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# Fixed point theory approach to boundary value problems for second-order difference equations on non-uniform lattices

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**Abstract**

In this paper, by means of the appropriate Green's function, an integral representation for the solutions of certain boundary value problems for second-order difference equations on (quadratic and $q$-quadratic) non-uniform lattices is presented. As a consequence, using fixed point theory, new results for the existence and uniqueness of the solution are proved on non-uniform lattices.

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## 1 Introduction

In this paper we solve certain boundary value problems for second-order difference equations on the most general (quadratic and $q$-quadratic) non-uniform lattices [1]. These problems appear in the discretization of differential operators when the convenient grid or mesh is not uniform. For our purposes, we shall use the Green's function approach [2, 3].

As indicated by Roach [2], 'Boundary value problems are an almost inevitable consequence of using mathematics to study problems in the real world'. In the monograph cited, the author examines in detail the one particular method which requires the construction of an auxiliary function known as a Green's function. Some historical developments of Green's functions, definitions, and applications to circuit theory, statics, wave equation, heat equation, quantum physics, finite elements, infinite products, Helmholtz equation or lattice Schrödinger operators can be found in [2–11].

A well-posed problem for an ordinary differential equation, partial differential equation or difference equation should have a unique solution that continuously depends on the sources. For linear equations, the operator that transforms the data into the solution is usually a linear integral operator [12]. The kernel is the Green's function. A list of formulas of such Green's functions for different situations and problems can be found in [6]. According to [13] the most appropriate way to solve a boundary value problem (BVP) is by calculating its Green's function and by means of the integral expression, it is also possible obtain some additional qualitative information about the solutions of the considered problem, such as their sign, oscillation properties, a priori bounds or their stability.

Let us consider the one-dimensional problem of a thin rod occupying the interval (0, 1) on the $x$ axis [3]. The boundary value problem reads

$$-\frac{d^2u(x)}{dx} = f(x), \quad 0 < x < 1; \qquad u(0) = \alpha, \qquad u(1) = \beta, \tag{1}$$

where $f(x)$ is the prescribed source density (per unit length of the rod) of heat and $\alpha$, $\beta$ are the prescribed end temperatures. The solution of the above problem can be written in the form [3, (1.1.2)]

$$u(x) = \int_0^1 G(x,\xi)f(\xi)\,d\xi + (1-x)\alpha + x\beta, \tag{2}$$

where the so-called Green's function $G(x,\xi)$ is a function of the real variables $x$ and $\xi$ defined on the square $0 \leq x, \xi \leq 1$ and is explicitly given by [3, (1.1.3), p.52]

$$G(x,\xi) = \begin{cases} x(1-\xi), & 0 < x < \xi, \\ \xi(1-x), & \xi < x < 1. \end{cases}$$

Since the Green's function $G(x,\xi)$ does not depend on the data, it is clear that (2) expresses in a very simple manner the dependence of the solution $u(x)$ on the data $f$, $\alpha$, $\beta$.

In many other applications, there appears the same differential equation with Cauchy boundary conditions $u(0) = u'(0) = 0$, whose solution has the following integral representation:

$$u(x) = u(0) + u'(0)x + \int_0^x (y-x)f(y)\,dy = \int_0^x (y-x)f(y)\,dy. \tag{3}$$

For specific $f$, the integration in (2) provides the solution of the boundary value problem (1). In other words, if we consider the differential operator $\mathcal{L}$ defined as

$$\mathcal{L}u(x) = -f(x),$$

the solution (2) provides a representation of the inverse operator $\mathcal{L}^{-1}$ as an integral operator with kernel the Green's function $G(x,\xi)$.

One of the main advantages of the Green's function is the fact that it is independent on the source and recently [13] an algorithm that calculates explicitly the Green's function related to a linear ordinary differential equation, with constant coefficients, coupled with two-point linear boundary conditions has been developed.

For these reasons the theory of Green's functions is a fundamental tool in the analysis of differential equations. It has been widely studied in the literature [2, 3, 6, 12] and it has a great importance for the use of monotone iterative techniques [14, 15], lower and upper solutions [16], fixed point theorems [17, 18], fractional calculus approach [19–23] or variational methods [24].

Although several results in the discrete case are similar to those already known in the continuous case, the adaptation from the continuous case to the discrete case is not direct but requires some special devices [25]. Despite the fact that BVP of differential equations has been studied by many authors using various methods and techniques, there are scarce techniques for studying the BVP of difference equations. In the last decades, the fixed point theorems have been improved and generalized in different directions for solving boundary value problems (see *e.g.* [26–31]). These results were usually obtained by analytic techniques and various fixed point theorems. For example, the upper and lower solution method, the conical shell fixed point theorems, the Brouwer and Schauder fixed point theorems or topological degree theory (see [32] and references therein).

As indicated at the beginning of this section, we use the Green's function to study some boundary value problems for second-order difference equations on non-uniform lattices. As in the classical case, separation of the variables and representation of solutions in terms of the basic Fourier series might also give explicit solutions of these boundary value problems [33]. Finally, fixed point theorems are used to prove the existence and uniqueness of solutions for the boundary value problems analyzed in this paper.

## 2 Basic definitions and notations

Let us introduce the following difference operator [33–37]:

$$\mathbb{D}_x f\big(x(s)\big) = \frac{f(x(s+1/2)) - f(x(s-1/2))}{x(s+1/2) - x(s-1/2)}. \tag{4}$$

This operator is the most general (divided difference) derivative having the following property: if $f(x(s))$ is a polynomial of degree $n$ in $x(s)$, then $\mathbb{D}_x f(x(s))$ is a polynomial of degree $n-1$ in $x(s)$, where for $s \in \mathbf{Z}$, the lattice function $x(s)$ has the form [1, 38, 39]

$$x(s) = c_1 q^s + c_2 q^{-s} + c_3, \quad \text{or} \tag{5}$$

$$x(s) = c_4 s^2 + c_5 s + c_6, \tag{6}$$

with $q \neq 1$, and $c_i$, $i = 1, \dots, 6$, are constants.

Depending on the constants $c_i$, there are four primary classes for the lattices $x(s)$ given in (5) and (6) which usually are called:

1. Linear lattices if we choose in (6) $c_4 = 0$ and $c_5 \neq 0$.
2. Quadratic lattices if we choose in (6) $c_4 \neq 0$.
3. $q$-linear lattices if we choose in (5) $c_2 = 0$ and $c_1 \neq 0$.
4. $q$-quadratic lattices if we choose in (5) $c_1 c_2 \neq 0$.

Some properties of these classical lattices are presented in [40]. Applications of the theory of basic Fourier series to $q$-analogues of several equations of mathematical physics such as the $q$-heat equation, $q$-wave equation and $q$-Laplace equation were studied in [33]. Besides, linear second-order partial $q$-difference equations of the hypergeometric type in two variables on $q$-linear lattices have been recently analyzed in [41].

The Green's function approach for difference equations on linear and $q$-linear lattices being well known (see *e.g.* [42–44]), we shall analyze certain boundary value problems for second-order difference equations on quadratic and $q$-quadratic lattices.

**Remark 1** The $q$-quadratic lattice, in its general non-symmetrical form, is the most general case and the other lattices can be found from this by limiting processes. For $c_1 = c_2 = 1/2$ and $c_3 = 0$, we obtain the Askey-Wilson lattice [45]. Moreover, if we choose $c_4 = 1$, $c_5 = \gamma + \delta + 1$, and $c_6 = 0$, we obtain the Racah lattice [46]. Askey-Wilson and Racah polynomials [46] both satisfy a second-order difference equation on non-uniform lattices [1, 38, 39].

The well-known relation in the continuous case

$$\frac{d}{dx}x^n = nx^{n-1}$$

has appropriate analogues [47] for $q$-quadratic and quadratic lattices, by using different systems of polynomials bases $\{\vartheta_n(x(s))\}$ with exact $\deg_{x(s)}(\vartheta_m(x(s))) = m$ as proposed in [39, 48]. In particular we have the following.

**Proposition 1**

1. *In the q-quadratic lattice* $x(s) = c_1q^s + c_2q^{-s} + c_3$, *with* $q \neq 1$, $c_1 > 0$ *and* $c_2 > 0$, *the basis* $\vartheta_n(s;q)$ *defined for* $n \geq 0$ *as*

$$\vartheta_n\bigl(x(s)\bigr) \equiv \vartheta_n(s;q) = \left(-\frac{c_1^{3/2}q^{1/4}}{\sqrt{c_2}}\right)^n \left(\frac{q^{-\frac{n}{2}+s+\frac{1}{4}}\sqrt{c_1}}{\sqrt{c_2}};q\right)_n \left(\frac{q^{-\frac{n}{2}-s+\frac{1}{4}}\sqrt{c_2}}{\sqrt{c_1}};q\right)_n \tag{7}$$

*satisfies, for* $n \geq 1$,

$$\mathbb{D}_x\vartheta_n(s;q) = \frac{\vartheta_n(s+1/2;q) - \vartheta_n(s-1/2;q)}{x(s+1/2) - x(s-1/2)} = k_n(c_1,c_2,c_3)\vartheta_{n-1}(s;q), \tag{8}$$

*where the constants* $k_n(c_1,c_2,c_3)$ *are explicitly given by*

$$k_n(c_1,c_2,c_3) = \frac{c_1q^{\frac{1-n}{2}}[n]_q}{c_2}, \tag{9}$$

*the q-Pochhammer symbol is given by*

$$(a;q)_0 = 1, \qquad (a;q)_n = \prod_{j=0}^{n-1}\bigl(1-aq^j\bigr), \quad n = 1,2,\ldots,$$

*and the q-number* $[z]_q$ *is defined by*

$$[z]_q = \frac{q^z-1}{q-1}, \qquad \lim_{q\to 1}[z]_q = z. \tag{10}$$

2. *In the quadratic lattice* $x(s) = c_4s^2 + c_5s + c_6$, *with* $c_4 \neq 0$, *the basis* $\vartheta_n(s)$ *is defined for* $n \geq 0$ *as*

$$\vartheta_n\bigl(x(s)\bigr) \equiv \vartheta_n(s) = 4^{-n}(-c_4)^n\left(-\frac{c_5}{c_4} - 2s + \frac{1}{2}\right)_n \left(\frac{c_5}{c_4} + 2s + \frac{1}{2}\right)_n, \tag{11}$$

*and satisfies, for* $n \geq 1$,

$$\mathbb{D}_x\vartheta_n(s) = \frac{\vartheta_n(s+1/2) - \vartheta_n(s-1/2)}{x(s+1/2) - x(s-1/2)} = k_n(c_4,c_5,c_6)\vartheta_{n-1}(s), \tag{12}$$

*where the constants* $k_n(c_4,c_5,c_6)$ *are explicitly given by*

$$k_n(c_4,c_5,c_6) = n, \tag{13}$$

*and* $(A)_n = A(A+1)\cdots(A+n-1)$ *denotes the Pochhammer symbol.*

As a consequence $\mathbb{D}_x$ is an exact lowering operator in these bases $\{\vartheta_n(x(s))\}$ where $k_n(c_i, c_{i+1}, c_{i+2})$, $i = 1$ or $i = 4$, are constants with respect to $s$ and they depend on the lattice type.

For a lattice $x(s)$ of the form (5) or (6), let [36]

$$L[x] = \left\{ x\left(\frac{s}{2} + s_1\right) : s \in \mathbf{Z} \right\}, \quad s_1 \in \mathbf{R}. \tag{14}$$

The $\mathbb{D}$-integral of a function [37]

$$f : L[x] \to \mathbf{R},$$

defined on $L[x]$ is defined by the Riemann sum over the lattice points

$$\int_{x(a)}^{x(b)} f\bigl(x(s)\bigr)\mathbb{D}x(s) := \sum_{s=a}^{b} f\bigl(x(s)\bigr)\Delta x(s-1/2), \tag{15}$$

where $x(a) < x(b)$, $a, b \in \mathbf{Z}$, and

$$\Delta x(s) = x(s+1) - x(s).$$

This definition reduces to the usual definition of the difference integral and the Thomae [49] and Jackson [50] $q$-integrals [46, 51, 52] in the canonical forms of the linear and $q$-linear lattices, respectively.

By using the notations

$$E_x^{\pm} f\bigl(x(s)\bigr) = f\bigl(x(s \pm 1/2)\bigr), \qquad E_x^{\pm} x(s) = x(s \pm 1/2), \quad \forall x(s) \in L[x],$$

the following properties follow from definition (15) [34, 37]:

1. An analogue of the fundamental theorem of calculus

$$\int_{x(a)}^{x(\omega)} \mathbb{D}_x f\bigl(x(s)\bigr)\mathbb{D}x(s) = f\bigl(E_x^+ x(\omega)\bigr) - f\bigl(E_x^- x(a)\bigr). \tag{16}$$

2. An analogue of integration by parts formula for two functions $f(x)$ and $g(x)$

$$\int_{x(a)}^{x(\omega)} f\bigl(x(s)\bigr)\mathbb{D}_x g(x)\mathbb{D}x(x)$$
$$= f\bigl(E_x^{+2} x(\omega)\bigr) g\bigl(E_x^+ x(\omega)\bigr) - f\bigl(x(a)\bigr) g\bigl(E_x^- x(a)\bigr) \tag{17}$$
$$- \int_{x(a)}^{x(\omega)} \mathbb{D}_x f\bigl(E_x^+ x(s)\bigr) g\bigl(E_x^+ x(s)\bigr)\mathbb{D}\bigl(E_x^+ x(s)\bigr). \tag{18}$$

Next we present two basic but illustrative computations which might help to clarify the notation used as well as the analogue of the fundamental theorem of calculus (16).

1. In the case of a $q$-quadratic lattice $x(s) = c_1 q^s + c_2 q^{-s} + c_3$, with $q \neq 1$, $c_1 > 0$ and $c_2 > 0$,

$$\begin{aligned}\int_{x(a)}^{x(b)} \vartheta_n(s;q)\mathbb{D}x(s) &= \sum_{s=a}^{b} \vartheta_n(s;q)\Delta x(s-1/2)\\ &= \sum_{s=a}^{b} \frac{c_2 q^{n/2}}{c_1[n+1]_q}\mathbb{D}_x\vartheta_{n+1}(s;q)\Delta x(s-1/2)\\ &= \frac{\vartheta_{n+1}(b+1/2;q)-\vartheta_{n+1}(a-1/2;q)}{k_{n+1}(c_1,c_2,c_3)},\end{aligned} \tag{19}$$

where $\vartheta_n(s;q)$ and $k_n(c_1,c_2,c_3)$ have been defined in (7) and (9).

2. In the case of a quadratic lattice $x(s) = c_4 s^2 + c_5 s + c_6$, with $c_4 \neq 0$, the following relation holds:

$$\begin{aligned}\int_{x(a)}^{x(b)} \vartheta_n(s)\mathbb{D}x(s) &= \sum_{s=a}^{b} \vartheta_n(s)\Delta x(s-1/2)\\ &= \sum_{s=a}^{b} \frac{1}{n+1}\mathbb{D}_x\vartheta_{n+1}(s)\Delta x(s-1/2)\\ &= \frac{\vartheta_{n+1}(b+1/2)-\vartheta_{n+1}(a-1/2)}{k_{n+1}(c_4,c_5,c_6)},\end{aligned} \tag{20}$$

where $\vartheta_n(s)$ and $k_n(c_4,c_5,c_6)$ have been defined in (11) and (13).

## 3 Boundary value problem on non-uniform lattices

Let us consider $f \in \mathcal{C}(L[x] \times \mathbf{R}, \mathbf{R})$, where $L[x]$ has been defined in (14). Analogues of trigonometric and exponential functions on $q$-quadratic lattices are given in [33, 40, 53]. In this section we explicitly obtain the Green's function $G(t,s;x(s))$ [2, 3, 6, 7] in quadratic and $q$-quadratic lattices $x(s)$ such that the solutions of the boundary value problem for a new class of second-order difference equation as

$$\mathbb{D}_x^2 u\bigl(x(s)\bigr) = f\bigl(x(s), u\bigl(x(s)\bigr)\bigr), \quad x(s) \in L[x], \tag{21}$$

with Cauchy boundary conditions

$$u\bigl(x(s_1)\bigr) = \alpha, \qquad \mathbb{D}_x u\bigl(x(s_1-1/2)\bigr) = \beta, \tag{22}$$

are given by the solutions

$$u\bigl(x(t)\bigr) = \alpha + \beta\left(\frac{\vartheta_1(x(s)) - \vartheta_1(x(s_1))}{k_1(c_i, c_{i+1}, c_{i+2})}\right) + \int_{x(s_1)}^{x(a)} G\bigl(t,s;x(s)\bigr) f\bigl(x(s), u\bigl(x(s)\bigr)\bigr)\mathbb{D}x(s), \tag{23}$$

where [40, Eq. (2.2.10)]

$$\mathbb{D}_x^2 u\bigl(x(s)\bigr) = \frac{1}{x(s+1/2)-x(s-1/2)}\left[\frac{u(x(s+1))-u(x(s))}{x(s+1)-x(s)} - \frac{u(x(s))-u(x(s-1))}{x(s)-x(s-1)}\right],$$

and $G(t,s;x(s))$ denotes the Green's function which shall be computed explicitly (see (24) and (25) below).

For $x(s) \in L[x]$, from (16) we have

$$g\big(x(s)\big) = g\big(x(a)\big) + \int_{x(a+1/2)}^{x(s-1/2)} \mathbb{D}_x g\big(x(r)\big)\mathbb{D}x(r).$$

Therefore, if $g(x(s)) = \mathbb{D}_x u(x(s))$ and $a = s_1$,

$$\mathbb{D}_x u\big(x(s)\big) = \mathbb{D}_x u\big(x(s_1)\big) + \int_{x(s_1+1/2)}^{x(s-1/2)} \mathbb{D}_x^2 u\big(x(r)\big)\mathbb{D}x(r),$$

or equivalently from (21)

$$\mathbb{D}_x u\big(x(y)\big) = \mathbb{D}_x u\big(x(s_1)\big) + \int_{x(s_1+1/2)}^{x(y-1/2)} f\big(x(z), u\big(x(z)\big)\big)\mathbb{D}x(z).$$

From (16) and the latter expression we have

$$\begin{aligned} u\big(x(s)\big) &= u\big(x(s_1)\big) + \int_{x(s_1+1/2)}^{x(s-1/2)} \mathbb{D}_x u\big(x(y)\big)\mathbb{D}x(y) \\ &= u\big(x(s_1)\big) + \int_{x(s_1+1/2)}^{x(s-1/2)} \left(\mathbb{D}_x u\big(x(s_1-1/2)\big) + \int_{x(s_1)}^{x(y-1/2)} f\big(x(z), u\big(x(z)\big)\big)\mathbb{D}x(z)\right)\mathbb{D}x(y) \\ &= u\big(x(s_1)\big) + \mathbb{D}_x u\big(x(s_1-1/2)\big)\left(\frac{\vartheta_1(x(s)) - \vartheta_1(x(s_1))}{k_1(c_i, c_{i+1}, c_{i+2})}\right) \\ &\quad + \int_{x(s_1+1/2)}^{x(s-1/2)} \int_{x(s_1)}^{x(y-1/2)} f\big(x(z), u\big(x(z)\big)\big)\mathbb{D}x(z)\mathbb{D}x(y), \end{aligned}$$

where the constants $k_1(c_i, c_{i+1}, c_{i+2})$, $i = 1$ or $i = 4$, depend on the lattice considered and have been explicitly given in (13) and (9) for quadratic and $q$-quadratic lattices, respectively. If we interchange the integrals, we obtain

$$\begin{aligned} u\big(x(s)\big) &= u\big(x(s_1)\big) + \mathbb{D}_x u\big(x(s_1-1/2)\big)\left(\frac{\vartheta_1(x(s)) - \vartheta_1(x(s_1))}{k_1(c_i, c_{i+1}, c_{i+2})}\right) \\ &\quad + \int_{x(s_1)}^{x(s-1)} f\big(x(z), u\big(x(z)\big)\big)\mathbb{D}x(z) \int_{x(z+s_1+1/2)}^{x(s-1/2)} \mathbb{D}x(y). \end{aligned}$$

Thus, we just need to compute the last integral, which is simple in terms of the basis associated with the operator as

$$\int_{x(z+s_1+1/2)}^{x(s-1/2)} \mathbb{D}x(y) = \frac{\vartheta_1(x(s)) - \vartheta_1(x(z))}{k_1(c_i, c_{i+1}, c_{i+2})},$$

and therefore

$$\begin{aligned} u\big(x(s)\big) &= u\big(x(s_1)\big) + \mathbb{D}_x u\big(x(s_1-1/2)\big)\left(\frac{\vartheta_1(x(s)) - \vartheta_1(x(s_1))}{k_1(c_i, c_{i+1}, c_{i+2})}\right) \\ &\quad + \int_{x(s_1)}^{x(s-1)} f\big(x(z), u\big(x(z)\big)\big)\left(\frac{\vartheta_1(x(s)) - \vartheta_1(x(z+s_1))}{k_1(c_i, c_{i+1}, c_{i+2})}\right)\mathbb{D}x(z) \end{aligned}$$

$$= \alpha + \beta\left(\frac{\vartheta_1(x(s)) - \vartheta_1(x(s_1))}{k_1(c_i, c_{i+1}, c_{i+2})}\right) + \int_{x(s_1)}^{x(s-1)} f\big(x(z), u\big(x(z)\big)\big)\left(\frac{\vartheta_1(x(s)) - \vartheta_1(x(z+s_1))}{k_1(c_i, c_{i+1}, c_{i+2})}\right)\mathbb{D}x(z).$$

If we impose $s_1 = 0$ and the boundary conditions (22) with $\alpha = \beta = 0$, we give the solution of the boundary value problem (21)-(22) for the two types of different non-uniform lattices considered in this article:

1. In the $q$-quadratic lattice $x(s) = c_1 q^s + c_2 q^{-s} + c_3$, with $q \neq 1$, $c_1 > 0$ and $c_2 > 0$, we have

$$\int_{x(z+1/2)}^{x(s-1/2)} \mathbb{D}x(y) = \frac{\vartheta_1(s;q) - \vartheta_1(z;q)}{k_1((c_1, c_2, c_3))} = c_1\big(q^s - q^z\big) + c_2\big(q^{-s} - q^{-z}\big)$$

and with the boundary conditions (22) we obtain

$$u\big(x(s)\big) = \int_{x(0)}^{x(s-1)} f\big(x(z), u\big(x(z)\big)\big)\left(\frac{\vartheta_1(s;q) - \vartheta_1(z;q)}{k_1(c_1, c_2, c_3)}\right)\mathbb{D}x(z) = \int_{x(0)}^{x(s-1)} f\big(x(z), u\big(x(z)\big)\big)\big(c_1\big(q^s - q^z\big) + c_2\big(q^{-s} - q^{-z}\big)\big)\mathbb{D}x(z). \tag{24}$$

2. In the quadratic lattice $x(s) = c_4 s^2 + c_5 s + c_6$, $c_4 \neq 0$,

$$\frac{\vartheta_1(s) - \vartheta_1(z)}{k_1(c_4, c_5, c_6)} = (s - z)\big(c_4(z + s) + c_5\big)$$

and therefore, with the boundary conditions (22) it yields

$$u\big(x(s)\big) = \int_{x(0)}^{x(s-1)} f\big(x(z), u\big(x(z)\big)\big)\left(\frac{\vartheta_1(s) - \vartheta_1(z)}{k_1(c_4, c_5, c_6)}\right)\mathbb{D}x(z) = \int_{x(0)}^{x(s-1)} f\big(x(z), u\big(x(z)\big)\big)\big((s - z)\big(c_4(s + z) + c_5\big)\big)\mathbb{D}x(z). \tag{25}$$

As a consequence, (24) and (25) provide an explicit representation of the Green's function for the BVP (21)-(22) for each type of non-uniform lattice considered in this paper (quadratic and $q$-quadratic).

Now, motivated by (23), let us introduce the following operator:

$$\mathbb{T}[u] = \int_{x(s_1)}^{x(s-1)} f\big(x(z), u\big(x(z)\big)\big)\left(\frac{\vartheta_1(s) - \vartheta_1(z + s_1)}{k_1(c_i, c_{i+1}, c_{i+2})}\right)\mathbb{D}x(z),$$

where the constants $k_1(c_i, c_{i+1}, c_{i+2})$ are given for $i = 1$ ($q$-quadratic lattice) in (9) and for $i = 4$ (quadratic lattice) in (13).

In this case, $\mathbb{T}$ is defined in the Banach space of all continuous functions from $L[x]$ to $\mathbf{R}$ with the norm defined by

$$\|u\| = \sup\big\{\big|u\big(x(s)\big)\big| : x(s) \in L[x]\big\}.$$

Let us define

$$\Lambda_1 = \sup_{x(s)\in L[x]} \left| \int_{x(s_1)}^{x(s-1)} \left( \frac{\vartheta_1(s) - \vartheta_1(z+s_1)}{k_1(c_i, c_{i+1}, c_{i+2})} \right) \mathbb{D}x(z) \right|. \tag{26}$$

In a similar way as in [43] we can prove the following result for the uniqueness of the solution of the boundary value problem, assuming that the function $f$ satisfies a Lipchitz bound in the second variable.

**Theorem 1** *Let $f : L[x] \times \mathbf{R} \to \mathbf{R}$ be a continuous function satisfying the condition*

$$\left| f(t,u) - f(t,v) \right| \le L|u-v|, \quad \forall t, u, v \in \mathbf{R},$$

*where $L$ is a Lipschitz constant. Then, the boundary value problem* (21)-(22) *has a unique solution provided $\Lambda = L\Lambda_1 < 1$, where $\Lambda_1$ is given by* (26).

*Proof* Let

$$\mathbb{T}\big[u\big(x(s)\big)\big] = \int_{x(s_1)}^{x(s-1)} f\big(x(z), u\big(x(z)\big)\big) \left( \frac{\vartheta_1(s) - \vartheta_1(z+s_1)}{k_1(c_i, c_{i+1}, c_{i+2})} \right) \mathbb{D}x(z),$$

for a continuous function $u$ and $x(s) \in L[x]$. Let us define

$$M_0 = \sup_{x(s)\in L[x]} \left| f\big(x(s), 0\big) \right|, \tag{27}$$

and consider

$$r \ge \frac{M_0 \Lambda_1}{1-\delta}, \tag{28}$$

where $\delta$ is chosen such that $\Lambda \le \delta < 1$. First, we prove that if

$$\mathcal{B}_r = \big\{ u \in \mathcal{C} : \|u\| \le r \big\},$$

then we have $\mathbb{T}\mathcal{B}_r \subset \mathcal{B}_r$. This property follows from

$$\begin{aligned}
\big\| \mathbb{T}[u] \big\| &= \sup_{x(s)\in L[x]} \left| \int_{x(s_1)}^{x(s-1)} f\big(x(z), u\big(x(z)\big)\big) \left( \frac{\vartheta_1(s) - \vartheta_1(z+s_1)}{k_1(c_i, c_{i+1}, c_{i+2})} \right) \mathbb{D}x(z) \right| \\
&\le \sup_{x(s)\in L[x]} \left| \int_{x(s_1)}^{x(s-1)} \Big( \big| f\big(x(z), u\big(x(z)\big)\big) - f\big(x(z), 0\big) \big| + \big| f\big(x(z), 0\big) \big| \Big) \right. \\
&\qquad \left. \times \left( \frac{\vartheta_1(s) - \vartheta_1(z+s_1)}{k_1(c_i, c_{i+1}, c_{i+2})} \right) \mathbb{D}x(z) \right| \\
&\le \big( L\|u\| + M_0 \big) \sup_{x(s)\in L[x]} \left| \int_{x(s_1)}^{x(s-1)} \left( \frac{\vartheta_1(s) - \vartheta_1(z+s_1)}{k_1(c_i, c_{i+1}, c_{i+2})} \right) \mathbb{D}x(z) \right| \le \big( L\|u\| + M_0 \big) \Lambda_1 \\
&\le (Lr + M_0)\Lambda_1 \le (\Lambda + 1 - \delta) r \le r.
\end{aligned}$$

Moreover,

$$\begin{aligned}\|\mathbb{T}[u]-\mathbb{T}[v]\| &= \sup_{x(s)\in L[x]}\left|\mathbb{T}\left[u\left(x(s)\right)\right]-\mathbb{T}\left[v\left(x(s)\right)\right]\right| \\ &\le \sup_{x(s)\in L[x]}\left|\int_{x(s_1)}^{x(s-1)}\left(\frac{\vartheta_1(s)-\vartheta_1(z+s_1)}{k_1(c_i,c_{i+1},c_{i+2})}\right)L\|u-v\|\mathbb{D}x(z)\right| \le \Lambda\|u-v\|.\end{aligned}$$

As $\Lambda < 1$, the conclusion follows from Banach's contraction mapping principle [54]. □

**Example 1** In [40] and references therein, the theory of basic analogs of Fourier series and their applications to quantum groups and mathematical physics is presented. Let us consider BVP which appears as a difference analog of equation for harmonic motion [40, (2.4.16)]

$$\begin{aligned}&\mathbb{D}_x^2 u\left(x(s)\right)+\frac{4q^{1/2}w^2}{(1-q)^2}u\left(x(s)\right)=0, \qquad u\left(x(1/4)\right)=\alpha, \\ &\mathbb{D}_x u\left(x(-1/4)\right)=\mathbb{D}_x u\left(x(1/4)\right)=\beta,\end{aligned} \tag{29}$$

where $s_1 = 1/4$, $q \neq 1$ and the non-uniform lattice is given by

$$x(s)=\frac{q^s+q^{-s}}{2}.$$

For $|\omega| < 1$, let us introduce the basic cosine $C_q(x(s);\omega)$ and basic sine $S_q(x(s);\omega)$ functions [33, 40] as

$$\begin{aligned}C_q\left(x(s);\omega\right)&=\frac{(-\omega^2;q^2)_\infty}{(-q\omega^2;q^2)_\infty}\,{}_2\varphi_1\left(\begin{matrix}-q^{2s+1},-q^{1-2s}\\ q\end{matrix}\,\middle|\,q^2;-\omega^2\right), \\ S_q\left(x(s);\omega\right)&=\frac{(-\omega^2;q^2)_\infty}{(-q\omega^2;q^2)_\infty}\frac{2q^{1/4}\omega}{1-q}x(s)\,{}_2\varphi_1\left(\begin{matrix}-q^{2s+2},-q^{2-2s}\\ q^3\end{matrix}\,\middle|\,q^2;-\omega^2\right),\end{aligned}$$

where the $q$-shifted factorial and the basic hypergeometric series are defined by [46, 51, 52]

$$\begin{aligned}&(a;q)_k=\prod_{j=0}^{k-1}\left(1-aq^j\right), \qquad (a;q)_\infty=\prod_{j=0}^{\infty}\left(1-aq^j\right), \\ &{}_2\varphi_1\left(\begin{matrix}a_1,a_2\\ b_1\end{matrix}\,\middle|\,q;t\right)=\sum_{k=0}^{\infty}\frac{(a_1;q)_k(a_2;q)_k}{(q;q)_k(b_1;q)_k}t^k.\end{aligned}$$

The basic sine and cosine functions satisfy [33, (2.4.17), (2.4.18)]

$$\mathbb{D}_x\left[S_q\left(x(s);\omega\right)\right]=\frac{2q^{1/4}}{1-q}\omega C_q\left(x(s);\omega\right), \qquad \mathbb{D}_x\left[C_q\left(x(s);\omega\right)\right]=\frac{-2q^{1/4}}{1-q}\omega S_q\left(x(s);\omega\right).$$

By using the tables of zeros of basic sine and cosine functions [40, Appendix C] if

$$\alpha=C_{q_1}\left(x(1/4);\omega_1\right), \qquad \beta=\frac{2q_1^{1/4}\omega_1}{1-q_1}C_{q_1}\left(x(1/4);\omega_1\right),$$

where $q_1 = 0.9$ and $\omega_1 = 0.32634395186574$, the explicit solution of the BVP (29) is

$$u\bigl(x(s)\bigr) = C_q\bigl(x(s);\omega\bigr) + S_q\bigl(x(s);\omega\bigr),$$

since $S_{q_1}(x(1/4);\omega_1) = S_{q_1}(x(-1/4);\omega_1) = 0$. Moreover, for these values

$$\frac{4q_1^{1/2}w_1^2}{(1-q_1)^2} \sim 40.4140507688415,$$

and since

$$\Lambda_1 = \sup_{x(s)\in L[x]} \left| \int_{x(1/4)}^{x(23/4)} \left( \frac{\vartheta_1(s) - \vartheta_1(z+s_1)}{k_1(c_i, c_{i+1}, c_{i+2})} \right) \mathbb{D}x(z) \right| = 0.02094573233035616,$$

we have $L\Lambda_1 < 1$, which ensures the unicity of the solution of the BVP for $x(1/4) \le x(s) \le x(23/4)$.

Moreover, if $f$ is a bounded function we have the following.

**Theorem 2** *If $f$ is a bounded function, then the operator $\mathbb{T}$ has a fixed point and, in consequence, the difference equation on a non-uniform lattice* (21) *with the boundary conditions* (22) *has a solution.*

*Proof* The operator $\mathbb{T}$ is compact since it is an integral operator [3] and the range of $\mathbb{T}$ is contained in a closed ball since $f$ is bounded.

The operator $\mathbb{T}$ has a fixed point in view the classical Schaefer's fixed point theorem [55, Theorem 4.3.2]. □

## Concluding remarks

We would like to mention that our goal here is not to exploit all possible boundary value problems covered by this approach, but to emphasize its systematic character and its simplicity, due to very recent results on bases associated with the difference operators considered in this article.

#### Competing interests

The authors declare that they have no competing interests.

#### Authors' contributions

Each of the authors, IA, EG, and JJN, contributed to each part of this study equally and read and approved the final version of the manuscript.

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