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Positive solutions of boundary value problem for singular positone and semi-positone third-order difference equations

Chengjun Yuan[1,2*], Gongqi Gai[2] and Yunhui Li[2]

* Correspondence: ycj7102@163.com
[1]School of Mathematics and Statistics, Northeast Normal University, Changchun, 130024, People's Republic of China
Full list of author information is available at the end of the article

Abstract

This article studies the boundary value problems for the third-order nonlinear singular difference equations

$$\Delta^3 u(i-2) + \lambda a(i) f(i, u(i)) = 0, \quad i \in [2, T+2],$$

satisfying five kinds of different boundary value conditions. This article shows the existence of positive solutions for positone and semi-positone type. The nonlinear term may be singular. Two examples are also given to illustrate the main results. The arguments are based upon fixed point theorems in a cone.

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1. Introduction

In this article, we consider the following dynamic equations:

$$\Delta^3 u(i-2) + \lambda a(i) f(i, u(i)) = 0, \quad i \in [2, T+2], \tag{1}$$

satisfying one of the following boundary value conditions:

$$u(0) = u(1) = u(T+3) = 0, \tag{2}$$

$$u(0) = u(1) = \Delta u(T+2) = 0, \tag{3}$$

$$u(0) = u(1) = \Delta^2 u(T+1) = 0, \tag{4}$$

$$u(0) = \Delta^2 u(0) = \Delta u(T+2) = 0, \tag{5}$$

$$\Delta u(0) = \Delta^2 u(0) = u(T+3) = 0, \tag{6}$$

where $a \in C([2, T+2], (0, +\infty))$.

The existence of positive solutions for nonlinear boundary value problems of difference equation have been studied by several authors. We refer the reader to [1-20] and references therein. In [16], the authors studied the following boundary value problem:

$$\Delta^3 u(i-2) + a(i)f(i, u(i)) = 0, \quad i \in [2, T+2] \tag{7}$$

satisfying one of the boundary value conditions (k) (k = 2, 3, ..., 6) with no singularity. The Green functions are constructed carefully, and some verifiable criteria for the existence of at least one positive solution and two positive solutions are obtained by using fixed point theorem.

Recently, some authors studied semi-positone boundary value problem of difference equations, for instance, see [17-20]

The author [17], studied the following second-order semi-positone boundary value problems:

$$\begin{cases} \Delta^2 u(i-1) + \lambda f(u(i)) = 0, \quad i \in [1,\ T], \\ u(0) = u(T+1) = 0, \end{cases} \tag{8}$$

where $\lambda > 0$ is a parameter, with no singularity, and where $f(t, u) \geq -M$ with M being a positive constant. They obtained nonexistence and multiplicity results on sublinear nonlinearities and an existence result on superlinear nonlinearities for (8), respectively.

In [18], the authors are concerned with the discrete third-order three-point boundary value problem:

$$\begin{cases} \Delta^3 u(i) = \lambda g(i) f(u(i)), \quad i = 0, 1, \ldots, n-2, \\ u(0) = \Delta u(p) = \Delta^2 u(n-1) = 0, \end{cases} \tag{9}$$

where $f: (0, \infty) \to (0, \infty)$ is a continuous function, and p and n are positive integers. The existence of positive solutions corresponding to the first eigenvalue of the problem is established, and an interval estimate for the first eigenvalue is obtained. In the nonlinear case, sufficient conditions for the existence and nonexistence of positive solutions are obtained.

It is noted that the boundary value problem (1) with boundary value condition (k) can be viewed as the discrete analogue of the following boundary value problems for ordinary differential equation:

$$u^{(3)}(t) + \lambda a(t) f(t, u(t)) = 0, \quad t \in (0, 1), \tag{10}$$

respectively satisfying the following boundary value conditions

$$u(0) = u'(0) = u(1) = 0, \tag{11}$$

$$u(0) = u'(0) = u'(1) = 0, \tag{12}$$

$$u(0) = u'(0) = u''(1) = 0, \tag{13}$$

$$u(0) = u''(0) = u'(1) = 0, \tag{14}$$

$$u'(0) = u''(0) = u(1) = 0. \tag{15}$$

In engineering, the equation (10) describes an elastic beam in an equilibrium state both the ends of which are simply supported.

Motivated by the results above mentioned, we study the boundary value problems (1), in which nonlinear term may be singularity. In this article, we shall prove our two existence results for the problem (1) using Krasnosel'skii's fixed point theorem. This article is organized as follows. In section 2, starting with some preliminary lemmas, we state the Krasnosel'skii's fixed point theorem. In Section 3, we give the sufficient conditions which state the existence of multiple positive solutions to the positone boundary value problem (1). In Section 4, we give the sufficient conditions which state the existence of at least one positive solutions to the semi-positone boundary value problem (1).

2. Preliminaries

In this section, we state the preliminary information that we need to prove the main results. From Definition 2.1 in [10], we have the following lemmas.

Lemma 2.1 $u(i)$ is a solution of equation (1) with boundary value condition (k) if only and if

$$u(i) = \sum_{j=2}^{T+2} G_k(i,j)a(j)f(j,u(j)), \quad i \in [0, T+3]; \tag{16}$$

where $k = 2, \ldots, 6$, and

$$\begin{aligned}
G_2(i,s) &= \begin{cases} \frac{i(i-1)(T+3-j)(T+4-j)}{2(T+3)(T+2)} - \frac{(i-j)(i-j+1)}{2}, & 0 \le j < i, \\ \frac{i(i-1)(T+3-j)(T+4-j)}{2(T+3)(T+2)}, & i \le j \le T+3; \end{cases}\\
G_3(i,j) &= \begin{cases} \frac{i(i-1)(T+3-j)}{2(T+2)} - \frac{(i-j)(i-j+1)}{2}, & 0 \le j < i, \\ \frac{i(i-1)(T+3-j)}{2(T+2)}, & i \le j \le T+3; \end{cases}\\
G_4(i,j) &= \begin{cases} \frac{i(i-1)}{2} - \frac{(i-s)(i-j+1)}{2}, & 0 \le j < i, \\ \frac{i(i-1)}{2}, & i \le j \le T+3; \end{cases}\\
G_5(i,j) &= \begin{cases} i(T+3-j) - \frac{(i-j)(i-j+1)}{2}, & 0 \le j < i, \\ i(T+3-j), & i \le j \le T+3; \end{cases}\\
G_6(i,j) &= \begin{cases} \frac{(T+3-j)(T+4-j)}{2} - \frac{(i-j)(i-j+1)}{2}, & 0 \le j < i, \\ \frac{(T+3-j)(T+4-j)}{2}, & i \le j \le T+3. \end{cases}
\end{aligned}$$

Lemma 2.2 [10] For $k = 2, \ldots, 6$, we have the conclusions:

$$0 \le G_k(i,j) \le g_k(j), \quad (i,j) \in [0, T+3] \times [2, T+2],$$

$$G_k(i,j) \ge M_k g_k(j), \quad (i,j) \in [2, T+2] \times [2, T+2],$$

where

$$\begin{aligned}
&g_2(j) = G_2(\tau(j), j), \quad \tau(s) = [\tfrac{4T^2+28T+48-4j}{8T+24-4j}], \quad M_2 = \tfrac{2}{(T+1)(T+2)},\\
&g_3(j) = \tfrac{(T+3-j)(j-1)}{2}, \qquad M_3 = \tfrac{2}{(T+1)(T+2)},\\
&g_4(j) = \tfrac{(2T+6-j)(j-1)}{2}, \qquad M_4 = \tfrac{1}{(T+1)(T+2)},\\
&g_5(j) = \tfrac{(T+3-j)(T+2+j)}{2}, \qquad M_5 = \tfrac{2}{T+2},\\
&g_6(j) = \tfrac{(T+3-j)(T+4-j)}{2}, \qquad M_6 = \tfrac{2}{(T+2)}.
\end{aligned}$$

From Lemma 2.2, it is easy to verify the following lemma.

Lemma 2.3 For k = 2, ..., 6, the Green's function $G_k(i, j)$ has properties

$$0 < M_0 h_k(i) g_k(j) \leq G_k(i,j) \leq h_k(i), \quad (i,j) \in [0, T+3] \times [2, T+2],$$

where $M_0 = \min_{2 \leq k \leq 6} \{M_k^*\}$ and

$$h_2(i) = i(i-1), \quad M_2^* = \frac{M_2}{(T+3)^2},$$
$$h_3(i) = i(i-1), \quad M_3^* = \frac{M_3}{(T+3)^2},$$
$$h_4(i) = i(i-1), \quad M_4^* = \frac{M_4}{(T+3)^2},$$
$$h_5(i) = (T+3)i, \quad M_5^* = \frac{M_5}{T+3},$$
$$h_6(i) = (T+4)(T+3-i), \quad M_6^* = \frac{M_6}{T+4}.$$

For our constructions, we shall consider the Banach space $E = C[0, T + 3]$ equipped with the standard norm $\|u\| = \max_{0 \leq i \leq T+3} |u(i)|, u \in E$. We define a cone P_k (k = 2, ..., 6) by

$$P_k = \{u \in X \,|u(i) \geq M_0 h_k(i) \|u\|, \quad i \in [0, T+3]\}.$$

We note that $u(i)$ is a solution of (1) with boundary value condition (k) (k = 2, ..., 6) if and only if

$$u(i) = \lambda \sum_{j=2}^{T+2} G_k(i,j) a(j) f(j, u(j)). \tag{17}$$

The following theorems will play major role in our next analysis.

Theorem 2.4 [21] Let X be a Banach space, and let $P \subset X$ be a cone in X. Let Ω_1, Ω_2 be open subsets of X with $0 \in \Omega_1 \subset \bar{\Omega}_1 \subset \Omega_2$, and let $S : P \to P$ be a completely continuous operator such that, either

1. $||Sw|| \leq ||w||$, $w \in P \cap \partial\Omega_1$, $||Sw|| \geq ||w||$, $w \in P \cap \partial\Omega_2$, or
2. $||Sw|| \geq ||w||$, $w \in P \cap \partial\Omega_1$, $||Sw|| \leq ||w||$ $w \in P \cap \partial\Omega_2$.

Then S has a fixed point in $P \cap \bar{\Omega}_2 \backslash \Omega_1$.

3. Singular positone problems

Theorem 3.1 Let us assume that the following conditions are satisfied,

(H1) $f \in C([2, T + 2] \times (0, +\infty), [0, +\infty))$;

(H2) $f(i, u) \leq K(i)(g(u) + h(u))$ on $[2, T + 2] \times (0, \infty)$ with $g > 0$ continuous and non-increasing on $(0, \infty)$, $h \geq 0$ continuous on $[0, \infty]$ and $\frac{h}{g}$ non-decreasing on $(0, \infty)$, $\exists K_0$ with $g(xy) \leq K_0 g(x) g(y)$ $\forall x > 0, y > 0$;

(H3) There exists $[\alpha, \beta] \subset [2, T + 2]$ such that $\lim_{u \to +\infty} \inf \frac{f(i,u)}{u} = +\infty$ for $i \in [\alpha, \beta]$; and

(H4) There exists $[\alpha_1, \beta_1] \subset [2, +2]$ such that $\lim_{u \to 0+} \inf \frac{f(i,u)}{u} = +\infty$ for $i \in [\alpha_1, \beta_1]$.

Then for each $r > 0$, there exists a positive number λ^* such that the problem (1) with boundary value condition (k) (k = 2, ..., 6) has at least two positive solutions for $0 < \lambda < \lambda^*$.

Proof. Now, we let $k \in [2,6]$ and define the integral operator $T_k : P_k \to E$ by

$$T_k u(i) = \lambda \sum_{j=2}^{T+2} G_k(i,j)a(j)f(j,u(j)),$$

where $P_k = \{u \in X \mid u(i) \geq M_0 h_k(i)||u||, i \in [0, T+3]\}$.

It is easy to check that $T_k(P_k) \subset P_k$. In fact, for each $u \in P_k$, we have by Lemma 2.2 that

$$T_k u(i) \leq \lambda \sum_{j=2}^{T+2} g_k(j)a(j)f(j, u(j)).$$

This implies $||T_k u|| \leq \lambda \sum_{j=2}^{T+2} g_k(j)a(j)f(u(j))$. On the other hand, we have

$$T_k u(i) \geq M_0 h_k(i)\lambda \sum_{j=2}^{T+2} g_k(j)a(j)f(j, u(j)).$$

Thus we have $T_k(P_k) \geq P_k$. In addition, standard argument show that T_k is completely continuous.

For any $r > 0$ given, and take $\Omega_r = \{u \in E \mid ||u|| < r\}$. Choose

$$\lambda^* = \frac{r}{K_0^2 g(M_0) \sum_{j=2}^{T+2} g_k(j)a(j)K(j)g(h_k(j))(g(r)+h(r))}. \tag{18}$$

For $u \in P \cap \partial\Omega_r$. From ($H2$) and (18), we have

$$\begin{aligned} T_k u(i) &= \lambda \sum_{j=2}^{T+2} G_k(i, j)a(j)f(j, u(j)) \\ &\leq \lambda \sum_{j=2}^{T+2} g_k(j)a(j)K(j)g(u)\left(1+\tfrac{h(u)}{g(u)}\right) \\ &\leq \lambda \sum_{j=2}^{T+2} g_k(j)a(j)K(j)g(M_0 h_k(j)r)\left(1+\tfrac{h(r)}{g(r)}\right) \\ &\leq \lambda K_0^2 g(M_0) \sum_{j=2}^{T+2} g_k(j)a(j)K(j)g(h_k(j))(g(r)+h(r)) \\ &< r. \end{aligned}$$

Thus,

$$||T_k u|| < ||u|| \text{for } u \in P_k \cap \partial\Omega_r. \tag{19}$$

Further, choose a constant $M_* > 0$ satisfying that

$$\lambda M^* M_0 \sigma \max_{0 \leq i \leq T+3} \{\sum_{j=\alpha}^{\beta} G_k(i, j)a(j)\} > 1, \tag{20}$$

where $\sigma = \min_{\alpha \leq i \leq \beta} \{h_k(i)\}$.

From $\lim_{u\to+\infty}\inf\frac{f(i,u)}{u} = +\infty$ for $i \in [\alpha, \beta]$, namely (H3), there is a constant $L > r$ such that

$$f(i,\ x) \geq M^{*}x, \quad \forall x \geq L,\ i \in [\alpha,\ \beta].$$

Let $R = r + \frac{L}{M_0\sigma}$ and $\Omega_R = \{u \in E \mid ||u|| < R\}$. For $u \in P_k \cap \partial\Omega_R$, we have that

$$u(i) \geq M_0 h_k(i)||u|| \geq M_0 R h_k(i) \geq M_0 R\sigma \geq L, \quad i \in [\alpha, \beta].$$

It follows that

$$f(i,\ u(i)) \geq M^{*}u(i) \geq M^{*}M_0 R\sigma, \quad i \in [\alpha,\ \beta].$$

Then, for $u \in P \cap \partial\Omega_R$, we have

$$\begin{aligned}
||T_k u|| &= \lambda \max_{0\leq i\leq T+3}\{\sum_{j=2}^{T+2} G_k(i,\ j)a(j)f(j,\ u(j))\} \\
&\geq \lambda \max_{0\leq i\leq T+3}\{\sum_{j=\alpha}^{\beta} G_k(i,\ j)a(j)f(j,\ u(j))\} \\
&\geq \lambda \max_{0\leq i\leq T+3}\{\sum_{j=\alpha}^{\beta} G_k(i,\ j)a(j)M^{*}M_0 R\sigma\} \\
&\geq \lambda M^{*}M_0 R\sigma \max_{0\leq i\leq T+3}\{\sum_{j=\alpha}^{\beta} G_k(i,j)a(j)\} \\
&\geq R.
\end{aligned}$$

Therefore, by the first part of the Fixed Point Theorem 2.4, T_k has a fixed point u_2 with $r \leq ||u_2|| \leq R$.

Finally, choose a constant $M_* > 0$ satisfying that

$$\lambda M_* M_0 \max_{0\leq i\leq T+3}\{\sum_{j=\alpha_1}^{\beta_1} G_k(i,\ j)a(j)h_k(j)\} > 1. \tag{21}$$

By (H4), i.e., $\lim_{u\to 0+}\inf\frac{f(i,u)}{u} = +\infty$ for $i \in [\alpha_1, \beta_1]$, there is a constant $\delta > 0$ and $\delta < r$ such that

$$f(i,\ u) \geq M_* u, \quad \forall u \leq \delta,\ i \in [\alpha_1,\ \beta_1].$$

Let $r_* = \frac{\delta}{2}$ and $\Omega_{r_*} = \{u \in E \mid ||u|| < r_*\}$. For $u \in P_k \cap \partial\Omega_{r_*}$, we have

$$u(i) \geq M_0 h_k(i)||u|| \geq M_0 r_* h_k(i).$$

It follows that

$$f(i,\ u(i)) \geq M_* u(i) \geq M_* M_0 r_* h_k(i), \quad i \in [\alpha_1,\ \beta_1].$$

Then, for $u \in P_k \cap \partial\Omega_{r_*}$, we have

$$\begin{aligned}
||T_k u|| &= \lambda \max_{0\le i\le T+3}\{\sum_{j=2}^{T+2} G_k(i,\ j)a(j)f(j,\ u(j))\}\\
&\ge \lambda \max_{0\le i\le T+3}\{\sum_{j=\alpha_1}^{\beta_1} G_k(i,\ j)a(j)f(j,\ u(j))\}\\
&\ge \lambda \max_{0\le i\le T+3}\{\sum_{j=\alpha_1}^{\beta_1} G_k(i,\ j)a(j)M_* M_0 r_* h_k(j)\}\\
&\ge \lambda M_* M_0 r_* \max_{0\le i\le T+3}\{\sum_{j=\alpha_1}^{\beta_1} G_k(i,j)a(j)h_k(j)\}\\
&\ge r_*.
\end{aligned}$$

Therefore, by the first part of the Fixed Point Theorem 2.4, T_k has a fixed point u_1 with $r_* \le ||u_1|| \le r$. It follows from (19) that $||u_1|| \ne r$.

Then for each $r > 0$, there exists a positive number λ^* such that the problem (1) with boundary value condition (k) (k = 2, ..., 6) has at least two positive solutions u_n (n = 1, 2) with $r_* \le ||u_1|| < r \le ||u_2|| \le R$ for $0 < \lambda < \lambda^*$.

This completes the proof of the theorem.

From the proof of Theorem 3.1, we have the following result.

Corollary 3.2 Assume that **(C1)**-**(C2)** hold. Further, suppose that (H1)-(H3) are satisfied. Then for each $r > 0$, there exists a positive number λ^* such that the problem (1) with boundary value condition (k) has at least one positive solution for $0 < \lambda < \lambda^*$.

Corollary 3.3 Assume that **(C1)**-**(C2)** hold. Further, suppose that (H1)-(H2) and (H4) are satisfied. Then for each $r > 0$, there exists a positive number λ^* such that the problem (1) with boundary value condition (k) has at least one positive solution for $0 < \lambda < \lambda^*$.

Remark Condition (H3) shows that f have the property $\lim_{u\to+\infty} \inf \frac{f(i,u)}{u} = +\infty$ for $i \in [\alpha, \beta]$; condition (H4) shows that f have the property $\lim_{u\to 0+} \inf \frac{f(i,u)}{u} = +\infty$ for $i \in [\alpha_1, \beta_1]$.

Example 3.1 Consider the boundary value problem:

$$-\Delta^3 u(i-2) = \lambda a(i)(c(i)u^{-a} + d(i)u^b(\sin^2 u + 1)), \quad i \in [2,\ T+2], \tag{22}$$

with boundary value condition (k), where $0 < a < 1 < b$ are constants, and

$$c(i) = \begin{cases} 1, & i \in [2,\ T], \\ 0, & i \in [T+1,\ T+2], \end{cases} \quad d(i) = \begin{cases} 0, & i \in [2,\ T-2], \\ 1, & i \in [T-2,\ T+2]. \end{cases}$$

Then for each $r >$, there exists a positive number λ^* such that the problem (22) has at least two positive solutions for $0 < \lambda < \lambda^*$.

In fact, it is clear that

$$f(i,\ u) = c(i)u^{-a} + d(i)u^b(\sin^2 u + 1)$$

and

$$\lim_{u\to 0+} \frac{f(i,u)}{u} = +\infty \text{ for } [\alpha,\ \beta] = [2,\ T] \subset [2,\ T+2];$$
$$\lim_{u\to +\infty} \frac{f(i,u)}{u} = +\infty \text{ for } [\alpha_1,\ \beta_1] = [T-2,\ T+2] \subset [2,\ T+2].$$

Let $K(i) = 1$, $g(u) = u^{-a}$ and $h(u) = 2u^{b}$, we have

$$f(i,\ u) \le K(i)(g(u) + h(u)), \quad K(i) = 1$$

and $g > 0$ continuous and non-increasing on $(0, \infty)$, $h \ge 0$ continuous on $(0, \infty)$ and $\frac{h}{g} = 2u^{a+b}$ non-decreasing on $(0, \infty)$; $K_0 = 1$ with $g(xy) = g(x)g(y) \le K_0 g(x)g(y)\ \forall x > 0$, $y > 0$;

Then, by Theorem 3.1, for each $r > 0$ given, we choose

$$\lambda^* = \frac{M_0^a r^{1+a}}{(1 + 2r^{a+b}) \sum_{j=2}^{T+2} g_k(j)a(j)h_k^{-a}(j)},$$

such that the problem (22) has at least two positive solutions for $0 < \lambda < \lambda^*$.

4. Singular semi-positone problems

Before we prove our next main result, we first state a result.

Lemma 4.1 The difference equation

$$-\Delta^3 u(i-2) = \lambda a(i)e(i), \quad i \in [2,\ T+2], \tag{23}$$

with boundary value condition (k) has a solution w with $w(t) \le c_0 h_k(i)$, where $c_0 = \sum_{j=2}^{T+2} a(j)e(j)$.

In fact, from Lemma 2.1, equation (23) has the solution:

$$w(t) = \sum_{j=2}^{T+2} G_k(i,\ j)a(j)e(j).$$

According to Lemma 2.3, we have

$$w(t) \le h_k(i) \sum_{j=2}^{T+2} a(j)e(j) = c_0 h_k(i).$$

Theorem 4.2 Assume that the following conditions are satisfied:

(B1) $f : [2, T+2] \times (0, \infty) \to R$ is continuous and there exists a function $e \in C([2, T+2], (0, +\infty))$ with $f(i, u) + e(i) \ge 0$ for $(i, u) \in [2, T+2] \times (0, \infty)$;

(B2) $f^*(i, u) = f(i, u) + e(i) \le K(i)(g(u) + h(u))$ on $[2, T+2] \times (0, \infty)$ with $g > 0$ continuous and non-increasing on $(0, \infty)$, $h \ge 0$ continuous on $[0, \infty)$ and $\frac{h}{g}$ non-decreasing on $(0, \infty)$;

(B3) $\exists\ K_0$ with $g(xy) \le K_0 g(x)g(y)\ \forall x > 0, y > 0$;

(B4) There exists $[\alpha, \beta] \subset [2, T+2]$ such that $\lim_{u\to+\infty} \inf \frac{f(i,u)}{u} = +\infty$ for $i \in [\alpha, \beta]$.

Then for each $r > 0$, there exists a positive number λ^* such that the problem (1) with boundary value condition (k) has at least one positive solution for $0 < \lambda < \lambda^*$.

Proof. To show (1) with boundary value condition that (k) has a non-negative solution, we will look at the equation:

$$-\Delta^3 y(i-2) = \lambda a(i) f^*(i, y(i) - \varphi(i)), \quad i \in [2,\ T+2], \tag{24}$$

with boundary value condition (k), where $\phi(i) = \lambda w(i)$; w is as in Lemma 4.1.

We let fixed $k \in [2,6]$. We will show, using Theorem 2.4, that there exists a solution y to (24) with $y(i) > \phi(i)$ for $i \in [2, T+2]$. If this is true, then $u(i) = y(i) - \phi(i)$ $(0 \le i \le T+4)$ is a non-negative solution (positive on [2, T+2]) of (1), since

$$\begin{aligned} -\Delta^3 u(i-2) &= -\Delta^3(y(i-2) - \varphi(i-2)) \\ &= \lambda a(i) f^*(i, y(i) - \varphi(i)) - \lambda a(i) e(i) \\ &= \lambda a(i)[f(i, y(i) - \varphi(i)) + e(i)] - \lambda a(i) e(i) \\ &= \lambda a(i) f(i, y(i) - \varphi(i)) \\ &= \lambda a(i) f(i, u(i)), \quad i \in [0, T+4]. \end{aligned}$$

Next let $T_k : K \to E$ be defined by

$$(T_k y)(i) = \lambda \sum_{j=2}^{T+2} G_k(i, j) a(j) f^*(s, y(s) - \varphi(s)), \quad 0 \le i \le T+3.$$

In addition, standard argument shows that $T_k(P_k) \subset P_k$ and T_k is completely continuous.

For any $r > 0$ given, let

$$\Omega_r = \{y \in E |\ ||y|| < r\}$$

and choose

$$\lambda^* = \min\left\{\frac{M_0 r}{2c_0}, \frac{r}{K_0^2 a_0(g(r) + h(r))}\right\}, \tag{25}$$

where $a_0 = g(\frac{M_0}{2}) \sum_{j=2}^{T+2} g_k(j) a(j) K(j) g(h_k(j))$.

We now show that

$$||T_k y|| \le ||y|| \mathit{for}\ y \in P_k \cap \partial\Omega_r. \tag{26}$$

To see this, let $y \in P_k \cap \partial\Omega_r$. Then $||y|| = r$ and $y(t) \ge M_0 h_k(i) r$ for $i \in [0, T+3]$. For $i \in [0, T+3]$, the Lemma 4.1 and (25) imply that

$$y(i) - \varphi(i) \ge M_0 r h_k(i) - \lambda c_0 h_k(i) \ge (M_0 r - \lambda c_0) h_k(i) \ge \tfrac{M_0 r}{2} h_k(i) > 0,$$

and hence, for $i \in [0, T+4]$, we have

$$\begin{aligned} (T_k y)(i) &= \lambda \sum_{j=2}^{T+2} G_k(i, j) a(j) f^*(j, y(j) - \varphi(j)) \\ &\le \lambda \sum_{j=2}^{T+2} g_k(j) a(j) K(j)[g(y(j) - \varphi(j)) + h(y(j) - \varphi(j))] \\ &= \lambda \sum_{j=2}^{T+2} g_k(j) a(j) K(j) g(y(j) - \varphi(j))\{1 + \tfrac{h(y(j) - \varphi(j))}{g(y(j) - \varphi(j))}\} \\ &\le \lambda \sum_{j=2}^{T+2} g_k(j) a(j) K(j) g(\tfrac{M_0 r}{2} h_k(j))\{1 + \tfrac{h(r)}{g(r)}\} ds \\ &\le \lambda K_0^2 g(\tfrac{M_0}{2})(g(r) + h(r)) \sum_{j=2}^{T+2} g_k(j) a(j) K(j) g(h_k(j)) \\ &= \lambda K_0^2 a_0(g(r) + h(r)) \le r. \end{aligned}$$

This yields $||T_k y|| \le r = ||y||$, and so (26) is satisfied.

Further, choose a constant $M^* > 0$ satisfying that

$$\lambda M^* \frac{M_0}{2}\sigma \max_{0\le i\le T+3}\{\sum_{j=\alpha}^{\beta} G_k(i,\ j)a(j)\} > 1, \tag{27}$$

where $\sigma = \min_{\alpha\le i\le\beta}\{h_k(i)\}$.

By $(B4)$, there is a constant $L > 0$ such that

$$f^*(i,\ x) \ge M^* x, \quad \forall x \ge L,\ i \in [\alpha,\ \beta].$$

Let $R = r + \frac{2L}{M_0\sigma}$ and $\Omega_R = \{y \in\ E \mid ||y|| < R\}$.

Next we show that

$$||T_k y|| \ \ge\ ||y|| \quad \textit{for } y \in P_k \cap \partial\Omega_R. \tag{28}$$

To verify this, let $y \in\ P_k \cap \partial\Omega_R$. Then have

$$y(i) - \varphi(i) \ge M_0 h_k(i)||y|| - \lambda c_0 h_k(i) \ge \frac{M_0}{2} R h_k(i) \ge \frac{M_0}{2} R\sigma \ge L, \quad i \in [\alpha,\ \beta].$$

It follows that, for $y \in\ P_k \cap \partial\Omega_R$, we have

$$f^*(i, y(i) - \varphi(i)) \ge M^*(y(i) - \varphi(i)) \ge M^* \frac{M_0}{2} R\sigma, \quad i \in [\alpha,\ \beta].$$

Then, we have

$$\begin{aligned}
||T_k y|| &= \lambda \max_{0\le i\le T+3}\{\sum_{j=2}^{T+2} G_k(i,\ j)a(j)f^*(j, y(j) - \varphi(j))\} \\
&\ge \lambda \max_{0\le i\le T+3}\{\sum_{j=\alpha}^{\beta} G_k(i,\ j)a(j)f^*(j, y(j) - \varphi(j))\} \\
&\ge \lambda \max_{0\le i\le T+3}\{\sum_{j=\alpha}^{\beta} G_k(i,\ j)a(j)M^* \tfrac{M_0}{2} R\sigma\} \\
&\ge \lambda M^* \tfrac{M_0}{2} R\sigma \max_{0\le i\le T+3}\{\sum_{j=\alpha}^{\beta} G_k(i,\ j)a(j)\} \\
&\ge R.
\end{aligned}$$

This yields $||T_k y|| \ge ||y||$, and so (28) holds.

Therefore, by the first part of the Fixed Point Theorem 2.4, T_k has a fixed point y with $r \le ||y|| \le R$. Since

$$y(i) - \varphi(i) \ge M_0 h_k(i) r - \lambda c_0 h_k(i) \ge (M_0 r - \lambda c_0) h_k(i) > 0, \quad i \in [0,\ T+3].$$

In other words, $u = y - \phi$ is a positive solution of the problem (1) with boundary value condition (k).

This completes the proof of the theorem.

Example 4.1. Consider the boundary value problem:

$$-\Delta^3 y(i-2) = \lambda a(i)(u^{-a} + c(i)u^b - \sin(iu + i^{\frac{1}{2}})) = 0, \quad i \in [2,\ T+2], \tag{29}$$

with boundary value condition (k). Where $0 < a < 1 < b$ are constants and

$$c(i) = \begin{cases} 1, & i \in [2,\ T], \\ 0, & \text{other.} \end{cases}$$

Then for each $r >$, there exists a positive number λ^* such that the problem (29) has at least one positive solution for $0 < \lambda < \lambda^*$.

To verify this, we will apply Theorem 4.2 (here $\lambda^* > 0$ as will be chosen later). Let

$$f(i,\ u) = u^{-a} + c(i)u^b - \sin(iu + i^{\frac{1}{2}}),$$

then condition (B1) holds. Next, we let

$$g(u) = u^{-a},\ h(u) = u^b + 2,\ K(i) = 1,\ e(i) = 1,\ K_0 = 1.$$

It is clear that $0 \le f(i,\ u) + e(i) \le K(i)(g(u) + h(u))$, $g(xy) \le K_0 g(x)g(y)$, and $\lim_{u\to+\infty} \inf \frac{f(i,u)}{u} = +\infty$, $i \in [\alpha,\ \beta] = [2,\ T] \subset [2,\ T+2]$ hold, i.e., conditions (B1)-(B4) hold. Thus, all the conditions of Theorem 4.2 are satisfied.

For each $r >$ given, let

$$\lambda^* = \min\left\{\frac{C_2 r}{2c_0 C_1},\ \frac{r}{K_0 a_0 (g(r) + h(r))}\right\},$$

where

$$c_0 = \sum_{j=2}^{T+2} a(j), \quad a_0 = g(\frac{M_0}{2}) \sum_{j=2}^{T+2} g_k(j)a(j)h_k^{-\alpha}(j).$$

Now Theorem 4.2 guarantees that the above equation has positive solution for $0 < \lambda < \lambda^*$.

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Author details
[1]School of Mathematics and Statistics, Northeast Normal University, Changchun, 130024, People's Republic of China
[2]School of Mathematics and computer, Harbin University, Harbin, 150086, People's Republic of China

Authors' contributions
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