].

[80] H. Bateman and A. Erdelyi, *Higher transcendental functions volume 2*, London, U.K. (1953).

[81] E.P. Verlinde, *Fusion rules and modular transformations in* 2D *conformal field theory*, *Nucl. Phys.* **B 300** (1988) 360 [INSPIRE].

[82] S. Gukov and E. Witten, *Rigid surface operators*, *Adv. Theor. Math. Phys.* **14** (2010) [arXiv:0804.1561] [INSPIRE].