Living Rev. Relativity, **12**, (2009), 6
http://www.livingreviews.org/lrr-2009-6

LIVING REVIEWS *in relativity*

Null Geodesic Congruences, Asymptotically-Flat Spacetimes and Their Physical Interpretation

Timothy M. Adamo
Mathematical Institute,
University of Oxford, U.K.
email: adamo@maths.ox.ac.uk

Carlos Kozameh
Facultad de Matemática, Astronomía y Física,
Universidad Nacional de Córdoba, Argentina
email: kozameh@famaf.unc.edu.ar

Ezra T. Newman
Department of Physics and Astronomy,
University of Pittsburgh, U.S.A.
email: newman@pitt.edu

Accepted on 5 August 2009
Published on 11 September 2009
(Revised on 25 June 2010)

Abstract

A priori, there is nothing very special about shear-free or asymptotically shear-free null geodesic congruences. Surprisingly, however, they turn out to possess a large number of fascinating geometric properties and to be closely related, in the context of general relativity, to a variety of physically significant effects. It is the purpose of this paper to try to fully develop these issues.

This work starts with a detailed exposition of the theory of shear-free and asymptotically shear-free null geodesic congruences, i.e., congruences with shear that vanishes at future conformal null infinity. A major portion of the exposition lies in the analysis of the space of regular shear-free and asymptotically shear-free null geodesic congruences. This analysis leads to the space of complex analytic curves in an auxiliary four-complex dimensional space, $\mathcal{H}$-space. They in turn play a dominant role in the applications.

The applications center around the problem of extracting interior physical properties of an asymptotically-flat spacetime directly from the asymptotic gravitational (and Maxwell) field itself, in analogy with the determination of total charge by an integral over the Maxwell field at infinity or the identification of the interior mass (and its loss) by (Bondi's) integrals of the Weyl tensor, also at infinity.

More specifically, we will see that the asymptotically shear-free congruences lead us to an asymptotic definition of the center-of-mass and its equations of motion. This includes a kinematic meaning, in terms of the center-of-mass motion, for the Bondi three-momentum. In addition, we obtain insights into intrinsic spin and, in general, angular momentum, including an angular-momentum–conservation law with well-defined flux terms. When a Maxwell field is present, the asymptotically shear-free congruences allow us to determine/define at infinity a center-of-charge world line and intrinsic magnetic dipole moment.

Imprint / Terms of Use

Living Reviews in Relativity is a peer reviewed open access journal published by the Max Planck Institute for Gravitational Physics, Am Mühlenberg 1, 14476 Potsdam, Germany. ISSN 1433-8351.

Because a *Living Reviews* article can evolve over time, we recommend to cite the article as follows:

Timothy M. Adamo, Carlos Kozameh and Ezra T. Newman,
"Null Geodesic Congruences, Asymptotically-Flat Spacetimes and Their Physical Interpretation",
Living Rev. Relativity, **12**, (2009), 6. [Online Article]: cited [<date>],
http://www.livingreviews.org/lrr-2009-6

The date given as <date> then uniquely identifies the version of the article you are referring to.

Article Revisions

Living Reviews supports two ways of keeping its articles up-to-date:

Fast-track revision A fast-track revision provides the author with the opportunity to add short notices of current research results, trends and developments, or important publications to the article. A fast-track revision is refereed by the responsible subject editor. If an article has undergone a fast-track revision, a summary of changes will be listed here.

Major update A major update will include substantial changes and additions and is subject to full external refereeing. It is published with a new publication number.

For detailed documentation of an article's evolution, please refer to the history document of the article's online version at http://www.livingreviews.org/lrr-2009-6.

25 June 2010: The two major changes are the addition of Appendix D, and that we have rewritten about a quarter of Section 7. The extra appendix was just to add new information while the changes in Section 7 were due to real errors on our part. In addition there were many very small changes, words changed here and there; a few small paragraphs were added as well. These word changes were needed to correct the errors (mainly language) in Section 7. Several misprints were also corrected and a new reference to Adamo and Newman (2010) was added.

Page 29: Corrected right-hand side from the original $\xi_0^a(\tau) = \tau\delta$.

Page 30: Corrected right-hand side from the original $v_R^0(\tau) = \sqrt{1 + (v_R^i)^2} \approx 1 + \frac{1}{2}v^{i2} + \dots$.

Page 32: Corrected right-hand side exponent from the original $D^i_{\mathcal{I}\mathbb{C}} = q\xi^a(s)$.

Page 39: Removed asterisks from right-hand side.

Page 41: Corrected right-hand side by adding $4i$.

Page 42: Changed original $u_{\mathrm{B}}^{(R)}$ to u_{B}.

Page 46: Replaced the two original r^* by r.

Page 48: Renamed subsection by adding 'Asymptotically'.

Page 51: Corrected first term from $T(u, \zeta, \bar{\zeta})$ to $T(u_{\mathrm{ret}}, \zeta, \bar{\zeta})$.

Page 52: Removed asterisks from right-hand side.

Page 59: Corrected $\mathcal{P}^i$ to Ξ^i.

Page 61: Corrected $\mathcal{P}^i$ to Ξ^i.

Page 64: About a quarter of Section 7 has been rewitten due to errors on our part.

Page 71: Changed this item from: "As we mentioned earlier, the complex analytic curves appear most naturally in $\mathcal{H}$-space. Nevertheless, we could not find any reasonable physical $\mathcal{H}$-space interpretation for them and had to rely on the Minkowski space interpretation."

Page 72: Added this last item.

Page 81: Added Appendix on the derivation of the $\mathcal{H}$-space metric.

Contents

List of Tables

1 Introduction

Though from the very earliest days of Lorentzian geometries, families of null geodesics (null geodesic congruences (NGCs)) were obviously known to exist, it nevertheless took many years for their significance to be realized. It was from the seminal work of Bondi [8], with the introduction of null surfaces and their associated null geodesics used for the study of gravitational radiation, that the importance of NGCs became recognized. To analyze the differential structure of such congruences, Sachs [56] introduced the fundamental 'tools', known as the optical parameters, namely, the divergence, the twist (or curl) and the shear of the congruence. From the optical parameters one then could classify congruences by the vanishing (or the asymptotic vanishing) of one or more of these parameters. All the different classes exist in flat space but, in general, only special classes exist in arbitrary spacetimes. For example, in flat space, divergence-free congruences always exist, but for nonflat vacuum spacetimes they exist only in the case of certain high symmetries. On the other hand, twist-free congruences (null surface-forming congruences) exist in all Lorentzian spacetimes. General vacuum spacetimes *do not* allow shear-free congruences, though all *asymptotically-flat spacetimes* do allow *asymptotically* shear-free congruences, a natural generalization of shear-free congruences, to exist.

Our primary topic of study will be the cases of shear-free and asymptotically shear-free NGCs. In flat space the general shear-free congruences have been extensively studied. However, only recently has the special family of *regular* congruences been investigated. In general, as mentioned above, vacuum (or Einstein–Maxwell) metrics do not possess shear-free congruences; the exceptions being the algebraically-special metrics, all of which contain one or two such congruences. On the other hand, all asymptotically-flat spacetimes possess large numbers of *regular asymptotically* shear-free congruences.

A priori there does not appear to be anything very special about shear-free or asymptotically shear-free NGCs. However, over the years, simply by observing a variety of topics, such as the classification of Maxwell and gravitational fields (algebraically-special metrics), twistor theory, $\mathcal{H}$-space theory and asymptotically-flat spacetimes, there have been more and more reasons to consider them to be of considerable importance. One of the earliest examples of this is Robinson's [54] demonstration that a necessary condition for a curved spacetime to admit a null solution of Maxwell's equation is that there be, in that space, a congruence of null, *shear-free geodesics*. Recent results have shown that the regular congruences – both the shear-free and the asymptotically shear-free congruences – have certain very attractive and surprising properties; each congruence is determined by a complex analytic curve in the auxiliary complex space that is referred to as $\mathcal{H}$-space. For asymptotically-flat spacetimes, some of these curves contain a great deal of physical information about the spacetime itself [29, 27, 28].

It is the main purpose of this work to give a relatively complete discussion of these issues. However, to do so requires a digression.

A major research topic in general relativity (GR) for many years has been the study of asymptotically-flat spacetimes. Originally, the term 'asymptotically flat' was associated with gravitational fields, arising from finite bounded sources, where infinity was approached along spacelike directions (e.g., [5, 58]). Then the very beautiful work of Bondi [8] showed that a richer and more meaningful idea to be associated with 'asymptotically flat' was to study gravitational fields in which infinity was approached along null directions. This led to an understanding of gravitational radiation via the Bondi energy-momentum loss theorem, one of the profound results in GR. The Bondi energy-momentum loss theorem, in turn, was the catalyst for the entire contemporary subject of gravitational radiation and gravitational wave detectors. The fuzzy idea of where and what is infinity was clarified and made more specific by the work of Penrose [46, 47] with the introduction of the conformal compactification (via the rescaling of the metric) of spacetime, whereby infinity was added as a boundary and brought into a finite spacetime region. Penrose's infinity or spacetime boundary, referred to as Scri or $\mathfrak{I}$, has many sub-regions: future null infinity, $\mathfrak{I}^+$; past null infinity, $\mathfrak{I}^-$; future and past timelike infinity, $\boldsymbol{I}^+$ and $\boldsymbol{I}^-$; and spacelike infinity, $\boldsymbol{I}^0$. In the present work, $\mathfrak{I}^+$ and its neighborhood will be our arena for study.

A basic question for us is what information about the interior of the spacetime can be obtained from a study of the asymptotic gravitational field; that is, what can be learned from the remnant of the full field that now 'lives' or is determined on $\mathfrak{I}^+$? This quest is analogous to obtaining the total interior electric charge or the electromagnetic multipole moments directly from the asymptotic Maxwell field, i.e., the Maxwell field at $\mathfrak{I}^+$, or the Bondi energy-momentum four-vector from the gravitational field (Weyl tensor) at $\mathfrak{I}^+$. However, the ideas described and developed here are not in the mainstream of GR; they lie outside the usual interest and knowledge of most GR workers. Nevertheless, they are strictly within GR; no new physics is introduced; only the vacuum Einstein or Einstein–Maxwell equations are used. The ideas come simply from observing (discovering) certain unusual and previously overlooked features of solutions to the Einstein equations and their asymptotic behavior.

These observations, as mentioned earlier, centered on the awakening realization of the remarkable properties and importance of the special families of null geodesics: the regular shear-free and asymptotically shear-free NGCs.

The most crucial and striking of these overlooked features (mentioned now but fully developed later) are the following: in flat space every regular shear-free NGC is determined by the *arbitrary choice of a complex analytic world line* in complex Minkowski space, $\mathbb{M}_\mathbb{C}$. Furthermore and more surprising, for *every asymptotically-flat spacetime*, every regular asymptotically shear-free NGC is determined by the given Bondi shear (given for the spacetime itself) and by the choice of an arbitrary complex analytic world line in an auxiliary complex four-dimensional space, $\mathcal{H}$-space, endowed with a complex Ricci-flat metric structure. In other words, the space of regular shear-free and asymptotically shear-free NGCs are both determined by arbitrary analytic curves in $\mathbb{M}_\mathbb{C}$ and $\mathcal{H}$-space respectively [29, 27, 26].

Eventually, a *unique* complex world line in this space is singled out, with both the real and imaginary parts being given physical meaning. The detailed explanation for the determination of this world line is technical and reserved for a later discussion. However, a rough intuitive idea can be given in the following manner.

The idea is a generalization of the trivial procedure in electrostatics of first defining the electric dipole moment, relative to an origin, and then shifting the origin so that the dipole moment vanishes and thus obtaining the center of charge. Instead, we define, on $\mathfrak{I}^+$, with specific Bondi coordinates and tetrad, the complex mass dipole moment (the real mass dipole plus 'i' times angular momentum) from certain components of the asymptotic Weyl tensor. (The choice of the specific Bondi system is the analogue of the choice of origin in the electrostatic case.) Then, knowing how the asymptotic Weyl tensor transforms under a change of tetrad and coordinates, one sees how the complex mass dipole moment changes when the tetrad is rotated to one defined from the asymptotically shear-free congruence. By setting the transformed complex mass dipole moment to zero, the unique complex world line, identified as the complex center of mass, is obtained. (In Einstein–Maxwell theory a similar thing is done with the asymptotic Maxwell field leading to the vanishing of the complex Maxwell dipole moment [electric plus 'i' times magnetic dipole moment], with a resulting complex center of charge.)

This procedure, certainly unusual and out of the mainstream and perhaps appearing to be ambiguous, does logically hold together. The real justification for these identifications comes not from its logical structure, but rather from the observed equivalence of the derived results from these identifications with well-known classical mechanical and electrodynamical relations. These derived results involve both kinematical and dynamical relations. Though they will be discussed at length later, we mention that they range from a kinematic expression for the Bondi momentum of the form, $P = Mv + \ldots$; a derivation of Newton's second law, $F = Ma$; a conservation law of angular momentum with a well-known angular momentum flux; to the prediction of the Dirac value of the gyromagnetic ratio. We note that, for the charged spinning particle metric [37], the imaginary part of the world line is indeed the spin angular momentum, a special case of our results.

A major early clue that shear-free NGCs were important in GR was the discovery of the (vacuum or Einstein–Maxwell) algebraically special metrics. These metrics are defined by the algebraic degeneracy in their principle null vectors, which form (by the Goldberg–Sachs theorem [18]) a null congruence which is both *geodesic and shear-free*. For the asymptotically-flat algebraically-special metrics, this shear-free congruence (a very special congruence from the set of asymptotically shear-free congruences) determines a unique world line in the associated auxiliary complex $\mathcal{H}$-space. This shear-free congruence (with its associated complex world line) is a special case of the above argument of transforming to the complex center of mass. Our general asymptotically-flat situation is, thus, a generalization of the algebraically-special case. Much of the analysis leading to the transformation of the complex dipoles in the case of the general asymptotically-flat spaces arose from generalizing the case of the algebraically-special metrics.

To get a rough feeling (first in flat space) of how the curves in $\mathbb{M}_{\mathbb{C}}$ are connected with the shear-free congruences, we first point out that the shear-free congruences are split into two classes: the twisting congruences and the twist-free ones. The regular twist-free ones are simply the null geodesics (the generators) of the light cones with apex on an arbitrary timelike Minkowski space world line. Observing backwards along these geodesics from afar, one 'sees' the world line. The regular twisting congruences are generated in the following manner: consider the complexification of Minkowski space, $\mathbb{M}_{\mathbb{C}}$. Choose an arbitrary complex (analytic) world line in $\mathbb{M}_{\mathbb{C}}$ and construct its family of *complex light cones*. The projection into the real Minkowski space, $\mathbb{M}$, of the complex geodesics (the generators of these complex cones), yields the real shear-free twisting NGCs [4]. The twist contains or 'remembers' the apex on the complex world line. Looking backwards via these geodesics, one appears 'to see' the complex world line. In the case of asymptotically shear-free congruences in curved spacetimes, one can not trace the geodesics back to a complex world line. However, one can have the illusion (i.e., a virtual image) that the congruence is coming from a complex world line. It is from this property that we can refer to the asymptotically shear-free congruences as lying on *generalized* light cones.

The analysis of the geometry of the asymptotically shear-free NGCs is greatly facilitated by the introduction of Good-Cut Functions (GCFs). Each GCF is a complex slicing of $\mathfrak{I}^+$ from which the associated asymptotically shear-free NGC and world line can be easily obtained. For the special world line and congruence that leads to the complex center of mass, there is a unique GCF that is referred to as the Universal-Cut Function, (UCF).

Information about a variety of objects is contained in and can be easily calculated from the UCF, e.g., the unique complex world line; the direction of each geodesic of the congruence arriving at $\mathfrak{I}^+$; and the Bondi asymptotic shear of the spacetime. The ideas behind the GCFs and UCF arose from some very pretty mathematics: from the 'good-cut equation' and its complex four-dimensional solution space, $\mathcal{H}$-space [34, 24]. In flat space almost every asymptotically vanishing Maxwell field determines its own Universal Cut Function, where the associated world line determines both the center of charge and the magnetic dipole moment. In general, for Einstein–Maxwell fields, there will be two different UCFs, (and hence two different world lines), one for the Maxwell field and one for the gravitational field. The very physically interesting special case where the two world lines coincide will be discussed.

In this work, we seek to provide a comprehensive overview of the theory of asymptotically shear-free NGCs, as well as their physical applications to both flat and asymptotically-flat spacetimes. The resulting theoretical framework unites ideas from many areas of relativistic physics and has a crossover with several areas of mathematics, which had previously appeared short of physical applications.

The main mathematical tool used in our description of $\mathfrak{I}^+$ is the Newman–Penrose (NP) formalism [39]. Spherical functions are expanded in spin-s tensor harmonics [43]; in our approximations only the $l = 0, 1, 2$ harmonics are retained. Basically, the detailed calculations should be considered as expansions around the Reissner–Nordström metric, which is treated as zeroth order; all other terms being small, i.e., at least first order. We retain terms only to second order.

In Section 2, we give a brief review of Penrose's conformal null infinity $\mathfrak{I}$ along with an exposition of the NP formalism and its application to asymptotically-flat spacetimes. There is then a description of $\mathfrak{I}^+$, the stage on which most of our calculations take place. The Bondi *mass aspect* (a function on $\mathfrak{I}^+$) is defined from the asymptotic Weyl tensor. From it we obtain the physical identifications of the Bondi mass and linear momentum. Also discussed is the asymptotic symmetry group of $\mathfrak{I}^+$, the Bondi–Metzner–Sachs (BMS) group [8, 56, 40, 49]. The Bondi mass and linear momentum become basic for the physical identification of the complex center-of-mass world line. For its pedagogical value and prominence in what follows, we review Maxwell theory in the spin-coefficient (SC) formalism.

Section 3 contains the detailed analysis of shear-free NGCs in Minkowski spacetime. This includes the identification of the flat space GCFs from which all regular shear-free congruences can be found. We also show the intimate connection between the flat space GCFs, the (homogeneous) good-cut equation, and $\mathbb{M}_\mathbb{C}$. As applications, we investigate the UCF associated with asymptotically-vanishing Maxwell fields and in particular the shear-free congruences associated with the Liénard–Wiechert (and complex Liénard–Wiechert) fields. This allows us to identify a real (and complex) center-of-charge world line, as mentioned earlier.

In Section 4, we give an overview of the machinery necessary to deal with twisting asymptotically shear-free NGCs in asymptotically-flat spacetimes. This involves a discussion of the theory of $\mathcal{H}$-space, the construction of the good-cut equation from the asymptotic Bondi shear and its complex four-parameter family of solutions. We point out how the simple Minkowski space of the preceding Section 3 can be seen as a special case of the more general theory outlined here. These results have ties to Penrose's twistor theory and the theory of Cauchy–Riemann (CR) structures; an explanation of these crossovers is given in Appendicies A and B.

In Section 5, as examples of the ideas developed here, linear perturbations off the Schwarzschild metric, the algebraically-special type II metrics and asymptotically-stationary spacetime are discussed.

In Section 6, the ideas laid out in the previous Sections 3, 4 and 5 are applied to the general class of asymptotically-flat spacetimes: vacuum and Einstein–Maxwell. Here, reviewing the material of the previous section, we apply the solutions to the good-cut equation to determine all regular asymptotically shear-free NGCs by first choosing arbitrary world lines in the solution space and then singling out a unique one; two world lines in the Einstein–Maxwell case, one for the gravitational field, the other for the Maxwell field. This identification of the unique lines comes from a study of the transformation properties, at $\mathfrak{I}^+$, of the asymptotically-defined mass and spin dipoles and the electric and magnetic dipoles. The work of Bondi, with the identification of energy-momentum and its evolution, allows us to make a series of surprising further physical identifications and predictions. In addition, with a slightly different approximation scheme, we discuss our ideas applied to the asymptotic gravitational field with an electromagnetic dipole field as the source.

Section 7 contains an analysis of the gauge (or BMS) invariance of our results.

Section 8, the Discussion/Conclusion section, begins with a brief history of the origin of the ideas developed here, followed by comments on alternative approaches, possible physical predictions from our results, a summary and open questions.

Finally, we conclude with four appendices, which contain several mathematical crossovers that were frequently used or referred to in the text: CR structures and twistors, a brief exposition of the tensorial spherical harmonics [43] and their Clebsch–Gordon product decompositions, and an overview of the metric construction on $\mathcal{H}$-space.

1.1 Notation and definitions

The following contains the notational conventions that will be in use throughout the course of this review.

- Throughout this work we use the symbols 'l', 'm', 'n' ... with several different 'decorations' but always meaning a null tetrad or a null tetrad field.

  a) Though in places, e.g., in Section 2.4, the symbols, l^a, m^a, n^a ..., i.e., with an $a, b, c\ldots$ *can be thought of* as the abstract representation of a null tetrad (i.e., Penrose's abstract index notation [50]), in general, our intention is to describe vectors in a coordinate representation.

  b) The symbols, l^a, $l^{\#\,a}, l^{*\,a}$ most often represent the coordinate versions of different null geodesic tangent fields, e.g., one-leg of a Bondi tetrad field or some rotated version.

  c) The symbol, $\hat{l}^a$, (with *hat*) has a *very different meaning from the others*. It is used to represent the Minkowski components of a normalized null vector giving the null directions on an arbitrary light cone:

$$\hat{l}^a = \frac{\sqrt{2}}{2(1+\zeta\bar{\zeta})}\left(1+\zeta\bar{\zeta}, \zeta+\bar{\zeta}, i\bar{\zeta}-i\zeta, -1+\zeta\bar{\zeta}\right) \equiv \left(\frac{\sqrt{2}}{2}Y_0^0, \frac{1}{2}Y_{1i}^0\right). \tag{1}$$

  As the complex stereographic coordinates $(\zeta, \bar{\zeta})$ sweep out the sphere, the $\hat{l}^a$ sweeps out the entire future null cone. The other members of the associated null tetrad are

$$\begin{aligned} \hat{m}^a &= \frac{\sqrt{2}}{2(1+\zeta\bar{\zeta})}\left(0, 1-\bar{\zeta}^2, -i(1+\bar{\zeta}^2), 2\bar{\zeta}\right), \\ \hat{n}^a &= \frac{\sqrt{2}}{2(1+\zeta\bar{\zeta})}\left(1+\zeta\bar{\zeta}, -(\zeta+\bar{\zeta}), i\zeta-i\bar{\zeta}, 1-\zeta\bar{\zeta}\right). \end{aligned} \tag{2}$$

- Several different timelike variables (u_{B}, $u_{\mathrm{ret}}, \tau, s$) and their derivatives are used.

  The Bondi time, u_{B}, is closely related to the retarded time, $u_{\mathrm{ret}} = \sqrt{2}u_{\mathrm{B}}$. The use of the retarded time, u_{ret}, is important in order to obtain the correct numerical factors in the expressions for the final physical results. Their derivatives are represented by

$$\begin{aligned} \partial_{u_{\mathrm{B}}} K &\equiv \dot{K}, \\ \partial_{u_{\mathrm{ret}}} K &\equiv K' = \frac{\sqrt{2}}{2}\dot{K}. \end{aligned} \tag{3}$$

  The $u_{\mathrm{ret}}, \tau, s$, derivatives are denoted by the same prime ($'$) since it is always applied to functions with the same functional argument. Though we are interested in real physical spacetime, often the time variables ($u_{\mathrm{ret}}, u_{\mathrm{B}}, \tau$) take complex values close to the real (s is always real). Rather than putting on 'decorations' to indicate when they are real or complex (which burdens the expressions with an overabundance of different symbols), we leave reality decisions to be understood from context. In a few places where the reality of the particular variable is manifestly first introduced (and is basic) we decorate the symbol by a superscript (R), i.e., $u_{\mathrm{B}}^{(R)}$ or $u_{\mathrm{ret}}^{(R)}$. After their introduction we revert to the undecorated symbol.

  Note: At this point we are taking the velocity of light as $c = 1$ and omitting it; later, when we want the correct units to appear explicitly, we restore the c. This entails, via $\tau \to c\tau$, $s \to cs$, changing the prime derivatives to include the c, i.e.,

$$\mathrm{K}' \to c^{-1}\mathrm{K}'. \tag{4}$$

- Often the angular (or sphere) derivatives, $\eth$ and $\bar{\eth}$, are used. The notation $\eth_{(\alpha)}K$ means, apply the $\eth$ operator to the function K while holding the variable (α) constant.

- The complex conjugate is represented by the overbar, e.g., $\bar{\zeta}$. When a complex variable, $\tilde{\zeta}$, is close to the complex conjugate of ζ, but independent, we use $\tilde{\zeta} \approx \bar{\zeta}$.

  Frequently, in this work, we use terms that are not in standard use. It seems useful for clarity to have some of these terms defined early.

- As mentioned earlier, we use the term "generalized light cones" to mean (real) NGCs that *appear to have their apexes* on a world line in the complexification of the spacetime. A detailed discussion of this will be given in Sections 3 and 4.

- The term "complex center of mass" (or "complex center of charge") is frequently used. Up to the choice of constants (to give correct units) this is basically the "real center of mass plus 'i' angular momentum" (or "real center of charge plus 'i' magnetic moment"). There will be two different types of these "complex centers ..."; one will be geometrically defined or *intrinsic*, i.e., independent of the choice of coordinate system, the other will be *relative*, i.e., it will depend on the choice of (Bondi) coordinates. The relations between them are nonlinear and nonlocal.

- A very important technical tool used throughout this work is a class of complex analytic functions, $u_{\mathrm{B}} = G(\tau, \zeta, \bar{\zeta})$, referred to as GCFs that are closely associated with shear-free NGCs. The details are given later. For any given asymptotically-vanishing Maxwell field with nonvanishing total charge, the Maxwell field itself allows one, on physical grounds, to choose a *unique member* of the class referred to as the (Maxwell) UCF. For vacuum asymptotically-flat spacetimes, the Weyl tensor allows the choice of a unique member of the class referred to as the (gravitational) UCF. For Einstein–Maxwell there will be two such functions, though in important cases they will coincide and be referred to as UCFs. When there is no ambiguity, in either case, they will simple be UCFs.

- A notational irritant arises from the following situation. Very often we expand functions on the sphere in spin-s harmonics, as, e.g.,

  $$\chi = \chi^0 Y_0 + \chi^i Y_{1i}(\zeta, \bar{\zeta}) + \chi^{ij} Y_{2ij}(\zeta, \bar{\zeta}) + \chi^{ijk} Y_{3ijk}(\zeta, \bar{\zeta}) + \dots,$$

  where the indices, $i, j, k \dots$ *represent three-dimensional Euclidean indices.* To avoid extra notation and symbols we write scalar products and cross-products without the use of an explicit Euclidean metric, leading to awkward expressions like

  $$\begin{aligned} \overrightarrow{\eta} \cdot \overrightarrow{\lambda} &\equiv \eta^i \lambda^i \equiv \eta^i \lambda_i, \\ \mu^k = (\overrightarrow{\eta} \times \overrightarrow{\lambda})^k &\equiv \eta^i \lambda^j \epsilon_{ijk}. \end{aligned}$$

  This, though easy to understand and keep track of, does run into the unpleasant fact that often the relativist four-vector,

  $$\chi^a = (\chi^0, \chi^i),$$

  appears as the $l = 0, 1$ harmonics in the harmonic expansions. Thus, care must be used when lowering or raising the relativistic index, i.e., $\eta_{ab}\chi^a = \chi_b = (\chi^0, -\chi^i)$.

Table 1: Glossary

Symbol/Acronym	*Definition*
$\mathfrak{I}^+$, $\mathfrak{I}^+_{\mathbb{C}}$	Future conformal null infinity, Complex future conformal null infinity
$\boldsymbol{I}^+$, $\boldsymbol{I}^-$, $\boldsymbol{I}^0$	Future, Past conformal timelike infinity, Conformal spacelike infinity
$\mathbb{M}$, $\mathbb{M}_{\mathbb{C}}$	Minkowski space, Complex Minkowski space
u_{B}, u_{ret}	Bondi time coordinate, Retarded Bondi time ($\sqrt{2}u_{\mathrm{B}} = u_{\mathrm{ret}}$)
$\partial_{u_{\mathrm{B}}} f = \dot{f}$	Derivation with respect to u_{B}
$\partial_{u_{\mathrm{ret}}} f = f'$	Derivation with respect to u_{ret}
r	Affine parameter along null geodesics
$(\zeta, \bar{\zeta})$	$(e^{i\phi}\cot(\theta/2),\ e^{-i\phi}\cot(\theta/2))$; stereographic coordinates on S^2
$Y^s_{li\ldots j}(\zeta, \bar{\zeta})$	Tensorial spin-s spherical harmonics
$\eth$, $\bar{\eth}$	$P^{1-s}\frac{\partial}{\partial\zeta}P^s$, $\ P^{1+s}\frac{\partial}{\partial\bar{\zeta}}P^{-s}$; spin-weighted operator on the two-sphere
P	Metric function on S^2; often $P = P_0 \equiv 1 + \zeta\bar{\zeta}$
$\eth_{(\alpha)} f$	Application of $\eth$-operator to f while the variable α is held constant
$\{l^a, n^a, m^a, \bar{m}^a\}$	Null tetrad system; $l^a n_a = -m^a \bar{m}_a = 1$
NGC	Null Geodesic Congruence
NP/SC	Newman–Penrose/Spin-Coefficient Formalism
$\{U, X^A, \omega, \xi^A\}$	Metric coefficients in the Newman–Penrose formalism
$\{\psi_0, \psi_1, \psi_2, \psi_3, \psi_4\}$	Weyl tensor components in the Newman-Penrose formalism
$\{\phi_0, \phi_1, \phi_2\}$	Maxwell tensor components in the Newman–Penrose formalism
ρ	Complex divergence of a null geodesic congruence
Σ	Twist of a null geodesic congruence
σ, σ^0	Complex shear, Asymptotic complex shear of a NGC
k	$\frac{8\pi G}{c^4}$; Gravitational constant
$u = G(\tau, \zeta, \bar{\zeta})$	Cut function on $\mathfrak{I}^+$
$\tau = s + i\lambda = T(u, \zeta, \bar{\zeta})$	Complex auxiliary (CR) potential function
$\partial_\tau f = f'$	Derivation with respect to τ
$\eth^2_{(\tau)} G(\tau, \zeta, \bar{\zeta}) = \sigma^0(u, \zeta, \bar{\zeta})$	Good-Cut Equation, describing asymptotically shear-free NGCs
GCF	Good-Cut Function
$L(u, \zeta, \bar{\zeta}) = \eth_{(\tau)} G$	Stereographic angle field for an asymptotically shear-free NGC at $\mathfrak{I}^+$
$\eth_{(u_{\mathrm{B}})} T + L\dot{T} = 0$	CR equation, describing the embedding of $\mathfrak{I}^+$ into $\mathbb{C}^2$
three-dimensional CR Structure	A class of one-forms describing a real three-manifold of $\mathbb{C}^2$
$\mathcal{H}$-space	Complex four-dimensional solution space to the Good-Cut Equation
$D^i_{\mathbb{C}} = D^i_E + iD^i_M = \frac{1}{2}\phi^0_{0i}$	Complex electromagnetic dipole
$\eta^a(u_{\mathrm{ret}})$	Complex center-of-charge world line, lives in $\mathcal{H}$-space

Table 1 – *Continued*

Symbol/Acronym	*Definition*
$D^i_{\mathbb{C}(\text{grav})} = D^i_{(\text{mass})} + ic^{-1}J^i$ $= -\frac{c^2}{6\sqrt{2}G}\psi_1^{0i}$	Complex gravitational dipole
$\xi^a(u_{\text{ret}})$	Complex center-of-mass world line, lives in $\mathcal{H}$-space
UCF	Universal Cut Function
$u = X(\tau, \zeta, \bar{\zeta})$	UCF; corresponding to the complex center-of-charge world line
$\Psi \equiv \psi_2^0 + \eth^2\overline{\sigma} + \sigma\dot{\overline{\sigma}} = \bar{\Psi}$	Bondi Mass Aspect
$M_{\text{B}} = -\frac{c^2}{2\sqrt{2}G}\Psi^0$	Bondi mass
$P^i = -\frac{c^3}{6G}\Psi^i$	Bondi linear three-momentum
$J^i = \frac{\sqrt{2}c^3}{12G}\ \text{Im}(\psi_1^{0i})$	Vacuum linear theory identification of angular momentum

2 Foundations

In this section, we review several of the key ideas and tools that are indispensable in our later discussions. We keep our explanations as concise as possible, and refrain from extensive proofs of any propositions. The reader will be directed to the appropriate references for the details. In large part, much of what is covered in this section should be familiar to many workers in GR.

2.1 Asymptotic flatness and $\mathfrak{I}^+$

Ever since the work of Bondi [8] illustrated the importance of null hypersurfaces in the study of outgoing gravitational radiation, the study of asymptotically-flat spacetimes has been one of the more important research topics in GR. Qualitatively speaking, a spacetime can be thought of as (future) asymptotically flat if the curvature tensor vanishes at an appropriate rate as infinity is approached along the future-directed null geodesics of the null hypersurfaces. The type of physical situation we have in mind is an arbitrary compact gravitating source (perhaps with an electric charge and current distribution), with the associated gravitational (and electromagnetic) field. The task is to gain information about the interior of the spacetime from the study of far-field features, multipole moments, gravitational and electromagnetic radiation, etc. [44]. The arena for this study is on what is referred to as future null infinity, $\mathfrak{I}^+$, the future boundary of the spacetime. The intuitive picture of this boundary is the set of all endpoints of future-directed null geodesics.

A precise definition of null asymptotic flatness and the boundary was given by Penrose [46, 47], whose basic idea was to rescale the spacetime metric by a conformal factor, which approaches zero asymptotically: the zero value defining future null infinity. This process leads to the boundary being a null hypersurface for the conformally-rescaled metric. When this boundary can be attached to the interior of the rescaled manifold in a regular way, then the spacetime is said to be asymptotically flat.

As the details of this formal structure are not used here, we will rely largely on the intuitive picture. A thorough review of this subject can be found in [12]. However, there are a number of important properties of $\mathfrak{I}^+$ arising from Penrose's construction that we rely on [44, 46, 47]:

(A): For both the asymptotically-flat vacuum Einstein equations and the Einstein–Maxwell equations, $\mathfrak{I}^+$ is a null hypersurface of the conformally rescaled metric.

(B): $\mathfrak{I}^+$ is topologically $S^2 \times \mathbb{R}$.

(C): The Weyl tensor C^a_{bcd} vanishes at $\mathfrak{I}^+$, with the peeling theorem describing the speed of its falloff (see below).

Property (B) allows an easy visualization of the boundary, $\mathfrak{I}^+$, as the past light cone of the point $\boldsymbol{I}^+$, future timelike infinity. As mentioned earlier, $\mathfrak{I}^+$ will be the stage for our study of asymptotically shear-free NGCs.

2.2 Bondi coordinates and null tetrad

Proceeding with our examination of the properties of $\mathfrak{I}^+$, we introduce, in the neighborhood of $\mathfrak{I}^+$, what is known as a Bondi coordinate system: $(u_\mathrm{B}, r, \zeta, \bar{\zeta})$. In this system, u_B, the Bondi time, labels the null surfaces, r is the affine parameter along the null geodesics of the constant u_B surfaces and $\zeta = e^{i\phi}\cot(\theta/2)$, the complex stereographic angle labeling the null geodesics of $\mathfrak{I}^+$. To reach $\mathfrak{I}^+$, we simply let $r \to \infty$, so that $\mathfrak{I}^+$ has coordinates $(u_\mathrm{B}, \zeta, \bar{\zeta})$. The time coordinate u_B, the topologically $\mathbb{R}$ portion of $\mathfrak{I}^+$, labels "cuts" of $\mathfrak{I}^+$. The stereographic angle ζ accounts for the topological generators of the S^2 portion of $\mathfrak{I}^+$, i.e., the null generators of $\mathfrak{I}^+$. The choice of a Bondi coordinate system is not unique, there being a variety of Bondi coordinate systems to choose from. The coordinate transformations between any two, known as Bondi–Metzner–Sachs (BMS) transformations or as the BMS group, are discussed later in this section.

Associated with the Bondi coordinates is a (Bondi) null tetrad system, $(l^a, n^a, m^a, \overline{m}^a)$. The first tetrad vector l^a is the tangent to the geodesics of the constant u_B null surfaces given by [44]

$$l^a = \frac{dx^a}{dr} = g^{ab}\nabla_b u_\mathrm{B}, \tag{5}$$

$$l^a \nabla_a l^b = 0, \tag{6}$$

$$l^a \frac{\partial}{\partial x^a} = \frac{\partial}{\partial r}. \tag{7}$$

The second null vector n^a is normalized so that:

$$l_a n^a = 1. \tag{8}$$

In Bondi coordinates, we have [44]

$$n^a \frac{\partial}{\partial x^a} = \frac{\partial}{\partial u_\mathrm{B}} + U\frac{\partial}{\partial r} + X^A \frac{\partial}{\partial x^A}, \tag{9}$$

for functions U and X^A to be determined, and $A = 2, 3$ with $x^2 = \zeta$ and $x^3 = \bar{\zeta}$. At $\mathfrak{I}^+$, n^a is tangent to the null generators of $\mathfrak{I}^+$.

The tetrad is completed with the choice of a complex null vector m^a, $(m^a m_a = 0)$ which is itself orthogonal to both l_a and n_a, initially tangent to the constant u_B cuts at $\mathfrak{I}^+$ and parallel propagated inward on the null geodesics. It is normalized by

$$m^a \bar{m}_a = -1. \tag{10}$$

Once more, in coordinates, we have [44]

$$m^a \frac{\partial}{\partial x^a} = \omega \frac{\partial}{\partial r} + \xi^A \frac{\partial}{\partial x^A}, \tag{11}$$

for some ω and ξ^A to be determined. All other scalar products in the tetrad are to vanish.

With the tetrad thus defined, the contravariant metric of the spacetime is given by

$$g^{ab} = l^a n^b + l^b n^a - m^a \bar{m}^b - m^b \bar{m}^a. \tag{12}$$

In terms of the *metric coefficients* U, ω, X^A, and ξ^A, the metric can be written as:

$$g^{ab} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & g^{11} & g^{1A} \\ 0 & g^{1A} & g^{AB} \end{pmatrix}, \tag{13}$$

$$g^{11} = 2(U - \omega\bar{\omega}),$$

$$g^{1A} = X^A - (\bar{\omega}\xi^A + \omega\bar{\xi}^A),$$

$$g^{AB} = -(\xi^A \bar{\xi}^B + \xi^B \bar{\xi}^A).$$

Thus, the spacetime metric is determined from the metric coefficients.

There remains the issue of both coordinate and tetrad freedom, i.e., local Lorentz transformations. Most of the time we work in one arbitrary but fixed Bondi coordinate system, though for special situations more general coordinate systems are used. The more general transformations are given, essentially, by choosing an arbitrary slicing of $\mathfrak{I}^+$, written as $u_\mathrm{B} = G(s, \zeta, \bar{\zeta})$ with s labeling the slices. To keep conventional coordinate conditions unchanged requires a rescaling of $r : r \to r' = (\partial_s G)^{-1} r$. It is also useful to be able to shift the origin of r by $r' = r - r_0(u_\mathrm{B}, \zeta, \bar{\zeta})$ with arbitrary $r_0(u_\mathrm{B}, \zeta, \bar{\zeta})$.

The tetrad freedom of null rotations around n^a, performed in the neighborhood of $\mathfrak{I}^+$, will later play a major role. For an arbitrary function $L(u_{\rm B},\zeta,\bar{\zeta})$ on $\mathfrak{I}^+$, the null rotation about the vector n^a [44] is given by

$$l^a \to l^{*a} = l^a - \frac{\bar{L}}{r}m^a - \frac{L}{r}\bar{m}^a + 0(r^{-2}), \tag{14}$$

$$m^a \to m^{*\,a} = m^a - \frac{L}{r}n^a + 0(r^{-2}), \tag{15}$$

$$n^a \to n^{*\,a} = n^a. \tag{16}$$

Eventually, by the appropriate choice of the function $L(u_{\rm B},\zeta,\bar{\zeta})$, the new null vector, l^{*a}, can be made into the tangent vector of an asymptotically shear-free NGC.

The second type of tetrad transformation is the rotation in the tangent $(m^a,\overline{m}^a)$ plane, which keeps l^a and n^a fixed:

$$m^a \to e^{i\lambda}m^a, \quad \lambda \in \mathbb{R}. \tag{17}$$

This latter transformation provides motivation for the concept of *spin weight.* A quantity $\eta_{(s)}(\zeta,\bar{\zeta})$ is said to have spin-weight s if, under the transformation, Equation (17), it transforms as

$$\eta \to \eta^{*}_{(s)}(\zeta,\bar{\zeta}) = e^{is\lambda}\eta_{(s)}(\zeta,\bar{\zeta}). \tag{18}$$

An example would be to take a vector on $\mathfrak{I}^+$, say η^a, and form the spin-weight-one quantity, $\eta_{(1)} = \eta^a m_a$.

Comment: For later use we note that $L(u_{\rm B},\zeta,\bar{\zeta})$ has spin weight, $s=1$.

For each s, spin-s functions can be expanded in a complete basis set, the spin-s harmonics, ${}_sY_{lm}(\zeta,\bar{\zeta})$ or spin-s tensor harmonics, $Y^{(s)}_{l\,i\ldots j}(\zeta,\bar{\zeta}) \Leftrightarrow {}_sY_{lm}(\zeta,\bar{\zeta})$.

A third tetrad transformation, the boosts, are given by

$$l^{\#\,a} = Kl^a, \qquad n^{\#\,a} = K^{-1}n^a. \tag{19}$$

These transformations induce the idea of conformal weight, an idea similar to spin weight. Under a boost transformation, a quantity, $\eta_{(w)}$, will have conformal weight w if

$$\eta_{(w)} \to \eta^{\#}_{(w)} = K^w \eta_{(w)}. \tag{20}$$

Sphere derivatives of spin-weighted functions $\eta_{(s)}(\zeta,\bar{\zeta})$ are given by the action of the operators $\eth$ and its conjugate operator $\bar{\eth}$, defined by [17]

$$\eth\eta_{(s)} = P^{1-s}\frac{\partial(P^s\eta_{(s)})}{\partial\zeta}, \tag{21}$$

$$\bar{\eth}\eta_{(s)} = P^{1+s}\frac{\partial(P^{-s}\eta_{(s)})}{\partial\bar{\zeta}}, \tag{22}$$

where the function P is the conformal factor defining the sphere metric,

$$ds^2 = \frac{4d\zeta d\bar{\zeta}}{P^2},$$

most often taken as

$$P = P_0 \equiv 1 + \zeta\bar{\zeta}.$$

2.3 The optical equations

Since this work concerns NGCs and, in particular, *shear-free* and *asymptotically shear-free* NGCs, it is necessary to first define them and then study their properties.

Given a Lorentzian manifold with local coordinates, x^a, and an NGC, i.e., a foliation by a three parameter family of null geodesics,

$$x^a = X^a(r, y^w), \tag{23}$$

with r the affine parameterization and the (three) y^w labeling the geodesics, the tangent *vector field* $l^a = DX^a \equiv \partial_r X^a$ satisfies the geodesic equation

$$l^a \nabla_a l^b = 0.$$

The two complex optical scalars (spin coefficients), ρ and σ, are defined by

$$\rho = \frac{1}{2}(-\nabla_a l^a + i \text{ curl } l^a), \tag{24}$$

$$\text{curl } l^a \equiv \sqrt{(\nabla_{[a} l_{b]} \nabla^a l^b)}$$

and

$$\sigma = \nabla_{(a} l_{b)} m^a m^b$$

with m^a an arbitrary complex (spacelike) vector satisfying $m^a m_a = m^a l_a = m^a \overline{m}_a - 1 = 0$. Equivalently σ can be defined by its norm,

$$\sigma\overline{\sigma} = \frac{1}{2}\left(\nabla_{(a} l_{b)} \nabla^a l^b - \frac{1}{2}(\nabla_a l^a)^2\right),$$

with an arbitrary phase.

The ρ and σ satisfy the *optical equations* of Sachs [56], namely,

$$\frac{\partial \rho}{\partial r} = \rho^2 + \sigma\bar{\sigma} + \Phi_{00}, \tag{25}$$

$$\frac{\partial \sigma}{\partial r} = 2\rho\sigma + \psi_0, \tag{26}$$

$$\Phi_{00} = R_{ab}\, l^a l^b,$$
$$\psi_0 = -C_{abcd}\, l^a m^b l^c m^d,$$

where Φ_{00} and Ψ_0 are, respectively, a Ricci and a Weyl tensor tetrad component (see below). In flat space, i.e., with $\Phi_{00} = \Psi_0 = 0$, excluding the degenerate case of $\rho\overline{\rho} - \sigma\overline{\sigma} = 0$, plane and cylindrical fronts, the general solution is

$$\rho = \frac{i\Sigma - r}{r^2 + \Sigma^2 - \sigma^0\overline{\sigma}^0}, \tag{27}$$

$$\sigma = \frac{\sigma^0}{r^2 + \Sigma^2 - \sigma^0\overline{\sigma}^0}. \tag{28}$$

The complex σ^0 (referred to as the asymptotic shear) and the real Σ (called the twist) are determined from the original congruence, Equation (23). Both are functions just of the parameters, y^w. Their behavior for large r is given by

$$\rho = -\frac{1}{r} + \frac{i\Sigma}{r^2} + \frac{\Sigma^2}{r^3} - \frac{\sigma^0\overline{\sigma}^0}{r^3} + O(r^{-4}), \tag{29}$$

$$\sigma = \frac{\sigma^0}{r^2} + O(r^{-4}). \tag{30}$$

From this, σ^0 gets its name as the asymptotic shear. In Section 3, we return to the issue of the explicit construction of NGCs in Minkowski space and in particular to the construction and detailed properties of *regular shear-free congruences*.

Note the important point that, in $\mathbb{M}$, the vanishing of the asymptotic shear forces the shear to vanish. The same is not true for asymptotically-flat spacetimes. Specifically, for future null asymptotically-flat spaces described in a Bondi tetrad and coordinate system, we have, from other considerations, that

$$\Phi_{00} = O(r^{-6}),$$
$$\psi_0 = O(r^{-5}),$$
$$\Sigma = 0,$$

which leads to the asymptotic behavior of ρ and σ,

$$\rho = \overline{\rho} = -\frac{1}{r} + \frac{\sigma_0 \overline{\sigma}_0}{r^3} + O(r^{-5}),$$
$$\sigma = \frac{\sigma_0}{r^2} + O(r^{-4}),$$

with the two order symbols explicitly depending on the leading terms in Φ_{00} and Ψ_0. The vanishing of σ_0 *does not*, in this nonflat case, imply that σ vanishes. This case, referred to as asymptotically shear-free, plays the major role later. It will be returned to in greater detail in Section 4.

2.4 The Newman–Penrose formalism

Though the NP formalism is the basic working tool for our analysis, this is not the appropriate venue for its detailed exposition. Instead we will simply give an outline of the basic ideas followed by the results found, from the application of the NP equations, to the problem of asymptotically-flat spacetimes.

The NP version [39, 44, 41] of the vacuum Einstein (or the Einstein–Maxwell) equations uses the tetrad components

$$\lambda^a_i = (l^a, n^a, m^a, \bar{m}^a), \tag{31}$$

$(i = 1, 2, 3, 4)$ rather than the metric, as the basic variable. (An alternate version, not discussed here, is to use a pair of two-component spinors.) The metric, Equation (12), can be written compactly as

$$g^{ab} = \eta^{ij}\lambda^a_i \lambda^b_j, \tag{32}$$

with

$$\eta^{ij} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \tag{33}$$

The complex spin coefficients, which play the role of the connection, are determined from the Ricci rotation coefficients [39, 44]:

$$\gamma^i_{jk} = \lambda^b_j \lambda^a_k \nabla_a \lambda^i_b \equiv \lambda^b_j \lambda^i_{b;k}, \tag{34}$$

via the linear combinations

$$\begin{aligned} \alpha &= \tfrac{1}{2}(\gamma_{124} - \gamma_{344}),\ \lambda = -\gamma_{244},\ \kappa = \gamma_{131}, \\ \beta &= \tfrac{1}{2}(\gamma_{123} - \gamma_{343}),\ \mu = -\gamma_{243},\ \rho = \gamma_{134}, \\ \gamma &= \tfrac{1}{2}(\gamma_{122} - \gamma_{342}),\ \nu = -\gamma_{242},\ \sigma = \gamma_{133}, \\ \varepsilon &= \tfrac{1}{2}(\gamma_{121} - \gamma_{341}),\ \pi = -\gamma_{241},\ \tau = \gamma_{132}. \end{aligned} \tag{35}$$

The third basic variable in the NP formalism is the Weyl tensor or, equivalently, the following five complex tetrad components of the Weyl tensor:

$$\psi_0 = -C_{abcd}\, l^a m^b l^c m^d, \quad \psi_1 = -C_{abcd}\, l^a n^b l^c m^d, \tag{36}$$

$$\psi_2 = -\frac{1}{2}\left(C_{abcd}\, l^a n^b l^c n^d - C_{abcd}\, l^a n^b m^c \overline{m}^d\right), \tag{37}$$

$$\psi_3 = C_{abcd}\, l^a n^b n^c \overline{m}^d, \quad \psi_4 = C_{abcd}\, n^a \overline{m}^b n^c \overline{m}^d. \tag{38}$$

When an electromagnetic field is present, we must include the complex tetrad components of the Maxwell field into the equations:

$$\begin{aligned} \phi_0 &= F_{ab}\, l^a m^b, \\ \phi_1 &= \frac{1}{2} F_{ab}\left(l^a n^b + m^a \overline{m}^b\right), \\ \phi_2 &= F_{ab}\, n^a \overline{m}^b, \end{aligned} \tag{39}$$

as well as the Ricci (or stress tensor) constructed from the three ϕ_i, e.g., $T_{ab}\, l^a l^b = k\phi_0\overline{\phi}_0, k = 2Gc^{-4}$.

Remark: We mention that much of the physical content and interpretations in the present work comes from the study of the lowest spherical harmonic coefficients in the leading terms of the far-field expansions of the Weyl and Maxwell tensors.

The NP version of the vacuum (or Einstein–Maxwell) equations consists of three sets (or four sets) of nonlinear first-order coupled partial differential equations for the variables: the tetrad components, the spin coefficients, the Weyl tensor (and Maxwell field when present). Though there is no hope that they can be solved in any general sense, many exact solutions have been found from them. Of far more importance, large classes of asymptotic solutions and perturbation solutions can be found. Our interest lies in the asymptotic behavior of the asymptotically-flat solutions. Though there are some subtle issues, integration in this class is not difficult [39, 45]. With no explanation of the integration process, except to mention that we use the Bondi coordinate and tetrad system of Equations (7), (9), and (11) and asymptotic flatness, we simply give the final results.

First, the radial behavior is described. The quantities with a zero superscript, e.g., σ^0, ψ_2^0, ..., are '*functions of integration*', i.e., functions only of $(u_{\mathrm{B}}, \zeta, \overline{\zeta})$.

- The Weyl tensor:

$$\begin{aligned} \psi_0 &= \psi_0^0 r^{-5} + O(r^{-6}), \\ \psi_1 &= \psi_1^0 r^{-4} + O(r^{-5}), \\ \psi_2 &= \psi_2^0 r^{-3} + O(r^{-4}), \\ \psi_3 &= \psi_3^0 r^{-2} + O(r^{-3}), \\ \psi_4 &= \psi_4^0 r^{-1} + O(r^{-2}). \end{aligned} \tag{40}$$

- The Maxwell tensor:

$$\begin{aligned} \phi_0 &= \phi_0^0 r^{-3} + O(r^{-4}), \\ \phi_1 &= \phi_1^0 r^{-2} + O(r^{-3}), \\ \phi_2 &= \phi_2^0 r^{-1} + O(r^{-2}). \end{aligned} \tag{41}$$

- The spin coefficients and metric variables:

$$\begin{aligned}
&\kappa = \pi = \epsilon = 0, \qquad \tau = \overline{\alpha} + \beta, \\
&\rho = \overline{\rho} = -r^{-1} - \sigma^0\overline{\sigma}^0 r^{-3} + O(r^{-5}), \\
&\sigma = \sigma^0 r^{-2} + \left((\sigma^0)^2\overline{\sigma}^0 - \psi_0^0/2\right) r^{-4} + O(r^{-5}), \\
&\alpha = \alpha^0 r^{-1} + O(r^{-2}), \qquad \beta = \beta^0 r^{-1} + O(r^{-2}), \\
&\gamma = \gamma^0 - \psi_2^0 (2r^2)^{-1} + O(r^{-3}), \qquad \lambda = \lambda^0 r^{-1} + O(r^{-2}), \\
&\mu = \mu^0 r^{-1} + O(r^{-2}), \qquad \nu = \nu^0 + O(r^{-1}),
\end{aligned} \tag{42}$$

$$\begin{aligned}
A &= \zeta \text{ or } \overline{\zeta}, \\
\xi^A &= \xi^{0A} r^{-1} - \sigma^0 \overline{\xi}^{0A} r^{-2} + \sigma^0\overline{\sigma}^0 \xi^{0A} r^{-3} + O(r^{-4}), \\
\omega &= \omega^0 r^{-1} - (\sigma^0\overline{\omega}^0 + \psi_1^0/2) r^{-2} + O(r^{-3}), \\
X^A &= (\psi_1^0 \overline{\xi}^{0A} + \overline{\psi}_1^0 \xi^{0A})(6r^3)^{-1} + O(r^{-4}), \\
U &= U^0 - (\gamma^0 + \overline{\gamma}^0) r - (\psi_2^0 + \overline{\psi}_2^0)(2r)^{-1} + O(r^{-2}).
\end{aligned}$$

- The functions of integration are determined, using coordinate conditions, as:

$$\xi^{0\zeta} = -P, \quad \overline{\xi}^{0\zeta} = 0, \tag{43}$$

$$\xi^{0\overline{\zeta}} = 0, \quad \overline{\xi}^{0\overline{\zeta}} = -P, \tag{44}$$

$$P = 1 + \zeta\overline{\zeta}, \tag{45}$$

$$\alpha^0 = -\overline{\beta}^0 = -\frac{\zeta}{2}, \tag{46}$$

$$\gamma^0 = \nu^0 = 0, \tag{47}$$

$$\omega^0 = -\overline{\eth}\sigma^0, \tag{48}$$

$$\lambda^0 = \dot{\overline{\sigma}}^0, \tag{49}$$

$$\mu^0 = U^0 = -1, \tag{50}$$

$$\psi_4^0 = -\ddot{\overline{\sigma}}^0, \tag{51}$$

$$\psi_3^0 = \eth\dot{\overline{\sigma}}^0, \tag{52}$$

$$\psi_2^0 - \overline{\psi}_2^0 = \overline{\eth}^2\sigma^0 - \eth^2\overline{\sigma}^0 + \overline{\sigma}^0\lambda^0 - \sigma^0\overline{\lambda}^0. \tag{53}$$

- The mass aspect,

$$\Psi \equiv \psi_2^0 + \eth^2\overline{\sigma}^0 + \sigma^0\dot{\overline{\sigma}}^0, \tag{54}$$

satisfies the physically *very important reality condition*:

$$\Psi = \overline{\Psi}. \tag{55}$$

- Finally, from the asymptotic Bianchi identities, we obtain the dynamical (or evolution) relations:

$$\dot{\psi}_2^0 = -\eth\psi_3^0 + \sigma^0\psi_4^0 + k\phi_2^0\bar{\phi}_2^0, \tag{56}$$
$$\dot{\psi}_1^0 = -\eth\psi_2^0 + 2\sigma^0\psi_3^0 + 2k\phi_1^0\bar{\phi}_2^0, \tag{57}$$
$$\dot{\psi}_0^0 = -\eth\psi_1^0 + 3\sigma^0\psi_2^0 + 3k\phi_0^0\bar{\phi}_2^0, \tag{58}$$
$$\dot{\phi}_1^0 = -\eth\phi_2^0, \tag{59}$$
$$\dot{\phi}_0^0 = -\eth\phi_1^0 + \sigma^0\phi_2^0; \tag{60}$$
$$k = 2Gc^{-4}. \tag{61}$$

Remark: These last five equations, the first of which contains the beautiful Bondi energy-momentum loss theorem, play the fundamental role in the dynamics of our physical quantities.

Remark: Using the mass aspect, Ψ, with Equations (51) and (52), the first of the asymptotic Bianchi identities can be rewritten in the concise form,

$$\dot{\Psi} = \dot{\sigma}\dot{\bar{\sigma}} + k\phi_2^0\overline{\phi}_2^0. \tag{62}$$

From these results, the characteristic initial problem can roughly be stated in the following manner. At $u_{\mathrm{B}} = u_{\mathrm{B}0}$ we choose the initial values for $(\psi_0^0, \psi_1^0, \psi_2^0)$, i.e., functions only of $(\zeta, \bar{\zeta})$. The characteristic data, the complex Bondi shear, $\sigma^0(u_{\mathrm{B}}, \zeta, \bar{\zeta})$, is then freely chosen. Since ψ_3^0 and ψ_4^0 are functions of σ^0, Equations (49), (51) and (52) and its derivatives, all the asymptotic variables can now be determined from Equations (56) – (60).

An important consequence of the NP formalism is that it allows simple proofs for many geometric theorems. Two important examples are the Goldberg–Sachs theorem [18] and the peeling theorem [57]. The peeling theorem is essentially given by the asymptotic behavior of the Weyl tensor in Equation (40) (and Equation (41)). The Goldberg–Sachs theorem essentially states that for an algebraically-special metric, the *degenerate principle null vector field is the tangent field to a shear-free NGC*. Both theorems are implicitly used later.

One of the immediate physical interpretations arising from the asymptotically-flat solutions was Bondi's [8] identifications, at $\mathfrak{I}^+$, of the interior spacetime four-momentum (energy/momentum). Given the mass aspect, Equation (54),

$$\Psi = \psi_2^0 + \eth^2\bar{\sigma}^0 + \sigma^0\dot{\bar{\sigma}}^0,$$

and the spherical harmonic expansion

$$\Psi = \Psi^0 + \Psi^i Y_{1i}^0 + \Psi^{ij} Y_{2ij}^0 + \dots, \tag{63}$$

Bondi identified the interior mass and three-momentum with the $l = 0$ and $l = 1$ harmonic contributions;

$$M_{\mathrm{B}} = -\frac{c^2}{2\sqrt{2}G}\Psi^0, \tag{64}$$

$$P^i = -\frac{c^3}{6G}\Psi^i. \tag{65}$$

The evolution of these quantities, (the Bondi mass/momentum loss) is then determined from Equation (62). The details of this will be discussed in Section 5.

The same clear cut asymptotic physical identification for interior angular momentum is not as readily available. In vacuum linear theory, the angular momentum is often taken to be

$$J^k = \frac{\sqrt{2}c^3}{12G}\,\mathrm{Im}\,(\psi_1^{0k}). \tag{66}$$

However, in the nonlinear treatment, correction terms quadratic in σ^0 and its derivatives are often included [59]. In the presence of a Maxwell field, this is again modified by the addition of electromagnetic multipole terms [29, 3].

In our case, where we consider *only quadrupole gravitational radiation*, the quadratic correction terms do in fact vanish and hence Equation (66), modified by the Maxwell terms, is correct as it is stated.

2.5 The Bondi–Metzner–Sachs group

The group of coordinate transformations at $\mathfrak{I}^+$ that preserves the Bondi coordinate conditions, the BMS group, is the same as the asymptotic symmetry group that arises from approximate solutions to Killing's equation as $\mathfrak{I}^+$ is approached. The BMS group has two parts: the homogeneous Lorentz group and the supertranslation group, which contains the Poincaré translation sub-group. Their importance to us lies in the fact that all the physical quantities arising from our identifications must transform appropriately under these transformations [49, 29].

Specifically, the BMS group is given by the supertranslations, with $\alpha(\zeta,\overline{\zeta})$ an arbitrary regular differentiable function on S^2:

$$\begin{aligned} \widehat{u}_B &= u_\mathrm{B} + \alpha(\zeta,\overline{\zeta}) \\ (\widehat{\zeta},\widehat{\overline{\zeta}}) &= (\zeta,\overline{\zeta}) \end{aligned} \tag{67}$$

and the Lorentz transformations, with (a,b,c,d) the complex parameters of $\mathrm{SL}(2,\mathbb{C})$,

$$\begin{aligned} \widehat{u}_B &= K u_\mathrm{B}, \\ K &= \frac{1+\zeta\overline{\zeta}}{(a\zeta+b)(\overline{a}\overline{\zeta}+\overline{b})+(c\zeta+d)(\overline{c}\overline{\zeta}+\overline{d})}, \\ \widehat{\zeta} &= \frac{a\zeta+b}{c\zeta+d}, \quad ad-bc=1. \end{aligned} \tag{68}$$

If $\alpha(\zeta,\overline{\zeta})$ is expanded in spherical harmonics,

$$\alpha(\zeta,\bar{\zeta}) = \sum \alpha^{ml} Y_{lm}(\zeta,\bar{\zeta}), \tag{69}$$

the $l = 0, 1$ terms represent the Poincaré translations, i.e.,

$$\alpha_{(P)}(\zeta,\bar{\zeta}) = d^a \hat{l}_a = \frac{\sqrt{2}}{2} d^0 Y_0^0 - \frac{1}{2} d^i Y_{1i}^0. \tag{70}$$

Details about the representation theory, with applications, are given later.

2.6 Algebraically-special metrics and the Goldberg–Sachs theorem

Among the most studied vacuum spacetimes are those referred to as 'algebraically-special' spacetimes, i.e., vacuum spacetimes that possess two or more coinciding principle null vectors (PNVs). PNV fields [50] (in general, four locally-independent fields exist) are defined by solutions, L^a, to the algebraic equation

$$L^b L_{[e} C_{a]bc[d} L_{f]} L^c = 0, \qquad L^a L_a = 0.$$

The Cartan–Petrov–Pirani–Penrose classification [52, 53, 50] describes the different degeneracies:

Alg. General	$[1, 1, 1, 1]$
Type II	$[2, 1, 1]$
Type D or degenerate	$[2, 2]$
Type III	$[3, 1]$
Type IV or Null	$[4]$.

In NP language, *if* the tetrad vector l^a is a principle null vector, i.e., $L_a = l_a$, then automatically,

$$\psi_0 = 0.$$

For the algebraically-special metrics, the special cases are

Type II	$\psi_0 = \psi_1 = 0$
Type III	$\psi_0 = \psi_1 = \psi_2 = 0$
Type IV	$\psi_0 = \psi_1 = \psi_2 = \psi_3 = 0$
Type D	$\psi_0 = \psi_1 = \psi_3 = \psi_4 = 0$ with both l^a and n^a PNVs.

An outstanding feature of the algebraically-special metrics is contained in the beautiful Goldberg–Sachs theorem [18].

Theorem. *For a nonflat vacuum spacetime, if there is an NGC that is shear-free, i.e., there is a null vector field with ($\kappa = 0, \sigma = 0$), then the spacetime is algebraically special and, conversely, if a vacuum spacetime is algebraically special, there is an NGC with ($\kappa = 0, \sigma = 0$).*

3 Shear-Free NGCs in Minkowski Space

The structure and properties of asymptotically shear-free NGCs (our main topic) are best understood by first looking at the special case of congruences that are shear-free everywhere (except at their caustics). Though shear-free congruences are also found in algebraically-special spacetimes, in this section only the shear-free NGCs in Minkowski spacetime, $\mathbb{M}$, are discussed [4]

3.1 The flat-space good-cut equation and good-cut functions

In Section 2, we saw that in the NP formalism, two of the complex spin coefficients, the *optical parameters* ρ and σ of Equations (27) and (28), play a particularly important role in their description of an NGC; namely, they carry the information of the divergence, twist and shear of the congruence.

From Equations (27) and (28), the radial behavior of the optical parameters for general *shear-free* NGCs, in Minkowski space, is given by

$$\rho = \frac{i\Sigma - r}{r^2 + \Sigma^2}, \qquad \sigma = 0, \tag{71}$$

where Σ is the twist of the congruence. A more detailed and much deeper understanding of the shear-free congruences can be obtained by first looking at the explicit coordinate expression, Equation (23), for all flat-space NGCs:

$$x^a = u_\mathrm{B}(\hat{l}^a + \hat{n}^a) - L\overline{\hat{m}}^a - \bar{L}\hat{m}^a + (r^* - r_0)\hat{l}^a, \tag{72}$$

where $L(u_\mathrm{B},\zeta,\bar{\zeta})$ is an arbitrary complex function of the parameters $y^w = (u_\mathrm{B},\zeta,\bar{\zeta})$; r_0, also an arbitrary function of $(u_\mathrm{B},\zeta,\bar{\zeta})$, determines the origin of the affine parameter; r^* can be chosen freely. Most frequently, to simplify the form of ρ, r_0 is chosen as

$$r_0 \equiv -\frac{1}{2}\left(\eth\bar{L} + \bar{\eth}L + L\dot{\bar{L}} + \bar{L}\dot{L}\right). \tag{73}$$

At this point, Equation (72) describes an arbitrary NGC with $(u_\mathrm{B},\zeta,\bar{\zeta})$ labeling the geodesics and r^* the affine distance along the individual geodesics; later $L(u_\mathrm{B},\zeta,\bar{\zeta})$ will be chosen so that the congruence is shear-free.

The tetrad $(\hat{l}^a,\hat{n}^a,\hat{m}^a,\overline{\hat{m}}^a)$ is given by [27]

$$\begin{aligned}
\hat{l}^a &= \frac{\sqrt{2}}{2(1+\zeta\bar{\zeta})}\left(1+\zeta\bar{\zeta}, \zeta+\bar{\zeta}, i\bar{\zeta}-i\zeta, -1+\zeta\bar{\zeta}\right), \\
\hat{n}^a &= \frac{\sqrt{2}}{2(1+\zeta\bar{\zeta})}\left(1+\zeta\bar{\zeta}, -(\zeta+\bar{\zeta}), i\zeta-i\bar{\zeta}, 1-\zeta\bar{\zeta}\right), \\
\hat{m}^a &= \frac{\sqrt{2}}{2(1+\zeta\bar{\zeta})}\left(0, 1-\bar{\zeta}^2, -i(1+\bar{\zeta}^2), 2\bar{\zeta}\right).
\end{aligned} \tag{74}$$

There are several important comments to be made about Equation (72). The first is that there is a simple geometric meaning to the parameters $(u_\mathrm{B},\zeta,\bar{\zeta})$: they are the values of the Bondi coordinates of $\mathfrak{I}^+$, where each geodesic of the congruence intersects $\mathfrak{I}^+$.

The second concerns the geometric meaning of L. At each point of $\mathfrak{I}^+$, consider the past light cone and its sphere of null directions. Coordinatize that sphere (of null directions) with stereographic coordinates. The function $L(u_\mathrm{B},\zeta,\bar{\zeta})$ *is the stereographic angle field* on $\mathfrak{I}^+$ that describes the null direction of each geodesic intersecting $\mathfrak{I}^+$ at the point $(u_\mathrm{B},\zeta,\bar{\zeta})$. The values

$L = 0$ and $L = \infty$ represent, respectively, the direction along the Bondi l^a and n^a vectors. This stereographic angle field completely determines the NGC.

The twist, Σ, of the congruence can be calculated in terms of $L(u_{\mathrm{B}}, \zeta, \bar{\zeta})$ directly from Equation (72) and the definition of the complex divergence, Equation (24), leading to

$$i\Sigma = \frac{1}{2}\left\{\eth\overline{L} + L\dot{\overline{L}} - \overline{\eth}L - \overline{L}\dot{L}\right\}. \tag{75}$$

We now demand that L be a *regular* function of its arguments, i.e., have no infinities, or, equivalently, that all members of the NGC come from the interior of the spacetime and not lie on $\mathfrak{I}^+$ itself.

It has been shown [6] that the condition on the stereographic angle field L for the NGC to be shear-free is that

$$\eth L + L\dot{L} = 0. \tag{76}$$

Our task is now to find the regular solutions of Equation (76). The key to doing this is via the introduction of a new complex variable τ and complex function [26, 27],

$$\tau = T(u_{\mathrm{B}}, \zeta, \bar{\zeta}). \tag{77}$$

T is related to L by the CR equation (related to the existence of a CR structure on $\mathfrak{I}^+$; see Appendix B):

$$\eth_{(u_{\mathrm{B}})}T + L\dot{T} = 0. \tag{78}$$

Remark: The following 'gauge' freedom becomes useful later. $\tau \Rightarrow \tau^* = F(\tau)$, with F analytic, leaving Equation (78) unchanged. In other words,

$$\tau^* = T^*\left(u_{\mathrm{B}}, \zeta, \bar{\zeta}\right) \equiv F\left(T(u_{\mathrm{B}}, \zeta, \bar{\zeta})\right), \tag{79}$$

leads to

$$\begin{aligned}
\eth_{(u_{\mathrm{B}})}T^* &= F'\eth_{(u_{\mathrm{B}})}T,\\
\dot{T}^* &= F'\dot{T},\\
\eth_{(u_{\mathrm{B}})}T^* + L\dot{T}^* &= 0.
\end{aligned}$$

We assume, in the neighborhood of real $\mathfrak{I}^+$, i.e., near the real u_{B} and $\tilde{\zeta} = \bar{\zeta}$, that $T(u_{\mathrm{B}}, \zeta, \tilde{\zeta})$ is analytic in the three arguments $(u_{\mathrm{B}}, \zeta, \tilde{\zeta})$. The inversion of Equation (77) yields the *complex analytic cut function*

$$u_{\mathrm{B}} = G(\tau, \zeta, \tilde{\zeta}). \tag{80}$$

Though we are interested in real values for u_{B}, from Equation (80) we see that for arbitrary τ in general it would take complex values. Shortly, we will also address the important issue of what values of τ are needed for real u_{B}.

Returning to the issue of integrating the shear-free condition, Equation (76), using Equation (77), we note that the derivatives of T, $\eth_{(u_{\mathrm{B}})}T$ and $\dot{T}$ can be expressed in terms of the derivatives of $G(\tau, \zeta, \bar{\zeta})$ by implicit differentiation. The u_{B} derivative of T is obtained by taking the u_{B} derivative of Equation (80):

$$1 = G'(\tau, \zeta, \bar{\zeta})\dot{T} \Rightarrow \dot{T} = \frac{1}{(G')}, \tag{81}$$

while the $\eth_{(u_{\mathrm{B}})}T$ derivative is found by applying $\eth_{(u_{\mathrm{B}})}$ to Equation (80),

$$\begin{aligned}
0 &= G'(\tau, \zeta, \bar{\zeta})\eth_{(u_{\mathrm{B}})}T + \eth_{(\tau)}G, \\
\eth_{(u_{\mathrm{B}})}T &= -\frac{\eth_{(\tau)}G}{G'(\tau, \zeta, \bar{\zeta})}.
\end{aligned} \tag{82}$$

When Equations (81) and (82) are substituted into Equation (78), one finds that L is given implicitly in terms of the cut function by

$$L(u_{\mathrm{B}},\zeta,\bar{\zeta}) = \eth_{(\tau)}G(\tau,\zeta,\bar{\zeta}), \tag{83}$$

$$u_{\mathrm{B}} = G(\tau,\zeta,\bar{\zeta}) \Leftrightarrow \tau = T(u_{\mathrm{B}},\zeta,\bar{\zeta}). \tag{84}$$

Thus, we see that all information about the NGC can be obtained from the cut function $G(\tau,\zeta,\bar{\zeta})$.

By further implicit differentiation of Equation (83), i.e.,

$$\eth_{(u_{\mathrm{B}})}L(u_{\mathrm{B}},\zeta,\bar{\zeta}) = \eth^2_{(\tau)}G(\tau,\zeta,\bar{\zeta}) + \eth_{(\tau)}G'(\tau,\zeta,\bar{\zeta})\cdot\eth_{(u_{\mathrm{B}})}T,$$

$$\dot{L}(u_{\mathrm{B}},\zeta,\bar{\zeta}) = \eth_{(\tau)}G'(\tau,\zeta,\bar{\zeta})\cdot\dot{T},$$

using Equation (78), the shear-free condition (76) becomes

$$\eth^2_{(\tau)}G(\tau,\zeta,\bar{\zeta}) = 0. \tag{85}$$

This equation will be referred to as the *homogeneous Good-Cut Equation* and its solutions as flat-space *GCFs*. In the next Section 4, an inhomogeneous version, the *Good-Cut Equation*, will be found for asymptotically shear-free NGCs. Its solutions will also be referred to as GCFs.

From the properties of the $\eth^2$ operator, the *general regular solution* to Equation (85) is easily found: G must contain only $l=0$ and $l=1$ spherical harmonic contributions; thus, any regular solution will be dependent on four arbitrary complex parameters, z^a. If these parameters are functions of τ, i.e., $z^a = \xi^a(\tau)$, then we can express any regular solution G in terms of the complex world line $\xi^a(\tau)$ [26, 27]:

$$u_{\mathrm{B}} = G(\tau,\zeta,\bar{\zeta}) = \xi^a(\tau)\hat{l}_a(\zeta,\bar{\zeta}) \equiv \frac{\sqrt{2}\xi^0}{2} - \frac{1}{2}\xi^i Y^0_{1i}. \tag{86}$$

The angle field $L(u_{\mathrm{B}},\zeta,\bar{\zeta})$ then has the form

$$L(u_{\mathrm{B}},\zeta,\bar{\zeta}) = \eth_{(\tau)}G(\tau,\zeta,\bar{\zeta}) = \xi^a(\tau)\hat{m}_a(\zeta,\bar{\zeta}), \tag{87}$$

$$u_{\mathrm{B}} = \xi^a(\tau)\hat{l}_a(\zeta,\bar{\zeta}). \tag{88}$$

Thus, we have our first major result: every regular shear-free NGC in Minkowski space is generated by the arbitrary choice of a complex world line in what turns out to be complex Minkowski space. See Equation (70) for the connection between the $l=(0,1)$ harmonics in Equation (86) and the Poincaré translations. We see in the next Section 4 how this result generalizes to regular asymptotically shear-free NGCs.

Remark: We point out that this construction of regular shear-free NGCs in Minkowski space is a special example of the Kerr theorem (cf. [51]). Writing Equations (87) and (88) as

$$u_{\mathrm{B}} = \frac{a + b\bar{\zeta} + \bar{b}\zeta + c\zeta\bar{\zeta}}{1+\zeta\bar{\zeta}},$$

$$L = \frac{(\bar{b}+c\bar{\zeta}) - \bar{\zeta}(a+b\bar{\zeta})}{1+\zeta\bar{\zeta}},$$

where the $(a(\tau), b(\tau), c(\tau), d(\tau))$ are simple combinations of the $\xi^a(\tau)$, we then find that

$$L + u\bar{\zeta} = \bar{b} + c\bar{\zeta},$$

$$u - L\zeta = a + b\bar{\zeta}.$$

Noting that the right-hand side of both equations are functions only of τ and $\overline{\zeta}$, we can eliminate the τ from the two equations, thereby constructing a function of three variables of the form

$$F(L + u\overline{\zeta}, u - L\zeta, \overline{\zeta}) = 0.$$

This is a special case of the general solution to Equation (76), which is the Kerr theorem.

In addition to the construction of the angle field, $L(u_{\mathrm{B}}, \zeta, \overline{\zeta})$, from the GCF, another quantity of great value in applications, obtained from the GCF, is the local change in u_{B} as τ changes, i.e.,

$$V(\tau, \zeta, \tilde{\zeta}) \equiv \partial_\tau G = G'. \tag{89}$$

3.2 Real cuts from the complex good cuts, I

Though our discussion of shear-free NGCs has relied, in an essential manner, on the use of the complexification of $\mathfrak{I}^+$ and the complex world lines in complex Minkowski space, it is the real structures that are of main interest to us. We want to find the intersection of the complex GCF with real $\mathfrak{I}^+$, i.e., what are the real points and real cuts of $u_{\mathrm{B}} = G(\tau, \zeta, \overline{\zeta})$, $(\tilde{\zeta} = \overline{\zeta})$, and what are the values of τ that yield *real* u_{B}.

To construct an associated family of real cuts from a GCF, we begin with

$$u_{\mathrm{B}} = G(\tau, \zeta, \overline{\zeta}) = \frac{\sqrt{2}}{2}\xi^0(\tau) - \frac{1}{2}\xi^i(\tau)Y^0_{1i}(\zeta, \overline{\zeta}) \tag{90}$$

and write

$$\tau = s + i\lambda \tag{91}$$

with s and λ real. The cut function can then be rewritten

$$\begin{aligned} u_{\mathrm{B}} &= G(\tau, \zeta, \overline{\zeta}) = G(s + i\lambda, \zeta, \overline{\zeta}) \\ &= G_R(s, \lambda, \zeta, \overline{\zeta}) + iG_I(s, \lambda, \zeta, \overline{\zeta}), \end{aligned} \tag{92}$$

with real $G_R(s, \lambda, \zeta, \overline{\zeta})$ and $G_I(s, \lambda, \zeta, \overline{\zeta})$. The $G_R(s, \lambda, \zeta, \overline{\zeta})$ and $G_I(s, \lambda, \zeta, \overline{\zeta})$ are easily calculated from $G(\tau, \zeta, \overline{\zeta})$ by

$$G_R(s, \lambda, \zeta, \overline{\zeta}) = \frac{1}{2}\left\{G(s + i\lambda, \zeta, \overline{\zeta}) + \overline{G(s + i\lambda, \zeta, \overline{\zeta})}\right\}, \tag{93}$$

$$G_I(s, \lambda, \zeta, \overline{\zeta}) = \frac{1}{2}\left\{G(s + i\lambda, \zeta, \overline{\zeta}) - \overline{G(s + i\lambda, \zeta, \overline{\zeta})}\right\}. \tag{94}$$

By setting

$$G_I(s, \lambda, \zeta, \overline{\zeta}) = 0 \tag{95}$$

and solving for

$$\lambda = \Lambda(s, \zeta, \overline{\zeta}) \tag{96}$$

we obtain the associated real slicing,

$$u^{(R)}_{\mathrm{B}} = G_R(s, \Lambda(s, \zeta, \overline{\zeta}), \zeta, \overline{\zeta}). \tag{97}$$

Thus, the values of τ that yield real values of u_{B} are given by

$$\tau = s + i\Lambda(s, \zeta, \overline{\zeta}). \tag{98}$$

As an example, using Equation (90), we find to first order in λ

$$u_{\mathrm{B}} = \frac{\sqrt{2}}{2}\xi_R^0(s) - \frac{\sqrt{2}}{2}\xi_I^0(s)'\lambda - \frac{1}{2}\left[\xi_R^i(s) - \xi_I^i(s)'\lambda\right]Y_{1i}^0(\zeta,\bar{\zeta}) \tag{99}$$
$$+i\left[\frac{\sqrt{2}}{2}\xi_I^0(s) + \frac{\sqrt{2}}{2}\xi_R^0(s)'\lambda\right] - i\frac{1}{2}\left[\xi_I^i(s) + \xi_R^i(s)'\lambda\right]Y_{1i}^0(\zeta,\bar{\zeta}),$$

$$u_{\mathrm{B}}^{(R)} = G_R(s,\Lambda,\zeta,\bar{\zeta}) \tag{100}$$
$$= \frac{\sqrt{2}}{2}\xi_R^0(s) - \frac{\sqrt{2}}{2}\xi_I^0(s)'\lambda - \frac{1}{2}\left[\xi_R^i(s) - \xi_I^i(s)'\lambda\right]Y_{1i}^0(\zeta,\bar{\zeta}),$$

$$\lambda = \Lambda(\zeta,\bar{\zeta}) = -\frac{\sqrt{2}\xi_I^0(s) + \xi_I^i(s)Y_{1i}^0(\zeta,\bar{\zeta})}{[\sqrt{2}\xi_R^0(s)' - \xi_R^i(s)'Y_{1i}^0(\zeta,\bar{\zeta})]}. \tag{101}$$

An Important Remark: We saw earlier that the shear-free angle field was given by

$$L(u_{\mathrm{B}},\zeta,\bar{\zeta}) = \eth_{(\tau)}G(\tau,\zeta,\bar{\zeta}), \tag{102}$$
$$u_{\mathrm{B}} = G(\tau,\zeta,\bar{\zeta}) \Leftrightarrow \tau = T(u_{\mathrm{B}},\zeta,\bar{\zeta}), \tag{103}$$

where real values of u_{B} should be used. If the real cuts, $u_{\mathrm{B}} = G_R\left(s,\Lambda(s,\zeta,\bar{\zeta}),\zeta,\bar{\zeta}\right)$, were used instead to calculate $L(u_{\mathrm{B}},\zeta,\bar{\zeta})$, the results would be wrong. The restriction of τ to yield real u_{B}, does not commute with the application of the $\eth$ operator, i.e.,

$$L(u_{\mathrm{B}},\zeta,\bar{\zeta}) \neq \eth G_R.$$

The $\eth$ differentiation *must be done first*, holding τ constant, before the reality of u_{B} is used. In other words, though we are interested in real $\mathfrak{I}^+$, it is essential that we consider its (local) complexification.

3.3 Approximations

Due to the difficulties involved in the intrinsic nonlinearities and the virtual impossibility of exactly inverting arbitrary analytic functions, it often becomes necessary to resort to approximations. The basic approximation will be to consider the complex world line $\xi^a(\tau)$ as being close to the straight line, $\xi_0^a(\tau) = \tau\delta_0^a$; deviations from this will be considered as first order. We retain terms up to second order, i.e., quadratic terms. Another frequently used approximation is to terminate spherical harmonic expansions after the $l = 2$ terms.

It is worthwhile to discuss some of the issues related to these approximations. One important issue is how to use the gauge freedom, Equation (79), $\tau \Rightarrow \tau^* = \Phi(\tau)$, to simplify the 'velocity vector',

$$v^a(\tau) = \xi^{a\prime}(\tau) \equiv \frac{d\xi^a}{d\tau}. \tag{104}$$

A Notational issue: Given a complex analytic function (or vector) of the complex variable τ, say $G(\tau)$, then $G(\tau)$ can be decomposed uniquely into two parts,

$$G(\tau) = \mathfrak{G}_R(\tau) + i\mathfrak{G}_I(\tau),$$

where all the coefficients in the Taylor series for $\mathfrak{G}_R(\tau)$ and $\mathfrak{G}_I(\tau)$ are real. With but a slight extension of conventional notation we refer to them as real analytic functions.

With this notation, we also write

$$v^a(\tau) = v_R^a(\tau) + iv_I^a(\tau).$$

By using the reparametrization of the world line, via $\tau^* = \Phi(\tau)$, with the decomposition

$$\Phi(\tau) = \Phi_R(\tau) + i\Phi_I(\tau),$$

the 'velocity' transforms as

$$v^{*a}(\tau^*) = v^a(\tau)[\Phi(\tau)']^{-1}.$$

One can easily check that by the appropriate choice of $\Phi_I(\tau)$ we can make $v_R^{*a}(\tau^*)$ and $v_I^{*a}(\tau^*)$ orthogonal, i.e.,

$$\eta_{ab}v_R^{*a}v_I^{*b} = 0, \tag{105}$$

and by the choice of $\Phi_R(\tau)$, the $v_R^{*a}(\tau^*)$ can be normalized to one,

$$\eta_{ab}v_R^{*a}v_R^{*b} = 1. \tag{106}$$

The remaining freedom in the choice of $\Phi(\tau)$ is simply an additive complex constant, which is used shortly for further simplification.

We now write $v_R^a(\tau) = \big(v_R^0(\tau), v_R^i(\tau)\big)$, which, with the slow motion approximation, yields, from the normalization,

$$v_R^0(\tau) = \sqrt{1 + (v_R^i)^2} \approx 1 + \frac{1}{2}v_R^{i2} + \dots . \tag{107}$$

From the orthogonality, we have

$$v_I^0(\tau) \approx v_R^i(\tau)v_I^i(\tau),$$

i.e., $v_I^0(\tau)$ is second order. Since $v_I^0(\tau) = \xi_I^0(\tau)'$ is second order, $\xi_I^0(\tau)$ is a constant plus a second-order term. Using the remaining complex constant freedom in $\Phi(\tau)$, the constant can be set to zero:

$$\xi_I^0(\tau) = \text{second order}. \tag{108}$$

Finally, from the reality condition on the u_B, Equations (94), (97) and (96) yield, with $\tau = s + i\lambda$ and λ treated as small,

$$u_\mathrm{B}^{(R)} = \xi_R^a(s)\hat{l}_a + v_I^a(s)\hat{l}_a\frac{\xi_I^b(s)\hat{l}_b}{\xi_R^{c\prime}(s)\hat{l}_c}, \tag{109}$$

$$\begin{aligned}\lambda = \Lambda(s,\zeta,\bar{\zeta}) &= -\frac{\xi_I^b(s)\hat{l}_b}{\xi_R^{c\prime}(s)\hat{l}_c},\\ &= \frac{\frac{\sqrt{2}}{2}\xi_I^i(s)Y_{1i}^0}{1 - \frac{\sqrt{2}}{2}\xi_R^{i\prime}(s)Y_{1i}^0}.\end{aligned} \tag{110}$$

Within this slow motion approximation scheme, we have from Equations (109) and (110),

$$u_\mathrm{ret}^{(R)} = \sqrt{2}u_\mathrm{B}^{(R)} = s - \frac{1}{\sqrt{2}}\xi_R^i(s)Y_{1i}^0 + 2v_I^a(s)\hat{l}_a\xi_I^b(s)\hat{l}_b, \tag{111}$$

$$\lambda \approx \frac{\sqrt{2}}{2}\xi_I^i(s)Y_{1i}^0\left(1 - \frac{\sqrt{2}}{2}v_R^j(s)Y_{1j}^0\right), \tag{112}$$

or, to first order, which is all that is needed,

$$\lambda = \frac{\sqrt{2}}{2}\xi_I^i(s)Y_{1i}^0.$$

We then have, to linear order,

$$\tau = s + i\frac{\sqrt{2}}{2}\xi_I^i(s)Y_{1i}^0, \qquad (113)$$
$$u_{\rm ret}^{(R)} = s - \frac{1}{\sqrt{2}}\xi_R^i(s)Y_{1i}^0.$$

3.4 Asymptotically-vanishing Maxwell fields

3.4.1 A prelude

The basic starting idea in this work is simple. It is in the generalizations and implementations where difficulties arise.

Starting in Minkowski space in a *fixed* given Lorentzian frame with spatial origin, the electric dipole moment $\overrightarrow{D}_E$ is calculated from an integral over the (localized) charge distribution. If there is a shift, $\overrightarrow{R}$, in the origin, the dipole transforms as

$$\overrightarrow{D_E^*} = \overrightarrow{D}_E - q\overrightarrow{R}. \qquad (114)$$

If $\overrightarrow{D}_E$ is time dependent, we obtain the center-of-charge world line by taking $\overrightarrow{D_E^*} = 0$, i.e., from $\overrightarrow{R} = \overrightarrow{D}_E q^{-1}$. It is this idea that we want to generalize and extend to gravitational fields.

The first generalization is formal and somewhat artificial: shortly it will become quite natural. We introduce, in addition to the electric dipole moment, the magnetic dipole moment $\overrightarrow{D}_M$ (also obtained by an integral over the current distribution) and write

$$\overrightarrow{D}_{\mathbb{C}} = \overrightarrow{D}_E + i\overrightarrow{D}_M.$$

By allowing the displacement $\overrightarrow{R}$ to take complex values, $\overrightarrow{R}_{\mathbb{C}}$, Equation (114), can be generalized to

$$\overrightarrow{D_{\mathbb{C}}^*} = \overrightarrow{D}_{\mathbb{C}} - q\overrightarrow{R}_{\mathbb{C}}, \qquad (115)$$

so that the complex center-of-charge is given by $\overrightarrow{D_{\mathbb{C}}^*} = 0$ or

$$\overrightarrow{R}_{\mathbb{C}} = \overrightarrow{D}_{\mathbb{C}} q^{-1}. \qquad (116)$$

We emphasize that this is done in a fixed Lorentz frame and only the origin is moved. In different Lorentz frames there will be different complex centers of charge.

Later, directly from the general *asymptotic Maxwell field itself* (satisfying the Maxwell equations), we define the *asymptotic* complex dipole moment and give *its* transformation law, including transformations between Lorentz frames. This yields a unique complex center of charge independent of the Lorentz frame.

3.4.2 Asymptotically-vanishing Maxwell fields: General properties

In this section, we describe how a complex center of charge for asymptotically vanishing Maxwell fields in flat spacetime can be found by using the shear-free NGCs congruences, constructed from solutions of the homogeneous good cut equation, to transform certain Maxwell field components to zero. Although this serves as a good example for our later methods in asymptotically flat spacetimes, the reader may wish to skip ahead to Section 4, where we go directly to gravitational fields in a setting of greater generality.

Our first set of applications of shear-free NGCs comes from Maxwell theory in Minkowski space. We review the general theory of the behavior of asymptotically-flat or vanishing Maxwell fields assuming throughout that there is a nonvanishing total charge, q. As stated in Section 2, the Maxwell field is described in terms of its complex tetrad components, (ϕ_0, ϕ_1, ϕ_2). In a Bondi coordinate/tetrad system the asymptotic integration is relatively simple [35, 25] resulting in the radial behavior (the peeling theorem):

$$\begin{aligned}\phi_0 &= \frac{\phi_0^0}{r^3} + O(r^{-4}), \\ \phi_1 &= \frac{\phi_1^0}{r^2} + O(r^{-3}), \\ \phi_2 &= \frac{\phi_2^0}{r} + O(r^{-2}),\end{aligned} \tag{117}$$

where the leading coefficients of r, $(\phi_0^0, \phi_1^0, \phi_2^0)$ satisfy the evolution equations:

$$\dot{\phi}_0^0 + \eth\phi_1^0 = 0, \tag{118}$$

$$\dot{\phi}_1^0 + \eth\phi_2^0 = 0. \tag{119}$$

The formal integration procedure is to take ϕ_2^0 as an arbitrary function of $(u_\mathrm{B}, \zeta, \bar{\zeta})$ (the free broadcasting data), then integrate the second, for ϕ_1^0, with a time-independent spin-weight $s = 0$ function of integration and finally integrate the first, for ϕ_0^0. Using a slight modification of this, namely from the spherical harmonic expansion, we obtain,

$$\phi_0^0 = \phi_{0i}^0 Y_{1i}^1 + \phi_{0ij}^0 Y_{2ij}^1 + \dots, \tag{120}$$

$$\phi_1^0 = q + \phi_{1i}^0 Y_{1i}^0 + \phi_{1ij}^0 Y_{2ij}^0 + \dots, \tag{121}$$

$$\phi_2^0 = \phi_{2i}^0 Y_{1i}^{-1} + \phi_{2ij}^0 Y_{2ij}^{-1} + \dots, \tag{122}$$

with the harmonic coefficients related to each other by the evolution equations:

$$\begin{aligned}\phi_0^0 &= 2q\eta^i(u_\mathrm{ret}) Y_{1i}^1 + Q_\mathbb{C}^{ij\prime} Y_{2ij}^1 + \dots, \\ \phi_1^0 &= q + \sqrt{2}q\eta^{i\prime}(u_\mathrm{ret}) Y_{1i}^0 + \frac{\sqrt{2}}{6} Q_\mathbb{C}^{ij\prime\prime} Y_{2ij}^0 + \dots, \\ \phi_2^0 &= -2q\eta^{i\prime\prime}(u_\mathrm{ret}) Y_{1i}^{-1} - \frac{1}{3} Q_\mathbb{C}^{ij\prime\prime\prime} Y_{2ij}^{-1} + \dots\end{aligned} \tag{123}$$

The physical meaning of the coefficients are

$$\begin{aligned}q &= \text{total electric charge}, \\ q\eta^i = D_\mathbb{C}^i &= \text{complex (electric \& magnetic) dipole moment} = D_E^i + iD_M^i, \\ Q_\mathbb{C}^{ij} &= \text{complex (electric \& magnetic) quadrupole moment},\end{aligned} \tag{124}$$

etc. For later use, the complex dipole is written as $D_\mathbb{C}^i(u_\mathrm{ret}) = q\eta^i(u_\mathrm{ret})$. Note that the $D_\mathbb{C}^i$ is defined relative to a given Bondi system. This is the analogue of a given origin for the calculations of the dipole moments of Equation (114).

Later in this section it will be shown that we can find a unique complex world line, $\xi^a(\tau) = (\xi^0, \xi^i)$, (the world line associated with a shear-free NGC), that is closely related to the $\eta^i(u_\mathrm{ret})$. From this complex world line we can define the *intrinsic* complex dipole moment, $D_{\mathcal{I}\mathbb{C}}^i = q\xi^i(s)$.

However, we first discuss a particular Maxwell field, F^{ab}, where one of its eigenvectors is a tangent field to a shear-free NGC. This solution, referred to as the complex Liénard–Wiechert field is the direct generalization of the ordinary Liénard–Wiechert field. Though it is a real solution in Minkowski space, *it can be thought of* as arising from a complex world line in complex Minkowski space.

3.4.3 A coordinate and tetrad system attached to a shear-free NGC

The parametric form of the general NGC was given earlier by Equation (72),

$$x^a = u_{\mathrm{B}}(\hat{l}^a + \hat{n}^a) - L\overline{\hat{m}}^a - \bar{L}\hat{m}^a + (r^* - r_0)\hat{l}^a. \tag{125}$$

The parameters $(u_{\mathrm{B}}, \zeta, \bar{\zeta})$ labeled the individual members of the congruence while r^* was the affine parameter along the geodesics. An alternative interpretation of the same equation is to consider it as the coordinate transformation between the coordinates, x^a (or the Bondi coordinates) and the *geodesic coordinates* $(u_{\mathrm{B}}, r^*, \zeta, \bar{\zeta})$. Note that these coordinates are not Bondi coordinates, though, in the limit, at $\mathfrak{I}^+$, they are. The associated (*geodesic*) tetrad is given as a function of these geodesic coordinates, but with Minkowskian components by Equations (74). We restrict ourselves to the special case of the coordinates and tetrad associated with the L from a shear-free NGC. Though we are dealing with a *real* shear-free twisting congruence, the congruence, as we saw, is generated by a complex analytic world line in the complexified Minkowski space, $z^a = \xi^a(\tau)$. The complex parameter, τ, must in the end be chosen so that the 'u_{B}' of Equation (90) is real. The Minkowski metric and the spin coefficients associated with this geodesic system can be calculated [27] in the $(u_{\mathrm{B}}, r^*, \zeta, \bar{\zeta})$ frame. Unfortunately, it must be stated parametrically, since the τ explicitly appears via the $\xi^a(\tau)$ and can not be directly eliminated. (An alternate choice of these geodesic coordinates is to use the τ instead of the u_{B}. Unfortunately, this leads to an analytic flat metric on the complexified Minkowski space, where the real spacetime is hard to find.)

The use and insight given by this coordinate/tetrad system is illustrated by its application to a special class of Maxwell fields. We consider, as mentioned earlier, the Maxwell field where one of its principle null vectors, l^{*a}, (an eigenvector of the Maxwell tensor, $F_{ab}l^{*a} = \lambda l_b^*$), is a tangent vector of a shear-free NGC. Thus, it depends on the choice of the complex world line and is therefore referred to as the complex Liénard–Wiechert field. (If the world line was real it would lead to the ordinary Liénard–Wiechert field.) We emphasize that though the source can formally be thought of as a charge moving on the complex world line, the Maxwell field is a real field on real Minkowski space. It will have a real (distributional) source at the caustics of the congruence. Physically, its behavior is very similar to real Liénard–Wiechert fields, the essential difference is that the electric dipole is now replaced by the combined electric and magnetic dipoles. The imaginary part of the world line determines the magnetic dipole moment.

In the spin-coefficient version of the Maxwell equations, using the geodesic tetrad, the choice of l^{*a} as the *principle null vector* 'congruence' is just the statement that

$$\phi_0^* = F_{ab}l^{*a}m^{*b} = 0.$$

This allows a very simple exact integration of the remaining Maxwell components [35].

3.4.4 Complex Liénard–Wiechert Maxwell field

The complex Liénard–Wiechert fields (*which we again emphasize are real Maxwell fields*) are formally given by the (geodesic) tetrad components of the Maxwell tensor in the null geodesic coordinate system $(u_{\mathrm{B}}, r^*, \zeta, \bar{\zeta})$, Equation (125). As the detailed calculations are long [35] and take us too far afield, we only give an outline here. The integration of the radial Maxwell equations leads to the asymptotic behavior,

$$\phi_0^* = 0, \tag{126}$$

$$\phi_1^* = \rho^2\phi_1^{*0}, \tag{127}$$

$$\phi_2^* = \rho\phi_2^{*0} + O(\rho^2), \tag{128}$$

with

$$\rho = -(r^* + i\Sigma)^{-1}, \qquad (129)$$
$$2i\Sigma = \eth\bar{L} + L\dot{\bar{L}} - \bar{\eth}L - \bar{L}\dot{L}.$$

The order expression is known in terms of $(\phi_1^{*0}, \phi_2^{*0})$. The function $L(u_\mathrm{B}, \zeta, \bar{\zeta})$ is given by

$$L(u_\mathrm{B}, \zeta, \bar{\zeta}) = \eth_{(\tau)} G(\tau, \zeta, \bar{\zeta}),$$
$$u_\mathrm{B} = G(\tau, \zeta, \bar{\zeta}) = \xi^a(\tau)\hat{l}_a(\zeta, \bar{\zeta}),$$

with $\xi^a(\tau)$ an arbitrary complex world line that determines the shear-free congruence whose tangent vectors are the Maxwell field eigenvectors.

Remark: In this case of the complex Liénard–Wiechert Maxwell field, the ξ^a determines the intrinsic center-of-charge world line, rather than the relative center-of-charge line.

The remaining unknowns, ϕ_1^{*0}, ϕ_2^{*0}, are determined by the last of the Maxwell equations,

$$\eth\phi_1^{*0} + L\dot{\phi}_1^{*0} + 2\dot{L}\phi_1^{*0} = 0, \qquad (130)$$
$$\eth\phi_2^{*0} + L\dot{\phi}_2^{*0} + \dot{L}\phi_2^{*0} = \dot{\phi}_1^{*0},$$

which have been obtained from Equations (59) and (60) via the null rotation between the Bondi and geodesic tetrads and the associated Maxwell field transformation, namely,

$$l^a \to l^{*a} = l^a - \frac{\bar{L}}{r} m^a - \frac{L}{r} \bar{m}^a + O(r^{*-2}), \qquad (131)$$

$$m^a \to m^{*a} = m^a - \frac{L}{r} n^a, \qquad (132)$$

$$n^a \to n^{*a} = n^a, \qquad (133)$$

with

$$\phi_0^{*0} = 0 = \phi_0^0 - 2L\phi_1^0 + L^2\phi_2^0, \qquad (134)$$

$$\phi_1^{*0} = \phi_1^0 - L\phi_2^0, \qquad (135)$$

$$\phi_2^{*0} = \phi_2^0. \qquad (136)$$

These remaining equations depend only on $L(u_\mathrm{B}, \zeta, \bar{\zeta})$, which, in turn, is determined by $\xi^a(\tau)$. In other words, the solution is driven by the complex line, $\xi^a(\tau)$. As they now stand, Equations (130) appear to be difficult to solve, partially due to the implicit description of the $L(u_\mathrm{B}, \zeta, \bar{\zeta})$.

Actually they are easily solved when the independent variables are changed, via Equation (86), from $(u_\mathrm{B}, \zeta, \bar{\zeta})$ to the complex $(\tau, \zeta, \bar{\zeta})$. They become, after a bit of work,

$$\eth_{(\tau)}(V^2\phi_1^0) = 0, \qquad (137)$$

$$\eth_{(\tau)}(V\phi_2^0) = \phi_1^{0\,\prime}, \qquad (138)$$

$$V = \xi^{a\prime}(\tau)\hat{l}_a(\zeta, \bar{\zeta}), \qquad (139)$$

with the solution

$$\phi_1^{*0} = \frac{q}{2} V^{-2}, \qquad (140)$$
$$\phi_2^{*0} = \frac{q}{2} V^{-1}\bar{\eth}_{(\tau)}[V^{-1}\partial_\tau V].$$

q being the Coulomb charge.

Though we now have the exact solution, unfortunately it is in complex coordinates where virtually every term depends on the complex variable τ, via $\xi^a(\tau)$. This is a severe impediment to a full description and understanding of the solution in the real Minkowski space.

In order to understand its asymptotic behavior and physical content, one must transform it, via Equations (131)–(136), back to a Bondi coordinate/tetrad system. This can only be done by approximations. After a lengthy calculation [35], we find the Bondi peeling behavior

$$\begin{aligned}\phi_0 &= r^{-3}\phi_0^0 + O(r^{-4}),\\ \phi_1 &= r^{-2}\phi_1^0 + O(r^{-3}),\\ \phi_2 &= r^{-1}\phi_2^0 + O(r^{-2}),\end{aligned} \tag{141}$$

with

$$\phi_0^0 = q\left(LV^{-2} + \frac{1}{2}L^2V^{-1}\overline{\eth}_{(\tau)}[V^{-1}V']\right), \tag{142}$$

$$\phi_1^0 = \frac{q}{2V^2}\left(1 + LV\overline{\eth}_{(\tau)}[V^{-1}V']\right), \tag{143}$$

$$\phi_2^0 = -\frac{q}{2}V^{-1}\overline{\eth}_{(\tau)}[V^{-1}V'], \tag{144}$$

$$V = \xi^{a\,\prime}\,\hat{l}_a(\zeta,\bar{\zeta}). \tag{145}$$

Next, treating the world line, as discussed earlier, as a *small deviation from the straight line*, $\xi^a(\tau) = \tau\delta_0^a$, i.e., by

$$\begin{aligned}\xi^a(\tau) &= \left(\tau + \delta\xi^0(\tau), \xi^i(\tau)\right),\\ \xi^i(\tau) &\ll 1, \quad \delta\xi^0(\tau) = \text{second order.}\end{aligned}$$

The GCF and its inverse (see Section 6) are given, to first order, by

$$u_{\mathrm{ret}} = \sqrt{2}u_{\mathrm{B}} = \sqrt{2}G = \tau - \frac{\sqrt{2}}{2}\xi^i(\tau)Y_{1i}^0(\zeta,\bar{\zeta}), \tag{146}$$

$$\tau = u_{\mathrm{ret}} + \frac{\sqrt{2}}{2}\xi^i(u_{\mathrm{ret}})Y_{1i}^0(\zeta,\bar{\zeta}). \tag{147}$$

Again to first order, Equations (142), (143) and (144) yield

$$\begin{aligned}\phi_0^0 &= 2q\xi^i(u_{\mathrm{ret}})Y_{1i}^1,\\ \phi_1^0 &= q + \sqrt{2}q\xi^{i\prime}(u_{\mathrm{ret}})Y_{1i}^0,\\ \phi_2^0 &= -2q\xi^{i\prime\prime}(u_{\mathrm{ret}})Y_{1i}^{-1},\end{aligned} \tag{148}$$

the known electromagnetic dipole field, with a Coulomb charge, q. One then has the physical interpretation of $\xi^a(u_{\mathrm{ret}}) : q\xi^a(u_{\mathrm{ret}})$ is the complex dipole moment; (the electric plus 'i' times magnetic dipole) and $\xi^a(u_{\mathrm{ret}})$ is the complex center of charge, the real part being the ordinary center of charge, while the imaginary part is the 'imaginary' magnetic center of charge. This simple relationship between the Bondi form of the complex dipole moment, $q\xi^i(u_{\mathrm{ret}})$, and the intrinsic complex center of charge, $\xi^a(\tau)$, is true only at linear order. The second-order relationship is given later.

Reversing the issue, if we had instead started with an exact complex Liénard–Wiechert field but now given in a Bondi coordinate/tetrad system and performed on it the transformations, Equations (14) and (134) to the geodesic system, it would have resulted in

$$\phi_0^* = 0.$$

This example was intended to show how physical meaning could be attached to the complex world line associated with a shear-free NGC. In this case and later in the case of asymptotically-flat spacetimes, when the GCF is singled out by either the Maxwell field or the gravitational field, it will be referred to it as a UCF. For either of the two cases, a flat-space asymptotically-vanishing Maxwell field (with nonvanishing total charge) and for a vacuum asymptotically-flat spacetime, there will be a unique UCF. In the case of the Einstein–Maxwell fields there, in general, will be two UCFs, one for each field.

3.4.5 Asymptotically vanishing Maxwell fields & shear-free NGCs

We return now to the general asymptotically-vanishing Maxwell field, Equations (117) and (120), and its transformation behavior under the null rotation around n^a,

$$\begin{aligned} l^a &\to l^{*a} = l^a - \frac{\bar{L}}{r}m^a - \frac{L}{r}\bar{m}^a + 0(r^{-2}), \qquad (149)\\ m^a &\to m^{*a} = m^a - \frac{L}{r}n^a,\\ n^a &\to n^{*a} = n^a, \end{aligned}$$

with $L(u_\mathrm{B},\zeta,\overline{\zeta}) = \xi^a(\tau)\hat{m}_a$, being one of our shear-free angle fields defined by a world line, $z^a = \xi^a(\tau)$. The leading components of the Maxwell fields transform as

$$\phi_0^{*0} = \phi_0^0 - 2L\phi_1^0 + L^2\,\phi_2^0, \qquad (150)$$

$$\phi_1^{*0} = \phi_1^0 - L\phi_2^0, \qquad (151)$$

$$\phi_2^{*0} = \phi_2^0. \qquad (152)$$

The ‘picture’ to adopt is that the new ϕ^*s are now given in a tetrad defined by the complex light cone (the generalized light cone) with origin on the complex world line. (This is obviously formal and physically nonsense, but mathematically quite sound, as the shear-free congruence can be thought of as having its origin on the complex line, $\xi^a(\tau)$.) From the physical identifications of charge, dipole moments, etc., of Equation (123), we can obtain the transformation law of these physical quantities. In particular, the $l = 1$ harmonic of ϕ_0^0, or, equivalently, the complex dipole, transforms as

$$\phi_{0i}^{0*} = \phi_{0i}^0 - 2(L\phi_1^0)|_i + (L^2\phi_2^0)|_i, \qquad (153)$$

where the notation $W|_i$ means *extract only* the $l = 1$ harmonic from a Clebsch–Gordon expansion of W. A subtlety and difficulty of this extraction process is explained/clarified below.

If by some accident the Maxwell field was a complex Liénard–Wiechert field, a world line $\xi^a(\tau)$ could be chosen so that $\phi_0^{*0} = 0$. However, though this cannot be done in general, the $l = 1$ harmonics of ϕ_0^{*0} can be made to vanish by the appropriate choice of the $\xi^a(\tau)$. This is the means by which a unique world line is chosen.

A Clarifying Comment:

An important observation, obvious but easily overlooked, concerning the spherical harmonic expansions, is that, in a certain sense, they lack uniqueness. As this issue is significant, its clarification is important.

Assume that we have a particular spin-s function on $\mathfrak{I}^+$, say, $\eta_{(s)}(u_\mathrm{B},\zeta,\overline{\zeta})$, given in a specific Bondi coordinate system, $(u_\mathrm{B},\zeta,\overline{\zeta})$, that has a harmonic expansion given, for constant u_B, by

$$\eta_{(s)}(u_\mathrm{B},\zeta,\overline{\zeta}) = \sum_{l,(ijk\ldots)} \eta_{(s)}^{l,(ijk\ldots)}(u_\mathrm{B})Y_{l,(ijk\ldots)}^{(s)}$$

If exactly the same function was given on different cuts or slices, say,

$$u_\mathrm{B} = G(s,\zeta,\overline{\zeta}), \qquad (154)$$

with

$$\eta^*_{(s)}(s,\zeta,\overline{\zeta}) = \eta_{(s)}\left(G(s,\zeta,\overline{\zeta}),\zeta,\overline{\zeta}\right),$$

the harmonic expansion at constant s would be different. The new coefficients are extracted by the two-sphere integral taken at constant s:

$$\eta^{*l,(ijk...)}_{(s)}(s) = \int_{S^2} \eta^*_{(s)}(s,\zeta,\overline{\zeta})\overline{Y}^{(s)}_{l,(ijk...)}d^2S. \tag{155}$$

It is in this rather obvious sense that the expansions are not unique. From Equation (97), using the real values of the complex cuts, i.e.,

$$u_{\mathrm{B}} = G_R\left(s,\Lambda(s,\zeta,\overline{\zeta}),\zeta,\overline{\zeta}\right), \tag{156}$$

(in place of $u_{\mathrm{B}} = G(s,\zeta,\overline{\zeta})$), the transformation, Equation (153), and harmonic extraction yields $\phi^{0*}_{0i}(s)$. This is proportional to the transformed complex dipole moment. This transformation, a functional of the form,

$$\phi^{0*}_{0i}(s) = \Gamma_i(\phi^0_0,\phi^0_1,\phi^0_2,\xi^a), \tag{157}$$

is decidedly nontensorial: in fact it is very nonlocal and nonlinear.

Though it is clear that extracting $\phi^{0*}_{0i}(s)$ with this relationship is available in principle, in practice it is impossible to do it exactly and all examples are done with approximations: essentially using slow motion for the complex world line.

3.4.6 The complex center of charge

The complex center of charge is defined by the vanishing of the complex dipole moment $\phi^{0*}_{0i}(s)$; in other words,

$$\Gamma_i(\phi^0_0,\phi^0_1,\phi^0_2,\xi^a) = 0 \tag{158}$$

determines the (up to this point) arbitrary complex world line, $\xi^a(\tau)$. In practice we do this only up to second order with the use of only the ($l = 0,1,2$) harmonics. The approximation we are using is to consider the charge q as zeroth order and the dipole moments and the spatial part of the complex world line as first order.

From Equation (153),

$$\phi^{0*}_{0i} = \Gamma_i \approx \phi^0_{0i} - 2L\phi^0_1|_i = 0 \tag{159}$$

with the identifications, Equation (123), for q and $D_{\mathbb{C}}$, we have to first order (with $s \approx u_{\mathrm{ret}} \approx \tau$),

$$D^i_{\mathbb{C}}(u_{\mathrm{ret}}) = q\eta^i(u_{\mathrm{ret}}) = q\xi^i(u_{\mathrm{ret}}). \tag{160}$$

This is exactly the same result as we obtained earlier in Equation (116), via the charge and current distributions in a fixed Lorentz frame.

Carrying this calculation [35] to second order, we find the second-order relationship between the intrinsic complex dipole and the intrinsic dipole,

$$D^i_{\mathcal{I}:\mathbb{C}} = q\xi^i(s), \qquad D^i_{\mathbb{C}} = q\eta^i(s), \tag{161}$$

$$\xi^k(s) = \eta^k(s) + i\eta^0_I(s)\eta^{k\prime}(s) - \frac{i}{2}\epsilon_{ijk}\eta^i(s)\eta^{j\prime}(s). \tag{162}$$

In Section 5, these ideas are applied to GR, with the complex electric and magnetic dipoles being replaced by the complex combination of the mass dipole and the angular momentum.

The GCF,

$$u_{\mathrm{B}} = \xi^a(\tau)\hat{l}_a(\zeta,\tilde{\zeta}),$$

with this uniquely determined world line is referred to as the Maxwell UCF.

4 The Good-Cut Equation and $\mathcal{H}$-Space

In Section 3, we discussed NGCs in Minkowski spacetime that were shear-free. In this section we consider *asymptotically* shear-free NGCs in asymptotically-flat spacetimes. That is to say, we consider NGCs that have nonvanishing shear in the interior of the spacetime but where, as null infinity is approached, the shear vanishes. In this case, whereas fully shear-free NGCs almost never occur, *asymptotically* shear-free congruences always exist. The case of algebraically-special spacetimes is the exception; they do allow one or two shear-free congruences.

We begin by reviewing the shear-free condition and follow with its generalization to the asymptotically shear-free case. From this we derive the generalization of the homogeneous good-cut equation to the inhomogeneous good-cut equation. Almost all the properties of the shear-free and asymptotically shear-free NGCs come from the study of these equations. We will see that virtually all the attributes of shear-free congruences are shared by the asymptotically shear-free congruences. It is from the use of these shared attributes that we will be able to extract physical identifications and information (e.g., complex center of mass/charge, Bondi mass, linear and angular momentum, equations of motion, etc.) from the asymptotic gravitational fields.

Though again the use of the formal complexification of $\mathfrak{I}^+$, i.e., $\mathfrak{I}^+_{\mathbb{C}}$, is essential for our analysis, it is the extraction of the real structures that is important.

4.1 Asymptotically shear-free NGCs and the good-cut equation

We saw in Section 3 that shear-free NGCs in Minkowski space could be constructed by looking at their properties near $\mathfrak{I}^+$, in one of two equivalent ways. The first was via the stereographic angle field, $L(u_{\mathrm{B}},\zeta,\bar{\zeta})$, which gives the directions the null rays make at their intersection with $\mathfrak{I}^+$. The condition for the congruence to be shear-free was that L must satisfy

$$\eth_{(u_{\mathrm{B}})}L + L\dot{L} = 0. \tag{163}$$

We required solutions that were all nonsingular (regular) on the $(\zeta,\bar{\zeta})$ sphere. (This equation has in the past most often been solved via twistor methods [19].)

The second was via the complex cut function, $u_{\mathrm{B}} = G(\tau,\zeta,\tilde{\zeta})$, that satisfied

$$\eth^2_{(\tau)}G = 0. \tag{164}$$

The regular solutions were easily given by

$$u_{\mathrm{B}} = G(\tau,\zeta,\tilde{\zeta}) = \xi^a(\tau)\hat{l}_a(\zeta,\tilde{\zeta}) \tag{165}$$

with inverse function,

$$\tau = T(u_{\mathrm{B}},\zeta,\tilde{\zeta}).$$

They determined the $L(u_{\mathrm{B}},\zeta,\bar{\zeta})$ that satisfies Equation (163) by the parametric relations

$$\begin{aligned} L(u_{\mathrm{B}},\zeta,\bar{\zeta}) &= \eth_{(\tau)}G(\tau,\zeta,\tilde{\zeta}), \\ u_{\mathrm{B}} &= \xi^a(\tau)\hat{l}_a(\zeta,\tilde{\zeta}), \end{aligned} \tag{166}$$

or by

$$L(u_{\mathrm{B}},\zeta,\bar{\zeta}) = \eth_{(\tau)}G(\tau,\zeta,\bar{\zeta})|_{\tau=T(u_{\mathrm{B}},\zeta,\tilde{\zeta})},$$

where $\xi^a(\tau)$ was an arbitrary complex world line in complex Minkowski space.

It is this pair of equations, (163) and (164), that will now be generalized to asymptotically-flat spacetimes.

In Section 2, we saw that the asymptotic shear of the (null geodesic) tangent vector fields, l^a, of the out-going Bondi null surfaces was given by the free data (the Bondi shear) $\sigma^0(u_{\rm B}, \zeta, \bar{\zeta})$. If, near $\mathfrak{I}^+$, a second NGC, with tangent vector l^{*a}, is chosen and then described by the null rotation from l^a to l^{*a} around n^a by

$$\begin{aligned} l^{*a} &= l^a + b\overline{m}^a + \bar{b}m^a + b\bar{b}n^a, \\ m^{*a} &= m^a + bn^a, \\ n^{*a} &= n^a, \\ b &= -L/r + O(r^{-2}), \end{aligned} \tag{167}$$

with $L(u_{\rm B}, \zeta, \bar{\zeta})$ an arbitrary stereographic angle field, then the asymptotic Weyl components transform as

$$\psi_0^{*0} = \psi_0^0 - 4L\psi_1^0 + 6L^2\psi_2^0 - 4L^3\psi_3^{*0} + L^4\psi_4^0, \tag{168}$$

$$\psi_1^{*0} = \psi_1^0 - 3L\psi_2^0 + 3L^2\psi_3^0 - L^3\psi_4^0, \tag{169}$$

$$\psi_2^{*0} = \psi_2^0 - 2L\psi_3^0 + L^2\psi_4^0, \tag{170}$$

$$\psi_3^{*0} = \psi_3^0 - L\psi_4^0, \tag{171}$$

$$\psi_4^{*0} = \psi_4^0, \tag{172}$$

and the (new) asymptotic shear of the null vector field l^{*a} is given by [6, 26]

$$\sigma^{0*} = \sigma^0 - \eth_{(u_{\rm B})}L - L\dot{L}. \tag{173}$$

By requiring that the new congruence be *asymptotically* shear-free, i.e., $\sigma^{0*} = 0$, we obtain the generalization of Equation (163) for the determination of $L(u_{\rm B}, \zeta, \bar{\zeta})$, namely,

$$\eth_{(u_{\rm B})}L + L\dot{L} = \sigma^0(u_{\rm B}, \zeta, \bar{\zeta}). \tag{174}$$

To solve this equation we again complexify $\mathfrak{I}^+$ to $\mathfrak{I}^+_{\mathbb{C}}$ by freeing $\bar{\zeta}$ to $\tilde{\zeta}$ and allowing $u_{\rm B}$ to take complex values close to the real.

Again we introduce the complex potential $\tau = T(u_{\rm B}, \zeta, \tilde{\zeta})$ that is related to L by

$$\eth_{(u_{\rm B})}T + L\dot{T} = 0, \tag{175}$$

with its inversion,

$$u_{\rm B} = G(\tau, \zeta, \tilde{\zeta}). \tag{176}$$

Equation (174) becomes, after the change in the independent variable, $u_{\rm B} \Rightarrow \tau = T(u_{\rm B}, \zeta, \bar{\zeta})$, and implicit differentiation (see Section 3.1 for the identical details),

$$\eth^2_{(\tau)}G = \sigma^0(G, \zeta, \tilde{\zeta}). \tag{177}$$

This, the inhomogeneous good-cut equation, is the generalization of Equation (164).

In Section 4.2, we will discuss how to construct solutions of Equation (177) of the form, $u_{\rm B} = G(\tau, \zeta, \tilde{\zeta})$; however, assuming we have such a solution, it determines the angle field $L(u_{\rm B}, \zeta, \bar{\zeta})$ by the parametric relations

$$\begin{aligned} L(u_{\rm B}, \zeta, \tilde{\zeta}) &= \eth_{(\tau)}G, \\ u_{\rm B} &= G(\tau, \zeta, \tilde{\zeta}). \end{aligned} \tag{178}$$

We return to the properties of these solutions in Section 4.2.

4.2 $\mathcal{H}$-space and the good-cut equation

Equation (177), written in earlier literature as

$$\eth^2 Z = \sigma^0(Z, \zeta, \tilde{\zeta}), \tag{179}$$

is a well-known and well-studied partial differential equation, often referred to as the "good-cut equation" [19, 20]. For sufficiently regular $\sigma^0(u_\mathrm{B}, \zeta, \bar{\zeta})$ (which is assumed here) it has been proven [20] that the solutions are determined by points in a complex four-dimensional space, z^a, referred to as $\mathcal{H}$-space, i.e., solutions are given as

$$u_\mathrm{B} = Z(z^a, \zeta, \tilde{\zeta}). \tag{180}$$

Later in this section, by choosing an arbitrary complex analytic world line in $\mathcal{H}$-space, $z^a = \xi^a(\tau)$, we describe how to construct the shear-free angle field, $L(u_\mathrm{B}, \zeta, \tilde{\zeta})$. First, however, we discuss properties and the origin of Equation (180).

Roughly or intuitively one can see how the four complex parameters enter the solution from the following argument. We can write Equation (179) as the integral equation

$$Z = z^a \hat{l}_a(\zeta, \tilde{\zeta}) + \oint \sigma^0(Z, \eta, \tilde{\eta}) K^+_{0,-2}(\eta, \tilde{\eta}, \zeta, \tilde{\zeta}) dS_\eta \tag{181}$$

with

$$K^+_{0,-2}(\zeta, \tilde{\zeta}, \eta, \tilde{\eta}) \equiv -\frac{1}{4\pi} \frac{(1+\tilde{\zeta}\eta)^2(\eta - \zeta)}{(1+\zeta\tilde{\zeta})(1+\eta\tilde{\eta})(\tilde{\eta} - \tilde{\zeta})},$$
$$dS_\eta = 4i \frac{d\eta \wedge d\tilde{\eta}}{(1+\eta\tilde{\eta})^2},$$

where $z^a \hat{l}_a(\zeta, \tilde{\zeta})$ is the kernel of the $\eth^2$ operator (the solution to the homogeneous good-cut equation) and $K^+_{0,-2}(\zeta, \tilde{\zeta}, \eta, \tilde{\eta})$ is the Green's function for the $\eth^2$ operator [23]. By iterating this equation, with the kernel being the zeroth iterate, i.e.,

$$Z_n(\zeta, \tilde{\zeta}) = z^a \hat{l}_a(\zeta, \tilde{\zeta}) + \int_{S^2} K^+_{0,-2}(\zeta, \tilde{\zeta}, \eta, \tilde{\eta}) \sigma(Z_{n-1}, \eta, \tilde{\eta}) dS_\eta, \tag{182}$$
$$Z_0(\zeta, \tilde{\zeta}) = z^a \hat{l}_a(\zeta, \tilde{\zeta}), \tag{183}$$

one easily sees how the four z^a enter the solution. Basically, the z^a come from the solution to the homogeneous equation.

It should be noted again that the $z^a \hat{l}_a(\zeta, \tilde{\zeta})$ is composed of the $l = (0, 1)$ harmonics,

$$z^a \hat{l}_a(\zeta, \tilde{\zeta}) = \frac{1}{\sqrt{2}} z^0 - \frac{1}{2} z^i Y^0_{1i}(\zeta, \tilde{\zeta}). \tag{184}$$

Furthermore, the integral term does not contribute to these lowest harmonics. This means that solutions can be written

$$u_\mathrm{B} = Z(z^a, \zeta, \tilde{\zeta}) \equiv z^a \hat{l}_a(\zeta, \tilde{\zeta}) + Z_{l \geq 2}(z^a, \zeta, \tilde{\zeta}), \tag{185}$$

with $Z_{l \geq 2}$ containing spherical harmonics $l = 2$ and higher.

We note that using this form of the solution implies that we have set stringent coordinate conditions on the $\mathcal{H}$-space by requiring that the first four spherical harmonic coefficients be the four $\mathcal{H}$-space coordinates. Arbitrary coordinates would just mean that these four coefficients were arbitrary functions of other coordinates. How these special coordinates change under the BMS group is discussed later.

Remark: It is of considerable interest that on $\mathcal{H}$-space there is a natural quadratic complex metric – demonstrated in Appendix D – that is given by the surprising relationship [34, 20]

$$ds^2_{(\mathcal{H})} = g_{(\mathcal{H})ab}\, dz^a dz^b \equiv \left(\frac{1}{8\pi} \int_{S^2} \frac{dS}{(dZ)^2} \right)^{-1}, \tag{186}$$

$$dZ \equiv \nabla_a Z\, dz^a, \tag{187}$$

$$dS = 4i \frac{d\zeta \wedge d\tilde{\zeta}}{(1+\zeta\tilde{\zeta})^2}. \tag{188}$$

Remarkably this turns out to be a Ricci-flat metric with a nonvanishing anti-self-dual Weyl tensor, i.e., it is intrinsically a complex vacuum metric. For vanishing Bondi shear, $\mathcal{H}$-space reduces to complex Minkowski space (i.e., $g_{(\mathcal{H})ab}|_{\sigma^0=0} = \eta_{ab}$).

4.2.1 Solutions to the shear-free equation

Returning to the issue of the solutions to the shear-free condition, i.e., Equation (174), $L(u_\mathrm{B}, \zeta, \tilde{\zeta})$, we see that they are easily constructed from the solutions to the good-cut equation, $u_\mathrm{B} = Z(z^a, \zeta, \tilde{\zeta})$. By choosing an arbitrary complex world line in the $\mathcal{H}$-space, i.e.,

$$z^a = \xi^a(\tau), \tag{189}$$

we write the GCF as

$$u_\mathrm{B} = G(\tau, \zeta, \tilde{\zeta}) \equiv Z(\xi^a(\tau), \zeta, \tilde{\zeta}), \tag{190}$$

or, from Equation (185),

$$u_\mathrm{B} = G(\tau, \zeta, \tilde{\zeta}) = \frac{1}{\sqrt{2}} \xi^0(\tau) - \frac{1}{2} \xi^i(\tau) Y^0_{1i}(\zeta, \tilde{\zeta}) + \xi^{ij}(\tau) Y^0_{2ij}(\zeta, \tilde{\zeta}) + \dots. \tag{191}$$

This leads immediately, via Equations (178) and (179), to the parametric description of the shear-free stereographic angle field $L(u_\mathrm{B}, \zeta, \tilde{\zeta})$, as well as the Bondi shear $\sigma^0(u_\mathrm{B}, \zeta, \tilde{\zeta})$:

$$u_\mathrm{B} = \frac{1}{\sqrt{2}} \xi^0(\tau) - \frac{1}{2} \xi^i(\tau) Y^0_{1i}(\zeta, \tilde{\zeta}) + \xi^{ij}(\tau) Y^0_{2ij}(\zeta, \tilde{\zeta}) + \dots, \tag{192}$$

$$L(u_\mathrm{B}, \zeta, \tilde{\zeta}) = \xi^i(\tau) Y^1_{1i}(\zeta, \tilde{\zeta}) - 6\xi^{ij}(\tau) Y^1_{2ij}(\zeta, \tilde{\zeta}) + \dots, \tag{193}$$

$$\sigma^0(u_\mathrm{B}, \zeta, \tilde{\zeta}) = 24\xi^{ij}(\tau) Y^2_{2ij} + \dots. \tag{194}$$

We denote the inverse to Equation (191) by

$$\tau = T(u_\mathrm{B}, \zeta, \tilde{\zeta}). \tag{195}$$

The asymptotic twist of the asymptotically shear-free NGC is exactly as in the flat-space case,

$$i\Sigma = \frac{1}{2} \left\{ \eth \overline{L} + L \dot{\overline{L}} - \overline{\eth} L - \overline{L} \dot{L} \right\}. \tag{196}$$

As in the flat-space case, the derived quantity

$$V(\tau, \zeta, \tilde{\zeta}) \equiv \partial_\tau G = G' \tag{197}$$

plays a large role in applications. (In the case of the Robinson–Trautman metrics [55, 28] V is the basic variable for the construction of the metric.)

Using the gauge freedom, $\tau \Rightarrow \tau^* = \Phi(\tau))$, in a slightly different way than in the Minkowski-space case, we impose the simple condition

$$\xi^0 = \tau. \tag{198}$$

A Brief Summary: The description and analysis of the asymptotically shear-free NGCs in asymptotically-flat spacetimes is remarkably similar to that of the flat-space regular shear-free NGCs. We have seen that all *regular* shear-free NGCs in Minkowski space and asymptotically-flat spaces are generated by solutions to the good-cut equation, with each solution determined by the choice of an arbitrary complex analytic world line in complex Minkowski space or $\mathcal{H}$-space. The basic governing variables are the complex GCF, $u_\mathrm{B} = G(\tau, \zeta, \tilde{\zeta})$, and the stereographic angle field on $\mathfrak{I}^+_\mathbb{C}$, $L(u_\mathrm{B}, \zeta, \tilde{\zeta})$, restricted to real $\mathfrak{I}^+$. In every sense, the flat-space case can be considered as a special case of the asymptotically-flat case.

In Sections 5 and 6, we will show that in every asymptotically flat spacetime a special complex-world line (along with its associated NGC and GCF) can be singled out using physical considerations. This special GCF is referred to as the (gravitational) UCF, and is denoted by

$$u_\mathrm{B} = X(\tau, \zeta, \tilde{\zeta}). \tag{199}$$

4.3 Real cuts from the complex good cuts, II

The construction of real structures from the complex structures, i.e., finding the complex values of τ that yield real values of u_B and the associated real cuts, is virtually identical to the flat-space construction of Section 3. The only difference is that we start with the GCF

$$u_\mathrm{B} = G(\tau, \zeta, \bar{\zeta}) = \xi^a(\tau)\hat{l}_a(\zeta, \bar{\zeta}) + G_{l\geq 2}(\tau, \zeta, \bar{\zeta}) \tag{200}$$

rather than the flat-space

$$u_\mathrm{B} = G(\tau, \zeta, \bar{\zeta}) = \xi^a(\tau)\hat{l}_a(\zeta, \bar{\zeta}).$$

Using $\tau = s + i\lambda$, Equation (200) is written

$$u_\mathrm{B} = G_R(s, \lambda, \zeta, \bar{\zeta}) + iG_I(s, \lambda, \zeta, \bar{\zeta}). \tag{201}$$

The reality of u_B, i.e.,

$$G_I(s, \lambda, \zeta, \bar{\zeta}) = 0, \tag{202}$$

again leads to

$$\lambda = \Lambda(s, \zeta, \bar{\zeta}) \tag{203}$$

and the real slicing,

$$u_\mathrm{B}^{(R)} = G_R\left(s, \Lambda(s, \zeta, \bar{\zeta}), \zeta, \bar{\zeta}\right). \tag{204}$$

Using Equation (191) and expanding to first order in λ, we have the expressions:

$$
\begin{aligned}
u_{\mathrm{B}} = & \frac{\sqrt{2}}{2}\xi_R^0(s) - \frac{\sqrt{2}}{2}\xi_I^0(s)'\lambda - \frac{1}{2}[\xi_R^i(s) - \xi_I^i(s)'\lambda]Y_{1i}^0(\zeta,\bar{\zeta}) \\
& +[\xi_R^{ij}(s) - \xi_I^{ij}(s)'\lambda]Y_{2ij}^0(\zeta,\bar{\zeta}) \\
& +i\left[\frac{\sqrt{2}}{2}\xi_I^0(s) + \frac{\sqrt{2}}{2}\xi_R^0(s)'\lambda\right] - \frac{i}{2}[\xi_I^i(s) + \xi_R^i(s)'\lambda]Y_{1i}^0(\zeta,\bar{\zeta}) \\
& +i[\xi_I^{ij}(s) + \xi_R^{ij}(s)'\lambda]Y_{2ij}^0(\zeta,\bar{\zeta}),
\end{aligned} \tag{205}
$$

$$
\begin{aligned}
u_{\mathrm{B}}^{(R)} = & G_R(s,\Lambda(s,\zeta,\bar{\zeta}),\zeta,\bar{\zeta}) \\
= & \frac{\sqrt{2}}{2}\xi_R^0(s) - \frac{\sqrt{2}}{2}\xi_I^0(s)'\lambda - \frac{1}{2}[\xi_R^i(s) - \xi_I^i(s)'\lambda]Y_{1i}^0(\zeta,\bar{\zeta}) \\
& +[\xi_R^{ij}(s) - \xi_I^{ij}(s)'\lambda]Y_{2ij}^0(\zeta,\bar{\zeta}),
\end{aligned} \tag{206}
$$

$$
\lambda = \Lambda(s,\zeta,\bar{\zeta}) = -\frac{\sqrt{2}\xi_I^0(s) + \xi_I^i(s)Y_{1i}^0(\zeta,\bar{\zeta}) + 2\xi_I^{ij}(s)Y_{2ij}^0(\zeta,\bar{\zeta})}{\{\sqrt{2}\xi_R^0(s)' - \xi_R^i(s)'Y_{1i}^0(\zeta,\bar{\zeta}) + 2\xi_R^{ij}(s)'Y_{2ij}^0(\zeta,\bar{\zeta})\}}. \tag{207}
$$

Since in all applications $\Lambda(s,\zeta,\bar{\zeta})$ is multiplied by a first-order quantity, only the first-order expression for $\Lambda(s,\zeta,\bar{\zeta})$ is needed. Using the slow motion assumption, we have,

$$
\lambda = \frac{\sqrt{2}}{2}\xi_I^i(s)Y_{1i}^0(\zeta,\bar{\zeta}) - \sqrt{2}\xi_I^{ij}(s)Y_{2ij}^0(\zeta,\bar{\zeta}). \tag{208}
$$

Substituting this into Equation (206), via the slow motion assumption, leads to the real cut function $u_{\mathrm{B}} = G_R(s,\Lambda(s,\zeta,\bar{\zeta}),\zeta,\bar{\zeta})$:

$$
\begin{aligned}
u_{\mathrm{ret}}^{(R)} = \sqrt{2}u_{\mathrm{B}}^{(R)} = & s - \frac{\sqrt{2}}{2}\xi_R^i(s)Y_{1i}^0(\zeta,\bar{\zeta}) + \sqrt{2}\xi_R^{ij}(s)Y_{2ij}^0(\zeta,\bar{\zeta}) \\
& +\frac{1}{3}\xi_I^i v_I^i + \frac{24}{5}v_I^{ij}\xi_I^{ij} + \frac{12}{5}(\xi_I^i v_I^{ki} - v_I^i\xi_I^{ki})Y_{1k}^0 + \frac{1}{6}\xi_I^k v_I^i Y_{2ij}^0 - v_I^{ij}\xi_I^{kj}Y_{2ik}^0.
\end{aligned} \tag{209}
$$

Later the linear versions are extensively used:

$$
u_{\mathrm{ret}}^{(R)} = \sqrt{2}u_{\mathrm{B}}^{(R)} = s - \frac{\sqrt{2}}{2}\xi_R^i(s)Y_{1i}^0(\zeta,\bar{\zeta}) + \sqrt{2}\xi_R^{ij}(s)Y_{2ij}^0(\zeta,\bar{\zeta}) \tag{210}
$$

$$
\tau = s + i\left(\frac{\sqrt{2}}{2}\xi_I^i(s)Y_{1i}^0(\zeta,\bar{\zeta}) - \sqrt{2}\xi_I^{ij}(s)Y_{2ij}^0(\zeta,\bar{\zeta})\right). \tag{211}
$$

5 Simple Applications

In this section we give four simple examples of the use of shear-free and asymptotically shear-free NGCs in GR. The first is for asymptotically-linearized perturbations off the Schwarzschild metric, while the next two are from the class of algebraically-special metrics, namely the Robinson–Trautman metric and the type II twisting metrics; the fourth is for asymptotically static/stationary metrics.

5.1 Linearized off Schwarzschild

As a first example, we describe how the shear-free NGCs are applied in linear perturbations off the Schwarzschild metric. The ideas used here are intended to clarify the more complicated issues in the full nonlinear asymptotic theory. We will see that these linear perturbations greatly resemble our results from the previous section on the determination of the intrinsic center of charge in Maxwell theory, when there were small deviations from the Coulomb field.

We begin with the Schwarzschild spacetime, treating the Schwarzschild mass, $M_{\mathrm{Sch}} \equiv M_{\mathrm{B}}$, as a zeroth-order quantity, and integrate the linearized Bianchi identities for the linear Weyl tensor corrections. Though we could go on and find the linearized connection and metric, we stop just with the Weyl tensor. The radial behavior is given by the peeling theorem, so that we can start with the linearized asymptotic Bianchi identities, Equations (56) – (58).

Our main variables for the investigation are the asymptotic Weyl tensor components and the Bondi shear, σ^0, with their related differential equations, i.e., the asymptotic Bianchi identities, Eq. (56), (57) and (55). Assuming the gravitational radiation is weak, we treat σ^0 and $\dot{\sigma}^0$ as small. Keeping only linear terms in the Bianchi identities, the equations for ψ_1^0 and Ψ (the mass aspect) become

$$\dot{\psi}_1^0 + \eth\Psi = \eth^3\overline{\sigma}^0, \tag{212}$$

$$\dot{\Psi} = 0, \tag{213}$$

$$\Psi = \overline{\Psi}, \tag{214}$$

$$\Psi \equiv \psi_2^0 + \eth^2\overline{\sigma}^0. \tag{215}$$

The ψ_1^0 is small (first order), while the

$$\Psi = \Psi^0 + \Psi^i Y_{1i}^0 + \Psi^{ij} Y_{2ij}^0 + \dots \tag{216}$$

has the zeroth-order Schwarzschild mass plus first-order terms

$$\Psi^0 = -\frac{2\sqrt{2}G}{c^2} M_{\mathrm{Sch}} + \delta\Psi^0, \tag{217}$$

$$\Psi^i = -\frac{6G}{c^3} P^i. \tag{218}$$

In *linear theory*, the complex (mass) dipole moment,

$$D^i_{\mathbb{C}(\mathrm{grav.})} = D^i_{(\mathrm{mass})} + ic^{-1}J^i \tag{219}$$

is given [59], on a particular Bondi cut with a Bondi tetrad (up to dimensional constants), by the $l = 1$ harmonic components of ψ_1^0, i.e., from the $\psi_1^{0\,i}$ in the expansion

$$\psi_1^0 = \psi_1^{0i} Y_{1i}^1 + \psi_1^{0ij} Y_{2ij}^1 + \dots \tag{220}$$

For a different cut and different tetrad, one needs the transformation law to the new ψ_1^{*0} and new $\psi_1^{*0\,i}$. Under the tetrad transformation (a null rotation around n^a) to the asymptotically shear-free vector field, l^{*a}, Equation (149),

$$l^a \to l^{*a} = l^a - \frac{\bar{L}}{r}m^a - \frac{L}{r}\bar{m}^a + O(r^{-2}),$$

with, from Equations (192) and (193),

$$u_{\mathrm{B}} = \xi^a(\tau)\hat{l}_a(\zeta,\bar{\zeta}) + \xi^{ij}(\tau)Y^0_{2ij}(\zeta,\bar{\zeta}) + \dots \tag{221}$$

$$= \frac{1}{\sqrt{2}}\xi^0(\tau) - \frac{1}{2}\xi^i(\tau)Y^0_{1i}(\zeta,\bar{\zeta}) + \xi^{ij}(\tau)Y^0_{2ij}(\zeta,\bar{\zeta}) + \dots \tag{222}$$

$$L(u_{\mathrm{B}},\zeta,\bar{\zeta}) = \xi^i(\tau)Y^1_{1i}(\zeta,\bar{\zeta}) - 6\xi^{ij}(\tau)Y^1_{2ij}(\zeta,\bar{\zeta}) + \dots \tag{223}$$

the linearized transformation is given by [6]

$$\psi_1^{0*} = \psi_1^0 - 3L\Psi. \tag{224}$$

The extraction of the $l = 1$ part of ψ_1^{0*} should, in principle, be taken on the new cut given by the real u_{B} obtained from $u_{\mathrm{B}} = \xi^a(\tau)\hat{l}_a(\zeta,\bar{\zeta}) + \xi^{ij}(\tau)Y^0_{2ij}(\zeta,\bar{\zeta}) + \dots$ with constant s in the expression, Equation (98), $\tau = s + i\Lambda(s,\zeta,\bar{\zeta})$. However, because of the linearization, the extraction can be taken on the u_{B} constant cuts. Following the same line of reasoning that led to the definition of center of charge, we demand the vanishing of the $l = 1$ part of ψ_1^{0*}.

This leads immediately to

$$\psi_1^0|_{l=1} = 3L\Psi|_{l=1}, \tag{225}$$

or, using the decomposition into real and imaginary parts, $\psi_1^{0i} = \psi_{1R}^{0i} + i\psi_{1I}^{0i}$ and $\xi^i(u_{\mathrm{ret}}) = \xi_R^i(u_{\mathrm{ret}}) + i\xi_I^i(u_{\mathrm{ret}})$,

$$\psi_{1R}^{0i} = -\frac{6\sqrt{2}G}{c^2}M_{\mathrm{Sch}}\xi_R^i(u_{\mathrm{ret}}), \tag{226}$$

$$\psi_{1I}^{0i} = -\frac{6\sqrt{2}G}{c^2}M_{\mathrm{Sch}}\xi_I^i(u_{\mathrm{ret}}). \tag{227}$$

Identifying [59, 37] the (intrinsic) *angular momentum*, either from the conventional linear identification or from the Kerr metric, as

$$J^i = S^i = M_{\mathrm{Sch}}c\xi_I^i \tag{228}$$

and the mass dipole as

$$D^i_{(\mathrm{mass})} = M_{\mathrm{Sch}}\xi_R^i, \tag{229}$$

we have

$$\psi_1^{0i} = -\frac{6\sqrt{2}G}{c^2}D^i_{\mathbb{C}(\mathrm{grav})} = -\frac{6\sqrt{2}G}{c^2}(D^i_{(\mathrm{mass})} + ic^{-1}J^i). \tag{230}$$

By inserting Equation (230) into Equation (212), taking, respectively, the real and imaginary parts, using Equation (218) and the reality of Ψ, we find

$$P^i = M_{\mathrm{Sch}}\xi_R^{i\,\prime} \equiv M_{\mathrm{Sch}}v_R^i, \tag{231}$$

the kinematic expression of linear momentum and

$$J^{i\,\prime} = 0, \tag{232}$$

the conservation of angular momentum.

Finally, from the $l = (0,1)$ parts of Equation (213), we have, at this approximation, that the mass and linear momentum remain constant, i.e., $M = M_{\text{Sch}} = M_{\text{B}}$ and $\delta\Psi^0 = 0$. Thus, we obtain the trivial equations of motion for the center of mass,

$$M_{\text{Sch}}\xi_R^{i\prime\prime} = 0. \tag{233}$$

It was the linearization that let to such simplifications. In Section 6, when nonlinear terms are included (in similar calculations), much more interesting and surprising physical results are found.

5.2 The Robinson–Trautman metrics

The algebraically-special type II Robinson–Trautman (RT) metrics are expressed in conventional RT coordinates, $(\tau, r, \zeta, \overline{\zeta})$, τ now real, by [55]

$$ds^2 = 2\left(K - \frac{V'}{V}r + \frac{\psi_2^0}{r}\right)d\tau^2 + 2d\tau dr - r^2\frac{2d\zeta d\overline{\zeta}}{V^2P_0^2}, \tag{234}$$

, with

$$K = 2V^2P_0^2\partial_{\overline{\zeta}}\partial_\zeta \log VP_0, \tag{235}$$
$$P_0 = 1 + \zeta\overline{\zeta}, \tag{236}$$
$$\psi_2^0 = \psi_2^0(\tau). \tag{237}$$

The unknowns are the Weyl component ψ_2^0, (closely related to the Bondi mass), which is a function only of (real) τ and the variable, $V(\tau, \zeta, \overline{\zeta})$, both of which satisfy the RT equation. (See below.) There remains the freedom

$$\tau \Rightarrow \tau^* = g(\tau), \tag{238}$$

which often is chosen so that $\psi_2^0(\tau) = \text{constant}$. However, we make a different choice. In the spherical harmonic expansion of V,

$$V = v^a\hat{l}_a(\zeta, \overline{\zeta}) + v^{ij}Y_{2ij}^0 + \dots, \tag{239}$$

the τ is chosen by normalizing the four-vector, v^a, to one, i.e., $v^av_a = 1$. The final field equation, the RT equation, is

$$\psi_2^{0\prime} - 3\psi_2^0\frac{V'}{V^3} - V^3\left(\eth_{(\tau)}^2\overline{\eth}_{(\tau)}^2V - V^{-1}\overline{\eth}_{(\tau)}^2V \cdot \eth_{(\tau)}^2V\right) = 0. \tag{240}$$

These spacetimes, via the Goldberg–Sachs theorem, possess a degenerate shear-free PNV field, l^a, that is surface-forming, (i.e., twist free). Using the tetrad constructed from l^a we have that the Weyl components are of the form

$$\psi_0 = \psi_1 = 0,$$
$$\psi_2 \neq 0.$$

Furthermore, the metric contains a "real timelike world line, $x^a = \xi^a(\tau)$," with normalized velocity vector $v^a = \xi^{a\prime}$. All of these properties allow us to identify the RT metrics as being analogous to the real Liénard–Wiechert solutions of the Maxwell equations.

Assuming for the moment that we have integrated the RT equation and know $V = V(\tau, \zeta, \overline{\zeta})$, then, by the integral

$$u = \int V(\tau, \zeta, \overline{\zeta})d\tau \equiv X_{\text{RT}}(\tau, \zeta, \bar{\zeta}), \tag{241}$$

the UCF for the RT metrics has been found. The freedom of adding $\alpha(\zeta,\bar{\zeta})$ to the integral is just the supertranslation freedom in the choice of a Bondi coordinate system. From $X_{\mathrm{RT}}(\tau,\zeta,\bar{\zeta})$ a variety of information can be obtained: the Bondi shear, σ^0, is given parametrically by

$$\sigma^0(u_{\mathrm{B}},\zeta,\bar{\zeta}) = \eth_{(\tau)}^2 X_{\mathrm{RT}}(\tau,\zeta,\bar{\zeta}), \\ u_{\mathrm{B}} = X_{\mathrm{RT}}(\tau,\zeta,\bar{\zeta}), \tag{242}$$

as well as the angle field L by

$$L(u_{\mathrm{B}},\zeta,\bar{\zeta}) = \eth_{(\tau)} X_{\mathrm{RT}}(\tau,\zeta,\bar{\zeta}), \\ u_{\mathrm{B}} = X_{\mathrm{RT}}(\tau,\zeta,\bar{\zeta}). \tag{243}$$

In turn, from this information the RT metric (in the neighborhood of $\mathfrak{I}^+$) can, in principle, be re-expressed in terms of the Bondi coordinate system. In practice one must revert to approximations. These approximate calculations lead, via the Bondi mass aspect evolution equation, to both Bondi mass loss and to equations of motion for the world line, $x^a = \xi^a(\tau)$. An alternate approximation for the mass loss and equations of motion is to insert the spherical harmonic expansion of V into the RT equation and look at the lowest harmonic terms. We omit further details aside from mentioning that we come back to these calculations in a more general context in Section 6.

5.3 Type II twisting metrics

It was pointed out in the previous section that the RT metrics are the general relativistic analogues of the (real) Liénard–Wiechert Maxwell fields. The type II algebraically-special twisting metrics are the gravitational analogues of the *complex* Liénard–Wiechert Maxwell fields described earlier. Unfortunately they are far more complicated than the RT metrics. In spite of the large literature and much effort there are very few known solutions and much still to be learned [28, 42, 33]. We give a very brief description of them, emphasizing only the items of relevance to us.

A null tetrad system (and null geodesic coordinates) can be adopted for the type II metrics so that the Weyl tetrad components are such that

$$\psi_0 = \psi_1 = 0, \\ \psi_2 \neq 0.$$

It follows from the Goldberg–Sachs theorem that the degenerate principal null congruence is geodesic and shear-free. Thus, from the earlier discussions it follows that there is a *unique* angle field, $L(u_{\mathrm{B}},\zeta,\bar{\zeta})$. As with the complex Liénard–Wiechert Maxwell fields, the type II metrics and Weyl tensors are given in terms of the angle field, $L(u_{\mathrm{B}},\zeta,\bar{\zeta})$. In fact, the entire metric and the field equations (the asymptotic Bianchi identities) can be written in terms of L and a Weyl tensor component (essentially the Bondi mass). Since $L(u_{\mathrm{B}},\zeta,\bar{\zeta})$ describes a unique shear-free NGC, it can be written, parametrically, in terms of a *unique* GCF, namely a UCF, $X_{(\text{type II})}(\tau,\zeta,\bar{\zeta})$, i.e., we have that

$$L(u_{\mathrm{B}},\zeta,\bar{\zeta}) = \eth_{(\tau)} X_{(\text{type II})}, \\ u_{\mathrm{B}} = X_{(\text{type II})}(\tau,\zeta,\bar{\zeta}).$$

Since $X_{(\text{type II})}(\tau,\zeta,\bar{\zeta})$ can be expanded in spherical harmonics, the $l = (0,1)$ harmonics can be identified with a (unique) complex world line in $\mathcal{H}$-space. The asymptotic Bianchi identities then yield both kinematic equations (for angular momentum and the Bondi linear momentum) and equations of motion for the world line, analogous to those obtained for the Schwarzschild perturbation and the RT metrics. As a kinematic example, the imaginary part of the world line is identified as the intrinsic spin, the same identification as in the Kerr metric,

$$S^i = M_{\mathrm{B}} c \xi_I^i. \tag{244}$$

In Section 6, a version of these results will be derived in a far more general context.

5.4 Asymptotically static and stationary spacetimes

By defining *asymptotically static or stationary* spacetimes as those asymptotically-flat spacetimes where the asymptotic variables are 'time' independent, i.e., u_{B} independent, we can look at our procedure for transforming to the complex center of mass (or complex center of charge). This example, though very special, has the huge advantage in that it can be done exactly, without the use of perturbations [2].

Imposing time independence on the asymptotic Bianchi identities, Equations (56) – (58),

$$\dot{\psi}_2^0 = -\eth\psi_3^0 + \sigma^0\psi_4^0, \tag{245}$$
$$\dot{\psi}_1^0 = -\eth\psi_2^0 + 2\sigma^0\psi_3^0, \tag{246}$$
$$\dot{\psi}_0^0 = -\eth\psi_1^0 + 3\sigma^0\psi_2^0, \tag{247}$$

and reality condition

$$\Psi \equiv \psi_2^0 + \eth^2\overline{\sigma} + \sigma\dot{\overline{\sigma}} = \overline{\Psi}, \tag{248}$$

we have, using Equations (51) and (52) with $\dot{\sigma}^0 = 0$, that

$$\psi_3^0 = \psi_4^0 = 0, \tag{249}$$
$$\eth\psi_2^0 = 0, \tag{250}$$
$$\eth\psi_1^0 = 3\sigma^0\psi_2^0, \tag{251}$$
$$\Psi \equiv \psi_2^0 + \eth^2\overline{\sigma} = \overline{\psi}_2^0 + \overline{\eth}^2\sigma = \overline{\Psi}. \tag{252}$$

From Equation (252), we find (after a simple calculation) that the imaginary part of ψ_2^0 is determined by the 'magnetic' [44] part of the Bondi shear (spin-weight $s = 2$) and thus must contain harmonics only of $l \geq 2$. But from Equation (250), we find that ψ_2^0 contains only the $l = 0$ harmonic. From this it follows that the 'magnetic' part of the shear must vanish. The remaining part of the shear, i.e., the 'electric' part, which by assumption is time independent, can be made to vanish by a supertranslation, via the Sachs theorem:

$$\begin{aligned} \widehat{u}_B &= u_{\mathrm{B}} + \alpha(\zeta,\overline{\zeta}), \\ \widehat{\sigma}(\zeta,\overline{\zeta}) &= \sigma(\zeta,\overline{\zeta}) + \eth^2\alpha(\zeta,\overline{\zeta}). \end{aligned} \tag{253}$$

In this Bondi frame, (i.e., frame with a vanishing shear), Equation (251), implies that

$$\psi_1^0 = \psi_1^{0i} Y_{1i}^1, \tag{254}$$
$$\psi_1^{0i} = -\frac{6\sqrt{2}G}{c^2} D_{\mathbb{C}(\mathrm{grav})}^i = -\frac{6\sqrt{2}G}{c^2}(D_{(\mathrm{mass})}^i + ic^{-1}J^i), \tag{255}$$

using the conventionally accepted physical identification of the complex gravitational dipole. (Since the shear vanishes, this agrees with probably all the various attempted identifications.)

From the mass identification, ψ_2^0 becomes

$$\psi_2^0 = -\frac{2\sqrt{2}G}{c^2} M_{\mathrm{B}}. \tag{256}$$

Since the Bondi shear is zero, the asymptotically shear-free congruences are determined by the same GCFs as in flat spaces, i.e., we have

$$L(u_{\mathrm{B}}, \zeta, \bar{\zeta}) = \eth_{(\tau)} G(\tau, \zeta, \bar{\zeta}) = \xi^a(\tau)\hat{m}_a(\zeta, \bar{\zeta}), \tag{257}$$
$$u_{\mathrm{B}} = \xi^a(\tau)\hat{l}_a(\zeta, \bar{\zeta}). \tag{258}$$

Our procedure for the identification of the complex center of mass, namely setting $\psi_1^{*0} = 0$ in the transformation, Equation (276),

$$\psi_1^{*0} = \psi_1^0 - 3L\psi_2^0 + 3L^2\psi_3^0 - L^3\psi_4^{*0}$$

leads, after using Equations (254), (249) and (257), to

$$\psi_1^0 = 3L\psi_2^0, \tag{259}$$
$$\psi_1^{0i} = -\frac{6\sqrt{2}G}{c^2} D^i_{\mathbb{C}(\mathrm{grav})},$$
$$D^i_{\mathbb{C}(\mathrm{grav})} = M_{\mathrm{B}}\xi^i.$$

From the time independence, ξ^i, the spatial part of the world line is a constant vector. By a (real) spatial Poincaré transformation (from the BMS group), the real part of ξ^i can be made to vanish, while by ordinary rotation the imaginary part of ξ^i can be made to point in the three-direction. Using the the gauge freedom in the choice of τ we set $\xi^0(\tau) = \tau$. Then pulling all these items together, we have for the complex world line, the UCF, $L(u_{\mathrm{B}}, \zeta, \bar{\zeta})$ and the angular momentum, J^i:

$$\xi^a(\tau) = (\tau, 0, 0, i\xi^3), \tag{260}$$
$$u_{\mathrm{B}} = \xi^a(\tau)\hat{l}_a(\zeta, \bar{\zeta}) \equiv \frac{\tau}{\sqrt{2}} - \frac{i}{2}\xi^3 Y^0_{1,3},$$
$$L(u_{\mathrm{B}}, \zeta, \bar{\zeta}) = i\xi_I^3 Y^1_{1,3},$$
$$J^i = S^i = M_{\mathrm{B}} c\xi^3\delta^i_3 = M_{\mathrm{B}} c(0, 0, \xi^3) = M_{\mathrm{B}} c\xi^i_I.$$

Thus, we have the complex center of mass on the complex world line, $z^a = \xi^a(\tau)$.

These results for the lower multipole moments, i.e., $l = 0, 1$, are *identical to those of the Kerr metric*. The higher moments are still present (appearing in higher r^{-1} terms in the Weyl tensor) and are not affected by these results.

6 Main Results

We saw in Sections 3 and 4 how shear-free and asymptotically shear-free NGCs determine arbitrary complex analytic world lines in the auxiliary complex $\mathcal{H}$-space (or complex Minkowski space). In the examples from Sections 3 and 5, we saw how, in each of the cases, one could pick out a special GCF, referred to as the UCF, and the associated complex world line by a transformation to the complex center of mass or charge by requiring that the complex dipoles vanish. In the present section we consider the same problem, but now perturbatively for the general situation of asymptotically-flat spacetimes satisfying either the vacuum Einstein or the Einstein–Maxwell equations in the neighborhood of future null infinity. Since the calculations are relatively long and complicated, we give the basic details only for the vacuum case, but then present the final results for the Einstein–Maxwell case without an argument.

We begin with the Reissner–Nordström metric, considering both the mass and the charge as zeroth-order quantities, and perturb from it. The perturbation data is considered to be first order and the perturbations themselves are general in the class of analytic asymptotically-flat spacetimes. Though our considerations are for arbitrary mass and charge distributions in the interior, we look at the fields in the neighborhood of $\mathfrak{I}^+$. The calculations are carried to second order in the perturbation data. Throughout we use expansions in spherical harmonics and their tensor harmonic versions, but terminate the expansions after $l = 2$. Clebsch–Gordon expansions are frequently used. See Appendix C.

6.1 A brief summary – Before continuing

Very briefly, for the purpose of organizing the many strands so far developed, we summarize our procedure for finding the complex center of mass. We begin with the gravitational radiation data, the Bondi shear, $\sigma^0(u_\mathrm{B},\zeta,\bar\zeta)$ and solve the good-cut equation,

$$\eth^2 Z = \sigma^0(Z,\zeta,\bar\zeta),$$

with solution $u_\mathrm{B} = Z(z^a,\zeta,\bar\zeta)$ and the four complex parameters z^a defining the solution space. Next we consider an arbitrary complex world line in the solution space, $z^a = \xi^a(\tau) = (\xi^0(\tau),\xi^i(\tau))$, so that $u_\mathrm{B} = Z(\xi^a(\tau),\zeta,\bar\zeta) = G(\tau,\zeta,\bar\zeta)$, a GCF, which can be expanded in spherical harmonics as

$$\begin{aligned} u_\mathrm{B} = G(\tau,\zeta,\bar\zeta) &= \xi^a(\tau)\hat{l}_a\zeta,\bar\zeta + \xi^{ij}(\tau)Y^0_{2ij} + \dots \\ &= \frac{\xi^0(\tau)}{\sqrt{2}} - \frac{1}{2}\xi^i(\tau)Y^0_{1i} + \xi^{ij}(\tau)Y^0_{2ij} + \dots \end{aligned} \tag{261}$$

Assuming slow motion and the gauge condition $\xi^0(\tau) = \tau$ (see Section 4), we have

$$u_\mathrm{B} = \frac{\tau}{\sqrt{2}} - \frac{1}{2}\xi^i(\tau)Y^0_{1i} + \xi^{ij}(\tau)Y^0_{2ij} + \dots \tag{262}$$

(Though the world line is arbitrary, the quadrupole term, $\xi^{ij}(\tau)$, and higher harmonics, are determined by both the Bondi shear and the world line.)

The inverse function,

$$\begin{aligned} \tau &= T(u_\mathrm{ret},\zeta,\bar\zeta), \\ u_\mathrm{ret} &= \sqrt{2}u_\mathrm{B}, \end{aligned}$$

can be found by the following iteration process [27]; writing Equation (262) as

$$\tau = u_\mathrm{ret} + F(\tau,\zeta,\bar\zeta), \tag{263}$$

with

$$F(\tau,\zeta,\bar{\zeta}) = \frac{\sqrt{2}}{2}\xi^i(\tau)Y^0_{1i}(\zeta,\bar{\zeta}) - \sqrt{2}\xi^{ij}(\tau)Y^0_{1ij}(\zeta,\bar{\zeta}) + \dots, \tag{264}$$

the iteration relationship, with the zeroth-order iterate, $\tau_0 = u_{\rm ret}$, is

$$\tau_n = u_{\rm ret} + F(\tau_{n-1},\zeta,\bar{\zeta}). \tag{265}$$

Though the second iterate easily becomes

$$\tau = T(u_{\rm ret},\zeta,\bar{\zeta}) = u_{\rm ret} + F\left(u_{\rm ret} + F(u_{\rm ret},\zeta,\bar{\zeta}),\zeta,\bar{\zeta}\right) \approx u_{\rm ret} + F + F\partial_{u_{\rm ret}}F. \tag{266}$$

For most of our calculations, all that is needed is the first iterate, given by

$$\tau = T(u_{\rm ret},\zeta,\bar{\zeta}) = u_{\rm ret} + \frac{\sqrt{2}}{2}\xi^i(u_{\rm ret})Y^0_{1i}(\zeta,\bar{\zeta}) - \sqrt{2}\xi^{ij}(u_{\rm ret})Y^0_{1ij}(\zeta,\bar{\zeta}). \tag{267}$$

This relationship is important later.

We also have the linearized *reality* relations – easily found earlier or from Equation (267):

$$\tau = s + i\lambda, \tag{268}$$

$$\lambda = \Lambda(s,\zeta,\bar{\zeta}) = \frac{\sqrt{2}}{2}\xi^i_I(s)Y^0_{1i} - \sqrt{2}\xi^{ij}_I(s)Y^0_{2ij}, \tag{269}$$

$$\tau = s + i\left(\frac{\sqrt{2}}{2}\xi^i_I(s)Y^0_{1i} - \sqrt{2}\xi^{ij}_I(s)Y^0_{2ij}\right), \tag{270}$$

$$u^{(R)}_{\rm ret} = \sqrt{2}G_R(s,\zeta,\bar{\zeta}) = \sqrt{2}u^{(R)}_{\rm B} = s - \frac{\sqrt{2}}{2}\xi^i_R(s)Y^0_{1i} + \sqrt{2}\xi^{ij}_R(s)Y^0_{2ij}. \tag{271}$$

The associated angle field, L, and the Bondi shear, σ^0, are given parametrically by

$$\begin{aligned} L(u_{\rm B},\zeta,\bar{\zeta}) &= \eth_{(\tau)}G(\tau,\zeta,\bar{\zeta}) \\ &= \xi^i(\tau)Y^1_{1i} - 6\xi^{ij}(\tau)Y^1_{1ij} + \dots \end{aligned} \tag{272}$$

and

$$\begin{aligned} \sigma^0(\tau,\zeta,\bar{\zeta}) &= \eth^2_{(\tau)}G(\tau,\zeta,\bar{\zeta}), \\ &= 24\xi^{ij}(\tau)Y^2_{2ij} + \dots, \end{aligned} \tag{273}$$

using $u_{\rm B} = G$, while the asymptotically shear-free NGC is given (again) by the null rotation

$$\begin{aligned} l^{*a} &= l^a + b\overline{m}^a + \bar{b}m^a + b\bar{b}n^a, \\ m^{*a} &= m^a + bn^a, \\ n^{*a} &= n^a, \\ b &= -L/r + O(r^{-2}). \end{aligned} \tag{274}$$

From Equation (274), the transformed asymptotic Weyl tensor becomes, Equations (275)–(279),

$$\psi^{*0}_0 = \psi^0_0 - 4L\psi^0_1 + 6L^2\psi^0_2 - 4L^3\psi^0_3 + L^4\psi^0_4, \tag{275}$$

$$\psi^{*0}_1 = \psi^0_1 - 3L\psi^0_2 + 3L^2\psi^0_3 - L^3\psi^0_4, \tag{276}$$

$$\psi^{*0}_2 = \psi^0_2 - 2L\psi^0_3 + L^2\psi^0_4, \tag{277}$$

$$\psi^{*0}_3 = \psi^0_3 - L\psi^0_4, \tag{278}$$

$$\psi^{*0}_4 = \psi^0_4. \tag{279}$$

The procedure is centered on Equation (276), where we search for and set to zero the $l = 1$ harmonic in ψ_1^{*0} on an $s =$ constant slice. This determines the complex center-of-mass world line and singles out a particular GCF referred to as the UCF,

$$X(\tau, \zeta, \bar{\zeta}) = G(\tau, \zeta, \bar{\zeta}),$$

with the real version,

$$X_R(s, \zeta, \bar{\zeta}) = G_R(s, \zeta, \bar{\zeta}), \tag{280}$$

for the gravitational field in the general asymptotically-flat case.

(For the case of the Einstein–Maxwell fields, in general, there will be two complex world lines, one for the center of charge, the other for the center of mass and the two associated UCFs. For later use we note that the gravitational world line will be denoted by ξ^a, while the electromagnetic world line by η^a. Later we consider the special case when the two world lines and the two UCFs coincide, i.e., $\xi^a = \eta^a$.)

From the assumption that σ^0 and L are first order and, from Equation (52), that $\psi_3^0 = \eth\dot{\bar{\sigma}}^0$, Equation (276), to second order, is

$$\psi_1^{*0} = \psi_1^0 - 3L[\Psi - \eth^2\bar{\sigma}^0], \tag{281}$$

where ψ_2^0 has been replaced by the mass aspect, Equation (54), $\Psi \approx \psi_2^0 + \eth^2\bar{\sigma}^0$.

Using the spherical harmonic expansions (see Equations (272) and (273)),

$$\Psi = \Psi^0 + \Psi^i Y_{1i}^0 + \Psi^{ij} Y_{2ij}^0 + \dots, \tag{282}$$

$$\psi_1^0 = \psi_1^{0i} Y_{1i}^1 + \psi_1^{0ij} Y_{2ij}^1 + \dots, \tag{283}$$

$$\psi_1^{*0} = \psi_1^{*0i} Y_{1i}^1 + \psi_1^{*0ij} Y_{2ij}^1 + \dots, \tag{284}$$

$$L(u_B, \zeta, \bar{\zeta}) = \xi^i(\tau) Y_{1i}^1 - 6\xi^{ij}(\tau) Y_{2ij}^1 + \dots, \tag{285}$$

$$\sigma^0(u_B, \zeta, \bar{\zeta}) = 24\xi^{ij}(\tau) Y_{2ij}^2 + \dots \tag{286}$$

Remembering that Ψ^0 is zeroth order, Equation (281), becomes

$$\begin{aligned}\psi_1^{*0} = {}& \psi_1^{0i} Y_{1i}^1 + \psi_1^{0ij} Y_{2ij}^1 \\ & -3[\xi^i(\tau) Y_{1i}^1 - 6\xi^{ij}(\tau) Y_{2ij}^1][\Psi^0 + \Psi^i Y_{1i}^0 + \{\Psi^{ij} - 24\bar{\xi}^{ij}(\tau)\} Y_{2ij}^0]\end{aligned}$$

or

$$\begin{aligned}\psi_1^{*0} = {}& \psi_1^{0i} Y_{1i}^1 + \psi_1^{0ij} Y_{2ij}^1 - 3\xi^i(\tau) Y_{1i}^1 \Psi^0 - 3\Psi^i \xi^j(\tau) Y_{1j}^1 Y_{1i}^0 \\ & -3\xi^k(\tau)[\Psi^{ij} - 24\bar{\xi}^{ij}(\tau)] Y_{1k}^1 Y_{2ij}^0 \\ & +18\xi^{ij}(\tau) Y_{2ij}^1 \Psi^0 + \Psi^k 18\xi^{ij}(\tau) Y_{2ij}^1 Y_{1k}^0 + 18\xi^{kl}(\tau)[\Psi^{ij} - 24\bar{\xi}^{ij}(\tau)] Y_{2kl}^1 Y_{2ij}^0.\end{aligned} \tag{287}$$

Note that the right-hand side of Equation (287) depends initially on both τ and u_{ret}, with $\tau = T(u_{\text{ret}}, \zeta, \bar{\zeta})$.

This equation, though complicated and unattractive, is our main source of information concerning the complex center-of-mass world line. Extracting this information, i.e., determining ψ_1^{*0i} at constant values of s by expressing τ and u_{ret} as functions of s and setting it equal to zero, takes considerable effort.

6.2 The complex center-of-mass world line

Before trying to determine ψ_1^{*0i}, several comments are in order:

1. As mentioned earlier, the right-hand side of Equation (287) is a function of both τ, via the the GC variables, ξ^i, ξ^{ij} and the u_{ret}, via the $\psi_1^{0i}, \psi_1^{0ij}$ and Ψ. The extraction of the $l = 1$ part of ψ_1^{*0} must be taken on the constant 's' cuts. In other words, both τ and u_{ret} must be eliminated by using Equations (268) – (271) and (267).

2. This elimination must be done in the linear terms, e.g., from Equation (271),

$$\begin{aligned}\eta(u_{\text{ret}}^{(R)}) &= \eta\left(s - \frac{\sqrt{2}}{2}\xi_R^i(s)Y_{1i}^0 + \sqrt{2}\xi_R^{ij}(s)Y_{2ij}^0\right)\\ &\approx \eta(s) - \frac{\sqrt{2}}{2}\eta(s)'\left[\xi_R^i(s)Y_{1i}^0 - 2\xi_R^{ij}(s)Y_{2ij}^0\right]\end{aligned}$$

   or, from Equation (270),

$$\begin{aligned}\chi(\tau) &= \chi\left(s + i\left(\frac{\sqrt{2}}{2}\xi_I^i(s)Y_{1i}^0 - \sqrt{2}\xi_I^{ij}(s)Y_{2ij}^0\right)\right)\\ &\approx \chi(s) + i\frac{\sqrt{2}}{2}\chi(s)'\left[\xi_I^i(s)Y_{1i}^0 - 2\xi_I^{ij}(s)Y_{2ij}^0\right].\end{aligned}$$

   In the nonlinear terms we can simply use

$$u_{\text{ret}} = \tau = s.$$

3. In the Clebsch–Gordon expansions of the harmonic products, though we need both the $l = 1$ and $l = 2$ terms in the calculation, we keep at the end only the $l = 1$ terms for the ψ_1^{0i}. (Note that there are no $l = 0$ terms since ψ_1^{*0} is spin weight $s = 1$.)

4. The calculation to determine the ψ_1^{*0i} for the constant 's' slices was probably the most tedious and lengthy in this work. The importance of the results necessitated the calculation be repeated several times.

Expanding and organizing Equation (287) with the linear terms given explicitly and the quadratic terms collected in the expression for A, we obtain the long expression with all terms functions of either τ or u_{ret}:

$$\psi_1^{0*} = \psi_1^{0i}Y_{1i}^1 + \psi_1^{0ij}Y_{2ij}^1 - 3\Psi^0\xi^iY_{1i}^1 + 18\Psi^0\xi^{ij}Y_{2ij}^1 + A. \tag{288}$$

$$\begin{aligned}A &= -3\xi^i\Psi^jY_{1i}^1Y_{1j}^0 + 18\xi^{ij}\Psi^kY_{2ij}^1Y_{1k}^0 - 3\xi^i\left[\Psi^{kj} - 24\bar{\xi}^{kj}\right]Y_{1i}^1Y_{2kj}^0 \\ &\quad +18\xi^{ij}\left[\Psi^{kl} - 24\bar{\xi}^{kl}(\tau)\right]Y_{2ij}^1Y_{2kl}^0 \\ &= A^iY_{1i}^1 + A^{ij}Y_{2ij}^1.\end{aligned} \tag{289}$$

Using Equations (269) and (271), the u_{ret} and τ in Equation (288) are replaced by s. On the right-hand side all the variables, e.g., $\psi_1^{0k}, \Psi^i, \xi^k, etc.$, are functions of '$s$'; their functional forms are the same as when they were functions of u_{ret} and τ; the linear terms are again explicitly given and the quadratic terms are collected in the $B_U + B_T + A$

$$\psi_1^{0*}(s,\zeta,\bar{\zeta}) = \psi_1^{0i}Y_{1i}^1 + \psi_1^{0ij}Y_{2ij}^1 - 3\Psi^0\xi^iY_{1i}^1 + 18\Psi^0\xi^{ij}Y_{2ij}^1 + B_U + B_T + A, \tag{290}$$

$$B_U = -\frac{\sqrt{2}}{2}\psi_1^{0i\prime}\xi_R^kY_{1k}^0Y_{1i}^1 + \sqrt{2}\psi_1^{0i\prime}\xi_R^{kl}Y_{2kl}^0Y_{1i}^1 - \frac{\sqrt{2}}{2}\psi_1^{0ij\prime}\xi_R^kY_{1k}^0Y_{2ij}^1 + \sqrt{2}\psi_1^{0ij\prime}\xi_R^{kl}Y_{2kl}^0Y_{2ij}^1,$$

$$B_T = i3\sqrt{2}\Psi^0\left\{-\frac{1}{2}\xi^{i\prime}\xi_I^k(s)Y_{1k}^0Y_{1i}^1 + \xi^{i\prime}\xi_I^{kl}(s)Y_{2kl}^0Y_{1i}^1 + 3\xi^{ij\prime}\xi_I^kY_{1k}^0Y_{2ij}^1 - 6\xi^{ij\prime}\xi_I^{kl}Y_{2kl}^0Y_{2ij}^1\right\}.$$

To proceed, we use the complex center-of-mass condition, namely, $\psi_1^{*0k}(s) = 0$, and solve for $\psi_1^{0k}(u_{\text{ret}})$. This is accomplished by first reversing the calculation via

$$\psi_1^0 = \psi_1^{0*} + 3L\psi_2^{0*} = \psi_1^{0*} + 3L\psi_2^0$$

and then, before extracting the $l = 1$ harmonic component, replacing the s by u_{ret}, via the inverse of Equation (271),

$$s = u_{\text{ret}}^{(R)} + \frac{\sqrt{2}}{2}\xi_R^i(u_{\text{ret}}^{(R)})Y_{1i}^0 - \sqrt{2}\xi_R^{ij}(u_{\text{ret}}^{(R)})Y_{2ij}^0,$$

using Equation 280

$$X_R(s,\zeta,\bar{\zeta}) = G_R(s,\zeta,\bar{\zeta}). \tag{291}$$

In this process several of the quadratic terms cancel out and new ones arise.

The final expression for ψ_1^{0j}, given in terms of the complex world line ξ^j expressed as a function of u_{ret}, then becomes our *basic* equation:

$$\psi_1^{0j} = 3\Psi^0\xi^j - \frac{18}{5}\xi^i\psi_2^{0ij} - \frac{108}{5}\Psi^i\xi^{ij} - \frac{3\sqrt{2}}{5}\psi_1^{0ij\prime}\xi_R^i + \frac{18\sqrt{2}}{5}\Psi^0(\xi_R^{ji} + i\xi_I^{ji})\xi^{i\prime} + i\left(\frac{3\sqrt{2}}{2}\xi^i\Psi^k + \frac{24}{5}\xi_R^{mk}\psi_1^{0mi\prime} - \frac{216\sqrt{2}}{5}\xi^{mk}\psi_2^{0mi} + \frac{3}{2}\Psi^0(\xi_R^k + i\xi_I^k)\xi^{i\prime}\right)\epsilon_{ikj}. \tag{292}$$

This, which becomes the analogue of the Newtonian dipole expression $\vec{D} = M\vec{R}$, is our *central relationship*. Almost all of our results in the following sections follow directly from it.

We emphasize that prior to this discussion/derivation, the ψ_1^{0j} and the ξ^j were independent quantities but in the final expression the ψ_1^{0j} is now a function of the ξ^j.

Note that the linear term

$$\psi_1^{0i} = 3\Psi^0\xi^i$$

coincides with the earlier results in the stationary case, Equation (259). From

$$\Psi^0 = -\frac{2\sqrt{2}G}{c^2}M_{\text{B}}$$

we have

$$\begin{aligned}\psi_1^{0i} &= -\frac{6\sqrt{2}G}{c^2}M_{\text{B}}\xi^i, \\ &= -\frac{6\sqrt{2}G}{c^2}D_{\mathbb{C}(\text{grav})}^i, \\ D_{\mathbb{C}(\text{grav})}^i &= M_{\text{B}}(\xi_R^i + i\xi_I^i) = (D_{(\text{mass})}^i + ic^{-1}J^i).\end{aligned} \tag{293}$$

We will see shortly that there is a great deal of physical content to be found in the nonlinear terms in Equation (292).

6.3 Results

6.3.1 Preliminaries

Before describing in details our results, it appears to be worthwhile to very roughly survey the results and describe the logical steps taken to reach them. Virtually everything, as we said earlier, follows from the equation for ψ_1^{0i}, i.e., Equation (292).

In the final results, we will include, from Section 3, the Maxwell field. Though we do consider the case where the two complex world lines (the complex center-of-mass and center-of-charge lines) differ from each other, some discussion will be directed to the special case of coinciding world lines.

1. The first step is to decompose the ψ_1^{0i} into its real and imaginary parts, identifying the real center of mass and the total angular momentum (as seen from infinity) described in the given Bondi coordinate system. In a different Bondi system they would undergo a specific transformation. These results are the analogues of Equation (115). It should be emphasized that there are alternative definitions, [59], but using our approximations all should reduce to our expression.

2. The second step is to look at the evolution equation for ψ_1^{0i}, i.e., insert our ψ_1^0 into the Bianchi identity, (57), now including the Maxwell field. We obtain, from the $l=1$ harmonic, the evolution of ψ_1^{0i}. After again decomposing it into the real and imaginary parts we find the kinematic description of the Bondi linear momentum, $P^i = M\xi^{i\prime} + \dots$ (i.e., the usual kinematic expression $P = Mv$ plus additional terms) and the evolution (conservation law) for the angular momentum including a flux expression, i.e., $J^{i\,\prime} = (Flux)^i$.

3. The third step is to reinsert the kinematic expression for the Bondi mass into the evolution equation for ψ_2^0, i.e., Equation (56). From the reality condition on the mass aspect, Equation (55) or Equation (53), only the real part is relevant. It leads to the evolution equation for the real part of the complex world line, a second-order ODE, that can be identified with Newton's second law, $F^i = M\xi^{i\,\prime\prime}$, with F^i being the recoil and radiation reaction forces. The $l=0$ harmonic term is the energy/mass loss equation of Bondi.

Before continuing we note that the $l=2$ coefficients in ψ_1^0 and ψ_2^0, i.e., ψ_1^{0ij} and ψ_2^{0ij}, appear frequently in second-order expressions, e.g., in Equation (292). Thus, knowing them, in terms of the free data, to first order is sufficient. By going to the *linearized Bianchi identities* (with the linearized Maxwell field) and the expression for the Bondi shear σ^0,

$$\dot{\psi}_2^0 = -\eth^2\dot{\bar{\sigma}}^0, \tag{294}$$

$$\dot{\psi}_1^0 = -\eth\psi_2^0 + 2kq\bar{\phi}_2^0, \tag{295}$$

$$\sigma^0(\tau,\zeta,\bar{\zeta}) = 24\xi^{ij}(\tau)Y_{2ij}^2 + \dots, \tag{296}$$

$$\phi_1^0 = q, \tag{297}$$

$$\bar{\phi}_2^0 = -2q\bar{\eta}^{i\,\prime\prime}(u_{\mathrm{ret}})Y_{1i}^1 - \frac{1}{3}\overline{Q}_{\mathbb{C}}^{ij\,\prime\prime\prime}Y_{2ij}^1, \tag{298}$$

we easily find

$$\psi_2^{0ij} = -24\bar{\xi}^{ij}(u_{\mathrm{ret}}), \tag{299}$$

$$\psi_1^{0ij\,\prime} = -72\sqrt{2}\bar{\xi}^{ij}(u_{\mathrm{ret}}) - \frac{\sqrt{2}}{3}c^{-3}kq\overline{Q}_{\mathbb{C}}^{ij\,\prime\prime\prime},$$

where a constant of integration was set to zero via initial conditions. These expressions are frequently used in the following. In future expressions we will restore explicitly 'c', via the derivative, $(') \to c^{-1}(')$ and replace the gravitational coupling constant by $k = 2Gc^{-4}$.

6.3.2 The real center of mass and the angular momentum

Returning to the basic relation, Equation (292), using Equations (299) we obtain

$$\begin{aligned}
\psi_1^{0j} &= 3\Psi^0\xi^j + \frac{3(12)^2}{5}\overline{\xi}^{ij}\xi^i + \frac{18\sqrt{2}}{5}c^{-1}\Psi^0\xi^{ji}v^i \\
&\quad -\frac{108}{5}\Psi^i\xi^{ij} + \frac{3(12)^2}{5}\xi_R^i\overline{\xi}^{ij} + \frac{4\sqrt{2}}{5}c^{-7}Gq\overline{Q}_{\mathbb{C}}^{ij\prime\prime\prime}\xi_R^i \\
&\quad +i\left(\frac{3}{2}c^{-1}\Psi^0\xi^k\xi^{i\,\prime} - \frac{(12)^3\sqrt{2}}{5}\xi_R^{mk}\overline{\xi}^{im} + \frac{3\sqrt{2}}{2}\xi^i\Psi^k\right)\epsilon_{ikj} \\
&\quad +i\left(\frac{3(12)^3\sqrt{2}}{5}\xi^{mk}\overline{\xi}^{im} - \frac{32}{5}Gc^{-7}q\overline{Q}_{\mathbb{C}}^{im\,\prime\prime\prime}\xi_R^{mk}\right)\epsilon_{ikj}.
\end{aligned} \tag{300}$$

By replacing Ψ^0 and Ψ^i, in terms of the Bondi mass and linear momentum, then decomposing the individual terms, e.g., $\xi^i = \xi_R^i + i\xi_I^i$, $Q_{\mathbb{C}}^{ij} = Q_E^{ij} + iQ_M^{ij}$, into their *real and imaginary* parts, the full expression is decomposed as

$$\begin{aligned}
\psi_1^{0i} &\equiv -\frac{6\sqrt{2}G}{c^2}(D_{(\mathrm{mass})}^i + ic^{-1}J^i) \\
&= -\frac{6\sqrt{2}G}{c^2}(M_{\mathrm{B}}\xi_R^i + iM_{\mathrm{B}}\xi_I^i) + \dots
\end{aligned} \tag{301}$$

The physical identifications – first from the real part, are, initially, a *tentative* definition of the mass dipole moment,

$$D_{(\mathrm{mass})}^{(T)\,j} = M_{\mathrm{B}}\xi_R^j + c^{-1}\left\{M_{\mathrm{B}}\xi_R^{k\prime}\xi_I^i + \frac{1}{2}M_{\mathrm{B}}\xi_R^k\xi_I^{i\,\prime}\right\}\epsilon_{kij} + \mathcal{D}_{(\mathrm{mass})}^{\#\,j}, \tag{302}$$

$$\begin{aligned}
\mathcal{D}_{(\mathrm{mass})}^{\#\,j} &= -\frac{54\sqrt{2}}{5}c^{-1}M_{\mathrm{B}}\xi_R^{i\prime}\xi_R^{ij} - \frac{6\sqrt{2}}{5}c^{-1}M_{\mathrm{B}}\xi_I^{ji}\xi_I^{i\,\prime} \\
&\quad -\frac{36\sqrt{2}}{5}G^{-1}c^2(\xi_R^i\xi_R^{ij} + \xi_I^i\xi_I^{ij}) \\
&\quad -\frac{36\sqrt{2}}{5}G^{-1}c^2\xi_R^i\xi_R^{ij} - \frac{\sqrt{2}}{15}c^{-5}q\xi_R^iQ_E^{ij\,\prime\prime\prime} \\
&\quad -\left\{\frac{2(12)^2}{5}G^{-1}c^2\xi_I^{im}\xi_R^{mk} - \frac{8}{15}c^{-5}q\xi_R^{mk}Q_M^{im\,\prime\prime\prime}\right\}\epsilon_{ikj} \\
&\quad +\frac{36\sqrt{2}}{5}c^{-4}q^2\eta_R^{i\prime\prime}\xi_R^{ij} + c^{-4}q^2\xi_I^i\eta_R^{k\prime\prime}\epsilon_{ikj},
\end{aligned} \tag{303}$$

and – from the imaginary part, again, a *tentative* definition of the total angular momentum,

$$J^{(T)\,j} = M_{\mathrm{B}} c \xi_I^j + \left\{ M_{\mathrm{B}} \xi_R^{k\,\prime} \xi_R^i + \frac{1}{2} M_{\mathrm{B}} \xi_I^i \xi_I^{k\,\prime} \right\} \epsilon_{ikj} + \mathcal{J}^{\#\,j}, \tag{304}$$

$$\begin{aligned} \mathcal{J}^{\#\,j} = & -\frac{3(12)\sqrt{2}}{5} \frac{c^3}{G} (\xi_I^i \xi_R^{ij} - 2\xi_R^i \xi_I^{ij}) - \frac{54\sqrt{2}}{5} M_{\mathrm{B}} \xi_R^{i\,\prime} \xi_I^{ij} \\ & + \frac{6\sqrt{2}}{5} M_{\mathrm{B}} \xi_R^{ji} \xi_I^{i\,\prime} + \frac{\sqrt{2}}{15} c^{-4} q \xi_R^i Q_M^{ij\prime\prime\prime} - \frac{8}{15} c^{-4} q \xi_R^{mk} Q_E^{im\,\prime\prime\prime} \epsilon_{ikj}. \\ & + \frac{36\sqrt{2}}{5} c^{-3} q^2 \eta_R^{i\,\prime\prime} \xi_I^{ij} + c^{-3} q^2 \xi_R^i \eta_R^{k\,\prime\prime} \epsilon_{ikj}. \end{aligned} \tag{305}$$

The reason for referring to these identifications as *tentative* is the following:

If there were no Maxwell field present, then the terms involving the electromagnetic dipole, $q\eta^i$, and quadruple, $Q_{\mathbb{C}}^{im}$, would not appear and these identifications, $D_{(\mathrm{mass})}^{(T)\,j}$ and $J^{(T)\,i}$, would then be considered to be firm; however, if a Maxwell field *is present*, we will see later that the identifications must be modified. Extra Maxwell terms are 'automatically' added to the above expressions when the conservation laws are considered.

As an important point we must mention a short cut that we have already taken in the interests of simplifying the presentation. When the *linearized* Equation (300) is substituted into the linearized Bianchi identity, $\dot{\psi}_1^0 = -\eth\psi_2^0$, we obtain the *linear* expression for the momentum, $P^i \equiv -\frac{c^3}{6G}\psi_2^{0i}$, in terms of the linearized expression for $\dot{\psi}_1^{0i}$, namely,

$$P^i = M_{\mathrm{B}} \xi_R^{i\prime} - \frac{2}{3} c^{-3} q^2 \eta_R^{i\prime\prime} = M_{\mathrm{B}} v_R^j - \frac{2}{3} c^{-3} q^2 \eta_R^{i\prime\prime}. \tag{306}$$

This expression is then 'fed' into the full nonlinear Equation (300) leading to the relations Equations (302) and (304).

Considering now *only the pure gravitational case*, there are several comments and observations to be made.

1. Equations (302) and (304) have been split into two types of terms: terms that contain only dipole information and terms that contain quadrupole information. The dipole terms are explicitly given, while the quadrupole terms are hidden in the $\mathcal{D}^{\#\,i}$ and $\mathcal{J}^{\#\,i}$.

2. In J^i we identify S^i as the intrinsic or spin angular momentum,

$$S^i = M_{\mathrm{B}} c \xi_I^i\,. \tag{307}$$

   This identification comes from the Kerr or Kerr–Newman metric [37]. The second term,

$$M_{\mathrm{B}} \xi_R^{k\,\prime} \xi_R^i \epsilon_{ikj} = (\overrightarrow{r} \times \overrightarrow{P})^j, \tag{308}$$

   is the orbital angular momentum. The third term,

$$\frac{1}{2} M_{\mathrm{B}} \xi_I^i \xi_I^{k\,\prime} \epsilon_{ikj} = \frac{1}{2M_{\mathrm{B}}c^2} S^i S^{k\prime} \epsilon_{ikj},$$

   though very small, represents a spin-spin contribution to the total angular momentum.

3. In the mass dipole expression $D_{(\mathrm{mass})}^{(T)\,i}$, the first term is the classical Newton mass dipole, while the next two are dynamical spin contributions.

In the following Sections 6.3.3 and 6.3.4 further physical results (with more comments and observations) will be found from the dynamic equations (asymptotic Bianchi identities) when they are applied to $D_{(\mathrm{mass})}^{(T)\,i}$ and $J^{(T)\,i}$.

6.3.3 The evolution of the complex center of mass

The evolution of the mass dipole and the angular momentum, defined from the ψ_1^{0i}, Equation (301), is determined by the Bianchi identity,

$$\dot{\psi}_1^0 = -\eth\psi_2^0 + 2\sigma^0\psi_3^0 + 2k\phi_1^0\bar{\phi}_2^0. \tag{309}$$

In the analysis of this relationship, the asymptotic Maxwell equations

$$\begin{aligned} \dot{\phi}_1^0 &= -\eth\phi_2^0, \\ \dot{\phi}_0^0 &= -\eth\phi_1^0 + \sigma^0\phi_2^0, \end{aligned} \tag{310}$$

and their solution, from Section 3, Equation (123) (needed only to first order),

$$\begin{aligned} \phi_0^0 &= 2q\eta^i(u_{\mathrm{ret}})Y_{1i}^1 + c^{-1}Q_{\mathbb{C}}^{ij\,\prime}Y_{2ij}^1 + \dots, \\ \phi_1^0 &= q + \sqrt{2}c^{-1}q\eta^{i\,\prime}(u_{\mathrm{ret}})Y_{1i}^0 + \frac{\sqrt{2}}{6}c^{-2}Q_{\mathbb{C}}^{ij\,\prime\prime}Y_{2ij}^0 + \dots, \\ \phi_2^0 &= -2c^{-2}q\eta^{i\,\prime\prime}(u_{\mathrm{ret}})Y_{1i}^{-1} - \frac{1}{3}c^{-3}Q_{\mathbb{C}}^{ij\,\prime\prime\prime}Y_{2ij}^{-1} + \dots, \end{aligned} \tag{311}$$

must be used. By extracting the $l=1$ harmonic from Equation (309), we find

$$\begin{aligned} \psi_1^{0i\,\prime} = \sqrt{2}c\Psi^i &+ \frac{i2(12)^3\sqrt{2}}{5}\xi^{kl}\bar{\xi}^{kj\,\prime}\epsilon_{lji} - 8\sqrt{2}Gc^{-5}q^2\bar{\eta}^{i\,\prime\prime} + i4\sqrt{2}Gc^{-6}q^2\bar{\eta}^{j\,\prime\prime}\eta^{m\,\prime}\epsilon_{mji} \\ &+ \frac{8}{5}Gc^{-7}q\bar{\eta}^{k\,\prime\prime}Q_{\mathbb{C}}^{ki\,\prime\prime} - \frac{8}{5}Gc^{-7}q\eta^{m\,\prime}\overline{Q}_{\mathbb{C}}^{im\,\prime\prime\prime} - i\frac{8\sqrt{2}}{15}Gc^{-8}Q_{\mathbb{C}}^{mj\,\prime\prime}\overline{Q}_{\mathbb{C}}^{lm\,\prime\prime\prime}\epsilon_{lji}. \end{aligned} \tag{312}$$

Using Equation (301),

$$\psi_1^{0i} \equiv -\frac{6\sqrt{2}G}{c^2}(D_{(\mathrm{mass})}^i + ic^{-1}J^i),$$

with the (real)

$$M_{\mathrm{B}} = -\frac{c^2}{2\sqrt{2}G}\Psi^0, \tag{313}$$

$$P^i = -\frac{c^3}{6G}\Psi^i, \tag{314}$$

we obtain, (1) from the real part, the *kinematic expression for the (real) linear momentum* and, (2) from the imaginary part, the *conservation or flux law for angular momentum.*

(1) Linear Momentum:

$$\begin{aligned} P^i &= D_{(\mathrm{mass})}^{i\prime} - \frac{2}{3}c^{-3}q^2\eta_R^{i\,\prime\prime} + \Pi^i, \\ \Pi^i &= -\frac{4(12)^2}{5}\frac{c^2}{G}\{\xi_I^{kl}\xi_R^{kj\prime} - \xi_R^{kl}\xi_I^{kj\prime}\}\epsilon_{lji} - \frac{2c^{-4}q^2}{3}\{\eta_R^{j\,\prime\prime}\eta_I^{l\,\prime} - \eta_I^{j\prime\prime}\eta_R^{l\prime}\}\epsilon_{lji} \\ &\quad + \frac{2\sqrt{2}}{15}c^{-5}q\{\eta_R^{k\prime\prime}Q_E^{ki\prime\prime} + \eta_I^{k\prime\prime}Q_M^{ki\prime\prime} - \eta_R^{m\prime}Q_E^{im\,\prime\prime\prime} + \eta_I^{m\prime}Q_M^{im\prime\prime\prime}\} \\ &\quad + \frac{4}{45}c^{-6}\{Q_M^{mj\,\prime\prime}Q_E^{ml\,\prime\prime\prime} - Q_E^{mj\,\prime\prime}Q_M^{ml\,\prime\prime\prime}\}\epsilon_{lji}. \end{aligned} \tag{315}$$

Using

$$D_{(\mathrm{mass})}^{(T)\,i\,\prime} = M_{\mathrm{B}}\xi_R^{i\,\prime} - c^{-1}\left\{M_{\mathrm{B}}\xi_R^{k\,\prime}\xi_I^j + \frac{1}{2}M_{\mathrm{B}}\xi_R^k\xi_I^{j\,\prime}\right\}^{\prime}\epsilon_{jki} + \mathcal{D}_{(\mathrm{mass})}^{\#\,i\,\prime},$$

we get the *kinematic expression* for the linear momentum,

$$\begin{aligned} P^i &= M_{\mathrm{B}}\xi_R^{i\,\prime} - \frac{2}{3}c^{-3}q^2\eta_R^{i\,\prime\prime} - c^{-1}\{M_{\mathrm{B}}\xi_R^{k\,\prime}\xi_I^{j\,\prime} + M_{\mathrm{B}}\xi_R^{k\,\prime\prime}\xi_I^j \\ &\quad + \frac{1}{2}M_{\mathrm{B}}(\xi_R^k\xi_I^{j\,\prime})'\}\epsilon_{jki} + \mathcal{D}_{(\mathrm{mass})}^{\#\,i\,\prime} + \Pi^i, \\ \Pi^i &= -\frac{4(12)^2}{5}\frac{c^2}{G}\{\xi_I^{kl}\xi_R^{kj\prime} - \xi_R^{kl}\xi_I^{kj\prime}\}\epsilon_{lji} - \frac{2c^{-4}q^2}{3}\{\eta_R^{j\prime\prime}\eta_I^{l\,\prime} - \eta_I^{j\prime\prime}\eta_R^{l\,\prime}\}\epsilon_{lji} \\ &\quad + \frac{2\sqrt{2}}{15}c^{-5}q\{\eta_R^{k\prime\prime}Q_E^{ki\prime\prime} + \eta_I^{k\prime\prime}Q_M^{ki\prime\prime} - \eta_R^{m\prime}Q_E^{im\prime\prime\prime} + \eta_I^{m\prime}Q_M^{im\prime\prime\prime}\} \\ &\quad + \frac{4}{45}c^{-6}\{Q_M^{mj\,\prime\prime}Q_E^{ml\,\prime\prime\prime} - Q_E^{mj\,\prime\prime}Q_M^{ml\,\prime\prime\prime}\}\epsilon_{lji}, \end{aligned} \tag{316}$$

or

$$\begin{aligned} P^i &= M_{\mathrm{B}}\xi_R^{i\prime} - \frac{2}{3}c^{-3}q^2\eta_R^{i\prime\prime} + c^{-1}M_{\mathrm{B}}\xi_R^{j\prime}\xi_I^{k\,\prime}\epsilon_{ijk} + \Xi^i, \\ \Xi^i &= -c^{-1}\{M_{\mathrm{B}}\xi_R^{k\,\prime\prime}\xi_I^j + \frac{1}{2}M_{\mathrm{B}}(\xi_R^k\xi_I^{j\,\prime})'\}\epsilon_{jki} + \mathcal{D}_{(\mathrm{mass})}^{\#\,i\,\prime} + \Pi^i, \end{aligned} \tag{317}$$

(2) Angular Momentum Flux:

$$J^{i\,\prime} = (Flux)^i, \tag{318}$$

$$J^i = J^{(T)\,i} + \frac{2}{3}q^2c^{-2}\eta_I^{i\,\prime}, \tag{319}$$

$$J^{(T)\,i} = M_{\mathrm{B}}c\xi_I^j + \{M_{\mathrm{B}}\xi_R^{k\prime}\xi_R^i + \frac{1}{2}M_{\mathrm{B}}\xi_I^i\xi_I^{k\,\prime}\}\epsilon_{ikj} + \mathcal{J}^{\#\,i}, \tag{320}$$

$$\begin{aligned} (Flux)^i &= -\frac{4(12)^2}{5}\frac{c^3}{G}\{\xi_R^{kl}\xi_R^{kj\prime} + \xi_I^{kl}\xi_I^{kj\prime}\}\epsilon_{lji} \\ &\quad -\frac{2}{3}q^2c^{-3}\{\eta_R^{j\prime\prime}\eta_R^{l\prime} + \eta_I^{j\prime\prime}\eta_I^{l\prime}\}\epsilon_{lji} \\ &\quad +\frac{4}{45}c^{-5}\{Q_E^{mj\prime\prime}Q_E^{ml\,\prime\prime\prime} + Q_M^{mj\,\prime\prime}Q_M^{ml\,\prime\prime\prime}\}\epsilon_{lji} \\ &\quad -\frac{2\sqrt{2}}{15}qc^{-4}\{\eta_R^{k\prime\prime}Q_M^{ki\prime\prime} - \eta_I^{k\prime\prime}Q_E^{ki\prime\prime}\} \\ &\quad +\frac{2\sqrt{2}}{15}qc^{-4}\{\eta_I^{m\prime}Q_E^{im\prime\prime\prime} - \eta_R^{m\prime}Q_M^{im\prime\prime\prime}\}. \end{aligned} \tag{321}$$

There are a variety of comments to be made about the physical content contained in these relations:

- The first term of P^i is the standard Newtonian kinematic expression for the linear momentum, $M_{\mathrm{B}}\xi_R^{k\,\prime}$.
- The second term, $-\frac{2}{3}c^{-3}q^2\eta_R^{i\,\prime\prime}$, which is a contribution from the second derivative of the electric dipole moment, $q\eta_R^i$, plays a special role for the case when the complex center of mass coincides with the complex center of charge, $\eta^a = \xi^a$. In this case, the second term is exactly the contribution to the momentum that yields the classical radiation reaction force of classical electrodynamics [30].
- The third term, $c^{-1}M_{\mathrm{B}}\xi_R^{j\prime}\xi_I^{k\,\prime}\epsilon_{ijk}$, is the classical Mathisson–Papapetrou spin-velocity contribution to the linear momentum. If the evolution equation (angular momentum conservation) is used, this term becomes third order.

- Many of the remaining terms in P^i, though apparently second order, are really of higher order when the dynamics are considered. Others involve quadrupole interactions, which contain high powers of c^{-1}. Though it is nice to see the Mathisson–Papapetrou term, it should be treated more as a suggestive result rather than a physical prediction.

  Aside: Later, in the discussion of the Bondi energy loss theorem, we will see that we can relate ξ^{ij}, i.e., the $l=2$ shear term, to the gravitational quadrupole by

$$\xi^{ij} = (\xi_R^{ij} + i\xi_I^{ij}) = \frac{\sqrt{2}Gc^{-4}}{24}(Q_{\text{Mass}}^{ij\,\prime\prime} + iQ_{\text{Spin}}^{ij\,\prime\prime}) = \frac{\sqrt{2}Gc^{-4}Q_{\text{Grav}}^{ij\,\prime\prime}}{24}. \tag{322}$$

- In the expression for J^i we have already identified, in the earlier discussion, the first two terms, $M_{\mathrm{B}}c\xi_I^j$ and $M_{\mathrm{B}}\xi_R^{k\,\prime}\xi_R^i\epsilon_{ikj}$ as the intrinsic spin angular momentum and the orbital angular momentum. The third term, $\frac{1}{2}M_{\mathrm{B}}\xi_I^i\xi_I^{k\,\prime}\epsilon_{ikj} = \frac{1}{2M_{\mathrm{B}}c^2}S^iS^{k\,\prime}\epsilon_{ikj}$, a spin-spin interaction term, considerably smaller, can be interpreted as a spin-precession contribution to the total angular momentum. An interesting contribution to the total angular momentum comes from the term, $\frac{2}{3}c^{-2}q^2\eta_I^{i\,\prime} = \frac{2}{3}c^{-2}qD_M^{i\,\prime}$, i.e., a contribution from the time-varying magnetic dipole.

- As mentioned earlier, our identification of $J^{(T)\,i}$ as the total angular momentum in the absence of a Maxwell field agrees with most other identifications (assuming our approximations). Very strong support of this view, with the Maxwell terms added in, comes from the flux law. In Equation (321) we see that there are five flux terms, the first is from the gravitational quadrupole flux, the second and third are from the classical electromagnetic dipole and electromagnetic quadrupole flux, while the fourth and fifth come from electromagnetic-gravitational coupling. *The Maxwell dipole part is identical to that derived from pure Maxwell theory* [30]. We emphasize that this angular momentum flux law has little to do directly with the chosen definition of angular momentum. The *imaginary part of the Bianchi identity*, Equation (309), *is the conservation law.* How to identify the different terms, i.e., identifying the time derivative of the angular momentum and the flux terms, comes from different arguments. The identification of the Maxwell contribution to total angular momentum and the flux contain certain arbitrary assignments: some terms on the left-hand side of the equation, i.e., terms with a time derivative, could have been moved onto the right-hand side and been called 'flux' terms. However, our assignments were governed by the question of what terms appeared most naturally to be explicit time derivatives, thereby being assigned to the time derivative of the angular momentum.

- The angular momentum conservation law can be considered as the evolution equation for the imaginary part of the complex world line, i.e., $\xi_I^i(u_{\text{ret}})$. The evolution for the real part is found from the Bondi energy-momentum loss equation.

- In the special case where the complex centers of mass and charge coincide, $\eta^a = \xi^a$, we have a rather attractive identification: since now the magnetic dipole moment is given by $D_M^i = q\xi_I^i$ and the spin by $S^i = M_{\mathrm{B}}c\xi_I^i$, we have that the gyromagnetic ratio is

$$\frac{|S^i|}{|D_M^i|} = \frac{M_{\mathrm{B}}c}{q}$$

  leading to the Dirac value of g, i.e., $g = 2$.

6.3.4 The evolution of the Bondi energy-momentum

Finally, to obtain the equations of motion, we substitute the kinematic expression for P^i into the Bondi evolution equation, the Bianchi identity, Equation (56);

$$\dot{\psi}_2^0 = -\eth\psi_3^0 + \sigma^0\psi_4^0 + k\phi_2^0\bar{\phi}_2^0, \tag{323}$$

or its much more useful and attractive (real) equivalent expression

$$\dot{\Psi} = \dot{\sigma}\dot{\overline{\sigma}} + 2Gc^{-4}\phi_2^0\overline{\phi}_2^0, \tag{324}$$

with

$$\Psi \equiv \psi_2^0 + \eth^2\overline{\sigma} + \sigma\dot{\overline{\sigma}} = \overline{\Psi}. \tag{325}$$

Remark: The Bondi mass, $M_\mathrm{B} = -\frac{c^2}{2\sqrt{2}G}\Psi^0$, and the original mass of the Reissner–Nordström (Schwarzschild) unperturbed metric, $M_\mathrm{RN} = -\frac{c^2}{2\sqrt{2}G}\psi_2^{0\,0}$, i.e., the $l=0$ harmonic of ψ_2^0, differ by a quadratic term in the shear, the $l=0$ part of $\sigma\dot{\overline{\sigma}}$. This suggests that the observed mass of an object is partially determined by its time-dependent quadrupole moment.

Looking only at the $l=0$ and $l=1$ spherical harmonics and switching to the ‘$'$’ derivatives with the c^{-1} inserted, we first obtain the Bondi mass loss theorem:

$$M_\mathrm{B}' = -\frac{2(12)^2c}{5G}\left(v_R^{ij}v_R^{ij} + v_I^{ij}v_I^{ij}\right) - \frac{2q^2}{3c^5}\left(\eta_R^{i\,\prime\prime}\eta_R^{i\,\prime\prime} + \eta_I^{i\,\prime\prime}\eta_I^{i\,\prime\prime}\right) - \frac{1}{180c^7}\left(Q_E^{ij\,\prime\prime\prime}Q_E^{ij\,\prime\prime\prime} + Q_M^{ij\,\prime\prime\prime}Q_M^{ij\,\prime\prime\prime}\right).$$

If we identify ξ^{ij} with the gravitational quadrupole moment Q^{ij}_Grav via

$$\xi^{ij} = (\xi_R^{ij} + i\xi_I^{ij}) = \frac{G}{12\sqrt{2}c^4}\left(Q^{ij\,\prime\prime}_\mathrm{Mass} + iQ^{ij\,\prime\prime}_\mathrm{Spin}\right) = \frac{GQ^{ij\,\prime\prime}_\mathrm{Grav}}{12\sqrt{2}c^4},$$

and the electric and magnetic dipole moments by

$$D^i_\mathbb{C} = q(\eta_R^i + i\eta_I^i) = D_E^i + iD_M^i,$$

the mass loss theorem becomes

$$\begin{aligned} M_\mathrm{B}' = &-\frac{G}{5c^7}\left(Q^{ij\,\prime\prime\prime}_\mathrm{Mass}Q^{ij\,\prime\prime\prime}_\mathrm{Mass} + Q^{ij\,\prime\prime\prime}_\mathrm{Spin}Q^{ij\,\prime\prime\prime}_\mathrm{Spin}\right) \\ &-\frac{2}{3c^5}\left(D_E^{i\,\prime\prime}D_E^{i\,\prime\prime} + D_M^{i\,\prime\prime}D_M^{i\,\prime\prime}\right) - \frac{1}{180c^7}\left(Q_E^{ij\,\prime\prime\prime}Q_E^{ij\,\prime\prime\prime} + Q_M^{ij\,\prime\prime\prime}Q_M^{ij\,\prime\prime\prime}\right). \end{aligned} \tag{326}$$

The mass/energy loss equation contains the classical energy loss due to electric and magnetic dipole radiation and electric and magnetic quadrupole (Q_E^{ij}, Q_M^{ij}) radiation. The gravitational energy loss is the conventional quadrupole loss by the above identification of ξ^{ij} with the gravitational quadrupole moment Q^{ij}_Grav.

The momentum loss equation, from the $l=1$ part of Equation (324), becomes

$$\begin{aligned} P^{k\,\prime} &= F^k_\mathrm{recoil}, \\ F^k_\mathrm{recoil} &\equiv \frac{2G}{15c^6}\left(Q^{lj\,\prime\prime\prime}_\mathrm{Spin}Q^{ij\,\prime\prime\prime}_\mathrm{Mass} - Q^{lj\,\prime\prime\prime}_\mathrm{Mass}Q^{ij\,\prime\prime\prime}_\mathrm{Spin}\right)\epsilon_{ilk} - \frac{2}{3c^4}D_M^{l\,\prime}D_E^{i\,\prime}\epsilon_{ilk} \\ &+\frac{1}{15c^5}\left(D_E^{j\,\prime}D_E^{jk\,\prime\prime\prime} + D_M^{j\,\prime}D_M^{jk\,\prime\prime\prime}\right) + \frac{1}{540c^6}\left(D_E^{lj\,\prime\prime\prime}D_M^{ij\,\prime\prime\prime} - D_M^{lj\,\prime\prime\prime}D_E^{ij\,\prime\prime\prime}\right)\epsilon_{ilk}. \end{aligned} \tag{327}$$

Finally, substituting the P^i from Equation (317), we have Newton's second law of motion:

$$M_\mathrm{B}\xi_R^{i\,\prime\prime} = F^i, \tag{328}$$

with

$$F^i = -M_\mathrm{B}'\xi_R^{i\,\prime} - c^{-1}M_\mathrm{B}(\xi_R^{j\,\prime}\xi_I^{k\,\prime})'\epsilon_{ijk} + \frac{2}{3}c^{-3}q^2\eta_R^{i\,\prime\prime\prime} + F^i_\mathrm{recoil} - \Xi^{i\,\prime}. \tag{329}$$

There are several things to observe and comment on concerning Equations (328) and (329):

- If the complex world line associated with the Maxwell center of charge coincides with the complex center of mass, i.e., if $\eta^i = \xi^i$, the term

$$\frac{2}{3}c^{-3}q\xi_R^{i\,\prime\prime\prime} \tag{330}$$

becomes the classical electrodynamic radiation reaction force.

- This result follows directly from the Einstein–Maxwell equations. There was no model building other than requiring that the two complex world lines coincide. Furthermore, there was no mass renormalization; the mass was simply the conventional Bondi mass as seen at infinity. The problem of the runaway solutions, though not solved here, is converted to the stability of the Einstein–Maxwell equations with the 'coinciding' condition on the two world lines. If the two world lines do not coincide, i.e., the Maxwell world line forms independent data, then there is no problem of unstable behavior. This suggests a resolution to the problem of the unstable solutions: one should treat the source as a structured object, not a point, and centers of mass and charge as independent quantities.
- The F^i_{recoil} is the recoil force from momentum radiation.
- The $\Xi^{i\,\prime} = -F^i_{\mathrm{RR}}$ can be interpreted as the gravitational radiation reaction.
- The first term in F^i, i.e., $-M_{\mathrm{B}}'\xi_R^{i\,\prime}$, is identical to a term in the classical Lorentz–Dirac equations of motion. Again it is nice to see it appearing, but with the use of the mass loss equation it is in reality third order.

6.4 Other related results

The ideas involved in the identification, at future null infinity, of interior physical quantities that were developed in the proceeding sections can also be applied to a variety of different perturbation schemes. Bramson, Adamo and Newman [9, 1, 3] have investigated how gravitational perturbations originating solely from a Maxwell radiation field can be carried through again using the asymptotic Bianchi identities to obtain, in a different context, the same identifications: a complex center-of-mass/charge world line, energy and momentum loss, as well as an angular momentum flux law that agrees exactly with the predictions of classical electromagnetic field theory. This scheme yields (up to the order of the perturbation) an approximation for the metric in the interior of the perturbed spacetime.

We briefly describe this procedure. One initially chooses as a background an exact solution of the Einstein equations; three cases were studied, flat Minkowski spacetime, the Schwarzschild spacetime with a 'small' mass and the Schwarzschild spacetime with a finite, 'zero order', mass. For such backgrounds, the set of spin coefficients is known and fixed. On this background the Maxwell equations were integrated to obtain the desired electromagnetic field that acts as the gravitational perturbation. Bramson has done this for a pure electric dipole solution [9, 1] on the Minkowski background. Recent work has used an electric and magnetic dipole field with a Coulomb charge [3]. The resulting Maxwell field, in each case, is then inserted into the asymptotic Bianchi identities, which, in turn, determine the behavior of the perturbed asymptotic Weyl tensor, i.e., the Maxwell field induces nontrivial changes to the gravitational field. Treating the Maxwell field as first order, the calculations were done to second order, as was done earlier in this review.

Using the just obtained Weyl tensor terms, one can proceed to the integration of the spin-coefficient equations and the second-order metric tensor. For example, one finds that the dipole Maxwell field induces a second-order Bondi shear, σ^0. (This in principle would lead to a fourth-order gravitational energy loss, which in our approximation is ignored.)

Returning to the point of view of this section, the perturbed Weyl tensor can now be used to obtain the same physical identifications described earlier, i.e., by employing a null rotation to set $\psi_1^{0*i} = 0$, equations of motion and asymptotic physical quantities, (e.g., center of mass and charge, kinematic expressions for momentum and angular momentum, etc.) for the interior of the system could be found. Although we will not repeat these calculations here, we present a few of the results. Though the calculations are similar to the earlier ones, they differ in two ways: there is no first-order freely given Bondi shear and the perturbation term orders are different.

For instance, the perturbations induced by a Coulomb charge and general electromagnetic dipole Maxwell field in a Schwarzschild background lead to energy, momentum, and angular momentum flux relations [3]:

$$\begin{aligned} M_{\mathrm{B}}' &= -\frac{2}{3c^5}\left(D_E^{i\,\prime\prime} D_E^{i\prime\prime} + D_M^{i\prime\prime} D_M^{i\,\prime\prime}\right), \\ P^{i\prime} &= \frac{1}{3c^4} D_E^{k\,\prime\prime} D_M^{j\,\prime\prime} \epsilon_{kji}, \\ J^{k\prime} &= \frac{2}{3c^3}\left(D_E^{i\,\prime\prime} D_E^{j\,\prime} + D_M^{i\,\prime\prime} D_M^{j\,\prime}\right)\epsilon_{ijk}, \end{aligned} \tag{331}$$

all of which agree exactly with predictions from classical field theory [30].

The familiarity of such results is an exhibit in favor of the physical identification methods described in this review, i.e., they are a confirmation of the consistency of the identification scheme.

7 Gauge (BMS) Invariance

The issue of gauge invariance, the understanding of which is not obvious or easy, must now be addressed. The claim is that the work described here is in fact gauge (or BMS) invariant.

First of all we have, $\mathfrak{I}^+_{\mathbb{C}}$, or its real part, $\mathfrak{I}^+$. On $\mathfrak{I}^+_{\mathbb{C}}$, for each choice of spacetime interior and solution of the Einstein–Maxwell equations, we have its UCF, either in its complex version, $u_B = X(\tau, \zeta, \tilde{\zeta})$, or its real version, Equation (291). The geometric picture of the UCF is a one-parameter family of slicings (complex or real) of $\mathfrak{I}^+_{\mathbb{C}}$ or $\mathfrak{I}^+$. This is a geometric construct that has a different appearance or description in different Bondi coordinate systems. It is this difference that we must investigate. We concentrate on the complex version.

Under the action of the supertranslation, Equation (67), we have:

$$X(\tau, \zeta, \tilde{\zeta}) \to \widehat{X}(\tau, \zeta, \tilde{\zeta}) = X(\tau, \zeta, \tilde{\zeta}) + \alpha(\zeta, \tilde{\zeta}), \tag{332}$$

with $\alpha(\zeta, \tilde{\zeta})$ an arbitrary complex smooth function on (complexified) S^2. Its effect is to add on a constant to each spherical harmonic coefficient. The special case of translations, with

$$\alpha(\zeta, \tilde{\zeta}) = d^a \hat{l}_a(\zeta, \tilde{\zeta}), \tag{333}$$

simply adds to the $l = (0, 1)$ harmonic components the complex constants d^a, so, via Equation (261), we have the (complex) Poincaré translations,

$$\xi^a \to \widehat{\xi}^a = \xi^a + d^a. \tag{334}$$

The action of the homogeneous Lorentz transformations, Equation (68),

$$\widehat{u}_B = K u_B, \tag{335}$$

$$K = \frac{1 + \zeta\overline{\zeta}}{(a\zeta + b)(\overline{a}\overline{\zeta} + \overline{b}) + (c\zeta + d)(\overline{c}\overline{\zeta} + \overline{d})}, \tag{336}$$

$$\widehat{\zeta} = \frac{a\zeta + b}{c\zeta + d}; \quad ad - bc = 1. \tag{337}$$

$$e^{i\lambda} = \frac{c\zeta + d}{\overline{c}\overline{\zeta} + \overline{d}} \tag{338}$$

is considerably more complicated. It leads to

$$\widehat{X}(\tau, \widehat{\zeta}, \widehat{\tilde{\zeta}}) = K X(\tau, \zeta, \tilde{\zeta}). \tag{339}$$

Before discussing the relevant effects of the Lorentz transformations on our considerations we first digress and describe an important technical issue concerning representation of the homogeneous Lorentz group.

The representation theory of the Lorentz group, developed and described by Gelfand, Graev and Vilenkin [16] used homogeneous functions of two complex variables (homogeneous of degrees, $n_1 - 1$ and $n_2 - 1$) as the representation space. Here we summarize these ideas via an equivalent method [21, 15] using spin-weighted functions on the sphere as the representation spaces. In the notation of Gelfand, Graev and Vilenkin, representations are labeled by the two numbers (n_1, n_2) or by (s, w), with $(n_1, n_2) = (w - s + 1, w + s + 1)$. The '$s$' is the same '$s$' as in the spin weighted functions and 'w' is the conformal weight [44] (sometimes called 'boost weight'). The different representations are written as $D_{(n_1, n_2)}$. The special case of irreducible unitary representations, which occur when (n_1, n_2) are not integers, play no role for us and will not be discussed. We consider only the case when (n_1, n_2) are integers so that the (s, w) take integer or half integer values. If n_1 and n_2 are both positive integers or both negative integers, we have, respectively, the positive or negative integer representations. The representation space, for each (s, w), are the functions on the sphere, $\eta_{(s,w)}(\zeta, \overline{\zeta})$, that can be expanded in spin-weighted spherical harmonics, ${}_sY_{lm}(\zeta, \overline{\zeta})$, so that

$$\eta_{(s,w)}(\zeta, \overline{\zeta}) = \sum_{l=s}^{\infty} \eta_{(lm)}\ {}_sY_{lm}(\zeta, \overline{\zeta}). \tag{340}$$

Under the action of the Lorentz group, Equation (335), they transform as

$$\widehat{\eta}_{(s,w)}(\widehat{\zeta}, \overline{\widehat{\zeta}}) = e^{is\lambda} K^w \eta_{(s,w)}(\zeta, \overline{\zeta}). \tag{341}$$

These representations, in general, are neither irreducible nor totally reducible. For us the important point is that many of these representations possess an invariant finite-dimensional subspace which (often) corresponds to the usual finite dimensional tensor representation space. Under the transformation, Equation (341), the finite number of coefficients in these subspaces transform among themselves. *It is this fact which we heavily utilize.* More specifically we have two related situations: (1) when the (n_1, n_2) are both positive integers, (or $w \geq |s|$), there will be finite dimensional invariant subspaces, $D^+_{(n_1,n_2)}$, which are spanned by the basis vectors ${}_sY_{lm}(\zeta, \overline{\zeta})$, with l given in the range, $|s| \leq l \leq w$. All the finite dimensional representations can be obtained in this manner. And (2) when the $(-n_1, -n_2)$ are both negative integers (i.e., we have a negative integer representation) there will be an *infinite* dimensional invariant subspace, $D^-_{(-n_1,-n_2)}$, described elsewhere [21]. One, however, can obtain a *finite* dimensional representation for each negative integer case by the following construction: One forms the factor space, $D_{(-n_1,-n_2)}/D^-_{(-n_1,-n_2)}$. This space is isomorphic to one of the finite dimensional spaces associated with the positive integers. The explicit form of the isomorphism, which is not needed here, is given in Held et al. [21, 16].

Of major interest for us is not so-much the invariant subspaces but instead their interactions with their compliments (the full vector space modulo the invariant subspace). Under the action of the Lorentz transformations applied to a general vector in the representation space, the components of the invariant subspaces remain in the invariant subspace but in addition components of the complement move into the invariant subspace. On the other hand, the components of the invariant subspaces do not move into the complement subspace: the transformed components of the compliment involve only the original compliment components. The transformation thus has a non-trival Jordan form.

Rather than give the full description of these invariant subspaces we confine ourselves to the few cases of relevance to us.

I. Though our interest is primarily in the negative integer representations, we first address the positive integer case of the $s = 0$ and $w = 1$, $[(n_1, n_2) = (2, 2)]$, representation. The harmonics, $l = (0, 1)$ form the invariant subspace. The cut function, $X(\tau, \zeta, \overline{\zeta})$, for each fix values of τ, lies in this space.

We write the GCF as

$$\begin{aligned} u &= X(\xi^a(\tau), \zeta, \overline{\zeta}), \\ &= \xi^a \hat{l}_a(\zeta, \overline{\zeta}) + \sum_{l,|m|\leq l} H^{lm}(\xi^a) Y_{lm}(\zeta, \overline{\zeta}). \end{aligned} \tag{342}$$

After the Lorentz transformation, the geometric slicings have not changed but their description in terms of $(u, \zeta, \overline{\zeta})$ has changed to that of $(u', \zeta', \overline{\zeta}')$. This leads to

$$\begin{aligned} u' &= KX, \\ &= X', \\ &= \xi'^a \hat{l}'_a(\zeta', \overline{\zeta}') + \sum_{l,|m|\le l} H'^{lm}(\xi'^a) Y_{lm}(\zeta', \overline{\zeta}'). \end{aligned} \tag{343}$$

Using the transformation properties of the invariant subspace and its compliment we see that the coordinate transformation must have the form;

$$\xi'^a = F^a(\xi^b, H^{lm}(\xi^b), \ldots), \tag{344}$$

in other words it moves the higher harmonic coefficients down to the $l = (0, 1)$ coefficients. The higher harmonic coefficients transform among themselves;

$$H'^{lm}(\xi'^a) = F^{lm}(\ldots, H^{l'm'}(\xi^b), \ldots). \tag{345}$$

Treating the ξ^a and ξ'^a as functions of τ, we have

$$\begin{aligned} V &= \partial_\tau X = v^a X,_a = v^a \hat{l}_a + v^a \sum_{l,|m|\le l} H^{lm},_a(\xi^a) Y_{lm}(\zeta, \overline{\zeta}), \\ v^a &= \frac{d\xi^a}{d\tau}, \qquad v'^a = \frac{d\xi'^a}{d\tau} = F^a, \qquad {}_b\frac{d\xi^b}{d\tau} = F^a,_b v^b, \\ V' &= KV, \\ &= v'^a \left(\hat{l}'_a + \sum_{l,|m|\le l} H'^{lm},\ _a Y_{lm}(\zeta', \overline{\zeta}') \right), \\ &= v^b F^a,\ _b \left(\hat{l}'_a + \sum_{l,|m|\le l} H'^{lm},\ _a Y_{lm}(\zeta', \overline{\zeta'}) \right). \end{aligned} \tag{346}$$

Our $\mathfrak{H}$-space coordinates, $z^a = \xi^a$, and their τ-derivatives, v^a, *are the coefficients* of the $l = (0, 1)$ harmonics in respectively the X and the V expansions. A Lorentz transformation induces a specific coordinate transformation and associated vector transformation on these coefficients.

II. A second important example concerns the mass aspect, (where we have introduced the Y_0^0 for simplicity of treatment of numerical factors)

$$\Psi \equiv \Psi_{(0,-3)} = \Psi^0 Y_0^0 + \Psi^i Y_{1i}^0 + \Psi^{ij} Y_{2ij}^0 + \ldots \tag{347}$$

Ψ is an $s = 0$ and $w = -3$, $[(n_1, n_2) = (-2, -2)]$ quantity. The invariant subspace has a basis set of the harmonics with $l \geq 2$. The factor space is *isomorphic* to the *finite dimensional* positive integer space, $[(n_1, n_2) = (2, 2)]$ and hence the harmonic coefficients of $l = (0, 1)$ lie in this finite dimensional representation space. From this isomorphism we know that functions of the form, $\Psi^0 Y_0^0 + \Psi^i Y_{1i}^0$, have four coefficients proportional to the Bondi four-momentum

$$P^a = (Mc, P^i), \tag{348}$$

a Lorentzian four-vector. Note that we have rescaled the Ψ^0 in Equation (347) by the Y_0^0, differing from that of Equation (63) in order to give the spherical harmonic coefficients immediate physical meaning without the use of the factors in equations Equation (64) and Equation (65).

This is the justification for calling the $l = (0, 1)$ harmonics of the mass aspect a Lorentzian four-vector. Technically, the Bondi four-momentum is a co-vector but we have allowed ourselves a slight notational irregularity.

III. The Weyl tensor component, ψ_1^0, has $s = 1$ and $w = -3$, $[(n_1, n_2) = (-3, -1)]$. The associated finite dimensional factor space is isomorphic to the finite part of the $s = -1, w = 1, [(n_1, n_2) = (3, 1)]$ representation. We have that

$$\psi_1^0 \equiv \psi_{1(1,-3)}^0 = \psi_{1i}^0 Y_{1i}^1 + \psi_{1ij}^0 Y_{2ij}^1 + \dots \tag{349}$$

leads to the finite-dimensional representation space

$$\text{finite space} = \psi_{1i}^0 Y_{1i}^{-1}. \tag{350}$$

The question of what finite tensor transformation this corresponds to is slightly more complicated than that of the previous examples of Lorentzian vectors. In fact, it corresponds to the Lorentz transformations applied to (complex) self-dual antisymmetric two-index tensors [29]. We clarify this with an example from Maxwell theory: from a given **E** and **B**, the Maxwell tensor, F^{ab}, and then its self-dual version can be constructed:

$$W^{ab+} = F^{ab} + iF^{*ab}.$$

A Lorentz transformation applied to the tensor, W^{ab+}, is equivalent [30] to the same transformation applied to

$$\psi_1^{0\ i} \leftrightarrow (\mathbf{E} + i\mathbf{B})_i. \tag{351}$$

These observations allow us to assign Lorentzian invariant physical meaning to our identifications of the Bondi momentum, P^a and the complex mass dipole moment and angular momentum vector, $D^i_{\text{Mass Dipole}} + iJ^i$.

IV. Our last example is a general discussion of how to construct Lorentzian invariants from the representation spaces. Though we will confine our remarks to just the cases of $s = 0$, it is easy to extend them to non-vanishing s by having the two functions have respectively spin-weight s and $-s$.

Consider pairs of conformally weighted functions ($s = 0$), $W_{(w)}, G_{(-w-2)}$, with weights respectively, $(w, -w-2)$. They are considered to be in dual spaces. Our claim is that the integrals of the form

$$I = \int G_{(-w-2)} W_{(w)} d\Omega \tag{352}$$

are Lorentz invariants.

We first point out that under the fractional linear transformation, $\zeta' \leftrightarrow \zeta$, Equation (337), the area element on the sphere

$$d\Omega = \frac{4i d\zeta \wedge d\bar{\zeta}}{(1 + \zeta\bar{\zeta})^2} \tag{353}$$

transforms as [21]

$$d\Omega' = K^2 d\Omega. \tag{354}$$

This leads immediately to

$$\begin{aligned} I &= \int G'_{(-w-2)} W'_{(w)} d\Omega' \\ &= \int K^{-w-2} G_{(-w-2)} K^w W_{(w)} K^2 d\Omega, \\ &= \int G_{(-w-2)} W_{(w)} d\Omega, \end{aligned} \tag{355}$$

the claimed result.

There are several immediate simple applications of Equation (352). By choosing an arbitrary $w = -2, s = 0$ function, say $G_{(-2)}(\zeta, \overline{\zeta})$, we immediately have a Lorentzian scalar,

$$N \equiv \int G_{(-2)} d\Omega = \int G'_{(-2)} d\Omega'. \tag{356}$$

If this is made more specific by chosing $G_{(-2)} = V^{-2}$, we have the remarkable result (proved in Appendix D) that this scalar yields the $\mathcal{H}$-space metric via

$$8\pi (g_{ab} v^a v^b)^{-1} \equiv \int V^{-2} d\Omega. \tag{357}$$

A simple variant of this arises by taking the derivative of (357) with respect to v^a, and multiplying by an arbitrary vector, w^a leading to

$$8\pi w^a v_a (g_{cd} v^c v^d)^{-2} \equiv \int V^{-3} w^a Z_{,a} \, d\Omega. \tag{358}$$

Many other versions can easily be found.

8 Discussion/Conclusion

8.1 History/background

The work reported in this document has had a very long gestation period. It began in 1965 [37] with the publication of a paper where a complex coordinate transformation was performed on the Schwarzschild/Reissner–Nordström solutions. This, in *a precise sense*, moved the 'point source' onto a complex world line in a complexified spacetime. It thereby led to a derivation of the spinning and the charged-spinning particle metrics. How and why this procedure worked was considered to be rather mysterious and a great deal of effort by a variety of people went into trying to unravel it. In the end, the use of the complex coordinate transformation for the derivation of these metrics appeared as if it was simply an accident, i.e., a trick with no immediate significance. Nevertheless, the idea of a complex world line, appearing in a natural manner, was an intriguing thought, which frequently returned. Some years later, working on an apparently unrelated subject, we studied and found the condition [6] for an NGC, in asymptotically-flat spacetime, to have a vanishing asymptotic shear. This condition (our previously discussed shear-free condition, Equation (174)), was closely related to Penrose's asymptotic twistor theory. In the flat-space case it lead to the Kerr theorem and totally shear-free NGCs. From a different point of view, searching for shear-free complex null surfaces, the good-cut equation was found with its four-complex parameter solution space. This lead to the theory of $\mathcal{H}$-space.

Years later, the different strands came together. The shear-free condition was found to be closely related to the good-cut equation; namely, that one equation could be transformed into the other. The major surprise came when we discovered that the *regular* solutions of either equation were *generated by complex world lines in an auxiliary* Minkowski space [26]. (These complex world lines could be thought of as being complex analytic curves in the associated $\mathcal{H}$-space. The deeper meaning of this remains a question still to be resolved; it is this issue which is partially addressed in the present work.)

The complex world line mentioned above, associated with the spinning, charged and uncharged particle metrics, now can be seen as just a special case of these regular solutions. Since these metrics were algebraically special, among the many possible asymptotically shear-free NGCs there was (at least) one *totally shear-free* (rather than asymptotically shear-free) congruence. This was the one we first discovered in 1965, that became the complex center-of-mass world line (which coincided with the complex center of charge in the charged case.). This observation was the clue for how to search for the generalization of the special world line associated with algebraically-special metrics and thus, in general, how to look for the special world line (and congruence) to be identified with the complex center of mass.

For the algebraically-special metrics, the null tetrad system at $\mathfrak{I}^+$ with one leg being the tangent null vector to the shear-free congruence leads to the vanishing of the asymptotic Weyl tensor component, i.e., $\psi_0^* = \psi_1^* = 0$. For the general case, no tetrad exists with that property but one can always find a null tetrad with one leg being tangent to the shear-free congruence so that the $l = 1$ harmonics of ψ_1^{0*} vanish. It is precisely that choice of tetrad that led to our definition of the complex center of mass.

8.2 Other choices for physical identification

The question of whether our definition of the complex center of mass is the best possible definition, or even a reasonable one, is not easy to answer. We did try to establish a criteria for choosing such a definition: (i) it should predict already known physical laws or reasonable new laws, (ii) it should have a clear geometric foundation and a logical consistency and (iii) it should agree with special cases, mainly the algebraically-special metrics or analogies with flat-space Maxwell theory. We did try out several other possible choices [29] and found them all failing. This clearly does not rule out others that we did not think of, but at the present our choice appears to be both natural and effective in making contact with physical phenomena.

8.3 Predictions

Our equations of motion are simultaneously satisfactory and unsatisfactory: they yield the equations of motion for an isolated object with a great deal of internal structure (time-dependent multipoles with the emission of gravitational and electromagnetic radiation) in the form of Newton's second law. In addition, they contain a definition of angular momentum with an angular momentum flux. The dipole part of the angular momentum flux agrees with classical E&M theory. Unfortunately, there appears to be no way to study or describe interacting particles in this manner.

However, there are some areas where these ideas might be tested, though the effects would be very small. For example, earlier we saw that there was a contribution to the Bondi mass (an addition to the Reissner–Nordström mass) from the quadrupole moment,

$$M_{\mathrm{B}} - M_{RN} \equiv \Delta M = -\frac{1}{5} G c^{-6} \,\mathrm{Re}\, Q^{ij\,\prime\prime}_{\mathrm{Grav}} \overline{Q}^{ij\,\prime\prime\prime}_{\mathrm{Grav}}. \tag{359}$$

There were predicted contributions to both the momentum and angular momentum flux from the gravitational and electromagnetic quadrupole radiation as well as new terms in the definition of the angular momentum, e.g., charge/magnetic-dipole coupling term

$$\frac{2}{3} q^2 c^{-2} \eta_I^{i\,\prime} = \frac{2}{3} q c^{-2} D_M^{i\,\prime}.$$

There are other unfamiliar terms that can be thought of as predictions of this theoretical construct. How to possibly measure them is not at all clear.

8.4 Summary of results

1. From the asymptotic Weyl and Maxwell tensors, with their transformation properties, we were able (via the asymptotically shear-free NGC) to obtain two complex world lines – a complex 'center of mass' and 'complex center of charge' in the auxiliary $\mathcal{H}$-space. When 'viewed' from a Bondi coordinate and tetrad system, this led to an expression for the real center of mass of the gravitating system and a kinematic expression for the total angular momentum (including intrinsic spin and orbital angular momentum), as seen from null infinity. It is interesting to observe that the *kinematical* expressions for the classical linear momentum and angular momentum came directly from *dynamical* laws (Bianchi identities) on the evolution of the Weyl tensor.

2. From the real parts of one of the asymptotic Bianchi identities, Equation (56), we found the standard kinematic expression for the Bondi linear momentum, $P = M\xi_R' + \dots$ with extra terms $\frac{2q^2}{3c^3} v_R^{k\,\prime}$ and the Mathisson–Papapetrou spin coupling, among others. The imaginary part was the angular momentum conservation law with a very natural looking flux expression of the form:

   $$J^{k\,\prime} = \mathrm{Flux}_{\mathrm{Grav}} + \mathrm{Flux}_{\mathrm{E\&M\,Quad}} + \mathrm{Flux}_{\mathrm{E\&M\,dipole}}$$

   with

   $$J = \text{spin} + \text{orbital} + \text{precession} + \text{varying magnetic dipole} + \text{quadrupole terms}.$$

   The last flux term is *identical* to that calculated from classical electromagnetic theory.

3. Using the kinematic expression for the Bondi momentum in a second Bianchi identity, Equation (57), we obtained a second-order ODE for the center of mass world line that could be identified with Newton's second law with radiation reaction forces and recoil forces, $M_{\mathrm{B}}\xi_R'' = F$.

4. From Bondi's mass/energy loss theorem we obtained the correct energy flux from the electromagnetic dipole and quadrupole radiation as well as the gravitational quadrupole radiation.

5. From the specialized case where the two world lines coincide and the definitions of spin and magnetic moment, we obtained the Dirac gyromagnetic ratio, $g = 2$. In addition, we find the classical electrodynamic radiation reaction term with the correct numerical factors. In this case we have the identifications of the meaning of the complex position vector: $\xi^i = \xi_R^i + i\xi_I^i$.

$$\begin{aligned} \xi_R^i &= \text{center-of-mass position} \\ S^i &= Mc\xi_I^i = \text{spin angular momentum} \\ D_E^i &= q\xi_R^i = \text{electric dipole moment} \\ D_M^i &= q\xi_I^i = \text{magnetic dipole moment} \end{aligned}$$

8.5 A variety of issues and questions

1. A particularly interesting issue raised by our equations is that of the run-away (unstable) behavior of the equations of motion for a charged particle (with or without an external field). We saw in Equation (328) that there was a driving term in the equation of motion depending on the electric dipole moment (or the real center of charge). This driving term was totally independent of the real center of mass and thus does not lead to the classical instability. However, if we restrict the complex center of charge to be the same as the complex center of mass (a severe, but very attractive restriction leading to $g = 2$), then the innocuous driving dipole term becomes the classical radiation reaction term – suggesting instability. (Note that in this coinciding case there was no model building – aside from the coinciding lines – and no mass renormalization.)

   A natural question then is: does this unstable behavior really remain? In other words, is it possible that the large number of extra terms in the gravitational radiation reaction or the momentum recoil force might stabilize the situation? Answering this question is extremely difficult. If the gravitational effects do not stabilize, then – at least in this special case – the Einstein–Maxwell equations are unstable, since the run-away behavior would lead to an infinite amount of electromagnetic dipole energy loss.

   An alternative possible resolution to the classical run-away problem is simply to say that the classical electrodynamic model is wrong; and that one must treat the center of charge as different from the center of mass with its own dynamics.

2. The interpretation and analysis of the complex analytic curves associated with the shear-free and asymptotically shear free null geodesic congruences naturally takes place in $\mathcal{H}$-space. Though we get extraordinarily attractive physical results – almost all coinciding with standard physical understanding – it nevertheless is a total mystery as to what it means or what is the physical significance of this complex Ricci-flat four-dimensional space.

3. In our approximations, it was assumed that the complex world line yielded cuts of $\mathfrak{I}^+$ that were close to Bondi cuts. At the present we do not have any straightforward means of finding the world lines and their associated cuts of $\mathfrak{I}^+$ that are far from the Bondi cuts.

4. As mentioned earlier, when the gravitational and electromagnetic world lines coincide we find the rather surprising result of the Dirac value for the gyromagnetic ratio. Unfortunately, though this appears to be a significant result, we do not have any deeper understanding of it. It was simply there for us to observe.

5. Is it possible that the complex structures that we have been seeing and using are more than just a technical device to organize ideas, and that they have a deeper significance? One direction to explore this is via Penrose's twistor and asymptotic twistor theory. It is known that much of the material described here is closely related to twistor theory; an example is the fact that asymptotic shear-free NGCs are really a special case of the Kerr theorem, an important application of twistor theory (see Appendix A). This connection is being further explored.

6. With much of the kinematics and dynamics of *ordinary classical mechanics* sitting in our results, i.e., in classical GR, is it possible that ordinary particle quantization could play a role in understanding quantum gravity? Attempts along this line have been made [14, 7] but, so far, without much success.

7. We reiterate that, *a priori*, there is no reason to suspect or believe that the world lines associated with shear-free congruences would allow the choice of a special congruence – and a special world line – to be singled out – and that furthermore it would be so connected with physical kinematics and dynamical laws. These results certainly greatly surprised and pleased us.

8. As a final remark, we want to point out that there is an issue that we have ignored, namely, do the asymptotic solutions of the Einstein equations that we have discussed and used throughout this work really exist. By 'really exist' we mean the following: Are there, in sufficiently general circumstances, Cauchy surfaces with physically-given data, such that their evolution yields these asymptotic solutions? We have tacitly assumed throughout, with physical justification but no rigorous mathematical justification, that the full interior vacuum Einstein equations do lead to these asymptotic situations. However, there has been a great deal of deep and difficult analysis [13, 10, 11] showing, in fact, that large classes of solutions to the Cauchy problem with physically-relevant data do lead to the asymptotic behavior we have discussed. Recently there has been progress made on the same problem for the Einstein–Maxwell equations.

9. An interesting issue, not yet explored but potentially important, is what can be said about the structure of $\mathcal{H}$-space where there are special points that are related to the real cuts of null infinity.

9 Acknowledgments

We are happy to note and acknowledge the great deal of detailed help and understanding that we have received over the years from Gilberto Silva-Ortigoza, our co-author on many of the earlier papers on the present subject. Many others have contributed to our understanding of asymptotically-flat spacetimes. Prime among them are Roger Penrose, Andrzej Trautman, Jerzy Lewandowski, Pawel Nurowski, Paul Tod, Lionel Mason, Jörg Frauendiener and Helmut Friedrich. We thank them all. We also want to thank the Center for Gravitational Wave Astronomy at the University of Texas at Brownsville for hospitality during the last stage of this work.

A Twistor Theory

Throughout this review, the study of the asymptotic gravitational field has been at the heart of all our investigations. Here we make contact with Penrose's asymptotic twistor theory (see, e.g., [51, 48, 22]). We give here a brief overview of asymptotic twistor theory and its connection to the good-cut equation and the study of asymptotically shear-free NGCs at $\mathfrak{I}^+$. For a more in depth exposition of this connection, see [36].

Let $\mathcal{M}$ be any asymptotically-flat spacetime manifold, with conformal future null infinity $\mathfrak{I}^+$, coordinatized by $(u_\mathrm{B}, \zeta, \bar{\zeta})$. We can consider the complexification of $\mathfrak{I}^+$, referred to as $\mathfrak{I}^+_\mathbb{C}$, which is in turn coordinatized by $(u_\mathrm{B}, \zeta, \tilde{\zeta})$, where now $u_\mathrm{B} \in \mathbb{C}$ and $\tilde{\zeta}$ is different, but close to $\bar{\zeta}$. Assuming analytic asymptotic Bondi shear $\sigma^0(u_\mathrm{B}, \zeta, \bar{\zeta})$, it can then be analytically continued to $\mathfrak{I}^+_\mathbb{C}$, i.e., we can consider $\sigma^0(u_\mathrm{B}, \zeta, \tilde{\zeta})$.

We have seen in Section 4 that solutions to the good-cut equation

$$\eth^2 G = \sigma^0(G, \zeta, \tilde{\zeta}) \tag{360}$$

yield a four complex parameter family of solutions, given by

$$u_\mathrm{B} = G(z^a; \zeta, \tilde{\zeta}). \tag{361}$$

In our prior discussions, we interpreted these solutions as defining a four (complex) parameter family of surfaces on $\mathfrak{I}^+_\mathbb{C}$ corresponding to each choice of the parameters z^a.

In order to force agreement with the conventional description of Penrose's asymptotic twistor theory we must use the *complex conjugate* good-cut equation

$$\overline{\eth}^2 \overline{G} = \overline{\sigma}^0(\overline{G}, \zeta, \tilde{\zeta}), \tag{362}$$

whose properties are identical to that of the good-cut equation. Its solutions, written as

$$u_\mathrm{B} = \overline{G}(\overline{z}^a; \zeta, \tilde{\zeta}), \tag{363}$$

define complex two-surfaces in $\mathfrak{I}^+_\mathbb{C}$ for fixed $\overline{z}^a$. If, in addition to fixing the $\overline{z}^a$, we fix $\zeta = \zeta_0 \in \mathbb{C}$, then Equation (362) becomes an ordinary second-order differential equation with solutions describing curves in $(u_\mathrm{B}, \tilde{\zeta})$ space. Hence, each solution to this ODE is given by specifying initial conditions for $\tilde{G}$ and $\partial_{\tilde{\zeta}}\tilde{G}$ at some arbitrary initial point, $\tilde{\zeta} = \tilde{\zeta}_0$. Note that it is *not necessary* that $\tilde{\zeta}_0 = \bar{\zeta}_0$ on $\mathfrak{I}^+_\mathbb{C}$. However, we chose this initial point to be the complex conjugate of the constant ζ_0, i.e., we take $\overline{G}$ and its first $\tilde{\zeta}$ derivative at $\tilde{\zeta} = \overline{\zeta}_0$ as the *initial* conditions.

Then the initial conditions for Equation (362) can be written as [36]

$$\begin{aligned} u_{B0} &= \overline{G}(\zeta_0, \tilde{\zeta}_0), \\ \overline{L}_0 &= \overline{\eth}\overline{G}(\zeta_0, \tilde{\zeta}_0) = P_0 \frac{\partial \overline{G}}{\partial \tilde{\zeta}_0}(\zeta_0, \tilde{\zeta}_0), \end{aligned} \tag{364}$$

with $P_0 = 1 + \zeta_0 \tilde{\zeta}_0$. Asymptotic projective twistor space, denoted $\mathbb{P}\mathfrak{T}$, is the space of all curves in $\mathfrak{I}^+_\mathbb{C}$ generated by initial condition triplets $(u_{B0}, \zeta_0, \overline{L}_0)$ [51]: an asymptotic projective twistor is the curve corresponding to $(u_{B0}, \zeta_0, \overline{L}_0)$.

A particular subspace of $\mathbb{P}\mathfrak{T}$, called null asymptotic projective twistor space ($\mathbb{P}\mathfrak{N}$), is the family of curves generated by initial conditions, which lie on (real) $\mathfrak{I}^+$; that is, at the initial point, $\tilde{\zeta}_0 = \bar{\zeta}_0$, the curve should cross the real $\mathfrak{I}^+$, i.e., should be real, $u_{B0} = \bar{u}_{B0}$. Equivalently, an element of $\mathbb{P}\mathfrak{N}$ can be said to intersect its dual curve (the solution generated by the complex conjugate initial conditions) at $\tilde{\zeta}_0 = \bar{\zeta}_0$. The effect of this is to reduce the three-dimensional complex twistor space to five real dimensions.

In standard notation, asymptotic projective twistors are defined in terms of their three complex twistor coordinates, (μ^0, μ^1, ζ) [51]. These twistor coordinates may be re-expressed in terms of the asymptotic twistor curves by

$$\begin{aligned} \mu^0 &= u_{B0} - \bar{L}_0 \bar{\zeta}_0, \\ \mu^1 &= \bar{L}_0 + \zeta_0 u_{B0}, \\ \zeta &= \zeta_0. \end{aligned} \tag{365}$$

By only considering the twistor initial conditions $\tilde{\zeta}_0 = \bar{\zeta}_0$, we can drop the initial value notation, and just let $u_{B0} = u_{\mathrm{B}}$ and $\tilde{\zeta} = \bar{\zeta}$.

The connection of twistor theory with *shear-free* NGCs takes the form of the *flat-space* Kerr theorem [51, 36]:

Theorem. *Any analytic function on $\mathbb{PT}$ (projective twistor space) generates a shear-free NGC in Minkowski space.*

Any analytic function on projective twistor space generates a shear-free NGC in Minkowski space, i.e., from $F(\mu^0, \mu^1, \zeta) \equiv F(u_{\mathrm{B}} - \overline{L\zeta}, \overline{L} + \zeta u_{\mathrm{B}}, \zeta) = 0$, one can construct a shear-free NGC in Minkowski space. The $\overline{L} = \overline{L}(u_{\mathrm{B}}, \zeta, \overline{\zeta})$, which defines the congruence, is obtained by solving the algebraic equation

$$F(u_{\mathrm{B}} - \overline{L\zeta}, \overline{L} + \zeta u_{\mathrm{B}}, \zeta) = 0.$$

It automatically satisfies the complex conjugate shear-free condition

$$\bar{\eth}\overline{L} + \overline{L}\dot{\overline{L}} = 0.$$

We are interested in a version of the Kerr theorem that yields the regular asymptotically shear-free NGCs. Starting with the general four-parameter solution to Equation (362), i.e., $u_{\mathrm{B}} = \overline{G}(\overline{z}^a; \zeta, \tilde{\zeta})$, we chose an arbitrary world line $\overline{z}^a = \xi^a(\tau)$, so that we have

$$\begin{aligned} u_{\mathrm{B}} &= \overline{G}(\xi^a(\tau), \zeta, \bar{\zeta}) = \overline{X}(\tau, \zeta, \bar{\zeta}), \\ \overline{L}(\tau, \zeta, \bar{\zeta}) &= \bar{\eth}_{(\tau)} \overline{X}(\tau, \zeta, \bar{\zeta}). \end{aligned} \tag{366}$$

By inserting these into the twistor coordinates, Equation (365), we find

$$\mu^0(\tau, \zeta, \bar{\zeta}) = u_{\mathrm{B}} - \bar{L}\bar{\zeta} = \overline{X} - \bar{\zeta}\bar{\eth}_{(\tau)}\overline{X}, \tag{367}$$

$$\mu^1(\tau, \zeta, \bar{\zeta}) = \bar{L} + \zeta u_{\mathrm{B}} = \bar{\eth}_{(\tau)}\overline{X} + \zeta\overline{X}. \tag{368}$$

The μ^0 and μ^1 are now functions of τ and ζ: the $\bar{\zeta}$ is now to be treated as a fixed quantity, the complex conjugate of ζ, and not as an independent variable.

By eliminating τ in Equations (367) and (368), we obtain a single function of μ^0, μ^1, and ζ: namely, $F(\mu^0, \mu^1, \zeta) = 0$. Thus, the *regular* asymptotically shear-free NGCs are described by a special class of twistor functions. This is a special case of a generalized version of the Kerr theorem [51, 36].

B CR Structures

A CR structure on a real three manifold $\mathcal{N}$, with local coordinates x^a, is given intrinsically by equivalence classes of one-forms, one real, one complex and its complex conjugate [31]. If we denote the real one-form by l and the complex one-form by m, then these are defined up to the transformations:

$$l \to a(x^a)l, \qquad (369)$$
$$m \to f(x^a)m + g(x^a)l.$$

The (a, f, g) are functions on $\mathcal{N}$: a is nonvanishing and real, f and g are complex function with f nonvanishing. We further require that there be a three-fold linear-independence relation between these one-forms [31]:

$$l \wedge m \wedge \bar{m} \neq 0. \qquad (370)$$

Any three-manifold with a CR structure is referred to as a three-dimensional CR manifold. There are special classes (referred to as embeddable) of three-dimensional CR manifolds that can be directly embedded into $\mathbb{C}^2$.

We show how the choice of any specific asymptotically shear-free NGC induces a CR structure on $\mathfrak{I}^+$. Though there are several ways of arriving at this CR structure, the simplest way is to look at the asymptotic null tetrad system associated with the asymptotically shear-free NGC, i.e., look at the $(l^{*a}, m^{*a}, \overline{m}^{*a}, n^{*a})$ of Equation (274). The associated dual one-forms, restricted to $\mathfrak{I}^+$ (after a conformal rescaling of m), become (with a slight notational dishonesty),

$$l^* = du_\mathrm{B} - \frac{L}{1+\zeta\bar{\zeta}}d\zeta - \frac{\bar{L}}{1+\zeta\bar{\zeta}}d\bar{\zeta}, \qquad (371)$$
$$m^* = \frac{d\bar{\zeta}}{1+\zeta\bar{\zeta}}, \qquad \overline{m}^* = \frac{d\zeta}{1+\zeta\bar{\zeta}},$$

with $L = L(u_\mathrm{B}, \zeta, \bar{\zeta})$, satisfying the shear-free condition. (This same result could have been obtained by manipulating the exterior derivatives of the twistor coordinates, Equation (365).)

The dual vectors – also describing the CR structure – are

$$\overline{\mathfrak{M}} = P\frac{\partial}{\partial\zeta} + L\frac{\partial}{\partial u_\mathrm{B}} = \eth_{(u_B)} + L\frac{\partial}{\partial u_\mathrm{B}}, \qquad (372)$$
$$\mathfrak{M} = P\frac{\partial}{\partial\bar{\zeta}} + \overline{L}\frac{\partial}{\partial u_\mathrm{B}} = \bar{\eth}_{(u_B)} + \overline{L}\frac{\partial}{\partial u_\mathrm{B}},$$
$$\mathfrak{L} = \frac{\partial}{\partial u_\mathrm{B}}.$$

Therefore, for the situation discussed here, where we have singled out a unique asymptotically shear-free NGC and associated complex world line, we have a uniquely chosen CR structure induced on $\mathfrak{I}^+$.

To see how our three manifold, $\mathfrak{I}^+$, can be imbedded into $\mathbb{C}^2$ we introduce the CR equation [32]

$$\overline{\mathfrak{M}}K \equiv \eth_{(u_\mathrm{B})}K + L\frac{\partial}{\partial u_\mathrm{B}}K = 0$$

and seek two independent (complex) solutions, $K_1 = K_1(u_\mathrm{B}, \zeta, \bar{\zeta}), K_2 = K_2(u_\mathrm{B}, \zeta, \bar{\zeta})$ that define the embedding of $\mathfrak{I}^+$ into $\mathbb{C}^2$ with coordinates (K_1, K_2).

We have immediately that $K_1 = \bar{\zeta} = x - iy$ is a solution. The second solution is also easily found; we see directly from Equation (175) [38],

$$\eth_{(u_B)}T + L\dot{T} = 0, \qquad (373)$$

that

$$\tau = T(u_{\mathrm{B}}, \zeta, \overline{\zeta}),$$

the inverse to $u_{\mathrm{B}} = X(\tau, \zeta, \overline{\zeta})$, is a CR function and that we can consider $\mathfrak{I}^+$ to be embedded in the $\mathbb{C}^2$ of $(\tau, \overline{\zeta})$.

C Tensorial Spin-s Spherical Harmonics

Some time ago, the generalization of ordinary spherical harmonics $Y_{lm}(\zeta,\bar{\zeta})$ to spin-weighted functions ${}_{(s)}Y_{lm}(\zeta,\bar{\zeta})$ (e.g., [21, 17, 40]) was developed to allow for harmonic expansions of spin-weighted functions on the sphere. In this paper we have instead used the tensorial form of these spin-weighted harmonics, the *tensorial spin-s spherical harmonics*, which are formed by taking appropriate linear combinations of the ${}_{(s)}Y_{lm}(\zeta,\bar{\zeta})$ [43]:

$$Y^{s}_{l\,i\ldots k} = \sum K^{sm}_{l\,i\ldots k\,(s)}Y_{lm},$$

where the indices obey $|s| \leq l$, and the number of spatial indices (i.e., $i\ldots k$) is equal to l. Explicitly, these tensorial spin-weighted harmonics can be constructed directly from the parameterized Lorentzian null tetrad, Equation (74):

$$\begin{aligned}
\hat{l}^a &= \frac{\sqrt{2}}{2(1+\zeta\bar{\zeta})}\left(1+\zeta\bar{\zeta}, \zeta+\bar{\zeta}, i\bar{\zeta}-i\zeta, -1+\zeta\bar{\zeta}\right), \\
\hat{n}^a &= \frac{\sqrt{2}}{2(1+\zeta\bar{\zeta})}\left(1+\zeta\bar{\zeta}, -(\zeta+\bar{\zeta}), i\zeta-i\bar{\zeta}, 1+\zeta\bar{\zeta}\right), \\
\hat{m}^a &= \frac{\sqrt{2}}{2(1+\zeta\bar{\zeta})}\left(0, 1-\bar{\zeta}^2, -i(1+\bar{\zeta}^2), 2\bar{\zeta}\right), \\
P &\equiv 1+\zeta\bar{\zeta}.
\end{aligned} \tag{374}$$

Taking the spatial parts of their duals, we obtain the one-forms

$$\begin{aligned}
l_i &= \frac{-1}{\sqrt{2}P}\left(\zeta+\bar{\zeta}, -i(\zeta-\bar{\zeta}), -1+\zeta\bar{\zeta}\right), \\
n_i &= \frac{1}{\sqrt{2}P}\left(\zeta+\bar{\zeta}, -i(\zeta+\bar{\zeta}), -1+\zeta\bar{\zeta}\right), \\
m_i &= \frac{-1}{\sqrt{2}P}\left(1-\bar{\zeta}^2, -i(1+\bar{\zeta}^2), 2\bar{\zeta}\right), \\
c_i &= l_i - n_i = -\sqrt{2}i\epsilon_{ijk}m_j\bar{m}_k.
\end{aligned} \tag{375}$$

From this we define $Y^{l}_{l\,i\ldots k}$ as [43]

$$\begin{aligned}
Y^{l}_{l\,i\ldots k} &= m_i m_j \ldots m_k, \\
Y^{-l}_{l\,i\ldots k} &= \bar{m}_i \bar{m}_j \ldots \bar{m}_k.
\end{aligned} \tag{376}$$

The other harmonics are determined by the action of the ð-operator on the forms, Equation (375), (with complex conjugates) via

$$\begin{aligned}
\eth l &= m, \\
\eth m &= 0, \\
\eth n &= -m, \\
\eth c &= 2m, \\
\eth \overline{m} &= n - l = -c.
\end{aligned} \tag{377}$$

Specifically, the spin-s harmonics are defined by

$$\begin{aligned}
Y^{s}_{l\,i\ldots k} &= \bar{\eth}^{l-s}\left(Y^{l}_{l\,i\ldots k}\right), \\
Y^{-|s|}_{l\,i\ldots k} &= \eth^{l-s}\left(Y^{-l}_{l\,i\ldots k}\right).
\end{aligned} \tag{378}$$

We now present a table of the tensorial spherical harmonics up to $l = 2$, in terms of the tetrad. Higher harmonics can be found in [43].

$l = 0$
$Y_0^0 = 1$
$l = 1$
$Y_{1i}^1 = m_i,$ $Y_{1i}^0 = -c_i,$ $Y_{1i}^{-1} = \bar{m}_i$
$l = 2$
$Y_{2ij}^2 = m_i m_j, \quad Y_{2ij}^1 = -(c_i m_j + m_i c_j), \quad Y_{2ij}^0 = 3c_i c_j - 2\delta_{ij}$ $Y_{2ij}^{-2} = \bar{m}_i \bar{m}_j, \quad Y_{2ij}^{-1} = -(c_i \bar{m}_j + \bar{m}_i c_j)$

In addition, it is useful to give the explicit relations between these different harmonics in terms of the ð-operator and its conjugate. Indeed, we can see generally that applying ð once raises the spin index by one, and applying $\bar{ð}$ lowers the index by one. This in turn means that

$$
\begin{aligned}
ðY^l_{l\,i\ldots k} &= 0, \\
\bar{ð}Y^{-l}_{l\,i\ldots k} &= 0.
\end{aligned}
$$

Other relations for $l \leq 2$ are given by

$$
\begin{aligned}
\bar{ð}Y^1_{1i} &= Y^0_{1i} = ðY^{-1}_{1i}, \\
ðY^0_{1i} &= -2Y^1_{1i}, \\
\bar{ð}Y^0_{1i} &= -2Y^{-1}_{1i},
\end{aligned}
$$

$$
\begin{aligned}
\bar{ð}Y^2_{2ij} &= Y^1_{2ij}, \\
\bar{ð}^2 Y^2_{2ij} &= Y^0_{2ij} = ð^2 Y^{-2}_{2ij}, \\
ðY^0_{2ij} &= -6Y^1_{2ij}, \\
ðY^1_{2ij} &= -4Y^2_{2ij}.
\end{aligned}
$$

Finally, due to the nonlinearity of the theory, we have been forced throughout this review to consider products of the tensorial spin-s spherical harmonics while expanding nonlinear expressions. These products can be expanded as a linear combination of individual harmonics using Clebsch–Gordon expansions. The explicit expansions for products of harmonics with $l = 1$ or $l = 2$ are given below (we omit higher products due to the complexity of the expansion expressions). Further products can be found in [43, 29].

C.1 Clebsch–Gordon expansions

$l = 1$ with $l = 1$

$$
\begin{aligned}
Y_{1i}^{1}Y_{1j}^{0} &= \frac{i}{\sqrt{2}}\epsilon_{ijk}Y_{1k}^{1} + \frac{1}{2}Y_{2ij}^{1},\\
Y_{1i}^{1}Y_{1j}^{-1} &= \frac{1}{3}\delta_{ij} - \frac{i\sqrt{2}}{4}\epsilon_{ijk}Y_{1k}^{0} - \frac{1}{12}Y_{2ij}^{0},\\
Y_{1i}^{0}Y_{1j}^{0} &= \frac{2}{3}\delta_{ij} + \frac{1}{3}Y_{2ij}^{0}
\end{aligned}
$$

$l = 1$ with $l = 2$

$$
\begin{aligned}
Y_{1i}^{1}Y_{2ij}^{2} &= Y_{3ijk}^{3},\\
Y_{1i}^{0}Y_{2jk}^{0} &= -\frac{4}{5}\delta_{kj}Y_{1i}^{0} + \frac{6}{5}\left(\delta_{ij}Y_{1k}^{0} + \delta_{ik}Y_{1j}^{0}\right) + \frac{1}{5}Y_{3ijk}^{0},\\
Y_{1i}^{1}Y_{2jk}^{0} &= \frac{2}{5}Y_{1i}^{1}\delta_{jk} - \frac{3}{5}Y_{1j}^{1}\delta_{ik} - \frac{3}{5}Y_{1k}^{1}\delta_{ij} + \frac{i}{\sqrt{2}}\left(\epsilon_{ikl}Y_{2jl}^{1} + \epsilon_{ijl}Y_{2kl}^{1}\right) + \frac{2}{5}Y_{3ijk}^{1},\\
Y_{1i}^{1}Y_{2jk}^{1} &= -\frac{1}{6}\eth\left(Y_{1i}^{1}Y_{2jk}^{0}\right),\\
Y_{2ij}^{-1}Y_{1k}^{1} &= \frac{3}{10}Y_{1i}^{0}\delta_{jk} + \frac{3}{10}Y_{1j}^{0}\delta_{ik} - \frac{1}{5}Y_{1k}^{0}\delta_{ij} + \frac{i\sqrt{2}}{12}\left(\epsilon_{jkl}Y_{2il}^{0} + \epsilon_{ikl}Y_{2lj}^{0}\right) - \frac{1}{30}Y_{3ijk}^{0},\\
Y_{1i}^{0}Y_{2jk}^{1} &= -\frac{2}{5}Y_{1i}^{1}\delta_{jk} + \frac{3}{5}Y_{1j}^{1}\delta_{ik} + \frac{3}{5}Y_{1k}^{1}\delta_{ij} - \frac{i}{3\sqrt{2}}\left(\epsilon_{ikl}Y_{2jl}^{1} + \epsilon_{ijl}Y_{2kl}^{1}\right) + \frac{4}{15}Y_{3ijk}^{1},\\
Y_{2ij}^{2}Y_{1k}^{-1} &= \frac{3}{10}Y_{1i}^{0}\delta_{jk} + \frac{3}{10}Y_{1j}^{0}\delta_{ik} - \frac{1}{5}Y_{1k}^{0}\delta_{ij} - \frac{i\sqrt{2}}{12}\left(\epsilon_{jkl}Y_{2il}^{0} + \epsilon_{ikl}Y_{2lj}^{0}\right) - \frac{1}{30}Y_{3ijk}^{0},\\
Y_{2ij}^{2}Y_{1k}^{0} &= \eth\left(Y_{2ij}^{2}Y_{1k}^{-1}\right)
\end{aligned}
$$

$l = 2$ with $l = 2$

The Clebsch–Gordon expansions involving two $l = 2$ harmonics have been used in the text. They are fairly long and are not given here but can be found in [43].

D $\mathcal{H}$-Space Metric

In the following the derivation of the $\mathcal{H}$-space metric, is given.

We begin with the cut function, $u_{\mathrm{B}} = Z(\xi^a(\tau), \zeta, \overline{\zeta}) = X(\tau, \zeta, \overline{\zeta})$ that satisfies the good cut eqation $\eth^2 Z = \sigma(Z, \zeta, \overline{\zeta})$. The $(\zeta, \overline{\zeta})$ are (for the time being) completely independent of each other though $\overline{\zeta}$ is to be treated as being "close" the complex conjugate of ζ.

(Later we will introduce ζ^* instead of ζ via

$$\zeta^* = \frac{\zeta + W}{1 - W\overline{\zeta}}, \tag{379}$$

for the purpose of simplifying an integration.)

Taking the gradient of $Z(z^a, \zeta, \overline{\zeta})$, multiplied by an arbitrary four vector v^a, (i.e., $V = v^a Z,_a$), we see that it satisfies the linear Good Cut equation,

$$\begin{aligned} \eth^2 Z,_a &= \sigma,_Z Z,_a \\ \eth^2 V &= \sigma,_Z V. \end{aligned} \tag{380}$$

Let V_0 be a particular solution, and assume for the moment that the general solution can be written as

$$Z,_a = V_0 l_a^* \tag{381}$$

with the four components of l_a^* to be determined. Substituting Equation (381) into the linearized GCE equation we have

$$\begin{aligned} \eth^2(V_0 l_a^*) &= \sigma,_Z V l_a^*, \\ \eth(l_a^* \eth(V_0) + V_0 \eth l_a^*) &= \sigma,_Z V l_a^*, \\ l_a^* \eth^2(V_0) + 2\eth V_0 \eth l_a^* + V_0 \eth^2 l_a^* &= \sigma,_Z V_0 l_a^*, \\ 2\eth V_0 \eth l_a^* + V_0 \eth^2 l_a^* &= 0, \\ 2V_0 \eth V_0 \eth l_a^* + V_0^2 \eth^2 l_a^* &= 0, \\ \eth V_0^2 \eth l_a^* + V_0^2 \eth^2 l_a^* &= 0, \\ \eth(V_0^2 \eth l_a^*) &= 0, \end{aligned}$$

which integrates immediately to

$$V_0^2 \eth l_a^* = m_a^* \tag{382}$$

The m_a^* are three independent $l = 1$, $s = 1$ functions. By taking linear combinations they can be written as

$$m_a^* = T_a^b \hat{m}_b = T_a^b \eth \hat{l}_b$$

where $\hat{l}_a$ is our usual $\hat{l}_a = \frac{\sqrt{2}}{2}\left(1, -\frac{\zeta+\overline{\zeta}}{1+\zeta\overline{\zeta}}, -\frac{i(\overline{\zeta}-\zeta)}{1+\zeta\overline{\zeta}}, \frac{1-\zeta\overline{\zeta}}{1+\zeta\overline{\zeta}}\right)$. The coefficients T_a^b are functions only of the coordinates, z^a.

Assuming that the monople term in V^2 is sufficiently large so that it has no zeros and then by rescaling V we can write V^{-2} as a monopole plus higher harmonics in the form

$${V_0}^{-2} = 1 + \eth W,$$

where W is a spin-wt $s = -1$ quantity. From Equation (382), we obtain

$$\begin{aligned} \eth l_a^* &= {V_0}^{-2} m_a^* = (1 + \eth W) m_a^*, \\ \eth l_a^* &= m_a^* + \eth W m_a^*, \\ \eth l_a^* &= m_a^* + \eth(W m_a^*), \\ \eth l_a^* &= T_a^b \eth \hat{l}_b + T_a^b \eth(W \hat{m}_b), \\ \eth l_a^* &= T_a^b \eth(\hat{l}_b + W \hat{m}_b), \end{aligned}$$

which integrates to

$$l^*_a = T^b_a(\hat{l}_a + W\hat{m}_a). \tag{383}$$

The general solution to the linearized GCE is thus

$$\begin{aligned} Z_{,a} &= V_0 l^*_a = V_0 T^b_a(\hat{l}_a + W\hat{m}_a), \\ V &= v^a Z_{,a} = V_0 v^a T^b_a(\hat{l}_a + W\hat{m}_a). \end{aligned} \tag{384}$$

We now demonstrate that

$$(g_{ab}v^a v^b)^{-1} = (8\pi)^{-1}\int V^{-2} d\Omega. \tag{385}$$

$$d\Omega = 4i\frac{d\zeta \wedge d\overline{\zeta}}{(1+\zeta\overline{\zeta})^2}. \tag{386}$$

In the integral, (385), we replace the independent variables $(\zeta, \overline{\zeta})$ by

$$\zeta^* = \frac{\zeta + W}{1 - W\overline{\zeta}}, \qquad \overline{\zeta}^* = \overline{\zeta} \tag{387}$$

after some algebraical manipulation we obtain

$$d\Omega^* = V_0^{-2} d\Omega, \tag{388}$$

and (surprisingly)

$$(\hat{l}_a + W\hat{m}_a) = L^*_a \equiv \frac{\sqrt{2}}{2}\left(1, -\frac{\zeta^* + \overline{\zeta}}{1+\zeta^*\overline{\zeta}}, -\frac{i(\overline{\zeta} - \zeta^*)}{1+\zeta^*\overline{\zeta}}, \frac{1-\zeta^*\overline{\zeta}}{1+\zeta^*\overline{\zeta}}\right), \tag{389}$$

so that

$$V = V_0 v^a T^b_a L^*_b. \tag{390}$$

Inserting Equations (387), (388) and (390) into (385) we obtain

$$\begin{aligned} (g_{ab}v^a v^b)^{-1} &= (8\pi)^{-1}\int (V_0 v^a T^b_a L^*_b)^{-2} V_0^2 d\Omega^*, \\ &= (8\pi)^{-1}\int (v^a T^b_a L^*_b)^{-2} d\Omega^*, \\ &= (8\pi)^{-1}\int (v^{*b} L^*_b)^{-2} d\Omega^*. \end{aligned} \tag{391}$$

Using the form Equation (389) the last integral can be easily evaluated (most easily done using θ and φ) leading to

$$\begin{aligned} (g_{ab}v^a v^b)^{-1} &= (\eta_{ab}v^{*a}v^{*b})^{-1} = (T^c_a T^d_b \eta_{cd} v^a v^b)^{-1}, \\ g_{ab} &= T^c_a T^d_b \eta_y cd, \end{aligned} \tag{392}$$

our sort for relationship.

We can go a step further. By taking the derivative of Equation (391) with respect to v^a, we easily find the covariant form of v, namely

$$\frac{v_a}{(g_{ab}v^a v^b)^2} = \frac{g_{ab}v^b}{(g_{ab}v^a v^b)^2} = (8\pi)^{-1}\int (v^a T^b_a L^*_b)^{-3} T^b_a L^*_b d\Omega^*.$$

References

[1] Adamo, T.M. and Newman, E.T., "The gravitational field of a radiating electromagnetic dipole", *Class. Quantum Grav.*, **25**, 245005, (2008). [DOI], [arXiv:0807.3537]. (Cited on page 62.)

[2] Adamo, T.M. and Newman, E.T., "Asymptotically stationary and static spacetimes and shear free null geodesic congruences", *Class. Quantum Grav.*, **26**, 155003, (2009). [DOI], [arXiv:0906.2409]. (Cited on page 48.)

[3] Adamo, T.M. and Newman, E.T., "Electromagnetically induced gravitational perturbations", *Class. Quantum Grav.*, **26**, 015004, (2009). [DOI], [arXiv:0807.3671]. (Cited on pages 23, 62, and 63.)

[4] Adamo, T.M. and Newman, E.T., "The real meaning of complex Minkowski-space world-lines", *Class. Quantum Grav.*, **27**, 075009, (2010). [DOI], [arXiv:0911.4205]. (Cited on pages 9 and 25.)

[5] Arnowitt, R., Deser, S. and Misner, C.W., "Energy and the Criteria for Radiation in General Relativity", *Phys. Rev.*, **118**, 1100–1104, (1960). [DOI], [ADS]. (Cited on page 7.)

[6] Aronson, B. and Newman, E.T., "Coordinate systems associated with asymptotically shear-free null congruences", *J. Math. Phys.*, **13**, 1847–1851, (1972). [DOI]. (Cited on pages 26, 39, 45, and 69.)

[7] Bergmann, P.G., "Non-Linear Field Theories", *Phys. Rev.*, **75**, 680–685, (1949). [DOI], [ADS]. (Cited on page 72.)

[8] Bondi, H., van der Burg, M.G.J. and Metzner, A.W.K., "Gravitational Waves in General Relativity. VII. Waves from Axi-Symmetric Isolated Systems", *Proc. R. Soc. London, Ser. A*, **269**, 21–52, (1962). [DOI], [ADS]. (Cited on pages 7, 10, 15, and 22.)

[9] Bramson, B., "Do electromagnetic waves harbour gravitational waves?", *Proc. R. Soc. London, Ser. A*, **462**, 1987–2000, (2006). [DOI]. (Cited on page 62.)

[10] Chruściel, P.T. and Friedrich, H., eds., *The Einstein Equations and the Large Scale Behavior of Gravitational Fields: 50 Years of the Cauchy Problem in General Relativity*, (Birkhäuser, Basel; Boston, 2004). [Google Books]. (Cited on page 72.)

[11] Corvino, J. and Schoen, R.M., "On the asymptotics for the vacuum Einstein constraint equations", *J. Differ. Geom.*, **73**, 185–217, (2006). [gr-qc/0301071]. (Cited on page 72.)

[12] Frauendiener, J., "Conformal Infinity", *Living Rev. Relativity*, **7**, lrr-2004-1, (2004). URL (accessed 31 July 2009):
http://www.livingreviews.org/lrr-2004-1. (Cited on page 15.)

[13] Friedrich, H., "On the Existence of n-Geodesically Complete or Future Complete Solutions of Einstein's Field Equations with Smooth Asymptotic Structure", *Commun. Math. Phys.*, **107**, 587–609, (1986). [DOI]. (Cited on page 72.)

[14] Frittelli, S., Kozameh, C.N., Newman, E.T., Rovelli, C. and Tate, R.S., "Fuzzy spacetime from a null-surface version of general relativity", *Class. Quantum Grav.*, **14**, A143–A154, (1997). [DOI], [gr-qc/9603061]. (Cited on page 72.)

[15] Frittelli, S. and Newman, E.T., "Pseudo-Minkowskian coordinates in asymptotically flat spacetimes", *Phys. Rev. D*, **55**, 1971–1976, (1997). [DOI], [ADS]. (Cited on page 64.)

[16] Gel'fand, I.M., Graev, M.I. and Vilenkin, N.Y., *Generalized Functions, Vol. 5: Integral geometry and representation theory*, (Academic Press, New York; London, 1966). (Cited on pages 64 and 65.)

[17] Goldberg, J.N., Macfarlane, A.J., Newman, E.T., Rohrlich, F. and Sudarshan, E.C.G., "Spin-s Spherical Harmonics and ð", *J. Math. Phys.*, **8**, 2155–2161, (1967). [DOI]. (Cited on pages 17 and 78.)

[18] Goldberg, J.N. and Sachs, R.K., "A Theorem on Petrov Types", *Acta Phys. Pol.*, **22**, 13–23, (1962). Republished as 10.1007/s10714-008-0722-5. (Cited on pages 9, 22, and 24.)

[19] Hansen, R.O. and Newman, E.T., "A complex Minkowski space approach to twistors", *Gen. Relativ. Gravit.*, **6**, 361–385, (1975). [DOI]. (Cited on pages 38 and 40.)

[20] Hansen, R.O., Newman, E.T., Penrose, R. and Tod, K.P., "The Metric and Curvature Properties of $\mathcal{H}$-Space", *Proc. R. Soc. London, Ser. A*, **363**, 445–468, (1978). [DOI], [ADS]. (Cited on pages 40 and 41.)

[21] Held, A., Newman, E.T. and Posadas, R., "The Lorentz Group and the Sphere", *J. Math. Phys.*, **11**, 3145–3154, (1970). [DOI]. (Cited on pages 64, 65, 67, and 78.)

[22] Hugget, S.A. and Tod, K.P., *An Introduction to Twistor Theory*, London Mathematical Society Student Texts, 4, (Cambridge University Press, Cambridge; New York, 1994), 2nd edition. [Google Books]. (Cited on page 74.)

[23] Ivancovich, J., Kozameh, C.N. and Newman, E.T., "Green's functions of the edh operators", *J. Math. Phys.*, **30**, 45–52, (1989). [DOI]. (Cited on page 40.)

[24] Ko, M., Newman, E.T. and Tod, K.P., "$\mathcal{H}$-Space and Null Infinity", in Esposito, F.P. and Witten, L., eds., *Asymptotic Structure of Space-Time*, Proceedings of a Symposium on Asymptotic Structure of Space-Time (SOASST), held at the University of Cincinnati, Ohio, June 14 – 18, 1976, pp. 227–271, (Plenum Press, New York, 1977). (Cited on page 9.)

[25] Kozameh, C.N. and Newman, E.T., "Electromagnetic dipole radiation fields, shear-free congruences and complex centre of charge world lines", *Class. Quantum Grav.*, **22**, 4667–4678, (2005). [DOI], [gr-qc/0504093]. (Cited on page 32.)

[26] Kozameh, C.N. and Newman, E.T., "The large footprints of H-space on asymptotically flat spacetimes", *Class. Quantum Grav.*, **22**, 4659–4665, (2005). [DOI], [gr-qc/0504022]. (Cited on pages 8, 26, 27, 39, and 69.)

[27] Kozameh, C.N., Newman, E.T., Santiago-Santiago, J.G. and Silva-Ortigoza, G., "The universal cut function and type II metrics", *Class. Quantum Grav.*, **24**, 1955–1979, (2007). [DOI], [gr-qc/0612004]. (Cited on pages 7, 8, 25, 26, 27, 33, and 50.)

[28] Kozameh, C.N., Newman, E.T. and Silva-Ortigoza, G., "On the physical meaning of the Robinson–Trautman–Maxwell fields", *Class. Quantum Grav.*, **23**, 6599–6620, (2006). [DOI], [gr-qc/0607074]. (Cited on pages 7, 41, and 47.)

[29] Kozameh, C.N., Newman, E.T. and Silva-Ortigoza, G., "On extracting physical content from asymptotically flat spacetime metrics", *Class. Quantum Grav.*, **25**, 145001, (2008). [DOI], [arXiv:0802.3314]. (Cited on pages 7, 8, 23, 67, 69, and 79.)

[30] Landau, L.D. and Lifshitz, E.M., *The classical theory of fields*, (Pergamon Press; Addison-Wesley, Oxford; Reading, MA, 1962), 2nd edition. (Cited on pages 59, 60, 63, and 67.)

[31] Lewandowski, J. and Nurowski, P., "Algebraically special twisting gravitational fields and CR structures", *Class. Quantum Grav.*, **7**, 309–328, (1990). [DOI]. (Cited on page 76.)

[32] Lewandowski, J., Nurowski, P. and Tafel, J., "Einstein's equations and realizability of CR manifolds", *Class. Quantum Grav.*, **7**, L241–L246, (1990). [DOI]. (Cited on page 76.)

[33] Lind, R.W., "Shear-free, twisting Einstein-Maxwell metrics in the Newman-Penrose formalism", *Gen. Relativ. Gravit.*, **5**, 25–47, (1974). [DOI]. (Cited on page 47.)

[34] Newman, E.T., "Heaven and Its Properties", *Gen. Relativ. Gravit.*, **7**, 107–111, (1976). [DOI]. (Cited on pages 9 and 41.)

[35] Newman, E.T., "Maxwell fields and shear-free null geodesic congruences", *Class. Quantum Grav.*, **21**, 3197–3221, (2004). [DOI]. (Cited on pages 32, 33, 35, and 37.)

[36] Newman, E.T., "Asymptotic twistor theory and the Kerr theorem", *Class. Quantum Grav.*, **23**, 3385–3392, (2006). [DOI], [gr-qc/0512079]. (Cited on pages 74 and 75.)

[37] Newman, E.T., Couch, E., Chinnapared, K., Exton, A., Prakash, A. and Torrence, R., "Metric of a Rotating, Charged Mass", *J. Math. Phys.*, **6**, 918–919, (1965). [DOI]. (Cited on pages 9, 45, 57, and 69.)

[38] Newman, E.T. and Nurowski, P., "CR structures and asymptotically flat spacetimes", *Class. Quantum Grav.*, **23**, 3123–3127, (2006). [DOI], [gr-qc/0511119]. (Cited on page 76.)

[39] Newman, E.T. and Penrose, R., "An Approach to Gravitational Radiation by a Method of Spin Coefficients", *J. Math. Phys.*, **3**, 566–578, (1962). [DOI], [ADS]. (Cited on pages 10, 19, and 20.)

[40] Newman, E.T. and Penrose, R., "Note on the Bondi–Metzner–Sachs Group", *J. Math. Phys.*, **7**, 863–870, (1966). [DOI], [ADS]. (Cited on pages 10 and 78.)

[41] Newman, E.T. and Penrose, R., "Spin-coefficient formalism", *Scholarpedia*, **4**(6), 7445, (2009). URL (accessed 30 July 2009): http://www.scholarpedia.org/article/Spin-coefficient_formalism. (Cited on page 19.)

[42] Newman, E.T. and Posadas, R., "Motion and Structure of Singularities in General Relativity", *Phys. Rev.*, **187**, 1784–1791, (1969). [DOI], [ADS]. (Cited on page 47.)

[43] Newman, E.T. and Silva-Ortigoza, G., "Tensorial spin-s harmonics", *Class. Quantum Grav.*, **23**, 497–509, (2006). [DOI], [gr-qc/0508028]. (Cited on pages 10, 78, 79, and 80.)

[44] Newman, E.T. and Tod, K.P., "Asymptotically flat space-times", in Held, A., ed., *General Relativity and Gravitation: One Hundred Years After the Birth of Albert Einstein*, 2, pp. 1–36, (Plenum Press, New York, 1980). (Cited on pages 15, 16, 17, 19, 48, and 64.)

[45] Newman, E.T. and Unti, T.W.J., "Behavior of Asymptotically Flat Empty Spaces", *J. Math. Phys.*, **3**, 891–901, (1962). [DOI], [ADS]. (Cited on page 20.)

[46] Penrose, R., "Asymptotic Properties of Fields and Space-Times", *Phys. Rev. Lett.*, **10**, 66–68, (1963). [DOI], [ADS]. (Cited on pages 7 and 15.)

[47] Penrose, R., "Zero Rest-Mass Fields Including Gravitation: Asymptotic Behaviour", *Proc. R. Soc. London, Ser. A*, **284**, 159–203, (1965). [DOI], [ADS]. (Cited on pages 7 and 15.)

[48] Penrose, R., "Twistor Algebra", *J. Math. Phys.*, **8**, 345–366, (1967). [DOI]. (Cited on page 74.)

[49] Penrose, R., "Relativistic symmetry groups", in Barut, A.O., ed., *Group Theory in Non-Linear Problems*, Proceedings of the NATO Advanced Study Institute, held in Istanbul, Turkey, August 7 – 18, 1972, NATO ASI Series C, 7, pp. 1–58, (Reidel, Dordrecht; Boston, 1974). (Cited on pages 10 and 23.)

[50] Penrose, R. and Rindler, W., *Spinors and space-time, Vol. 1: Two-spinor calculus and relativistic fields*, Cambridge Monographs on Mathematical Physics, (Cambridge University Press, Cambridge; New York, 1984). [Google Books]. (Cited on pages 11, 23, and 24.)

[51] Penrose, R. and Rindler, W., *Spinors and space-time, Vol. 2: Spinor and twistor methods in space-time geometry*, Cambridge Monographs on Mathematical Physics, (Cambridge University Press, Cambridge; New York, 1986). [Google Books]. (Cited on pages 27, 74, and 75.)

[52] Petrov, A.Z., "The Classification of Spaces Defining Gravitational Fields", *Gen. Relativ. Gravit.*, **32**, 1665–1685, (2000). [DOI]. (Cited on page 24.)

[53] Pirani, F.A.E., "Invariant Formulation of Gravitational Radiation Theory", *Phys. Rev.*, **105**(3), 1089–1099, (1957). [DOI]. (Cited on page 24.)

[54] Robinson, I., "Null Electromagnetic Fields", *J. Math. Phys.*, **2**, 290–291, (1961). [DOI]. (Cited on page 7.)

[55] Robinson, I. and Trautman, A., "Some spherical gravitational waves in general relativity", *Proc. R. Soc. London, Ser. A*, **265**, 463–473, (1962). [DOI]. (Cited on pages 41 and 46.)

[56] Sachs, R.K., "Gravitational Waves in General Relativity. VIII. Waves in Asymptotically Flat Space-Time", *Proc. R. Soc. London, Ser. A*, **270**, 103–126, (1962). [DOI], [ADS]. (Cited on pages 7, 10, and 18.)

[57] Sachs, R.K., "Gravitational radiation", in DeWitt, C.M. and DeWitt, B., eds., *Relativity, Groups and Topology*, Lectures delivered at Les Houches during the 1963 session of the Summer School of Theoretical Physics, University of Grenoble, pp. 523–562, (Gordon and Breach, New York, 1964). (Cited on page 22.)

[58] Sommers, P., "The geometry of the gravitational field at spacelike infinity", *J. Math. Phys.*, **19**, 549–554, (1978). [DOI], [ADS]. (Cited on page 7.)

[59] Szabados, L.B., "Quasi-Local Energy-Momentum and Angular Momentum in General Relativity", *Living Rev. Relativity*, **12**, lrr-2009-4, (2009). URL (accessed 31 July 2009): http://www.livingreviews.org/lrr-2009-4. (Cited on pages 23, 44, 45, and 55.)