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Existence and monotone iteration of symmetric positive solutions for integral boundary-value problems with ϕ-Laplacian operator

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Abstract
The purpose of this paper is to investigate the existence and iteration of symmetric positive solutions for integral boundary-value problems. An existence result of positive, concave and symmetric solutions and its monotone iterative scheme are established by using the monotone iterative technique. An example is worked out to demonstrate the main result.
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Keywords: integral boundary conditions; monotone iterative technique; positive solutions

1 Introduction

The existence and multiplicity of positive solutions for linear and nonlinear multi-point boundary-value problems have been widely studied by many authors using a fixed point theorem in cones. To identify a few, we refer a reader to [1–4] and references therein.

At the same time, boundary-value problems with integral boundary conditions constitute a very interesting and important class of problems. They include two-, three-, multi-point and nonlocal boundary-value problems as special cases. Hence, increasing attention is paid to boundary-value problems with integral boundary conditions. For an overview of the literature on integral boundary-value problems, see [5–9].

Recently an increasing interest has been observed in investigating the existence of positive solutions of boundary-value problems for differential equations by using the monotone iterative method. In [10–14], the authors used the monotone iterative method to get a positive solution of multi-point boundary-value problems for differential equations. In particular, we would like to mention some excellent results.

In [15], Pang and Tong concentrated on the following problem:

$$\begin{cases} u''(x) + f(x, u(x), u'(x)) = 0, \quad 0 < x < 1, \\ u(0) = u(1) = \int_0^1 p(s)u(s)\,ds. \end{cases}$$

They constructed a specific form of the symmetric upper and lower solutions, and by applying monotone iterative techniques, they constructed successive iterative schemes for approximating solutions.

In [12], Sun *et al.* considered the positive solutions to the p-Laplacian boundary-value problem

$$\begin{cases} (\phi_p(u'(t)))' + q(t)f(t,u(t),u'(t)) = 0, & t \in (0,1), \\ u'(0) = \sum_{i=1}^{n} \alpha_i u'(\xi_i), & u'(1) = \sum_{i=1}^{n} \beta_i u'(\xi_i). \end{cases}$$

They investigated the iteration and existence of positive solutions for the multi-point boundary-value problem with p-Laplacian. By applying monotone iterative techniques, they constructed some successive iterative schemes to approximate the solutions.

In [16], Ding considered the integral boundary-value problem

$$\begin{cases} (\phi(u'(t)))' + f(t,u(t),u'(t)) = 0, & t \in [0,1], \\ u(0) = \int_0^1 u(r)g(r)\,dr, & u(1) = \int_0^1 u(r)h(r)\,dr. \end{cases}$$

By applying classical monotone iterative techniques, they not only obtained the existence of positive solutions, but also gave iterative schemes for approximating the solutions.

So, motivated by all the works above, in this study, we consider the following second-order multi-point integral boundary-value problem (BVP):

$$\begin{cases} (\phi(u'(t)))' + f(t,u(t),u'(t)) = 0, & t \in (0,1), \\ u(0) = \sum_{i=1}^{m-1} \alpha_i \int_{\eta_{i-1}}^{\eta_i} g(r)u(r)\,dr, & u(1) = \sum_{i=1}^{m-1} \alpha_i \int_{\xi_{i-1}}^{\xi_i} g(r)u(r)\,dr, \end{cases} \tag{1.1}$$

where $\phi : \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism and homomorphism with $\phi(0) = 0$. A projection $\phi : \mathbb{R} \to \mathbb{R}$ is called an increasing homeomorphism and homomorphism if the following conditions are satisfied:

(a) If $x \leq y$, then $\phi(x) \leq \phi(y)$ for all $x, y \in \mathbb{R}$;

(b) ϕ is a continuous bijection and its inverse mapping is also continuous;

(c) $\phi(xy) = \phi(x)\phi(y)$ for all $x, y \in \mathbb{R}$.

We will assume that the following assumptions are satisfied:

(C1) $f \in \mathcal{C}([0,1] \times [0,+\infty) \times \mathbb{R}, \mathbb{R})$, $f(t,u,v) > 0$ for all $(t,u,v) \in [0,\frac{1}{2}] \times [0,+\infty) \times \mathbb{R}$ and $f(t,u,v) = f(1-t,u,-v)$ for all $(t,u,v) \in [\frac{1}{2},1] \times [0,+\infty) \times \mathbb{R}$;

(C2) $0 = \eta_0 < \eta_1 < \cdots < \eta_{m-2} < \eta_{m-1} = 1$, $\eta_i + \xi_{m-1-i} = 1$ for $i \in \{0,\ldots,m-1\}$ and $\alpha_i \geq 0$ with $\alpha_i = \alpha_{m-i}$ for $i \in \{1,\ldots,m-1\}$;

(C3) $g \in \mathcal{C}([0,1],[0,+\infty))$ and g is symmetric about $\frac{1}{2}$ on $[0,1]$. In addition $0 < \sum_{i=1}^{m-1} \alpha_i \int_{\eta_{i-1}}^{\eta_i} g(r)\,dr \leq \frac{1}{2}$.

We note that the m-point boundary condition is related to $m-1$ intervals of the area under the curve of the solution $u(t)$ from $t = \eta_{i-1}$ to $t = \eta_i$ for $i = 1,\ldots,m-1$. We investigate here the iteration and existence of symmetric positive solutions for the multi-point integral boundary-value problems with ϕ-Laplacian (1.1). We do not require the existence of lower and upper solutions. By applying monotone iterative techniques, we construct successive iterative schemes for approximating solutions. To the best of our knowledge, no contribution exists concerning the existence of symmetric positive solutions for multi-point boundary-value problems with integral boundary conditions by applying monotone iterative techniques.

2 Preliminaries

In this section, we give some lemmas which help to simplify the presentation of our main result.

Lemma 2.1 *If $h \in \mathcal{C}[0,1]$ is nonnegative on $[0,1]$ and $h(t) \neq 0$ on any subinterval of $[0,1]$, then the BVP*

$$\begin{cases} (\phi(u'(t)))' + h(t) = 0, \quad t \in (0,1), \\ u(0) = \sum_{i=1}^{m-1} \alpha_i \int_{\eta_{i-1}}^{\eta_i} g(r)u(r)\,dr, \qquad u(1) = \sum_{i=1}^{m-1} \alpha_i \int_{\xi_{i-1}}^{\xi_i} g(r)u(r)\,dr, \end{cases} \tag{2.1}$$

has a unique solution $u(t)$ given by

$$\begin{aligned} u(t) = {}& \frac{1}{1 - \sum_{i=1}^{m-1} \alpha_i \int_{\eta_{i-1}}^{\eta_i} g(r)\,dr} \sum_{i=1}^{m-1} \alpha_i \int_{\eta_{i-1}}^{\eta_i} g(r) \int_0^r \phi^{-1}\biggl(A_x - \int_0^s h(\tau)\,d\tau\biggr)\,ds\,dr \\ & + \int_0^t \phi^{-1}\biggl(A_x - \int_0^s h(\tau)\,d\tau\biggr)\,ds \end{aligned}$$

or

$$\begin{aligned} u(t) = {}& -\frac{1}{1 - \sum_{i=1}^{m-1} \alpha_i \int_{\xi_{i-1}}^{\xi_i} g(r)\,dr} \sum_{i=1}^{m-1} \alpha_i \int_{\xi_{i-1}}^{\xi_i} g(r) \int_r^1 \phi^{-1}\biggl(A_x - \int_0^s h(\tau)\,d\tau\biggr)\,ds\,dr \\ & - \int_t^1 \phi^{-1}\biggl(A_x - \int_0^s h(\tau)\,d\tau\biggr)\,ds, \end{aligned}$$

where A_x satisfies

$$\begin{aligned} & \frac{1}{1 - \sum_{i=1}^{m-1} \alpha_i \int_{\eta_{i-1}}^{\eta_i} g(r)\,dr} \sum_{i=1}^{m-1} \alpha_i \int_{\eta_{i-1}}^{\eta_i} g(r) \int_0^r \phi^{-1}\biggl(A_x - \int_0^s h(\tau)\,d\tau\biggr)\,ds\,dr \\ & \quad + \frac{1}{1 - \sum_{i=1}^{m-1} \alpha_i \int_{\xi_{i-1}}^{\xi_i} g(r)\,dr} \sum_{i=1}^{m-1} \alpha_i \int_{\xi_{i-1}}^{\xi_i} g(r) \int_r^1 \phi^{-1}\biggl(A_x - \int_0^s h(\tau)\,d\tau\biggr)\,ds\,dr \\ & \quad + \int_0^1 \phi^{-1}\biggl(A_x - \int_0^s h(\tau)\,d\tau\biggr)\,ds = 0. \end{aligned} \tag{2.2}$$

Proof Suppose that u is a solution of BVP (1.1). By integrating both sides of

$$u'(t) = \phi^{-1}\biggl(A_x - \int_0^t h(s)\,ds\biggr),$$

we have

$$u(t) = u(0) + \int_0^t \phi^{-1}\biggl(A_x - \int_0^s h(\tau)\,d\tau\biggr)\,ds$$

or

$$u(t) = u(1) \int_t^1 \phi^{-1}\biggl(A_x - \int_0^s h(\tau)\,d\tau\biggr)\,ds.$$

According to boundary conditions (2.1), we have

$$\begin{aligned} u(t) = {}& \frac{1}{1 - \sum_{i=1}^{m-1} \alpha_i \int_{\eta_{i-1}}^{\eta_i} g(r)\,dr} \sum_{i=1}^{m-1} \alpha_i \int_{\eta_{i-1}}^{\eta_i} g(r) \int_0^r \phi^{-1}\biggl(A_x - \int_0^s h(\tau)\,d\tau\biggr)\,ds\,dr \\ & + \int_0^t \phi^{-1}\biggl(A_x - \int_0^s h(\tau)\,d\tau\biggr)\,ds \end{aligned}$$

or

$$u(t) = -\frac{1}{1-\sum_{i=1}^{m-1}\alpha_i\int_{\xi_{i-1}}^{\xi_i}g(r)\,dr}\sum_{i=1}^{m-1}\alpha_i\int_{\xi_{i-1}}^{\xi_i}g(r)\int_r^1\phi^{-1}\left(A_x-\int_0^s h(\tau)\,d\tau\right)ds\,dr$$
$$-\int_t^1\phi^{-1}\left(A_x-\int_0^s h(\tau)\,d\tau\right)ds,$$

where A_x satisfies (2.2). □

Lemma 2.2 *If* $h\in\mathcal{C}[0,1]$ *is nonnegative on* $[0,1]$ *and* $h(t)\neq 0$ *on any subinterval of* $[0,1]$, *then there exists a unique* $A_x\in(-\infty,+\infty)$ *satisfying* (2.2). *Moreover, there is a unique* $\delta_x\in(0,1)$ *such that* $A_x=\int_0^{\delta_x}h(\tau)\,d\tau$.

Proof For any h in Lemma 2.1, define

$$\begin{aligned}H_x(c) = {}&\frac{1-\sum_{i=1}^{m-1}\alpha_i\int_{\xi_{i-1}}^{\xi_i}g(r)\,dr}{1-\sum_{i=1}^{m-1}\alpha_i\int_{\eta_{i-1}}^{\eta_i}g(r)\,dr}\\ &\times\sum_{i=1}^{m-1}\alpha_i\int_{\eta_{i-1}}^{\eta_i}g(r)\int_0^r\phi^{-1}\left(c-\int_0^s h(\tau)\,d\tau\right)ds\,dr\\ &+\left(1-\sum_{i=1}^{m-1}\alpha_i\int_{\xi_{i-1}}^{\xi_i}g(r)\,dr\right)\int_0^1\phi^{-1}\left(c-\int_0^s h(\tau)\,d\tau\right)ds\\ &+\sum_{i=1}^{m-1}\alpha_i\int_{\xi_{i-1}}^{\xi_i}g(r)\int_r^1\phi^{-1}\left(c-\int_0^s h(\tau)\,d\tau\right)ds\,dr.\end{aligned}\tag{2.3}$$

From the expression of $H_x(c)$ it is easy to see that $H_x:\mathbb{R}\to\mathbb{R}$ is continuous and strictly increasing, and

$$H_x(0)<0,\qquad H_x\left(\int_0^1 h(\tau)\,d\tau\right)>0.$$

Therefore there exists a unique $A_x\in(0,\int_0^1 h(\tau)\,d\tau)\subset(-\infty,+\infty)$ satisfying (2.2). Furthermore, let

$$F(t)=\int_0^t h(\tau)\,d\tau,$$

then $F(t)$ is continuous and strictly increasing on $[0,1]$, and $F(0)=0$, $F(1)=\int_0^1 h(\tau)\,d\tau$. Thus

$$0=F(0)<A_x<F(1)=\int_0^1 h(\tau)\,d\tau$$

implies that there exists a unique $\delta_x\in(0,1)$ such that

$$A_x=\int_0^{\delta_x}h(\tau)\,d\tau.$$ □

Remark 2.1 By Lemmas 2.1 and 2.2 the unique solution of BVP (2.1) is given by

$$u(t) = \begin{cases} \frac{1}{1-\sum_{i=1}^{m-1}\alpha_i \int_{\eta_{i-1}}^{\eta_i} g(r)\,dr} \\ \quad \times \sum_{i=1}^{m-1}\alpha_i \int_{\eta_{i-1}}^{\eta_i} g(r)\int_0^r \phi^{-1}(\int_s^{\delta_x} h(\tau)\,d\tau)\,ds\,dr \\ \quad + \int_0^t \phi^{-1}(\int_s^{\delta_x} h(\tau)\,d\tau)\,ds, & 0 \le t \le \delta_x, \\ \frac{1}{1-\sum_{i=1}^{m-1}\alpha_i \int_{\xi_{i-1}}^{\xi_i} g(r)\,dr} \\ \quad \times \sum_{i=1}^{m-1}\alpha_i \int_{\xi_{i-1}}^{\xi_i} g(r)\int_r^1 \phi^{-1}(\int_{\delta_x}^{s} h(\tau)\,d\tau)\,ds\,dr \\ \quad + \int_t^1 \phi^{-1}(\int_{\delta_x}^{s} h(\tau)\,d\tau)\,ds, & \delta_x \le t \le 1. \end{cases} \tag{2.4}$$

Lemma 2.3 *Suppose that* (C1)-(C3) *hold. If* $h \in \mathcal{C}[0,1]$ *is symmetric, nonnegative on* $[0,1]$ *and* $h(t) \not\equiv 0$ *on any subinterval of* $[0,1]$, *then the unique solution* $u(t)$ *of BVP* (2.1) *is concave and symmetric with* $u(t) \ge 0$ *on* $[0,1]$.

Proof Let $u(t)$ be a solution of BVP (2.1). Then

$$\bigl(\phi\bigl(u'(t)\bigr)\bigr)' = -h(t) < 0, \quad t \in [0,1].$$

Therefore, $\phi(u'(t))$ is strictly decreasing. It follows that $u'(t)$ is also strictly decreasing. Thus, $u(t)$ is strictly concave on $[0,1]$. For symmetry of $h(t)$, it is easy to show that $H_x(\int_0^{\delta_x} h(\tau)\,d\tau) = 0$, *i.e.*, $\delta_x = \frac{1}{2}$. So from (2.4) and for $t \in [0,\frac{1}{2}]$, then $1-t \in [\frac{1}{2},1]$, by the transformation $r = 1-\hat{r}$, we have

$$\begin{aligned} u(t) &= \frac{1}{1-\sum_{i=1}^{m-1}\alpha_i \int_{1-\eta_i}^{1-\eta_{i-1}} g(\hat{r})\,d\hat{r}} \sum_{i=1}^{m-1}\alpha_i \int_{1-\eta_i}^{1-\eta_{i-1}} g(\hat{r}) \int_0^{1-\hat{r}} \phi^{-1}\biggl(\int_s^{\frac{1}{2}} h(\tau)\,d\tau\biggr)\,ds\,d\hat{r} \\ &\quad + \int_0^t \phi^{-1}\biggl(\int_s^{\frac{1}{2}} h(\tau)\,d\tau\biggr)\,ds. \end{aligned}$$

Again let $s = 1-\hat{s}$ and by (C2), $\eta_i + \xi_{m-1-i} = 1$ for $i \in \{0,\ldots,m-1\}$. Then

$$\begin{aligned} u(t) &= \frac{1}{1-\sum_{i=1}^{m-1}\alpha_i \int_{\xi_{m-1-i}}^{\xi_{m-i}} g(\hat{r})\,d\hat{r}} \sum_{i=1}^{m-1}\alpha_i \int_{\xi_{m-1-i}}^{\xi_{m-i}} g(\hat{r}) \int_{\hat{r}}^{1} \phi^{-1}\biggl(\int_{1-\hat{s}}^{\frac{1}{2}} h(\tau)\,d\tau\biggr)\,d\hat{s}\,d\hat{r} \\ &\quad + \int_{1-t}^{1} \phi^{-1}\biggl(\int_{1-\hat{s}}^{\frac{1}{2}} h(\tau)\,d\tau\biggr)\,d\hat{s}. \end{aligned}$$

Finally, $\tau = 1-\hat{\tau}$,

$$\begin{aligned} u(t) &= \frac{1}{1-\sum_{i=1}^{m-1}\alpha_i \int_{\xi_{i-1}}^{\xi_i} g(r)\,dr} \sum_{i=1}^{m-1}\alpha_i \int_{\xi_{i-1}}^{\xi_i} g(r) \int_r^1 \phi^{-1}\biggl(\int_{\frac{1}{2}}^{s} h(\tau)\,d\tau\biggr)\,ds\,dr \\ &\quad + \int_{1-t}^{1} \phi^{-1}\biggl(\int_{\frac{1}{2}}^{s} h(\tau)\,d\tau\biggr)\,ds \\ &= u(1-t). \end{aligned} \tag{2.5}$$

We note that $\alpha_i = \alpha_{m-i}$ for $i \in 1,\dots,m-1$. So,

$$\begin{aligned}\sum_{i=1}^{m-1}\alpha_i\int_{\xi_{m-1-i}}^{\xi_{m-i}} g(r)\,dr &= \alpha_1\int_{\xi_{m-2}}^{\xi_{m-1}} g(r)\,dr+\cdots+\alpha_{m-1}\int_{\xi_{m-1-(m-1)}}^{\xi_{m-(m-1)}} g(r)\,dr\\ &= \alpha_{m-1}\int_{\xi_{m-2}}^{\xi_{m-1}} g(r)\,dr+\cdots+\alpha_1\int_{\xi_0}^{\xi_1} g(r)\,dr\\ &= \sum_{i=1}^{m-1}\alpha_i\int_{\xi_{i-1}}^{\xi_i} g(r)\,dr.\end{aligned}$$

If $t\in[\frac{1}{2},1]$, then $1-t\in[0,\frac{1}{2}]$. From (2.5), it follows that $u(1-t)=u(1-(1-t))=u(t)$, $t\in[\frac{1}{2},1]$. This together with (2.5) implies that $u(t)=u(1-t)$, $t\in[0,1]$. Without loss of generality, we assume that $u(0)=\min\{u(0),u(1)\}$. By the concavity of u, we know that $u(t)\ge u(0)$, $t\in[0,1]$. So we obtain

$$u(0)=\sum_{i=1}^{m-1}\alpha_i\int_{\eta_{i-1}}^{\eta_i} g(r)u(r)\,dr \ge u(0)\sum_{i=1}^{m-1}\alpha_i\int_{\eta_{i-1}}^{\eta_i} g(r)\,dr.$$

By $0<\sum_{i=1}^{m-1}\alpha_i\int_{\eta_{i-1}}^{\eta_i} g(r)\,dr<\frac{1}{2}$, it is obvious that $u(0)\ge 0$. Hence, $u(t)\ge 0$, $t\in[0,1]$. □

Let $\mathbb{B}=\mathcal{C}^1[0,1]$ be the Banach space with the norm

$$\|u\| = \max_{t\in[0,1]}\bigl(u^2(t)+u'^2(t)\bigr)^{1/2}.$$

Define the cone $P\subset\mathbb{B}$ by

$$P=\bigl\{u\in\mathbb{B} : u(t) \text{ is nonnegative, symmetric and concave on } [0,1]\bigr\}.$$

We can define an operator $T : P\to\mathbb{B}$ by

$$(Tu)(t) = \begin{cases} \frac{1}{1-\sum_{i=1}^{m-1}\alpha_i\int_{\eta_{i-1}}^{\eta_i} g(r)\,dr} \\ \quad\times \sum_{i=1}^{m-1}\alpha_i\int_{\eta_{i-1}}^{\eta_i} g(r)\int_0^r \phi^{-1}(\int_s^{1/2} f(\tau,u(\tau),u'(\tau))\,d\tau)\,ds\,dr \\ \quad + \int_0^t \phi^{-1}(\int_s^{1/2} f(\tau,u(\tau),u'(\tau))\,d\tau)\,ds, & 0\le t\le 1/2,\\ \frac{1}{1-\sum_{i=1}^{m-1}\alpha_i\int_{\xi_{i-1}}^{\xi_i} g(r)\,dr} \\ \quad\times \sum_{i=1}^{m-1}\alpha_i\int_{\xi_{i-1}}^{\xi_i} g(r)\int_r^1 \phi^{-1}(\int_{1/2}^{s} f(\tau,u(\tau),u'(\tau))\,d\tau)\,ds\,dr \\ \quad + \int_t^1 \phi^{-1}(\int_{1/2}^{s} f(\tau,u(\tau),u'(\tau))\,d\tau)\,ds, & 1/2\le t\le 1. \end{cases} \tag{2.6}$$

It is easy to prove that each fixed point of T is a solution for (1.1).

Lemma 2.4 *Let* (C1)-(C3) *hold. Then* $T : P\to P$ *is completely continuous.*

Proof From the definition of T, it is easy to check that Tu is nonnegative on $[0,1]$ and satisfies boundary conditions (1.1) for all $u\in P$. Furthermore, since $(\phi((Tu)'(t)))'\le 0$, we obtain the concavity of $(Tu)(t)$ on $[0,1]$. From the definition of the symmetry of f, $(Tu)(t)=$

$(Tu)(1-t)$ holds for $t \in [0,1]$. In summary, $Tu \in P$, and $TP \subset P$. Next, by standard methods and the Arzela-Ascoli theorem, one can easily prove that the operator T is completely continuous. □

Lemma 2.5 *Assume that* (C1)-(C3) *hold. Suppose also that there exists* $a > 0$ *such that for* $0 \le t \le 1/2$, $0 \le u_1 \le u_2 \le a$, $0 \le |v_1| \le |v_2| \le a$,

$$f(t,u_1,v_1) \le f(t,u_2,v_2).$$

Then, for $u_1, u_2 \in \overline{P}_a$ *with* $u_1(t) \le u_2(t)$, $|u_1'(t)| \le |u_2'(t)|$, $t \in [0,1]$, *we have*

$$(Tu_1)(t) \le (Tu_2)(t), \qquad \bigl|(Tu_1)'(t)\bigr| \le \bigl|(Tu_2)'(t)\bigr|, \quad t \in [0,1].$$

Proof First we prove that, for all $t \in [0,1/2]$,

$$(Tu_1)(t) \le (Tu_2)(t), \qquad \bigl|(Tu_1)'(t)\bigr| \le \bigl|(Tu_2)'(t)\bigr|.$$

From our assumptions, we have

$$f\bigl(\tau,u_1(\tau),u_1'(\tau)\bigr) \le f\bigl(\tau,u_2(\tau),u_2'(\tau)\bigr), \quad \tau \in [0,1/2],$$

and hence

$$\int_s^{1/2} f\bigl(t,u_1(\tau),u_1'(\tau)\bigr)\,d\tau \le \int_s^{1/2} f\bigl(t,u_2(\tau),u_2'(\tau)\bigr)\,d\tau, \quad s \in [0,1/2].$$

Since ϕ^{-1} is increasing on $\mathbb{R}$, then for all $s \in [0,1/2]$, we obtain

$$\phi^{-1}\biggl(\int_s^{1/2} f\bigl(\tau,u_1(\tau),u_1'(\tau)\bigr)\,d\tau\biggr) \le \phi^{-1}\biggl(\int_s^{1/2} f\bigl(\tau,u_2(\tau),u_2'(\tau)\bigr)\,d\tau\biggr).$$

Thus, for $t \in [0,1/2]$,

$$\begin{aligned}
&(Tu_1)(t) - (Tu_2)(t) \\
&\quad = \frac{1}{1-\sum_{i=1}^{m-1}\alpha_i \int_{\eta_{i-1}}^{\eta_i} g(r)\,dr} \sum_{i=1}^{m-1} \alpha_i \int_{\eta_{i-1}}^{\eta_i} g(r) \int_0^r \biggl[\phi^{-1}\biggl(\int_s^{1/2} f\bigl(\tau,u_1(\tau),u_1'(\tau)\bigr)\,d\tau\biggr) \\
&\qquad - \phi^{-1}\biggl(\int_s^{1/2} f\bigl(\tau,u_2(\tau),u_2'(\tau)\bigr)\,d\tau\biggr)\biggr]\,ds\,dr \\
&\qquad + \int_0^t \biggl[\phi^{-1}\biggl(\int_s^{1/2} f\bigl(\tau,u_1(\tau),u_1'(\tau)\bigr)\,d\tau\biggr) - \phi^{-1}\biggl(\int_s^{1/2} f\bigl(\tau,u_2(\tau),u_2'(\tau)\bigr)\,d\tau\biggr)\biggr]\,ds \\
&\quad \le 0
\end{aligned}$$

and

$$\begin{aligned}
&\bigl|(Tu_1)'(t)\bigr| - \bigl|(Tu_2)'(t)\bigr| \\
&\quad = \phi^{-1}\biggl(\int_t^{1/2} f\bigl(\tau,u_1(\tau),u_1'(\tau)\bigr)\,d\tau\biggr) - \phi^{-1}\biggl(\int_t^{1/2} f\bigl(\tau,u_2(\tau),u_2'(\tau)\bigr)\,d\tau\biggr) \le 0.
\end{aligned}$$

Therefore, $(Tu_1)(t) \leq (Tu_2)(t)$, $|(Tu_1)'(t)| \leq |(Tu_2)'(t)|$ hold for $t \in [0,1/2]$. In fact, if $t \in [1/2,1]$, then $1-t \in [0,1/2]$ and hence from the fact that $(Tu_1)(t)$ and $(Tu_2)(t)$ are symmetric about 1/2 on $[0,1]$, it follows that, for $t \in [1/2,1]$,

$$\begin{aligned}
&(Tu_1)(t) - (Tu_2)(t) = (Tu_1)(1-t) - (Tu_2)(1-t) \leq 0, \\
&\left|(Tu_1)'(t)\right| - \left|(Tu_2)'(t)\right| = \left|(Tu_1)'(1-t)\right| - \left|(Tu_2)'(1-t)\right| \leq 0.
\end{aligned}$$

In summary,

$$(Tu_1)(t) \leq (Tu_2)(t), \qquad \left|(Tu_1)'(t)\right| \leq \left|(Tu_2)'(t)\right|, \quad t \in [0,1]. \qquad \square$$

3 Main results

In this section, we establish our existence result of positive, concave, and symmetric solutions and its monotone iterative scheme for BVP (1.1). For convenience, we denote

$$\begin{aligned}
&A_1 = \max\left\{\frac{1}{1-\sum_{i=1}^{m-1}\alpha_i \int_{\eta_{i-1}}^{\eta_i} g(r)\,dr}, \frac{1}{1-\sum_{i=1}^{m-1}\alpha_i \int_{\xi_{i-1}}^{\xi_i} g(r)\,dr}\right\}, \\
&A = \max\{\sqrt{2}A_1, 2\sqrt{2}\}.
\end{aligned}$$

Theorem 3.1 *Assume that* (C1)-(C3) *hold. If there exists $a > 0$ such that*

(i) *$f(t,u_1,v_1) \leq f(t,u_2,v_2)$ for any $0 \leq t \leq \frac{1}{2}$, $0 \leq u_1 \leq u_2 \leq a$, $0 \leq |v_1| \leq |v_2| \leq a$;*

(ii) *$\max_{0 \leq t \leq \frac{1}{2}} f(t,a,a) \leq \phi(\frac{a}{A})$.*

Then (1.1) *has at least two positive, concave, and symmetric solutions w^* and v^* satisfying*

$$\begin{aligned}
&0 < w^* \leq \frac{a\sqrt{2}}{2}, \qquad 0 < \left|(w^*)'\right| \leq \frac{a\sqrt{2}}{2}, \\
&\lim_{n\to\infty} w_n = \lim_{n\to\infty} T^n w_0 = w^*, \\
&\lim_{n\to\infty} (w_n)' = \lim_{n\to\infty} \left(T^n w_0\right)' = \left(w^*\right)',
\end{aligned}$$

where

$$w_0(t) = \begin{cases} \frac{a\sqrt{2}}{4}\left(\frac{\sum_{i=1}^{m-1}\alpha_i \int_{\eta_{i-1}}^{\eta_i} g(r)\,dr}{1-\sum_{i=1}^{m-1}\alpha_i \int_{\eta_{i-1}}^{\eta_i} g(r)\,dr} + 2t\right), & 0 \leq t \leq 1/2, \\ \frac{a\sqrt{2}}{4}\left(\frac{\sum_{i=1}^{m-1}\alpha_i \int_{\xi_{i-1}}^{\xi_i} g(r)\,dr}{1-\sum_{i=1}^{m-1}\alpha_i \int_{\xi_{i-1}}^{\xi_i} g(r)\,dr} + 2(1-t)\right), & 1/2 \leq t \leq 1 \end{cases} \tag{3.1}$$

and

$$\begin{aligned}
&0 < v^* \leq a, \qquad 0 < \left|(v^*)'\right| \leq a, \\
&\lim_{n\to\infty} v_n = \lim_{n\to\infty} T^n v_0 = v^*, \\
&\lim_{n\to\infty} (v_n)' = \lim_{n\to\infty} \left(T^n v_0\right)' = \left(v^*\right)',
\end{aligned}$$

where $v_0(t) = 0$, $0 \leq t \leq 1$.

Proof Let $\overline{P}_a = \{u \in P \mid \|u\| \le a\}$. Define two functionals on $\mathbb{B}$ as follows:

$$\alpha(u) = \max_{0\le t\le 1}\left|u(t)\right|, \qquad \beta(u) = \max_{0\le t\le 1}\left|u'(t)\right|.$$

Then

$$\|u\| \le \sqrt{2}\max\left\{\alpha(u), \beta(u)\right\}.$$

If $u \in \overline{P}_a$, then $\|u\| \le a$. Hence,

$$0 \le u(t) \le \max_{0\le t\le 1}\left|u(t)\right| \le \|u\| \le a, \qquad 0 \le \left|u'(t)\right| \le \max_{0\le t\le 1}\left|u'(t)\right| \le \left\|u'\right\| \le a.$$

From (i) and (ii), we have that

$$0 < f\left(t, u(t), u'(t)\right) \le f(t, a, a) \le \max_{0\le t\le 1/2} f(t, a, a) \le \phi\left(\frac{a}{A}\right).$$

Then,

$$\begin{aligned}
\alpha(Tu) &= \max_{0\le t\le 1}\left|(Tu)(t)\right| = (Tu)\left(\frac{1}{2}\right) \\
&= \frac{1}{1-\sum_{i=1}^{m-1}\alpha_i\int_{\eta_{i-1}}^{\eta_i} g(r)\,dr} \\
&\quad \times \sum_{i=1}^{m-1}\alpha_i\int_{\eta_{i-1}}^{\eta_i} g(r)\int_0^r \phi^{-1}\left(\int_s^{1/2} f\left(\tau, u(\tau), u'(\tau)\right)d\tau\right)ds\,dr \\
&\quad + \int_0^{1/2}\phi^{-1}\left(\int_s^{1/2} f\left(\tau, u(\tau), u'(\tau)\right)d\tau\right)ds \\
&= \frac{1}{1-\sum_{i=1}^{m-1}\alpha_i\int_{\xi_{i-1}}^{\xi_i} g(r)\,dr} \\
&\quad \times \sum_{i=1}^{m-1}\alpha_i\int_{\xi_{i-1}}^{\xi_i} g(r)\int_r^1 \phi^{-1}\left(\int_{1/2}^{s} f\left(\tau, u(\tau), u'(\tau)\right)d\tau\right)ds\,dr \\
&\quad + \int_{1/2}^{1}\phi^{-1}\left(\int_{1/2}^{s} f\left(\tau, u(\tau), u'(\tau)\right)d\tau\right)ds \\
&\le \max\left\{\frac{1}{1-\sum_{i=1}^{m-1}\alpha_i\int_{\eta_{i-1}}^{\eta_i} g(r)\,dr}\right. \\
&\quad \times \sum_{i=1}^{m-1}\alpha_i\int_{\eta_{i-1}}^{\eta_i} g(r)\int_0^1 \phi^{-1}\left(\int_0^1 f\left(\tau, u(\tau), u'(\tau)\right)d\tau\right)ds\,dr \\
&\quad + \int_0^1\phi^{-1}\left(\int_0^1 f\left(\tau, u(\tau), u'(\tau)\right)d\tau\right)ds, \\
&\quad \frac{1}{1-\sum_{i=1}^{m-1}\alpha_i\int_{\xi_{i-1}}^{\xi_i} g(r)\,dr} \\
&\quad \times \sum_{i=1}^{m-1}\alpha_i\int_{\xi_{i-1}}^{\xi_i} g(r)\int_0^1 \phi^{-1}\left(\int_0^1 f\left(\tau, u(\tau), u'(\tau)\right)d\tau\right)ds\,dr
\end{aligned}$$

$$\begin{aligned}
&\quad + \int_0^1 \phi^{-1}\left(\int_0^1 f\bigl(\tau, u(\tau), u'(\tau)\bigr)\, d\tau\right) ds\Biggr\} \\
&\leq \frac{a}{A} \max\left\{\frac{1}{1-\sum_{i=1}^{m-1}\alpha_i \int_{\eta_{i-1}}^{\eta_i} g(r)\, dr}, \frac{1}{1-\sum_{i=1}^{m-1}\alpha_i \int_{\xi_{i-1}}^{\xi_i} g(r)\, dr}\right\} \\
&\leq \frac{a\sqrt{2}}{2}.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\beta(Tu) &= \max_{0\leq t\leq 1}\bigl|(Tu)'(t)\bigr| = (Tu)'(0) \\
&= \phi^{-1}\left(\int_0^{1/2} f\bigl(\tau, u(\tau), u'(\tau)\bigr)\, d\tau\right) \\
&\leq \frac{a}{A}\frac{A}{\sqrt{2}} = \frac{a\sqrt{2}}{2}.
\end{aligned}$$

Hence

$$\|Tu\| \leq \sqrt{2}\max\bigl\{\alpha(Tu), \beta(Tu)\bigr\} \leq \sqrt{2}\frac{\sqrt{2}}{2}a = a.$$

So we have $T(\overline{P}_a) \subset \overline{P}_a$. Let

$$w_0(t) = \begin{cases} \frac{a\sqrt{2}}{4}\left(\frac{\sum_{i=1}^{m-1}\alpha_i \int_{\eta_{i-1}}^{\eta_i} g(r)\, dr}{1-\sum_{i=1}^{m-1}\alpha_i \int_{\eta_{i-1}}^{\eta_i} g(r)\, dr} + 2t\right), & 0 \leq t \leq 1/2, \\ \frac{a\sqrt{2}}{4}\left(\frac{\sum_{i=1}^{m-1}\alpha_i \int_{\xi_{i-1}}^{\xi_i} g(r)\, dr}{1-\sum_{i=1}^{m-1}\alpha_i \int_{\xi_{i-1}}^{\xi_i} g(r)\, dr} + 2(1-t)\right), & 1/2 \leq t \leq 1. \end{cases} \tag{3.2}$$

Then $\|w_0\| \leq \frac{a\sqrt{2}}{2}$ and $w_0(t) \in \overline{P}_a$. Let $w_1(t) = (Tw_0)(t)$, then $w_1 \in \overline{P}_a$. Now we denote a sequence $\{w_n\}$ by the iterative scheme

$$w_{n+1} = Tw_n = T^n w_0, \quad n = 0, 1, 2, \ldots. \tag{3.3}$$

Since $T(\overline{P}_a) \subset \overline{P}_a$ and $w_0(t) \in \overline{P}_a$, we have $w_n \in \overline{P}_a$, $n = 0, 1, 2, \ldots$. From Lemma 2.4, T is compact, we assert that $\{w_n\}_{n=1}^{\infty}$ has a convergent subsequence $\{w_{n_k}\}_{k=1}^{\infty}$ and there exists $w^* \in \overline{P}_a$ such that $w_{n_k} \to w^*$. On the other hand, since

$$\begin{aligned}
w_1(t) &= (Tw_0)(t) \\
&= \begin{cases} \frac{1}{1-\sum_{i=1}^{m-1}\alpha_i \int_{\eta_{i-1}}^{\eta_i} g(r)\, dr} \\ \quad \times \sum_{i=1}^{m-1}\alpha_i \int_{\eta_{i-1}}^{\eta_i} g(r) \int_0^r \phi^{-1}(\int_s^{1/2} f(\tau, w_0(\tau), w_0'(\tau))\, d\tau)\, ds\, dr \\ \quad + \int_0^t \phi^{-1}(\int_s^{1/2} f(\tau, w_0(\tau), w_0'(\tau))\, d\tau)\, ds, & 0 \leq t \leq 1/2, \\ \frac{1}{1-\sum_{i=1}^{m-1}\alpha_i \int_{\xi_{i-1}}^{\xi_i} g(r)\, dr} \\ \quad \times \sum_{i=1}^{m-1}\alpha_i \int_{\xi_{i-1}}^{\xi_i} g(r) \int_r^1 \phi^{-1}(\int_{1/2}^s f(\tau, w_0(\tau), w_0'(\tau))\, d\tau)\, ds\, dr \\ \quad + \int_t^1 \phi^{-1}(\int_{1/2}^s f(\tau, w_0(\tau), w_0'(\tau))\, d\tau)\, ds, & 1/2 \leq t \leq 1 \end{cases} \\
&\leq \begin{cases} \frac{a}{A}\left(\frac{\sum_{i=1}^{m-1}\alpha_i \int_{\eta_{i-1}}^{\eta_i} g(r)\, dr}{1-\sum_{i=1}^{m-1}\alpha_i \int_{\eta_{i-1}}^{\eta_i} g(r)\, dr} + t\right), & 0 \leq t \leq 1/2, \\ \frac{a}{A}\left(\frac{\sum_{i=1}^{m-1}\alpha_i \int_{\xi_{i-1}}^{\xi_i} g(r)\, dr}{1-\sum_{i=1}^{m-1}\alpha_i \int_{\xi_{i-1}}^{\xi_i} g(r)\, dr} + (1-t)\right), & 1/2 \leq t \leq 1 \end{cases}
\end{aligned}$$

$$\begin{aligned} &\le \begin{cases} \frac{a\sqrt{2}}{4}\left(\frac{\sum_{i=1}^{m-1}\alpha_i\int_{\eta_{i-1}}^{\eta_i}g(r)\,dr}{1-\sum_{i=1}^{m-1}\alpha_i\int_{\eta_{i-1}}^{\eta_i}g(r)\,dr}+2t\right), & 0\le t\le 1/2,\\ \frac{a\sqrt{2}}{4}\left(\frac{\sum_{i=1}^{m-1}\alpha_i\int_{\xi_{i-1}}^{\xi_i}g(r)\,dr}{1-\sum_{i=1}^{m-1}\alpha_i\int_{\xi_{i-1}}^{\xi_i}g(r)\,dr}+2(1-t)\right), & 1/2\le t\le 1\end{cases}\\ &= w_0(t),\quad 0\le t\le 1\end{aligned}$$

and

$$\begin{aligned} \left|w_1'(t)\right| &= \left|(Tw_0)'(t)\right|\\ &= \begin{cases} \phi^{-1}\left(\int_t^{1/2}f(\tau,w_0(\tau),w_0'(\tau))\,d\tau\right), & 0\le t\le 1/2,\\ \phi^{-1}\left(\int_{1/2}^{t}f(\tau,w_0(\tau),w_0'(\tau))\,d\tau\right), & 1/2\le t\le 1\end{cases}\\ &\le \frac{a}{A}\\ &\le \frac{a\sqrt{2}}{2} = \left|w_0'(t)\right|,\quad 0\le t\le 1.\end{aligned}$$

We note that

$$\alpha(w_1) := \max_{0\le t\le 1}\left|w_1(t)\right| \le \frac{a\sqrt{2}}{2},\qquad \beta(w_1) := \max_{0\le t\le 1}\left|w_1'(t)\right| \le \frac{a\sqrt{2}}{2}.$$

Consequently,

$$\|w_1\| \le \sqrt{2}\max\left\{\alpha(w_1),\beta(w_1)\right\} \le a.$$

By Lemma 2.5, we know that T is increasing, it follows that

$$\begin{aligned} &w_2(t) = (Tw_1)(t) \le (Tw_0)(t) = w_1(t),\quad 0\le t\le 1,\\ &\left|w_2'(t)\right| = \left|(Tw_1)'(t)\right| \le \left|(Tw_0)'(t)\right| = \left|w_1'(t)\right|,\quad 0\le t\le 1.\end{aligned}$$

Moreover, we have

$$w_{n+1}(t) \le w_n(t),\qquad \left|w_{n+1}'(t)\right| \le \left|w_n'(t)\right|,\quad 0\le t\le 1, n=0,1,2,\ldots.$$

Therefore, $w_n \to w^*$. Let $n\to\infty$ in (3.3) to obtain $Tw^* = w^*$ since T is continuous. It is well known that the fixed point of the operator T is a solution of BVP (1.1). Therefore, w^* is a concave, symmetric, positive solution of BVP (1.1).

Let $v_0(t) = 0$, $0\le t\le 1$, then $v_0(t)\in\overline{P}_a$. Let $v_1 = Tv_0$, then $v_1\in\overline{P}_a$. We denote

$$v_{n+1} = Tv_n = T^n v_0,\quad n=0,1,2,\ldots.$$

Similar to $\{w_n\}_{n=1}^{\infty}$, we assert that $\{v_n\}_{n=1}^{\infty}$ has a convergent subsequence $\{v_{n_k}\}_{k=1}^{\infty}$ and there exists $v^*\in\overline{P}_a$ such that $v_{n_k}\to v^*$, that is,

$$\begin{aligned} &v_{n_k}(t) \quad\Rightarrow\quad v^*(t)\quad (k\to\infty)\text{ on }[0,1],\\ &v_{n_k}'(t) \quad\Rightarrow\quad v^*(t)\quad (k\to\infty)\text{ on }[0,1].\end{aligned}$$

Since $v_1 = Tv_0 = T0 \in \overline{P}_a$, then

$$v_1(t) = (Tv_0)(t) = (T0)(t) \geq 0, \quad 0 \leq t \leq 1,$$
$$\left|v_1'(t)\right| = \left|(Tv_0)'(t)\right| = \left|(T0)'(t)\right| \geq 0, \quad 0 \leq t \leq 1,$$

we have

$$v_2(t) = (Tv_1)(t) \geq (T0)(t) = v_1(t), \quad 0 \leq t \leq 1,$$
$$\left|v_2'(t)\right| = \left|(Tv_1)'(t)\right| \geq \left|(T0)'(t)\right| = \left|v_1'(t)\right|, \quad 0 \leq t \leq 1.$$

By induction, it is easy to see that for $n = 1, 2, \ldots,$

$$v_{n+1}(t) \geq v_n(t), \qquad \left|v_{n+1}'(t)\right| \geq \left|v_n'(t)\right|, \quad 0 \leq t \leq 1.$$

Thus $v_n \to v^*$ and $Tv^* = v^*$.

Since every fixed point of T in P is a solution of BVP (1.1), then w^* and v^* are two positive, concave, and symmetric solutions of BVP (1.1). □

Corollary 3.1 *Assume that* (C1)-(C3) *and Theorem* 3.1(i) *hold*. *If there exists* $0 < a_1 < a_2 < \cdots < a_n$ *such that*

(iii) $\max_{0\leq t\leq 1/2} f(t, a_k, a_k) \leq \phi(\frac{a_k}{A})$, $k = 1, 2, \ldots, n$.

Then BVP (1.1) *has at least* $2n$ *positive, concave solutions* w_k^* *and* v_k^* *satisfying*

$$0 < w_k^* \leq \frac{a_k\sqrt{2}}{2}, \qquad 0 < \left|\left(w_k^*\right)'\right| \leq \frac{a_k\sqrt{2}}{2},$$
$$\lim_{n\to\infty} w_{k_n} = \lim_{n\to\infty} T^n w_{k_0} = w_k^*,$$
$$\lim_{n\to\infty} (w_{k_n})' = \lim_{n\to\infty} \left(T^n w_{k_0}\right)' = \left(w_k^*\right)',$$

where

$$w_{k_0} = \begin{cases} \frac{a_k\sqrt{2}}{4}\left(\frac{\sum_{i=1}^{m-1}\alpha_i\int_{\eta_{i-1}}^{\eta_i} g(r)\,dr}{1-\sum_{i=1}^{m-1}\alpha_i\int_{\eta_{i-1}}^{\eta_i} g(r)\,dr} + 2t\right), & 0 \leq t \leq 1/2, \\ \frac{a_k\sqrt{2}}{4}\left(\frac{\sum_{i=1}^{m-1}\alpha_i\int_{\xi_{i-1}}^{\xi_i} g(r)\,dr}{1-\sum_{i=1}^{m-1}\alpha_i\int_{\xi_{i-1}}^{\xi_i} g(r)\,dr} + 2(1-t)\right), & 1/2 \leq t \leq 1 \end{cases}$$

and

$$0 < v_k^* \leq a_k, \qquad 0 < \left|\left(v_k^*\right)'\right| \leq a_k,$$
$$\lim_{n\to\infty} v_{k_n} = \lim_{n\to\infty} T^n v_{k_0} = v_k^*,$$
$$\lim_{n\to\infty} (v_{k_n})' = \lim_{n\to\infty} \left(T^n v_{k_0}\right)' = \left(v_k^*\right)',$$

where $v_{k_0}(t) = 0$, $0 \leq t \leq 1$.

4 Example

Consider the following second-order boundary-value problem with integral boundary conditions:

$$\begin{cases} (\phi(u'(t)))' + f(t,u(t),u'(t)) = 0, \quad t \in (0,1), \\ u(0) = \frac{1}{2}\int_0^{1/4} \frac{1}{2}u(r)\,dr + \frac{1}{2}\int_{1/3}^1 \frac{1}{2}u(r)\,dr, \\ u(1) = \frac{1}{2}\int_0^{2/3} \frac{1}{2}u(r)\,dr + \frac{1}{2}\int_{3/4}^1 \frac{1}{2}u(r)\,dr, \end{cases} \tag{4.1}$$

where

$$f(t,u,v) = t(1-t) + \sqrt{2}\left(\frac{u}{2} + \frac{v^2}{13}\right) + 39, \quad (t,u,v) \in [0,1] \times [0,\infty) \times (-\infty,\infty),$$

and $\phi(u) = u|u|$, $g(t) = \frac{1}{2}$, $a_1 = \frac{1}{2}$, $a_2 = 0$, $a_3 = \frac{1}{2}$, $n_0 = 0$, $n_1 = \frac{1}{4}$, $n_2 = \frac{1}{3}$, $n_3 = 1$. We note that $0 = \eta_0 < \eta_1 < \cdots < \eta_{m-2} < \eta_{m-1} = 1$, $\eta_i + \xi_{m-1-i} = 1$ for $i \in 0,\dots,m-1$ and $\alpha_i = \alpha_{m-i}$ for $i \in 1,\dots,m-1$. Then $\xi_0 = 1$, $\xi_1 = \frac{2}{3}$, $\xi_2 = \frac{3}{4}$, $\xi_3 = 1$. Choosing $a = 26\sqrt{2}$, by calculations we obtain $A = 2\sqrt{2}$. It is easy to verify that $f(t,u,v)$ satisfies

(1) $f(t,u_1,v_1) \leq f(t,u_2,v_2)$ for any $0 \leq t \leq \frac{1}{2}$, $0 \leq u_1 \leq u_2 \leq 26\sqrt{2}$, $0 \leq |v_1| \leq |v_2| \leq 26\sqrt{2}$;

(2) $\max_{0 \leq t \leq 1/2} f(t,a,a) = f(\frac{1}{2}, 26\sqrt{2}, 26\sqrt{2}) \leq \phi(\frac{a}{A}) = 169$.

Hence, by Theorem 3.1, (4.1) has two positive solutions w^* and v^*. For $n = 0,1,2,\dots$, the two iterative schemes are:

$$w_0(t) = \begin{cases} 11 + 26t, & 0 \leq t \leq \frac{1}{2}, \\ 11 + 26(1-t), & \frac{1}{2} \leq t \leq 1, \end{cases}$$

$$w_{n+1}(t)$$

$$= \begin{cases} \frac{24}{37}\sum_{i=1}^3 \alpha_i \int_{\eta_{i-1}}^{\eta_i} \int_0^r \phi^{-1}(\int_s^{1/2} (\tau(1-\tau) \\ \quad + \sqrt{2}(\frac{w_n(\tau)}{2} + \frac{w_n'^2(\tau)}{13}) + 39)\,d\tau)\,ds\,dr \\ \quad + \int_0^t \phi^{-1}(\int_s^{1/2} (\tau(1-\tau) + \sqrt{2}(\frac{w_n(\tau)}{2} + \frac{w_n'^2(\tau)}{13}) + 39)\,d\tau)\,ds, & 0 \leq t \leq 1/2, \\ \frac{24}{37}\sum_{i=1}^3 \alpha_i \int_{\xi_{i-1}}^{\xi_i} \int_r^1 \phi^{-1}(\int_{1/2}^s (\tau(1-\tau) \\ \quad + \sqrt{2}(\frac{w_n(\tau)}{2} + \frac{w_n'^2(\tau)}{13}) + 39)\,d\tau)\,ds\,dr \\ \quad + \int_t^1 \phi^{-1}(\int_{1/2}^s (\tau(1-\tau) + \sqrt{2}(\frac{w_n(\tau)}{2} + \frac{w_n'^2(\tau)}{13}) + 39)\,d\tau)\,ds, & 1/2 \leq t \leq 1, \end{cases}$$

$$v_0(t) = 0,$$

$$v_{n+1}(t)$$

$$= \begin{cases} \frac{24}{37}\sum_{i=1}^3 \alpha_i \int_{\eta_{i-1}}^{\eta_i} \int_0^r \phi^{-1}(\int_s^{1/2} (\tau(1-\tau) \\ \quad + \sqrt{2}(\frac{v_n(\tau)}{2} + \frac{v_n'^2(\tau)}{13}) + 39)\,d\tau)\,ds\,dr \\ \quad + \int_0^t \phi^{-1}(\int_s^{1/2} (\tau(1-\tau) + \sqrt{2}(\frac{v_n(\tau)}{2} + \frac{v_n'^2(\tau)}{13}) + 39)\,d\tau)\,ds, & 0 \leq t \leq 1/2, \\ \frac{24}{37}\sum_{i=1}^3 \alpha_i \int_{\xi_{i-1}}^{\xi_i} \int_r^1 \phi^{-1}(\int_{1/2}^s (\tau(1-\tau) \\ \quad + \sqrt{2}(\frac{v_n(\tau)}{2} + \frac{v_n'^2(\tau)}{13}) + 39)\,d\tau)\,ds\,dr \\ \quad + \int_t^1 \phi^{-1}(\int_{1/2}^s (\tau(1-\tau) + \sqrt{2}(\frac{v_n(\tau)}{2} + \frac{v_n'^2(\tau)}{13}) + 39)\,d\tau)\,ds, & 1/2 \leq t \leq 1. \end{cases}$$

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to the manuscript and typed, read and approved the final manuscript.

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