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Variational methods for fractional q -Sturm-Liouville problems

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ArabiaDepartment of Mathematics,
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Giza, Egypt**Abstract**

In this paper, we formulate a regular q -fractional Sturm-Liouville problem (qFSLP) which includes the left-sided Riemann-Liouville and the right-sided Caputo q -fractional derivatives of the same order α , $\alpha \in (0, 1)$. We introduce the essential q -fractional variational analysis needed in proving the existence of a countable set of real eigenvalues and associated orthogonal eigenfunctions for the regular qFSLP when $\alpha > 1/2$ associated with the boundary condition $y(0) = y(a) = 0$. A criterion for the first eigenvalue is proved. Examples are included. These results are a generalization of the integer regular q -Sturm-Liouville problem introduced by Annaby and Mansour in (J. Phys. A, Math. Gen. 38:3775-3797, 2005; J. Phys. A, Math. Gen. 39:8747, 2006).

MSC: 39A13; 26A33; 49R05**Keywords:** left- and right-sided Riemann-Liouville and Caputo q -derivatives; eigenvalues and eigenfunctions; q -fractional variational calculus**1 Introduction**

In the joint paper of Sturm and Liouville [3], they studied the problem

$$-\frac{d}{dx} \left(p \frac{dy}{dx} \right) + r(x)y(x) = \lambda w y(x), \quad x \in [a, b], \quad (1.1)$$

with certain boundary conditions at a and b . Here, the functions p , w are positive on $[a, b]$ and r is a real valued function on $[a, b]$. They proved the existence of non-zero solutions (eigenfunctions) only for special values of the parameter λ which are called eigenvalues. For a comprehensive study of the contribution of Sturm and Liouville to the theory, see [4]. Recently, many mathematicians have become interested in a fractional version of (1.1), *i.e.*, when the derivative is replaced by a fractional derivative like Riemann-Liouville derivative or Caputo derivative; see [5–10]. Iterative methods, variational method, and the fixed point theory are three different approaches used in proving the existence and uniqueness of solutions of Sturm-Liouville problems, *cf.* [4, 11, 12]. The calculus of variations has recently been developed to calculate the extremum of a functional that contains fractional derivatives, which is called the fractional calculus of variations; see for example [13–20]. In [5], Klimek *et al.* applied the methods of fractional variational calculus to prove the existence of a countable set of orthogonal solutions and corresponding eigenvalues. In [1, 2],

Annaby and Mansour introduced a q -version of (1.1), *i.e.*, when the derivative is replaced by Jackson q -derivative. Their results are applied and developed in different respects; for example, see [21–26]. Throughout this paper q is a positive number less than 1. The set of non-negative integers is denoted by \mathbb{N}_0 , and the set of positive integers is denoted by \mathbb{N} . For $t > 0$,

$$A_{q,t} := \{tq^n : n \in \mathbb{N}_0\}, \quad A_{q,t}^* := A_{q,t} \cup \{0\},$$

and

$$\mathcal{A}_{q,t} := \{\pm tq^n : n \in \mathbb{N}_0\}.$$

When $t = 1$, we simply use A_q, A_q^* , and \mathcal{A}_q to denote $A_{q,1}, A_{q,1}^*$, and $\mathcal{A}_{q,1}$, respectively. In the following, we state the basic q -notations and notions we use in this article, *cf.* [27, 28].

For $n \in \mathbb{N}_0$, the q -shifted factorial $(a; q)_n$ of $a \in \mathbb{C}$ is defined by

$$(a; q)_0 := 1 \quad \text{and} \quad \text{for } n \in \mathbb{N}, \quad (a; q)_n := \prod_{k=1}^n (1 - aq^{k-1}). \tag{1.2}$$

The multiple q -shifted factorial for complex numbers a_1, \dots, a_k is defined by

$$(a_1, a_2, \dots, a_k; q)_n := \prod_{j=1}^k (a_j; q)_n. \tag{1.3}$$

The limit $\lim_{n \rightarrow \infty} (a; q)_n$ exists and is denoted by $(a; q)_\infty$. For $\alpha \in \mathbb{R}$,

$$(a; q)_\alpha = \frac{(a; q)_\infty}{(aq^\alpha; q)_\infty}.$$

The q -gamma function, [28, 29], is defined for $z \in \mathbb{C}, z \neq -n, n \in \mathbb{N}$ by

$$\Gamma_q(z) := \frac{(q; q)_\infty}{(q^z; q)_\infty} (1 - q)^{1-z}, \quad 0 < |q| < 1. \tag{1.4}$$

Here we take the principal values of q^z and $(1 - q)^{1-z}$. Then $\Gamma_q(z)$ is a meromorphic function with poles at $z = -n, n \in \mathbb{N}$.

Let $\mu \in \mathbb{R}$ be fixed. A set $A \subseteq \mathbb{R}$ is called a μ -geometric set if for $x \in A, \mu x \in A$. If f is a function defined on a q -geometric set $A \subseteq \mathbb{R}$, the q -difference operator, D_q , is defined by

$$D_q f(x) := \frac{f(x) - f(qx)}{x - qx}, \quad x \in A / \{0\}. \tag{1.5}$$

If $0 \in A$, we say that f has q -derivative at zero if

$$\lim_{n \rightarrow \infty} \frac{f(xq^n) - f(0)}{xq^n}, \quad x \in A, \tag{1.6}$$

exists and does not depend on x . In this case, we shall denote this limit by $D_q f(0)$. In some literature the q -derivative at zero is defined to be $f'(0)$ if it exists, *cf.* [30, 31], but the above

definition is more suitable for our approach. The non-symmetric Leibniz rule

$$D_q(fg)(x) = g(x)D_qf(x) + f(qx)D_qg(x) \tag{1.7}$$

holds. Equation (1.7) can be symmetrized using the relation $f(qx) = f(x) - x(1 - q)D_qf(x)$, giving the additional term $-x(1 - q)D_qf(x)D_qg(x)$. The q -integration of Jackson [32] is defined for a function f defined on a q -geometric set A to be

$$\int_a^b f(t) d_qt := \int_0^b f(t) d_qt - \int_0^a f(t) d_qt, \quad a, b \in A, \tag{1.8}$$

where

$$\int_0^x f(t) d_qt := \sum_{n=0}^{\infty} xq^n(1 - q)f(xq^n), \quad x \in A, \tag{1.9}$$

provided that the series converges. A function f defined on X is called q -regular at zero if

$$\lim_{n \rightarrow \infty} f(xq^n) = f(0) \quad \text{for all } x \in X.$$

Let $C(X)$ denote the space of all q -regular at zero functions defined on X with values in \mathbb{R} . $C(X)$ associated with the norm function

$$\|f\| = \sup\{|f(xq^n)| : x \in X, n \in \mathbb{N}_0\},$$

is a normed space. The q -integration by parts rule [33] is

$$\int_a^b f(x)D_qg(x) = f(x)g(x)|_a^b + \int_a^b D_qf(x)g(qx) d_qx, \quad a, b \in X, \tag{1.10}$$

where f, g are q -regular at zero functions.

For $p > 0$, and Y is $A_{q,t}$ or $A_{q,t}^*$, the space $L_q^p(Y)$ is the normed space of all functions defined on Y such that

$$\|f\|_p := \left(\int_0^t |f(u)|^p d_qu \right)^{1/p} < \infty.$$

If $p = 2$, then $L_q^2(Y)$ associated with the inner product

$$\langle f, g \rangle := \int_0^t f(u)\overline{g(u)} d_qu \tag{1.11}$$

is a Hilbert space. A weighted $L_q^2(Y, w)$ space is the space of all functions f defined on Y , such that

$$\int_0^t |f(u)|^2 w(u) d_qu < \infty,$$

where w is a positive function defined on Y . $L_q^2(Y, w)$ associated with the inner product

$$\langle f, g \rangle := \int_0^t f(u) \overline{g(u)} w(u) d_q u$$

is a Hilbert space. The space of all q -absolutely functions on $A_{q,t}^*$ is denoted by $\mathcal{AC}_q(A_{q,t}^*)$ and defined as the space of all q -regular at zero functions f satisfying

$$\sum_{j=0}^{\infty} |f(uq^j) - f(uq^{j+1})| \leq K \quad \text{for all } u \in A_{q,t}^*,$$

and K is a constant depending on the function f , cf. [33], Definition 4.3.1. That is,

$$\mathcal{AC}_q(A_{q,t}^*) \subseteq C_q(A_{q,t}^*).$$

The space $\mathcal{AC}_q^{(n)}(A_{q,t}^*)$ ($n \in \mathbb{N}$) is the space of all functions defined on X such that $f, D_q f, \dots, D_q^{n-1} f$ are q -regular at zero and $D_q^{n-1} f \in \mathcal{AC}_q(A_{q,t}^*)$, cf. [33], Definition 4.3.2. Also it has been proved in [33], Theorem 4.6, that a function $f \in \mathcal{AC}_q^{(n)}(A_{q,t}^*)$ if and only if there exists a function $\phi \in L_q^1(A_{q,t}^*)$ such that

$$f(x) = \sum_{k=0}^{n-1} \frac{D_q^k f(0)}{\Gamma_q(k+1)} x^k + \frac{x^{n-1}}{\Gamma_q(n)} \int_0^x (qu/x; q)_{n-1} \phi(u) d_q u, \quad x \in A_{q,t}^*.$$

In particular, $f \in \mathcal{AC}(A_{q,t}^*)$ if and only if f is q -regular at zero such that $D_q f \in L_q^1(A_{q,t}^*)$. It is worth noting that in [33], all the definitions and results we have just mentioned are defined and proved for functions defined on the interval $[0, a]$ instead of $A_{q,t}^*$. In [34], Mansour studied the problem

$$D_{q,a}^\alpha p(x) {}^c D_{q,0^+}^\alpha y(x) + (r(x) - \lambda w_\alpha(x)) y(x) = 0, \quad x \in A_{q,a}^*, \tag{1.12}$$

where $p(x) \neq 0$ and $w_\alpha > 0$ for all $x \in A_{q,a}^*$, p, r, w_α are real valued functions defined in $A_{q,a}^*$ and the associated boundary conditions are

$$c_1 y(0) + c_2 [I_{q,a}^{1-\alpha} p {}^c D_{q,0^+}^\alpha y](0) = 0, \tag{1.13}$$

$$d_1 y(a) + d_2 [I_{q,a}^{1-\alpha} p {}^c D_{q,0^+}^\alpha y] \left(\frac{a}{q} \right) = 0, \tag{1.14}$$

with $c_1^2 + c_2^2 \neq 0$ and $d_1^2 + d_2^2 \neq 0$. it is proved that the eigenvalues are real and the eigenfunctions associated to different eigenvalues are orthogonal in the Hilbert space $L_q^2(A_{q,a}^*, w_\alpha)$. A sufficient condition on the parameter λ to guarantee the existence and uniqueness of the solution is introduced by using the fixed point theorem, also a condition is imposed on the domain of the problem in order to prove the existence and uniqueness of solution for any λ . This paper is organized as follows. Section 2 is on the q -fractional operators and their properties which we need in the sequel. Cardoso [35] introduced basic Fourier series for functions defined on a q -linear grid of the form $\{\pm q^n : n \in \mathbb{N}_0\} \cup \{0\}$. In Section 3, we reformulate Cardoso's results for functions defined on a q -linear grid of the

form $\{\pm aq^n : n \in \mathbb{N}_0\} \cup \{0\}$. In Section 4, we introduce a fractional q -analog for Euler-Lagrange equations for functionals defined in terms of Jackson q -integration and the integrand contains the left-sided Caputo fractional q -derivative. We also introduce a fractional q -isoperimetric problem. In Section 5, we use the variational q -calculus developed in Section 4 to prove the existence of a countable number of eigenvalues and orthogonal eigenfunctions for the fractional q -Sturm-Liouville problem with the boundary condition $y(0) = y(a) = 0$. We also define the Rayleigh quotient and prove a criterion for the smallest eigenvalue.

2 Fractional q -calculus

This section includes the definitions and properties of the left-sided and right-sided Riemann-Liouville q -fractional operators which we need in our investigations.

The left-sided Riemann-Liouville q -fractional operator is defined by

$$I_{q,a^+}^\alpha f(x) = \frac{x^{\alpha-1}}{\Gamma_q(\alpha)} \int_a^x (qt/x; q)_{\alpha-1} f(t) d_q t. \tag{2.1}$$

This definition was introduced by Agarwal in [36] when $a = 0$ and by Rajković *et al.* [37] for $a \neq 0$. The right-sided Riemann-Liouville q -fractional operator is defined by

$$I_{q,b^-}^\alpha f(x) = \frac{1}{\Gamma_q(\alpha)} \int_{qx}^b t^{\alpha-1} (qx/t; q)_{\alpha-1} f(t) d_q t; \tag{2.2}$$

see [34]. The left-sided Riemann-Liouville q -fractional operator satisfies the semigroup property

$$I_{q,a^+}^\alpha I_{q,a^+}^\beta f(x) = I_{q,a^+}^{\alpha+\beta} f(x).$$

The case $a = 0$ is proved in [36], while the case $a > 0$ is proved in [37].

The right-sided Riemann-Liouville q -fractional operator satisfies the semigroup property [34]

$$I_{q,b^-}^\alpha I_{q,b^-}^\beta f(x) = I_{q,b^-}^{\alpha+\beta} f(x), \quad x \in A_{q,b}^* \tag{2.3}$$

for any function defined on $A_{q,b}$ and for any values of α and β .

For $\alpha > 0$ and $\lceil \alpha \rceil = m$, the left- and right-sided Riemann-Liouville fractional q -derivatives of order α are defined by

$$D_{q,a^+}^\alpha f(x) := D_q^m I_{q,a^+}^{m-\alpha} f(x), \quad D_{q,b^-}^\alpha f(x) := \left(\frac{-1}{q}\right)^m D_{q^{-1}}^m I_{q,b^-}^{m-\alpha} f(x),$$

the left- and right-sided Caputo fractional q -derivatives of order α are defined by

$${}^c D_{q,a^+}^\alpha f(x) := I_{q,a^+}^{m-\alpha} D_q^m f(x), \quad {}^c D_{q,b^-}^\alpha := \left(\frac{-1}{q}\right)^m I_{q,b^-}^{m-\alpha} D_{q^{-1}}^m f(x);$$

see [34]. From now on, we shall consider left-sided Riemann-Liouville and Caputo fractional q -derivatives when the lower point $a = 0$ and right-sided Riemann-Liouville and

Caputo fractional q -derivatives when $b = a$. According to [33], pp.124, 148, $D_{q,0^+}^\alpha f(x)$ exists if

$$f \in L^1_q(A_{q,a}^*) \quad \text{such that} \quad I_{q,0^+}^{m-\alpha} f \in \mathcal{AC}_q^{(m)}(A_{q,a}^*),$$

and ${}^c D_{q,a^+}^\alpha f$ exists if

$$f \in \mathcal{AC}_q^{(m)}(A_{q,a}^*).$$

The following proposition was proved in [34] but we add the proof here for convenience of the reader.

Proposition 2.1 *Let $\alpha \in (0, 1)$.*

(i) *If $f \in L^1_q(A_{q,a}^*)$ such that $I_{q,0^+}^\alpha f \in \mathcal{AC}_q(A_{q,a}^*)$ then*

$${}^c D_{q,0^+}^\alpha I_{q,0^+}^\alpha f(x) = f(x) - \frac{I_{q,0^+}^\alpha f(0)}{\Gamma_q(1-\alpha)} x^{-\alpha}. \tag{2.4}$$

Moreover, if f is bounded on $A_{q,a}^$ then*

$${}^c D_{q,0^+}^\alpha I_{q,0^+}^\alpha f(x) = f(x). \tag{2.5}$$

(ii) *For any function f defined on $A_{q,a}^*$,*

$${}^c D_{q,a^-}^\alpha I_{q,a^-}^\alpha f(x) = f(x) - \frac{a^{-\alpha}}{\Gamma_q(1-\alpha)} (qx/a; q)_{-\alpha} (I_{q,a^-}^\alpha f) \left(\frac{a}{q} \right). \tag{2.6}$$

(iii) *If $f \in L^1_q(A_{q,a})$ then*

$$D_{q,0^+}^\alpha I_{q,0^+}^\alpha f(x) = f(x). \tag{2.7}$$

(iv) *For any function f defined on $A_{q,a}^*$,*

$$D_{q,a^-}^\alpha I_{q,a^-}^\alpha f(x) = f(x). \tag{2.8}$$

(v) *If $f \in \mathcal{AC}_q(A_{q,a}^*)$ then*

$$I_{q,0^+}^\alpha {}^c D_{q,0^+}^\alpha f(x) = f(x) - f(0). \tag{2.9}$$

(vi) *If f is a function defined on $A_{q,a}^*$ then*

$$I_{q,a^-}^\alpha D_{q,a^-}^\alpha f(x) = f(x) - \frac{a^{\alpha-1}}{\Gamma_q(\alpha)} (qx/a; q)_{\alpha-1} (I_{q,a^-}^{1-\alpha} f) \left(\frac{a}{q} \right). \tag{2.10}$$

(vii) *If f is defined on $[0, a]$ such that $D_q f$ is continuous on $[0, a]$ then*

$${}^c D_{q,0^+}^\alpha f(x) = D_{q,0^+}^\alpha [f(x) - f(0)]. \tag{2.11}$$

Proof The proof of (2.4) is a special case of [33], Eq. (5.7), but note that there is a misprint in Eq. (5.7); the summation should start from $i = 1$. If f is bounded on $A_{q,a}$, then $I_{q,0^+}^{1-\alpha} f(0) = 0$, and (2.5) follows at once from (2.4). Now we prove (2.6). We have

$${}^c D_{q,a}^\alpha - I_{q,a}^\alpha f(x) = \frac{1}{\Gamma_q(1-\alpha)} \int_{qx}^a t^{-\alpha} (qx/t; q)_{-\alpha} D_q(I_{q,a}^\alpha f) \left(\frac{t}{q}\right) d_q t,$$

where we used $-\frac{1}{q} D_{q^{-1}} f(x) = D_{q,x} f(\frac{x}{q})$. Then applying the q -integration by parts formula (1.10) and using

$$D_{q,t} t^\beta (qx/t; q)_\beta = -[\beta] t^{\beta-1} (qx/t; q)_{\beta-1}, \quad \beta \in \mathbb{R}, [\beta] := \frac{1-q^\beta}{1-q},$$

we obtain

$${}^c D_{q,a}^\alpha - I_{q,a}^\alpha f(x) = \frac{a^{-\alpha}}{\Gamma_q(1-\alpha)} (qx/a; q)_{-\alpha} (I_{q,a}^{1-\alpha} f) \left(\frac{a}{q}\right) - I_{q,a}^\alpha - I_{q,a}^\alpha f(x).$$

Hence, the result follows from the semigroup property (2.3). Equation (2.7) was proved in [33], Eq. (4.66). The proof of (2.8) follows from the fact that

$$D_{q,a}^\alpha - I_{q,a}^\alpha f(x) = -\frac{1}{q} I_{q,a}^{1-\alpha} - I_{q,a}^\alpha f(x) = -\frac{1}{q} D_{q^{-1}} I_{q,a} f(x) = f(x),$$

where we used the semigroup property (2.3). The proof of (2.9) is a special case of [33], Eq. (5.6). The proof of (2.10) is similar to the proof of (2.8) and is omitted. Finally, the proof of (2.11) is a special case of [33], Eq. (5.8). □

Set $X = A_{q,a}$ or $A_{q,a}^*$. Then

$$C(X) \subseteq L_q^2(X) \subseteq L_q^1(X).$$

Moreover, if $f \in C(X)$ then

$$\|f\|_1 \leq \sqrt{a} \|f\|_2 \leq a \|f\|.$$

We also have the following inequalities:

1. If $f \in C(A_{q,a}^*)$ then $I_{q,0^+}^\alpha f \in C(A_{q,a}^*)$ and

$$\|I_{q,0^+}^\alpha f\| \leq \frac{a^\alpha}{\Gamma_q(\alpha + 1)} \|f\|. \tag{2.12}$$

2. If $f \in L_q^1(X)$ then $I_{q,0^+}^\alpha f \in L_q^1(X)$ and

$$\|I_{q,0^+}^\alpha f\|_1 \leq M_{\alpha,1} \|f\|_1, \quad M_{\alpha,1} := \frac{a^\alpha (1-q)^\alpha}{(1-q^\alpha)(q; q)_\infty}. \tag{2.13}$$

3. If $f \in L_q^2(X)$ then $I_{q,0^+}^\alpha f \in L_q^2(X)$ and

$$\|I_{q,0^+}^\alpha f\|_2 \leq M_{\alpha,2} \|f\|_2, \tag{2.14}$$

where

$$M_{\alpha,2} := \frac{a^\alpha}{\Gamma_q(\alpha)} \sqrt{\frac{(1-q)}{(1-q^{2\alpha})}} \left(\int_0^1 (q\xi; q)_{\alpha-1}^2 d_q \xi \right)^{1/2}.$$

4. If $\alpha > \frac{1}{2}$ and $f \in L_q^2(X)$ then $I_{q,0^+}^\alpha f \in C(X)$ and

$$\|I_{q,0^+}^\alpha f\| \leq \tilde{M}_\alpha \|f\|, \quad \tilde{M}_\alpha := \frac{a^{\alpha-\frac{1}{2}}}{\Gamma_q(\alpha)} \left(\int_0^1 (q\xi; q)_{\alpha-1}^2 d_q \xi \right)^{1/2}. \tag{2.15}$$

5. Since $\|f\|_2 \leq \sqrt{a} \|f\|$, we conclude that if $f \in C(X)$ then $I_{q,0^+}^\alpha f \in L_q^2(X)$ and

$$\|I_{q,0^+}^\alpha f\|_2 \leq K_\alpha \|f\|, \quad K_\alpha := \sqrt{a} M_{\alpha,2}. \tag{2.16}$$

6. If $f \in C(A_{q,a}^*)$ then $I_{q,a^-}^\alpha f \in C(A_{q,a}^*)$ and

$$\|I_{q,a^-}^\alpha f\| \leq c_{\alpha,0} \|f\|, \quad c_{\alpha,0} := \frac{a^\alpha (1-q)^\alpha}{(1-q^\alpha)(q; q)_\infty}.$$

7. If $f \in L_q^1(X)$ then $I_{q,a^-}^\alpha f \in L_q^1(X)$ and

$$\|I_{q,a^-}^\alpha f\|_1 \leq \begin{cases} \frac{(1-q)^\alpha a^\alpha}{(1-q^\alpha)(q; q)_\infty} \|f\|_1, & \text{if } \alpha < 1, \\ \frac{(1-q)^{\alpha-1} a^{\alpha-1}}{(q; q)_\infty} \|f\|_1, & \text{if } \alpha \geq 1. \end{cases}$$

8. If $\alpha \neq \frac{1}{2}$ and $f \in L_q^2(X)$ then $I_{q,a^-}^\alpha f \in L_q^1(X)$ and

$$\|I_{q,a^-}^\alpha f\|_2 \leq \begin{cases} \frac{(1-q)^{\alpha-\frac{1}{2}} a^\alpha}{\sqrt{1-q^{2\alpha-1}}(q; q)_\infty} \|f\|_2, & \text{if } \alpha < \frac{1}{2}, \\ \frac{(1-q)^\alpha a^\alpha}{(q; q)_\infty \sqrt{(1-q^{2\alpha-1})(1-q^{2\alpha})}} \|f\|_2, & \text{if } \alpha > \frac{1}{2}. \end{cases}$$

The following lemmas are needed in the remaining sections.

Lemma 2.2 *Let $\alpha > 0$. If*

(a) $f \in L_q^1(X)$ and g is a bounded function on $A_{q,a}$,

or

(b) $\alpha \neq \frac{1}{2}$ and f, g are $L_q^2(X)$ functions,

then

$$\int_0^a g(x) I_{q,0^+}^\alpha f(x) d_q x = \int_0^a f(x) I_{q,a^-}^\alpha g(x) d_q x. \tag{2.17}$$

Proof The condition (a) or (b) of the present lemma ensures the convergence of the q -integrals in (2.17). Since

$$\int_0^a g(x) I_{q,0^+}^\alpha f(x) d_q x = \frac{1}{\Gamma_q(\alpha)} \int_0^a g(x) x^{\alpha-1} \int_0^x (qt/x; q)_{\alpha-1} f(t) d_q t d_q x,$$

from the conditions on the functions f and g , the double q -integral is absolutely convergent, therefore we can interchange the order of the q -integrations to obtain

$$\begin{aligned} \int_0^a g(x) I_{a^+}^\alpha f(x) d_q x &= \int_0^a f(t) \frac{1}{\Gamma_q(\alpha)} \int_{qt}^a x^{\alpha-1} (qt/x; q)_{\alpha-1} g(x) d_q x d_q t \\ &= \int_0^a f(t) I_{q, a^-}^\alpha g(t) d_q t. \end{aligned} \quad \square$$

Lemma 2.3 *Let $\alpha \in (0, 1)$.*

(a) *If $g \in L^1_q(A^*_{q,a})$ such that $I^{1-\alpha}_q g \in AC_q(A^*_{q,a})$, and $D^i_q f \in C(A^*_{q,a})$ ($i = 0, 1$) then*

$$\int_0^a f(x) D^{\alpha}_{q,0^+} g(x) d_q x = -f\left(\frac{x}{q}\right) I^{1-\alpha}_{q,0^+} g(x) \Big|_{x=0}^a + \int_0^a g(x) {}^c D^{\alpha}_{q,a^-} f(x) d_q x. \quad (2.18)$$

(b) *If $f \in AC_q(A^*_{q,a})$, and g is a bounded function on $A^*_{q,a}$ such that $D^{\alpha}_{q,a^-} g \in L^1_q(A^*_{q,a})$ then*

$$\int_0^a g(x) {}^c D^{\alpha}_{q,0^+} f(x) d_q x = (I^{1-\alpha}_q g) \left(\frac{x}{q}\right) f(x) \Big|_{x=0}^a + \int_0^a f(x) D^{\alpha}_{q,a^-} g(x) d_q x. \quad (2.19)$$

Proof The conditions on the functions f and g guarantee the convergence of the q -integrals in (2.18) and (2.19), and their proofs follow from Lemma 2.2 and the q -integration by parts rule (1.10). □

3 Basic Fourier series on q -linear grid and some properties

The purpose of this section is to reformulate Cardoso’s results of Fourier series expansions for functions defined on the q -linear grid $\mathcal{A}_q := \{q^n, n \in \mathbb{N}_0\}$ to functions defined on q -linear grids $\mathcal{A}_{q,a} := \{\pm aq^n, n \in \mathbb{N}_0\}$, $a > 0$.

Cardoso in [35] defined the space of all q -linear Hölder functions on the q -linear grid \mathcal{A}_q . We generalize his definition for functions defined on a q -linear grid of the form $\mathcal{A}_{q,a}$, $a > 0$.

Definition 3.1 A function f defined on $\mathcal{A}_{q,a}$, $a > 0$, is called a q -linear Hölder of order λ if there exists a constant $M > 0$ such that

$$|f(\pm aq^{n-1}) - f(\pm aq^n)| \leq Mq^{n\lambda} \quad \text{for all } n \in \mathbb{N}.$$

Definition 3.2 The q -trigonometric functions $S_q(z)$ and $C_q(z)$ are defined for $z \in \mathbb{C}$ by [35, 38]

$$\begin{aligned} S_q(z) &= \sum_{n=0}^{\infty} (-1)^n \frac{q^{n(n+\frac{1}{2})} z^{2n+1}}{(q; q)_{2n+1}} = \frac{z}{1-q} {}_1\phi_1(0; q^3; q^2, q^{3/2} z^2), \\ C_q(z) &= \sum_{n=0}^{\infty} (-1)^n \frac{q^{n(n-\frac{1}{2})} z^{2n}}{(q; q)_{2n}} = {}_1\phi_1(0; q; q^2, q^{1/2} z^2). \end{aligned}$$

One can verify that

$$D_{q,z} S_q(wz) = \frac{w}{1-q} C_q(\sqrt{q} wz),$$

$$D_{q,z}C_q(wz) = -\frac{w}{1-q}S_q(\sqrt{q}wz),$$

where $z \in \mathbb{C}$ and $w \in \mathbb{C}$ is a fixed parameter. A modification of the orthogonality relation given in [38], Theorem 4.1, is the following.

Theorem 3.3 *Let w and w' be roots of $S_q(z)$, and $\mu(w) := (1 - q)C_q(q^{1/2}w)S'_q(w)$. Then*

$$\int_{-a}^a C_q\left(\frac{q^{\frac{1}{2}}wx}{a}\right)C_q\left(\frac{q^{\frac{1}{2}}w'x}{a}\right)d_qx = \begin{cases} 0, & \text{if } w \neq w', \\ 2a, & \text{if } w = w' = 0, \\ a\mu(w), & \text{if } w = w' \neq 0, \end{cases}$$

$$\int_{-a}^a S_q\left(\frac{qw_x}{a}\right)S_q\left(\frac{qw'_x}{a}\right)d_qx = \begin{cases} 0, & \text{if } w \neq w', \\ aq^{-1/2}\mu(w), & \text{if } w = w'. \end{cases}$$

Cardoso introduced a sufficient condition for the uniform convergence of the basic Fourier series

$$S_q(f) := \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k C_q(q^{1/2}w_k x) + b_k S_q(qw_k x),$$

where $a_0 = \int_{-1}^1 f(t) d_q t$ and for $k = 1, 2, \dots$,

$$a_k = \frac{1}{\mu_k} \int_{-1}^1 f(t) C_q(q^{1/2}w_k t) d_q t, \quad b_k = \frac{1}{\mu_k} \int_{-1}^1 f(t) S_q(qw_k t) d_q t,$$

$$\mu_k = (1 - q)C_q(q^{1/2}w_k)S'_q(w_k)$$

on the q -linear grid \mathcal{A}_q , where $\{w_k : k \in \mathbb{N}\}$ is the set of positive zeros of $S_q(z)$. Cardoso proved that $\mu_k = O(q^{-2k^2})$ as $k \rightarrow \infty$ for any $q \in (0, 1)$. In the following we give a modified version of Cardoso's result for any function defined on the q -linear grid $\mathcal{A}_{q,a}$, $a > 0$.

Theorem 3.4 *If $f \in C(\mathcal{A}_{q,a}^*)$ is a q -linear Hölder function of order $\lambda > \frac{1}{2}$, then the q -Fourier series*

$$S_q(f) := \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k C_q\left(q^{1/2} \frac{w_k x}{a}\right) + b_k S_q\left(q \frac{w_k x}{a}\right), \tag{3.1}$$

where $a_0 = \frac{1}{a} \int_{-a}^a f(t) d_q t$ and, for $k = 1, 2, \dots$,

$$a_k(f) = \frac{1}{a\mu_k} \int_{-a}^a f(t) C_q\left(q^{1/2} \frac{w_k t}{a}\right) d_q t, \quad b_k(f) = \frac{\sqrt{q}}{a\mu_k} \int_{-a}^a f(t) S_q\left(q \frac{w_k t}{a}\right) d_q t,$$

converges uniformly to the function f on the q -linear grid $\mathcal{A}_{q,a}$.

Proof The proof is a modification of the proof of [35], Theorem 4.1, and is omitted. \square

Remark 3.5 We replaced the condition

$$f(0^+) = f(0^-), \tag{3.2}$$

where

$$f(0^+) := \lim_{x \rightarrow 0^+} f(x), \quad f(0^-) := \lim_{x \rightarrow 0^-} f(x),$$

in [35], Theorem 4.1, by the weakest condition that f is q -regular at zero because (3.2) is only needed to guarantee that $\lim_{n \rightarrow \infty} f(q^{n-1/2}) = \lim_{n \rightarrow \infty} f(-q^{n-1/2})$ and this holds if f is q -regular at zero. See [33], Eq. (1.22), for a function which is q -regular at zero but not continuous at zero.

A modified version of [35], Theorem 3.5, is the following.

Theorem 3.6 *If there exists $c > 1$ such that*

$$\int_{-a}^a f(t) C_q \left(\sqrt{q} \frac{w_k t}{a} \right) = O(q^{ck}) \quad \text{and} \quad \int_{-a}^a f(t) S_q \left(q \frac{w_k t}{a} \right) = O(q^{ck}) \quad \text{as } k \rightarrow \infty,$$

then the q -Fourier series (3.1) converges uniformly on $\mathcal{A}_{q,a}$.

A modified version of [35], Corollary 4.3, is the following.

Corollary 3.7 *If f is continuous and piecewise smooth on a neighborhood of the origin, then the corresponding q -Fourier series $S_q(f)$ converges uniformly to f on the q -linear grid $\mathcal{A}_{q,a}$.*

Theorem 3.8 *If $f \in C(\mathcal{A}_{q,a}^*)$ is a q -linear Hölder odd function of order $\lambda > \frac{1}{2}$ and satisfying $f(0) = f(a) = 0$, then the q -Fourier series*

$$S_q(f) := \sum_{k=1}^{\infty} c_k S_q \left(\frac{w_k x}{a} \right),$$

where

$$c_k(f) = c_k = \frac{2}{a \sqrt{q} \mu_k} \int_0^a f(t) S_q \left(\frac{w_k t}{a} \right) d_q t,$$

converges uniformly to the function f on the q -linear grid $\mathcal{A}_{q,a}$.

Proof The proof follows from (3.4) by considering the function $g(x) := f(qx)$, $x \in \mathcal{A}_{q,a}$. Since it is odd, we have $a_k = 0$ for $k = 0, 1, \dots$, and

$$b_k(f) = \sqrt{q} \mu_k \int_{-a}^a g(t) S_q \left(\frac{q w_k t}{a} \right) d_q t,$$

making the substitution $u = qt$ and using the fact that g is an odd function, we obtain the required result. □

Definition 3.9 Let $(f_n)_n$ be a sequence of functions in $C(\mathcal{A}_{q,a}^*)$. We say that f_n converges to a function f in q -mean if

$$\lim_{n \rightarrow \infty} \sqrt{\int_{-a}^a |f_n(x) - f(x)|^2 d_q x} = 0.$$

Proposition 3.10 *If $g \in C(A_{q,a}^*)$ is an odd function satisfying $D_q^k g$ ($k = 0, 1, 2$) is a continuous and piecewise smooth function in a neighborhood of zero, satisfying the boundary condition*

$$g(0) = g(a) = 0, \tag{3.3}$$

then g can be approximated in the q -mean by a linear combination

$$g_n(x) = \sum_{r=0}^n c_r^{(n)} S_q \left(\frac{w_r x}{a} \right),$$

where at the same time $D_q^k g_n$ ($k = 1, 2$) converges in q -mean to the $D_q^k g$. Moreover, the coefficients $c_r^{(n)}$ need not depend on n and can be written simply as c_r .

Proof We consider the q -sine Fourier transform of $D_q^2 g$. Hence

$$D_q^2 g(x) = \sum_{k=1}^{\infty} b_k S_q \left(\frac{q w_k x}{a} \right) = \lim_{n \rightarrow \infty} \gamma_n(x), \quad x \in A_{q,a}, \tag{3.4}$$

where

$$\gamma_n(x) = \sum_{k=1}^n b_k S_q \left(\frac{q w_k x}{a} \right), \quad b_k = \frac{\sqrt{q}}{a \mu_k} \int_0^a D_q^2 g(x) S_q \left(\frac{q w_k x}{a} \right) d_q x.$$

Consequently,

$$\lim_{n \rightarrow \infty} \int_0^a |D_q^2 g(x) - \gamma_n(x)|^2 d_q x = 0.$$

Hence

$$D_q g(x) - D_q g(0) = \int_0^x D_q^2 g(x) d_q x = \frac{a(1-q)}{\sqrt{q}} \sum_{k=1}^{\infty} \frac{b_k}{w_k} \left(-C_q \left(\frac{q^{1/2} w_k x}{a} \right) + 1 \right).$$

Applying the q -integration by parts rule (1.10) gives

$$a_k(D_q g) = -\frac{a(1-q)}{\sqrt{q} w_k} b_k(D_q^2 g).$$

That is,

$$D_q g(x) - D_q g(0) = \sum_{k=1}^{\infty} a_k(D_q g) \left(C_q \left(\frac{q^{1/2} w_k x}{a} \right) - 1 \right).$$

Hence

$$D_q g(x) = \sum_{k=1}^{\infty} a_k(D_q g) C_q \left(\frac{q^{1/2} w_k x}{a} \right), \quad x \in A_{q,a}^*. \tag{3.5}$$

Note that $a_0(D_qg) = 0$ because $g(0) = g(a) = 0$. Again by q -integrating the two sides of (3.5), we obtain

$$g(x) = \sum_{k=1}^{\infty} a_k(D_qg) \frac{a(1-q)}{w_k} S_q\left(\frac{w_k x}{a}\right), \quad x \in A_{q,a}^*. \tag{3.6}$$

One can verify that

$$b_k(g) = \frac{a(1-q)}{w_k} a_k(D_qg).$$

Hence the right-hand sides of (3.5) and (3.6) are the q -Fourier series of D_qg and g , respectively. Hence the convergence is uniform in $C(\mathcal{A}_{q,a}^*)$ and $L_{q^2}(\mathcal{A}_{q,a}^*)$ norms. \square

4 q -Fractional variational problems

The calculus of variations is as old as the calculus itself, and has many applications in physics and mechanics. As the calculus has two forms, the continuous calculus with the power concept of limits, and the discrete calculus which also is called the calculus of finite differences, the calculus of variations has also both the discrete and the continuous forms. For a brief history of the continuous calculus of variations, see [39]. The discrete calculus of variations started in 1948 by Fort in his book [40] where he devoted a chapter to the finite analog of the calculus of variations, and he introduced a necessary condition analog to the Euler equation and also a sufficient condition. The paper of Cadzow [41], 1969, was the first paper published in this field, then Logan developed the theory in his PhD thesis [42], 1970, and in a series of papers [43–46]. See also the PhD thesis of Harmsen [47] for a brief history for the discrete variational calculus; and for the developments in the theory, see [48–54]. In 2004, a q -version of the discrete variational calculus is introduced by Bangerezako in [55] for functions defined in the form

$$J(y(x)) = \int_{q^\alpha}^{q^\beta} xF(x, y(x), D_qy(x), \dots, D_q^k y(x)) d_qx,$$

where q^α and q^β are in the uniform lattice $A_{q,a}^*$ for some $a > 0$ such that $\alpha > \beta$, provided that the boundary conditions

$$D_q^j y(q^\alpha) = D_q^j y(q^{\beta+1}) = c_j \quad (j = 0, 1, \dots, k - 1).$$

He introduced a q -analog of the Euler-Lagrange equation which he applied to solve certain isoperimetric problem. Then, in 2005, Bangerezako [56] introduced certain q -variational problems on a nonuniform lattice. In [57, 58], Malinowska, and Torres introduced the Hahn quantum variational calculus. They derived the Euler-Lagrange equation associated with the variational problem

$$J(y) = \int_a^b F(t, y(qt + w), D_{q,w}y(t)) d_{q,w}t,$$

under the boundary condition $y(a) = \alpha, y(b) = \beta$ where α and β are constants and $D_{q,w}$ is the Hahn difference operator defined by

$$D_{q,w}f(t) = \begin{cases} \frac{f(qt+w)-f(t)}{(qt+w)-t}, & \text{if } t \neq \frac{w}{1-q}, \\ f'(0), & \text{if } t = \frac{w}{1-q}. \end{cases}$$

Problems of the classical calculus of variations with integrand depending on fractional derivatives instead of ordinary derivatives are first introduced by Agrawal [17] in 2002. Then he extended his result for variational problems including Riesz fractional derivatives in [18]. Numerous works have been dedicated to the subject since Agrawal’s work. See for example [5, 13–16, 59–61].

In this section, we shall derive Euler-Lagrange equation for a q -variational problem when the integrand includes a left-sided q -Caputo fractional derivative and we also solve a related isoperimetric problem. From now on, we fix $\alpha \in (0, 1)$, and define a subspace of $C(A_{q,a}^*)$ by

$${}_0E_a^\alpha = \{y \in \mathcal{AC}(A_{q,a}^*) : {}^cD_{q,0^+}^\alpha y \in C(A_{q,a}^*)\},$$

and the space of variations ${}^c\text{Var}(0, a)$ for the Caputo q -derivative by

$${}^c\text{Var}(0, a) = \{h \in {}_0E_a^\alpha : h(0) = h(a) = 0\}.$$

For a function $f(x_1, x_2, \dots, x_n)$ ($n \in \mathbb{N}$) by $\partial_i f$ we mean the partial derivative of f with respect to the i th variable, $i = 1, 2, \dots, n$. In the sequel, we shall need the following definition from [62].

Definition 4.1 Let $A \subseteq \mathbb{R}$ and $g : A \times]-\bar{\theta}, \bar{\theta}[\rightarrow \mathbb{R}$. We say that $g(t, \cdot)$ is continuous at θ_0 uniformly in t , if and only if $\forall \epsilon > 0, \exists \delta > 0$ such that

$$|\theta - \theta_0| < \delta \implies |g(t, \theta) - g(t, \theta_0)| < \epsilon \quad \text{for all } t \in A.$$

Furthermore, we say that $g(t, \cdot)$ is differentiable at θ_0 uniformly in t if and only if $\forall \epsilon > 0, \exists \delta > 0$ such that

$$|\theta - \theta_0| < \delta \implies \left| \frac{g(t, \theta) - g(t, \theta_0)}{\theta - \theta_0} - \delta_2 g(t, \theta_0) \right| < \epsilon \quad \text{for all } t \in A.$$

We now present first order necessary conditions of optimality for functionals, defined on ${}_0E_a^\alpha$, of the type

$$J(y) = \int_0^a F(x, y, {}^cD_{q,0^+}^\alpha y) d_q x, \quad 0 < \alpha < 1, \tag{4.1}$$

where $F : A_{q,a}^* \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is a given function. We assume that:

1. The functions $(u, v) \rightarrow F(t, u, v)$ and $(u, v) \rightarrow \partial_i F(t, u, v)$ ($i = 2, 3$) are continuous functions uniformly on $A_{q,a}$.
2. $F(\cdot, y(\cdot), {}^cD_{q,0^+}^\alpha(\cdot)), \delta_i F(\cdot, y(\cdot), {}^cD_{q,0^+}^\alpha(\cdot))$ ($i = 2, 3$) are q -regular at zero.

- 3. $\delta_3 F$ has a right Riemann-Liouville fractional q -derivative of order α which is q -regular at zero.

Definition 4.2 Let $y_0 \in {}_0E_a^\alpha$. Then J has a local maximum at y_0 if

$$\exists \delta > 0 \text{ such that } J(y) \leq J(y_0) \text{ for all } y \in {}_0E_a^\alpha \text{ with } \|y - y_0\| < \delta,$$

and J has a local minimum at y_0 if

$$\exists \delta > 0 \text{ such that } J(y) \geq J(y_0) \text{ for all } y \in S \text{ with } \|y - y_0\| < \delta.$$

J is said to have a local extremum at y_0 if it has either a local maximum or local minimum.

Lemma 4.3 Let $\gamma \in L_q^2(A_{q,a}^*)$.

(i) If

$$\int_0^a \gamma(x)h(x) d_q x = 0 \tag{4.2}$$

for every $h \in L_q^2(A_{q,a})$ then

$$\gamma(x) \equiv 0 \text{ on } A_{q,a}. \tag{4.3}$$

(ii) If (4.2) holds only for all functions $h \in L_q^2(A_{q,a}^*)$ satisfying $h(a) = 0$ then

$$\gamma(x) \equiv 0 \text{ on } A_{q,qa}. \tag{4.4}$$

Moreover, in the two cases, if γ is q -regular at zero, then $\gamma(0) = 0$.

Proof To prove (i), we fix $k \in \mathbb{N}_0$ and set $h_k(x) = \begin{cases} 1, & x = aq^k, \\ 0, & \text{otherwise.} \end{cases}$ Then $h_k \in L_q^2(0, a)$. Substituting in (4.2) yields

$$aq^k(1 - q)\gamma(aq^k) = 0, \quad \forall k \in \mathbb{N}_0.$$

Thus, $\gamma(aq^k) = 0$ for all $k \in \mathbb{N}_0$. Clearly if γ is q -regular at zero, then

$$\gamma(0) := \lim_{k \rightarrow \infty} \gamma(aq^k) = 0.$$

The proof of (ii) is similar and is omitted. □

Lemma 4.4 If $\alpha \in C(A_{q,a}^*)$ and

$$\int_0^a \alpha(x)D_q h(x) d_q x = 0$$

for any function h satisfying

- 1. h and $D_q h$ are q -regular at zero,
- 2. $h(0) = h(a) = 0$,

then $\alpha(x) = c$ for all $x \in A_{q,a}^*$ where c is a constant.

Proof Let c be the constant defined by the relation $c = \frac{1}{a} \int_0^a \alpha(x) d_q x$. Let

$$h(x) := \int_0^x [\alpha(\xi) - c] d_q \xi, \quad x \in A_{q,a}^*$$

So, h and $D_q h$ are q -regular at zero functions such that $h(0) = h(a) = 0$. We have

$$\int_0^a [\alpha(x) - c] D_q h(x) d_q x = \int_0^a \alpha(x) D_q h(x) d_q x + [\alpha(x) - c] h(x) \Big|_{x=0}^a = 0,$$

on the other hand,

$$\int_0^a \alpha(x) D_q h(x) d_q x = \int_0^a [\alpha(x) - c]^2 d_q x = 0.$$

Therefore, $\alpha(x) = c$ for all $x \in A_{q,a}$. But α is q -regular at zero, hence $\alpha(0) = 0$. This yields the required result. □

Theorem 4.5 *Let $y \in {}^c \text{Var}(0, a)$ be a local extremum of J . Then y satisfies the Euler-Lagrange equation*

$$\partial_2 F(x) + D_{q,a}^\alpha \partial_3 F(x) = 0, \quad \forall x \in A_{q,q,a}^* \tag{4.5}$$

Proof Let y be a local extremum of J and let η be arbitrary but fixed variation function of y . Define

$$\Phi(\epsilon) = J(y + \epsilon \eta).$$

Since y is a local extremum for J , and $J(y) = \Phi(0)$, it follows that 0 is a local extremum for ϕ . Hence $\phi'(0) = 0$. But

$$0 = \phi'(0) = \lim_{\epsilon \rightarrow 0} \frac{d}{d\epsilon} \phi(y + \epsilon \eta) = \int_0^a (\partial_2 F \eta + \partial_3 F^c D_{q,0^+}^\alpha \eta) d_q x.$$

Using (2.18), we obtain

$$0 = \int_0^a (\partial_2 F + {}^c D_{q,a}^\alpha \partial_3 F) \eta d_q x + I_{q,a}^{1-\alpha} \partial_3 F(x) \eta(x) \Big|_{x=0}^a.$$

Since η is a variation function, $\eta(0) = \eta(a) = 0$, and we have

$$\int_0^a (\partial_2 F + D_{q,a}^\alpha \partial_3 F) \eta d_q x = 0$$

for any $\eta \in {}^c \text{Var}(0, a)$. Consequently, from Lemma 4.3, we obtain (4.5) and this completes the proof. □

4.1 A q -fractional isoperimetric problem

In the following, we shall solve the q -fractional isoperimetric problem: Given a functional J as in (4.1), find which functions minimize (or maximize) J , when subject to the boundary

conditions

$$y(0) = y_0, \quad y(a) = y_a \tag{4.6}$$

and the q -integral constraint

$$I(y) = \int_0^a G(x, y, {}^c D_{q,0^+}^\alpha y) d_q x = l, \tag{4.7}$$

where l is a fixed real number. Here, similarly to before:

1. The functions $(u, v) \rightarrow G(t, u, v)$ and $(u, v) \rightarrow \partial_i G(t, u, v)$ ($i = 2, 3$) are continuous functions uniformly on $A_{q,a}$.
2. $G(\cdot, y(\cdot), {}^c D_{q,0^+}^\alpha(\cdot))$, $\delta_i G(\cdot, y(\cdot), {}^c D_{q,0^+}^\alpha(\cdot))$ ($i = 2, 3$) are q -regular at zero.
3. $\delta_3 G$ has a right Riemann-Liouville fractional q -derivative of order α which is q -regular at zero.

A function $y \in E$ that satisfies (4.6) and (4.7) is called admissible.

Definition 4.6 An admissible function y is an extremal for I in (4.7) if it satisfies the equation

$$\partial_2 G(x) + D_{q,a}^\alpha \partial_3 G(x) = 0, \quad \forall x \in A_{q,qa}^* \tag{4.8}$$

Theorem 4.7 Let y be a local extremum for J given by (4.1), subject to the conditions (4.6) and (4.7). If y is not an extremal of the function I , then there exists a constant λ such that y satisfies

$$\partial_2 H(x) + D_{q,a}^\alpha \partial_3 H(x) = 0, \quad \forall x \in A_{q,qa}^* \tag{4.9}$$

where $H := F - \lambda G$.

Proof Let $\eta_1, \eta_2 \in {}^c \text{Var}(0, a)$ be two functions, and let ϵ_1 and ϵ_2 be two real numbers, and consider the new function of two parameters

$$\check{y} = y + \epsilon_1 \eta_1 + \epsilon_2 \eta_2. \tag{4.10}$$

The reason why we consider two parameters is that we can choose one of them as a function of the other in order for \check{y} to satisfy the q -integral constraint (4.7). Let

$$\check{I}(\epsilon_1, \epsilon_2) = \int_0^a G(x, \check{y}, {}^c D_{q,0^+}^\alpha \check{y}) d_q x - l.$$

It follows by the q -integration by parts rule (1.10) that

$$\left. \frac{\partial \check{I}}{\partial \epsilon_2} \right|_{(0,0)} = \int_0^a (\partial_2 G(x) + D_{q,a}^\alpha \partial_3 G(x)) \eta_2 d_q x.$$

Since y is not an extremal of I , there exists a function η_2 satisfying the condition $\left. \frac{\partial \check{I}}{\partial \epsilon_2} \right|_{(0,0)} \neq 0$. Hence, from the fact that $\check{I}(0, 0) = 0$ and the implicit function theorem, there

exists a C^1 function $\epsilon_2(\cdot)$, defined in some neighborhood of zero, such that

$$\check{I}(\epsilon_1, \epsilon_2(\epsilon_1)) = 0.$$

Therefore, there exists a family of variations of type (4.10) satisfying the q -integral constraint. To prove the theorem, we define a new function $\check{J}(\epsilon_1, \epsilon_2) = J(\check{y})$. Since $(0, 0)$ is a local extremum of \check{J} subject to the constraint $\check{I}(0, 0) = 0$, and $\nabla \check{I}(0, 0) \neq (0, 0)$, by the Lagrange multiplier rule, see [63], there exists a constant λ for which the following holds:

$$\nabla \check{J}(0, 0) \neq (0, 0) - \lambda \nabla \check{I}(0, 0) = (0, 0).$$

Simple calculation shows that

$$\left. \frac{\partial \check{J}}{\partial \epsilon_1} \right|_{(0,0)} = \int_0^a (\partial_2 F(x) + D_{q,a^-}^\alpha \partial_3 F(x)) \eta_1 d_q x$$

and

$$\left. \frac{\partial \check{I}}{\partial \epsilon_1} \right|_{(0,0)} = \int_0^a (\partial_2 G(x) + D_{q,a^-}^\alpha \partial_3 G(x)) \eta_1 d_q x.$$

Consequently,

$$\int_0^a [\partial_2 F(x) + D_{q,a^-}^\alpha \partial_3 F(x) - \lambda (\partial_2 G(x) + D_{q,a^-}^\alpha \partial_3 G(x))] \eta_1 d_q x.$$

Since η_1 is arbitrary, from Lemma 4.3, we obtain

$$\partial_2 F(x) + D_{q,a^-}^\alpha \partial_3 F(x) - \lambda (\partial_2 G(x) + D_{q,a^-}^\alpha \partial_3 G(x)) = 0$$

for all $x \in A_{q,q,a}^*$. This is equivalent to (4.9) and completes the proof. □

The functions

$$e_{\alpha,\beta}(z; q) := \sum_{n=0}^\infty \frac{z^n}{\Gamma_q(\alpha n + 1)}; \quad |z(1-q)^\alpha| < 1,$$

$$E_{\alpha,\beta}(z; q) := \sum_{n=0}^\infty q^{\frac{\alpha}{2}n(n-1)} \frac{z^n}{\Gamma_q(\alpha n + 1)}; \quad z \in \mathbb{C},$$

are q -analogs of the Mittag-Leffler function

$$E_{\alpha,\beta}(z) = \sum_{n=0}^\infty \frac{z^n}{\Gamma_q(\alpha n + 1)}, \quad z \in \mathbb{C};$$

see [33]. We have

$${}^c D_{q,0^+}^\alpha e_{\alpha,1}(z; q) := e_{\alpha,1}(z; q); \quad {}^c D_{q,0^+}^\alpha E_{\alpha,1}(z; q) = E_{\alpha,1}(qz; q). \tag{4.11}$$

Example 4.8 Consider the fractional q -isoperimetric problem:

$$\begin{aligned}
 J(y) &= \int_0^a ({}^c D_{q,0^+}^\alpha y(x))^2 d_q x, \\
 I(y) &= \int_0^a e_{\alpha,1}(x^\alpha; q) {}^c D_{q,0^+}^\alpha y(x) d_q x = l, \\
 y(0) &= 1, \quad y(a) = e_{\alpha,1}(a^\alpha; q),
 \end{aligned}
 \tag{4.12}$$

where $0 < a(1 - q) < 1$. Then

$$H = ({}^c D_{q,0^+}^\alpha y)^2 - \lambda e_{\alpha,1}(x; q) {}^c D_{q,0^+}^\alpha y$$

and

$$\partial_2 H + D_{q,a}^\alpha \partial_3 H = D_{q,a}^\alpha (2 {}^c D_{q,0^+}^\alpha y(x) - \lambda e_{\alpha,1}(x; q)).$$

Therefore a solution of the problem is $\lambda = 2$ and $y(x) = e_{\alpha,1}(x^\alpha; q)$. Similarly a solution of the problem

$$\begin{aligned}
 J(y) &= \int_0^a ({}^c D_{q,0^+}^\alpha y(x))^2 d_q x, \\
 I(y) &= \int_0^a E_{\alpha,1}((qx)^\alpha; q) {}^c D_{q,0^+}^\alpha y(x) d_q x = l, \\
 y(0) &= 1, \quad y(a) = E_{\alpha,1}(a^\alpha; q),
 \end{aligned}
 \tag{4.13}$$

where $a > 0$ is $y(x) = E_{\alpha,1}(x^\alpha; q)$.

5 Existence of discrete spectrum for a fractional q -Sturm-Liouville problem

In this section, we use the q -calculus of variations we developed in Section 4 to investigate the existence of solutions of the qFSLP

$$D_{q,a}^\alpha p(x) {}^c D_{q,0^+}^\alpha y(x) + r(x)y(x) = \lambda w_\alpha y(x), \quad x \in A_{q,qa}^* \tag{5.1}$$

under the boundary condition

$$y(0) = y(a) = 0. \tag{5.2}$$

The proof of the main result of this section depends on the Arzelà-Ascoli theorem [64], p.156. The setting of this theorem is a compact metric space X . Let $C(X)$ denote the space of all continuous functions on X with values in \mathbb{C} or \mathbb{R} . $C(X)$ is associated with the metric function

$$d(f, g) = \max\{|f(x) - g(x)| : x \in X\}.$$

Theorem 5.1 (Arzelà-Ascoli theorem) *If a sequence $\{f_n\}_n$ in $C(X)$ is bounded and equicontinuous then it has a uniformly convergent subsequence.*

In our q -setting, we take $X = A_{q,a}^*$. Hence $f \in C(A_{q,a}^*)$ if and only if f is q -regular at zero, *i.e.*,

$$f(0) := \lim_{n \rightarrow \infty} f(aq^n).$$

Remark 5.2 A question may be raised as to why in (5.1) we have only $x \in A_{q,qa}^*$ instead of $A_{q,a}^*$. The reason for that is that the qFSLP (5.1)-(5.2) will be solved by using the q -fractional isoperimetric problem developed in Theorem 4.7, and its q -Euler-Lagrange equation (4.9) holds only for $x \in A_{q,qa}^*$. Also, in order for (5.1) to hold at $x = a$, we should have $D_{q,a}^\alpha (p(\cdot)^c D_{q,0^+}^\alpha y(\cdot))(a) = 0$ and this holds only if $p(a)^c D_{q,0^+}^\alpha y(a) = 0$, which may not hold.

Theorem 5.3 Let $\frac{1}{2} < \alpha < 1$. Assume that the functions p, r, w_α are defined on $A_{q,a}^*$ and satisfying the conditions:

- (i) w_α is a positive continuous function on $[0, a]$ such that $D_q^k \frac{1}{w_\alpha}$ ($k = 0, 1, 2$) are bounded functions on $A_{q,a}$,
- (ii) r is a bounded function on $A_{q,a}$,
- (iii) $p \in C(A_{q,a}^*)$ such that $\inf_{x \in A_{q,a}} p(x) > 0$, and $\sup_{x \in A_{q,a}} |\frac{r(x)}{w_\alpha(x)}| < \infty$.

The q -fractional Sturm-Liouville problem (5.1)-(5.2) has an infinite number of eigenvalues $\lambda^{(1)}, \lambda^{(2)}, \dots$, and to each eigenvalue $\lambda^{(n)}$ there is a corresponding eigenfunction $y^{(n)}$, which is unique up to a constant factor. Furthermore, the eigenfunctions $y^{(n)}$ form an orthogonal set of solutions in the Hilbert space $L_q^2(A_{q,a}^*, w_\alpha)$.

Proof As we mentioned in Remark 5.2, the qFSLP (5.1)-(5.2) can be recast as the q -fractional variational isoperimetric problem: Find the extremal of the functional

$$J(y) := \int_0^a [p(x)({}^c D_{q,0^+}^\alpha y)^2 + r(x)y^2] d_q x \tag{5.3}$$

subject to the boundary condition

$$y(0) = y(a) = 0, \tag{5.4}$$

and the isoperimetric constraint

$$I(y) = \int_0^a w_\alpha(x)y^2 d_q x = 1. \tag{5.5}$$

The q -fractional Euler-Lagrange equation for the functional I is

$$2w_\alpha(x)y(x) = 0 \quad \text{for all } x \in A_{q,a},$$

which is satisfied only for the trivial solution $y = 0$, because w_α is positive on $A_{q,a}$. So, no extremals for I can satisfy the q -isoperimetric condition. If y is an extremal for the q -fractional isoperimetric problem, then from Theorem 4.7, there exists a constant λ such that y satisfies the q -fractional Euler-Lagrange equation (4.9) in $A_{q,qa}^*$ but this is equivalent to the qFSLP (5.1).

In the following, we shall derive a method for approximating the eigenvalues and the eigenfunctions at the same time similar to the technique in [5, 11]. The proof follows in six steps.

Step 1. First let us point out that functional J defined in (5.3) is bounded from below. Indeed, since p, w_α are positive on $A_{q,a}$,

$$\begin{aligned} J(y) &= \int_0^a [p(x)({}^c D_{q,0^+}^\alpha y)^2 + r(x)y^2] \\ &\geq \inf_{x \in A_{q,a}} \frac{r(x)}{w_\alpha(x)} \int_0^a w_\alpha(x)y^2(x) d_q x \\ &= \inf_{x \in A_{q,a}} \frac{r(x)}{w_\alpha(x)} =: M > -\infty. \end{aligned}$$

According to the Ritz method [11], p.201, we approximate a solution of (5.3)-(5.4) using the following q -trigonometric functions with the coefficients depending on w_α :

$$y_m(x) = \frac{1}{\sqrt{w_\alpha}} \sum_{k=1}^m \frac{\beta_k}{\sqrt{\mu_k}} S_q \left(\frac{w_k x}{a} \right). \tag{5.6}$$

Observe that $y_m(0) = y_m(a) = 0$. By substituting (5.6) into (5.3) and (5.5) we obtain

$$\begin{aligned} J_m(\beta_1, \dots, \beta_m) &= J_m([\beta]) \\ &= \sum_{k,j=1}^m \frac{\beta_j \beta_k}{\sqrt{\mu_j} \sqrt{\mu_k}} \int_0^a \left[p(x) {}^c D_{q,0^+}^\alpha \frac{S_q(\frac{w_k x}{a})}{\sqrt{w_\alpha}} {}^c D_{q,0^+}^\alpha \frac{S_q(\frac{w_j x}{a})}{\sqrt{w_\alpha}} \right. \\ &\quad \left. + \frac{r(x)}{w_\alpha(x)} S_q \left(\frac{w_k x}{a} \right) S_q \left(\frac{w_j x}{a} \right) \right] d_q x \end{aligned} \tag{5.7}$$

subject to the condition

$$I_m(\beta_1, \beta_2, \dots, \beta_m) = I_m([\beta]) = \frac{a\sqrt{q}}{2} \sum_{k=1}^m \beta_k^2 = 1. \tag{5.8}$$

The functions defined in (5.7) and (5.8) are functions of the m variables $\beta_1, \beta_2, \dots, \beta_m$. Thus, in terms of the variables β_1, \dots, β_m , our problem is to minimize $J_m(\beta_1, \beta_2, \dots, \beta_m)$ on the surface σ_m of the m dimensional sphere defined in (5.8). Since σ_m is a compact set and $J_m(\beta_1, \beta_2, \dots, \beta_m)$ is continuous on σ_m , $J_m(\beta_1, \beta_2, \dots, \beta_m)$ has a minimum λ_m^1 at some point $(\beta_1^{(1)}, \dots, \beta_m^{(1)})$ of σ_m . Let

$$y_m^{(1)} = \frac{1}{\sqrt{w_\alpha}} \sum_{k=1}^m \frac{\beta_k^{(1)}}{\mu_k} S_q \left(\frac{w_k x}{a} \right).$$

If this procedure is carried out for $m = 1, 2, \dots$, we obtain a sequence of numbers $\lambda_1^{(1)}, \lambda_2^{(1)}, \dots$, and a corresponding sequence of functions

$$y_1^{(1)}(x), \quad y_2^{(1)}(x), \quad y_3^{(1)}(x), \quad \dots$$

Noting that σ_m is the subset of σ_{m+1} obtained by setting $\beta_{m+1} = 0$, while

$$J_m(\beta_1, \dots, \beta_m) = J_{m+1}(\beta_1, \dots, \beta_m, 0),$$

consequently,

$$\lambda_{m+1}^{(1)} \leq \lambda_m^{(1)}. \tag{5.9}$$

Increasing the domain of definition of a function can only decrease its minimum. It follows from (5.9) and the fact that $J(y)$ is bounded from below that its limit

$$\lambda^{(1)} = \lim_{m \rightarrow \infty} \lambda_m^{(1)}$$

exists.

Step 2. We shall prove that the sequence $(y_m^{(1)})_{m \in \mathbb{N}}$ contains a uniformly convergent subsequence. From now on, for simplicity, we shall write y_m instead of $y_m^{(1)}$. Recall that

$$\lambda_m^{(1)} = \int_0^a [p(x)({}^c D_{q,0^+}^\alpha y_m)^2 + r(x)y_m^2] d_q x$$

is convergent, so it must be bounded, *i.e.*, there exists a constant $M_0 > 0$ such that

$$\int_0^a [p(x)({}^c D_{q,0^+}^\alpha y_m)^2 + r(x)y_m^2] d_q x \leq M_0, \quad m \in \mathbb{N}.$$

Therefore, for all $m \in \mathbb{N}$ we have the inequality

$$\begin{aligned} \int_0^a p(x)({}^c D_{q,0^+}^\alpha y_m)^2 d_q x &\leq M_0 + \left| \int_0^a r(x)y_m^2(x) d_q x \right| \\ &\leq M_0 + \sup_{x \in A_{q,a}} \left| \frac{r(x)}{w_\alpha(x)} \right| \int_0^a w_\alpha(x)y_m^2(x) d_q x \\ &:= M_0 + \sup_{x \in A_{q,a}} \left| \frac{r(x)}{w_\alpha(x)} \right| =: M_1. \end{aligned}$$

Moreover, since $\inf_{x \in A_{q,a}} p(x) > 0$ we have

$$\left(\inf_{x \in A_{q,a}} p(x) \right) \int_0^a ({}^c D_{q,0^+}^\alpha y_m)^2 d_q x \leq \int_0^a p(x)({}^c D_{q,0^+}^\alpha y_m)^2 d_q x \leq M_1,$$

and hence

$$\int_0^a ({}^c D_{q,0^+}^\alpha y_m)^2 d_q x \leq \frac{M_1}{\inf_{x \in A_{q,a}} p(x)} =: M_2. \tag{5.10}$$

Since $y_m(0) = 0$, from (2.15) and (5.10)

$$\begin{aligned} \|y_m\| &= \|I_{q,0^+}^\alpha {}^c D_{q,0^+}^\alpha y_m\| \leq \tilde{M}_\alpha \|{}^c D_{q,0^+}^\alpha y_m\|_2 \\ &\leq \tilde{M}_\alpha M_2 \end{aligned}$$

for $\alpha > 1/2$. Hence, $(y_m)_m$ is uniformly bounded on $A_{q,a}^*$. Now we prove that the sequence $(y_m)_m$ is equicontinuous. Let $x_1, x_2 \in A_{q,a}$. Assume that $x_1 < x_2$. Applying the Schwarz inequality and (2.9)

$$\begin{aligned} & \Gamma_q(\alpha) |y_m(x_2) - y_m(x_1)| \\ &= \Gamma_q(\alpha) |I_{q,0^+}^\alpha {}^c D_{q,0^+}^\alpha y_m(x_2) - I_{q,0^+}^\alpha {}^c D_{q,0^+}^\alpha y_m(x_2)| \\ &= \left| x_2^{\alpha-1} \int_0^{x_2} (qt/x_2; q)_{\alpha-1} {}^c D_{q,0^+}^\alpha y_m(t) d_q t - x_1^{\alpha-1} \int_0^{x_1} (qt/x_1; q)_{\alpha-1} {}^c D_{q,0^+}^\alpha y_m(t) d_q t \right| \\ &\leq \left| x_2^{\alpha-1} \int_{x_1}^{x_2} (qt/x_2; q)_{\alpha-1} {}^c D_{q,0^+}^\alpha y_m(t) d_q t \right| \\ &\quad + \left| \int_0^{x_1} \{x_2^{\alpha-1} (qt/x_2; q)_{\alpha-1} - x_2^{\alpha-1} (qt/x_1; q)_{\alpha-1}\} {}^c D_{q,0^+}^\alpha y_m(t) d_q t \right| \\ &\leq M_2 \left(x_2^{2\alpha-2} \int_{x_1}^{x_2} (qt/x_2; q)_{\alpha-1}^2 d_q t \right)^{1/2} \\ &\quad + M_2 \left(\int_0^{x_1} (x_2^{\alpha-1} (qt/x_2; q)_{\alpha-1} - x_1^{\alpha-1} (qt/x_1; q)_{\alpha-1})^2 d_q t \right)^{1/2}. \end{aligned}$$

Since $x_1 < x_2$, we have

$$x_2^{\alpha-1} (qt/x_2; q)_{\alpha-1} \leq x_1^{\alpha-1} (qt/x_1; q)_{\alpha-1} \quad \text{for all } t < x_1 < x_2.$$

Using the inequality

$$t_1 \geq t_2 \geq 0 \rightarrow (t_1 - t_2)^2 \leq t_1^2 - t_2^2,$$

we obtain

$$\begin{aligned} & \int_0^{x_1} (x_2^{\alpha-1} (qt/x_2; q)_{\alpha-1} - x_1^{\alpha-1} (qt/x_1; q)_{\alpha-1})^2 d_q t \\ &\leq \int_0^{x_1} x_1^{2\alpha-2} (qt/x_1; q)_{\alpha-1}^2 d_q t - \int_0^{x_1} x_2^{2\alpha-2} (qt/x_2; q)_{\alpha-1}^2 d_q t \\ &= \int_{x_1}^{x_2} x_2^{2\alpha-2} (qt/x_2; q)_{\alpha-1}^2 d_q t + \int_0^{x_1} x_1^{2\alpha-2} (qt/x_1; q)_{\alpha-1}^2 d_q t \\ &\quad - \int_0^{x_2} x_2^{2\alpha-2} (qt/x_2; q)_{\alpha-1}^2 d_q t \\ &= \int_{x_1}^{x_2} x_2^{2\alpha-2} (qt/x_2; q)_{\alpha-1}^2 d_q t + (x_1^{2\alpha-1} - x_2^{2\alpha-1}) \int_0^1 (q\xi; q)_{\alpha-1}^2 d_q \xi \\ &\leq \int_{x_1}^{x_2} x_2^{2\alpha-2} (qt/x_2; q)_{\alpha-1}^2 d_q t \end{aligned}$$

for $\alpha > \frac{1}{2}$. Hence, we have

$$\begin{aligned} |y_m(x_2) - y_m(x_1)| &\leq \frac{2M_2}{\Gamma_q(\alpha)} x_2^{\alpha-1} \left(\int_{x_1}^{x_2} (qt/x_2; q)_{\alpha-1}^2 d_q t \right)^{1/2} \\ &\leq \frac{2M_2}{\Gamma_q(\alpha) (q^\alpha; q)_\infty^2} x_2^{\alpha-1} \sqrt{x_2 - x_1} \leq \frac{2M_2}{\Gamma_q(\alpha)^2 (q^\alpha; q)_\infty^2} (x_2 - x_1)^{\alpha - \frac{1}{2}}. \end{aligned}$$

Hence $\{y_m\}$ is equicontinuous. Therefore, from the Arzelà-Ascoli theorem for metric spaces, a uniformly convergent subsequence $(y_{m_n})_{n \in \mathbb{N}}$ exists. It means that we can find $y^{(1)} \in C(A_{q,a}^*)$ such that

$$y^{(1)} = \lim_{n \rightarrow \infty} y_{m_n}. \tag{5.11}$$

Step 3. From the Lagrange multiplier at $[\beta] = (\beta_1^{(1)}, \dots, \beta_m^{(1)})$, we have

$$\frac{\delta}{\delta \beta_j} [I_m([\beta]) - \lambda_m^{(1)} I_m([\beta])] |_{[\beta]=[\beta^{(1)}]}, \quad j = 1, 2, \dots, m.$$

By multiplying each equation by an arbitrary constant c_j and summing from 1 to m , we obtain

$$0 = \sum_{j=1}^m c_j \frac{\delta}{\delta \beta_j} [I_m([\beta]) - \lambda_m^{(1)} I_m([\beta])] |_{[\beta]=[\beta^{(1)}]}. \tag{5.12}$$

For $m \in \mathbb{N}$, set

$$h_m(x) := \frac{1}{\sqrt{w_\alpha}} g_m(x); \quad g_m(x) := \sum_{j=1}^m \frac{c_j}{\sqrt{\mu_j}} S_q \left(\frac{w_j x}{a} \right).$$

According to Proposition 3.10, we can choose the coefficients c_j such that there exists a function g satisfying

$$\lim_{m \rightarrow \infty} D_q^k g_m = D_q^k g \quad (k = 0, 1, 2)$$

and the convergence is in $L_q^2(A_{q,a}^*)$ norm. Hence

$$\lim_{m \rightarrow \infty} \|D_q^k h_m - D_q^k h\|_2 = 0 \quad (k = 0, 1, 2). \tag{5.13}$$

We can write (5.12) in the form

$$0 = \int_0^a [p(x) {}^c D_{q,0^+}^\alpha y_m {}^c D_{q,0^+}^\alpha h_m + (r(x) - \lambda_m^1 w_\alpha(x)) y_m h_m] d_q x. \tag{5.14}$$

Since $y_m(0) = 0$, from (2.11)

$${}^c D_{q,0^+}^\alpha y_m = D_{q,0^+}^\alpha y_m = D_q I_{q,0^+}^{1-\alpha} y_m.$$

Then replacing ${}^c D_{q,0^+}^\alpha y_m$ by $D_{q,0^+}^\alpha y_m$ in (5.14) and applying the q -integration by parts rule (1.10), we obtain

$$\begin{aligned} 0 &= I_m \\ &:= - \int_0^a D_q p(x) {}^c D_{q,0^+}^\alpha h_m(x) (I_{q,0^+}^{1-\alpha} y_m)(qx) d_q x \\ &\quad - \int_0^a p(qx) D_q {}^c D_{q,0^+}^\alpha h_m(x) (I_{q,0^+}^{1-\alpha} y_m)(qx) d_q x \end{aligned}$$

$$\begin{aligned}
 &+ \int_0^a [r(x) - \lambda_m^{(1)} w_\alpha(x)] y_m h_m d_q x \\
 &+ p(x) {}^c D_{q,0^+}^\alpha h_m(x) I_{q,0^+}^{1-\alpha} y_m(x) \Big|_{x=0}^{x=a}.
 \end{aligned}$$

In the following we shall prove that

$$\begin{aligned}
 I &:= \lim_{m \rightarrow \infty} I_m \\
 &= \int_0^a -D_q p(x) {}^c D_{q,0^+}^\alpha h(x) (I_{q,0^+}^{1-\alpha} y^{(1)})(qx) d_q x \\
 &\quad - \int_0^a p(qx) D_q {}^c D_{q,0^+}^\alpha h(x) (I_{q,0^+}^{1-\alpha} y^{(1)})(qx) d_q x \\
 &\quad + p(x) {}^c D_{q,0^+}^\alpha h(x) I_{q,0^+}^{1-\alpha} y(x) \Big|_{x=0}^{x=a} + \int_0^a [r(x) - \lambda^{(1)} w_\alpha(x)] y^{(1)} h d_q x.
 \end{aligned} \tag{5.15}$$

Indeed,

$$\begin{aligned}
 &|I_m - I| \\
 &\leq \int_0^a |D_q p(x) [{}^c D_{q,0^+}^\alpha h_m(x) (I_{q,0^+}^{1-\alpha} y_m)(qx) - {}^c D_{q,0^+}^\alpha h(x) (I_{q,0^+}^{1-\alpha} y^{(1)})(qx)]| d_q x \\
 &\quad + \int_0^a |p(qx) [D_q {}^c D_{q,0^+}^\alpha h_m(x) (I_{q,0^+}^{1-\alpha} y_m)(qx) - D_q {}^c D_{q,0^+}^\alpha h(x) (I_{q,0^+}^{1-\alpha} y^{(1)})(qx)]| d_q x \\
 &\quad + |p(x) {}^c D_{q,0^+}^\alpha h_m(x) I_{q,0^+}^{1-\alpha} y_m(x) - p(x) {}^c D_{q,0^+}^\alpha h(x) I_{q,0^+}^{1-\alpha} y^{(1)}(x) \Big|_{x=0}^{x=a} \\
 &\quad + \int_0^a |[r(x) - \lambda_m^{(1)} w_\alpha(x)] y_m h_m - [r(x) - \lambda^{(1)} w_\alpha(x)] y^{(1)} h| d_q x.
 \end{aligned} \tag{5.16}$$

For the first q -integral in (5.16), by adding and subtracting the term

$$D_q p(x) {}^c D_{q,0^+}^\alpha h(x) (I_{q,0^+}^{1-\alpha} y_m)(qx)$$

to the integrand, we obtain

$$\begin{aligned}
 &\int_0^a |D_q p(x) [{}^c D_{q,0^+}^\alpha h_m(x) (I_{q,0^+}^{1-\alpha} y_m)(qx) - {}^c D_{q,0^+}^\alpha h(x) (I_{q,0^+}^{1-\alpha} y^{(1)})(qx)]| d_q x \\
 &\leq \|D_q p\| \|{}^c D_{q,0^+} h\|_\infty \| (I_{q,0^+}^{1-\alpha} y_m)(qx) - (I_{q,0^+}^{1-\alpha} y^{(1)})(qx) \|_1 \\
 &\quad + \|D_q p\| M_3 K_{1-\alpha} \|{}^c D_{q,0^+} (h_m - h)\|_2 \\
 &\leq \frac{\|D_q p\|}{q} \{ \|{}^c D_{q,0^+} h\|_\infty \| I_{q,0^+}^{1-\alpha} y_m - I_{q,0^+}^{1-\alpha} y^{(1)} \|_1 + M_3 K_{1-\alpha} \|{}^c D_{q,0^+} (h_m - h)\|_2 \},
 \end{aligned}$$

where $K_{1-\alpha}$ is the constant defined in (2.13) and $M_3 := \sup_{m \in \mathbb{N}} \|y_m\|_\infty$. From (5.11) and (5.13)

$$\lim_{m \rightarrow \infty} \|y_m - y^{(1)}\| = \lim_{m \rightarrow \infty} \|D_q h_m - D_q h\|_2 = 0,$$

then applying (2.12)-(2.14), we obtain

$$\lim_{m \rightarrow \infty} \| I_{q,0^+}^{1-\alpha} y_m - I_{q,0^+}^{1-\alpha} y^{(1)} \|_1 = \lim_{m \rightarrow \infty} \|{}^c D_{q,0^+} (h_m - h)\|_2 = 0$$

and the first q -integral vanishes as $m \rightarrow \infty$. As for the second q -integral, we add and subtract the term $p(qx)D_q^c D_{q,0^+}^\alpha h(x)(I_{q,0^+}^{1-\alpha} y_m)(qx)$. This gives

$$\begin{aligned} & \int_0^a |p(qx)[D_q^c D_{q,0^+}^\alpha h_m(x)(I_{q,0^+}^{1-\alpha} y_m)(qx) - D_q^c D_{q,0^+}^\alpha h(x)(I_{q,0^+}^{1-\alpha} y^{(1)})(qx)]| d_q x \\ & \leq \|p\| \|D_q^c D_{q,0^+}^\alpha h\|_2 \|(I_{q,0^+}^{1-\alpha} y_m)(qx) - (I_{q,0^+}^{1-\alpha} y^{(1)})(qx)\|_2 \\ & \quad + \|p\| M_3 K_{1-\alpha} \|D_q^c D_{q,0^+}^\alpha (h_m - h)\|_2 \\ & \leq \frac{\|p\|}{q} \{ \|D_q^c D_{q,0^+}^\alpha h\|_2 \|I_{q,0^+}^{1-\alpha} y_m - I_{q,0^+}^{1-\alpha} y^{(1)}\|_2 + M_3 K_{1-\alpha} \|D_q^c D_{q,0^+}^\alpha (h_m - h)\|_2 \}. \end{aligned}$$

Since $D_q^c D_{q,0^+}^\alpha f(x) = I_q^{1-\alpha} D_q^2 f$ if $D_q f(0) = 0$, and since $\lim_{m \rightarrow \infty} \|D_q^2 h_m - D_q^2 h\|_2 = 0$, from (2.14), the second q -integral tends to zero as m tends to ∞ . For the next two terms, we have for $x = 0, a$,

$$(I_{q,0^+}^{1-\alpha} y_m)(qx) = q^{1-\alpha} I_{q,0^+}^\alpha y_m(qx) \rightarrow q^{1-\alpha} I_{q,0^+}^\alpha y^{(1)}(qx)$$

resulting from the convergence of the sequence $\|y_m - y\| \rightarrow 0$, and at the points $x = 0, x = a$, we obtain

$$D_{q,0^+}^\alpha h_m(0) \rightarrow D_{q,0^+}^\alpha h(0), \quad D_{q,0^+}^\alpha h_m(a) \rightarrow D_{q,0^+}^\alpha h(a).$$

Therefore,

$$|p(x)D_{q,0^+}^\alpha h_m(x)I_{q,0^+}^{1-\alpha} y_m(x) - p(x)D_{q,0^+}^\alpha h(x)I_{q,0^+}^{1-\alpha} y^{(1)}(x)|_{x=0}^{x=a} = 0.$$

Similarly, the last term in the estimation (5.16) vanishes as $m \rightarrow \infty$.

Step 4. We have

$$I = \int_0^a p(x)^c D_{q,0^+}^\alpha y(x)^c D_{q,0^+}^\alpha h(x) + (r(x) - \lambda w_\alpha)y(x)h(x) d_q x = 0. \tag{5.17}$$

Set

$$\begin{aligned} \gamma_1(x) & := p(x)^c D_{q,0^+}^\alpha y(x), \\ \gamma_2(x) & := (r(x) - \lambda w_\alpha)y(x). \end{aligned} \tag{5.18}$$

Thus, since $h(0) = h(a) = 0$,

$$I = \int_0^a [(I_{q,a^-}^{1-\alpha} \gamma_1)(x) - (I_{q,0^+} \gamma_2)(qx)] D_q h(x) d_q x = 0.$$

Hence, from Lemma 4.4 there is a constant c such that

$$(I_{q,a^-}^{1-\alpha} \gamma_1)(x) - (I_{q,0^+} \gamma_2)(qx) = c, \quad \forall x \in A_{q,a}^*. \tag{5.19}$$

Acting on the two sides of (5.19) by $-\frac{1}{q}D_{q^{-1}}$, we obtain

$$D_{q,a^-}^\alpha \gamma_1(x) + \gamma_2(x) = 0, \quad x \in A_{q,qa}^*.$$

Hence, y is a solution of the qFSLP.

Step 5. In the following, we show that $(y_m^{(1)})_{m \in \mathbb{N}}$ itself converges to $y^{(1)}$. First, from Theorem 3.12 of [34], for a given λ the solution of

$$[D_{q,a}^\alpha - p(x) {}^c D_{q,0^+}^\alpha + r(x)]y(x) = \lambda w_\alpha(x)y(x), \tag{5.20}$$

subject to the boundary conditions

$$y(0) = y(a) = 0 \tag{5.21}$$

and the normalization condition

$$\int_0^a w_\alpha(x)y^2(x) d_q x = 1, \tag{5.22}$$

is unique except for a sign. Let us assume that $y^{(1)}$ solves (5.20) and the corresponding eigenvalue is $\lambda = \lambda^{(1)}$. Suppose that $y^{(1)}$ is nontrivial, *i.e.*, there exists $x_0 \in A_{q,qa}^*$ such that $y(x_0) \neq 0$ and choose the sign so that $y^{(1)}(x_0) > 0$. Similarly, for all $m \in \mathbb{N}$, let $y_m^{(1)}$ solve (5.20) with corresponding eigenvalue $\lambda = \lambda_m^{(1)}$, and let us choose the sign so that $y_m^{(1)}(x_0) \geq 0$. Now, suppose that $(y_m^{(1)})$ does not converge to $y^{(1)}$. It means that we can find another subsequence of $y_m^{(1)}$ such that it converges to another solution $\tilde{y}^{(1)}$. But for $\lambda = \lambda^{(1)}$, the solution of (5.20)-(5.22) is unique except for a sign, hence

$$\tilde{y}^{(1)} = -y^{(1)}$$

and we must have $\tilde{y}^{(1)}(x_0) < 0$. However, this is impossible because for all $m \in \mathbb{N}$, $y_m^{(1)}(x_0) \geq 0$. This is a contradiction, hence the solution is unique.

Step 6. In order to find the eigenfunction $y^{(2)}$ and the corresponding eigenvalue $\lambda^{(2)}$, we minimize the functional (5.3) subject to (5.4) and (5.5) but now with an extra orthogonality condition,

$$\int_0^a y(x)y^{(1)}(x)w_\alpha(x) d_q x = 0.$$

If we approximate the solution by

$$y_m(x) = \frac{1}{\sqrt{w_\alpha}} \sum_{k=1}^m \frac{\beta_k}{\sqrt{\mu_k}} S_q\left(\frac{w_k x}{a}\right), \quad y_m(0) = y_m(a) = 0,$$

then we again obtain the quadratic form (5.7). However, admissible solutions are satisfying (5.8) together with

$$\frac{a\sqrt{q}}{2} \sum_{k=1}^m \beta_k \beta_k^{(1)} = 0, \tag{5.23}$$

i.e., they lie in the $(m - 1)$ -dimensional sphere. As before, we find that the function $\tilde{J}([\beta])$ has a minimum $\lambda_m^{(2)}$ and there exists $\lambda^{(2)}$ such that

$$\lambda^{(2)} = \lim_{m \rightarrow \infty} \lambda_m^{(2)},$$

because $J(y)$ is bounded from below. Moreover, it is clear that

$$\lambda^{(1)} \leq \lambda^{(2)}. \tag{5.24}$$

The function $y_m^{(2)}$ defined by

$$y_m^{(2)}(x) := \frac{1}{\sqrt{w_\alpha}} \sum_{k=1}^m \frac{\beta_k^{(2)}}{\sqrt{\mu_k}} S_q \left(\frac{w_k x}{a} \right),$$

achieves its minimum $\lambda_m^{(2)}$, where $\beta^{(2)} = (\beta_1^{(2)}, \dots, \beta_m^{(2)})$ is the point satisfying (5.8) and (5.23). By the same argument as before, we can prove that the sequence $(y_m^{(2)})$ converges uniformly to a limit function $y^{(2)}$, which satisfies the qFSLP (5.1) with $\lambda^{(2)}$, boundary conditions (5.4) and orthogonality condition (5.5). Therefore, the solution $y^{(2)}$ of the qFSLP corresponding to the eigenvalue $\lambda^{(2)}$ exists. Furthermore, because orthogonal functions cannot be linearly dependent, and since only one eigenfunction corresponds to each eigenvalue (except for a constant factor) we have the strict inequality

$$\lambda^{(1)} < \lambda^{(2)}$$

instead of (5.24). Finally, if we repeat the above procedure with similar modifications, we can obtain the eigenvalues $\lambda^{(3)}, \lambda^{(4)}, \dots$ and the corresponding eigenvectors $y^{(3)}, y^{(4)}, \dots$ □

5.1 The first eigenvalue

Definition 5.4 The Rayleigh quotient for the q -fractional Sturm-Liouville problem (5.1)-(5.2) is defined by

$$R(y) := \frac{J(y)}{I(y)},$$

where $J(y)$ and $I(y)$ are given by (5.3) and (5.5), respectively.

Theorem 5.5 *Let y be a non-zero function satisfying y and ${}^c D_{q,0^+}^\alpha y$ be in $C(A_{q,a}^*)$ and $y(0) = y(a) = 0$. Then y is a minimizer of $R(y)$ and $R(y) = \lambda$ if and only if y is an eigenfunction of problem (5.1)-(5.2) associated with λ . That is, the minimum value of R at y is the first eigenvalue $\lambda^{(1)}$.*

Proof First, we prove the necessity. Assume that y is a non-zero minimizer of $R(y)$ and $R(y) = \lambda$. Consider the one parameter family of curves

$$y = y + h\eta, \quad |h| \leq \epsilon,$$

where η and ${}^c D_{q,0^+}^\alpha \eta$ are $C(A_{q,a}^*)$ functions and $\eta(0) = \eta(a) = 0$ and $\eta \neq 0$. Define functions ϕ, ψ, ξ on $[-\epsilon, \epsilon]$ by

$$\phi(h) := I(y + h\eta), \quad \psi(h) := J(y + h\eta), \quad \xi(h) = R(y + h\eta) = \frac{\psi(h)}{\phi(h)}, \quad h \in [-\epsilon, \epsilon].$$

Hence ξ is C^1 function on $[-\epsilon, \epsilon]$. Since $\xi(0) = R(y)$, 0 is a minimum value of ξ . Consequently, $\xi'_i(0) = 0$. But

$$\xi'(h) = \frac{1}{\phi(h)} \left(\psi'(h) - \frac{\psi(h)}{\phi(h)} \phi'(h) \right)$$

and

$$\psi'(0) = 2 \int_0^a [p(x)^c D_{q,0^+}^\alpha y^c D_{q,0^+}^\alpha \eta + r(x)y\eta] d_q x,$$

$$\phi'(0) = 2 \int_0^a w_\alpha(x)y(x)\eta(x) d_q x,$$

$$\frac{\psi(0)}{\phi(0)} = R(y) = \lambda.$$

Therefore,

$$\xi'(0) = \frac{2}{I(y)} \left(\int_0^a [p(x)^c D_{q,0^+}^\alpha y^c D_{q,0^+}^\alpha \eta + (r(x)y - \lambda w_\alpha)\eta] d_q x \right).$$

Using (2.19), we obtain

$$\int_0^a [D_{q,a^-}^\alpha p(x)^c D_{q,0^+}^\alpha y(x) + (r(x) - \lambda)w_\alpha(x)y(x)] \eta(x) d_q x = 0.$$

Applying Lemma 4.3, we obtain

$$D_{q,a^-}^\alpha p(x)^c D_{q,0^+}^\alpha y(x) + r(x)y(x) = \lambda w_\alpha(x)y(x), \quad x \in A_{q,qa}^*.$$

This proves the necessity. Now we prove the sufficiency. Assume that y is an eigenfunction of (5.1)-(5.2) associated with an eigenvalue λ . Then

$$D_{q,a^-}^\alpha p(x)^c D_{q,0^+}^\alpha y(x) + r(x)y(x) = \lambda w_\alpha(x)y(x), \quad x \in A_{q,qa}^*. \tag{5.25}$$

Multiplying (5.25) by y and calculating the q -integration from 0 to a , we obtain

$$\int_0^a [y(x)D_{q,a^-}^\alpha p(x)^c D_{q,0^+}^\alpha y(x) + r(x)y^2(x)] d_q x = \lambda \int_0^a w_\alpha(x)y^2(x) d_q x.$$

Since $y \neq 0$, $\int_0^a w_\alpha(x)y^2(x) d_q x > 0$ and

$$\frac{\int_0^a [y(x)D_{q,a^-}^\alpha p(x)^c D_{q,0^+}^\alpha y(x) + r(x)y^2(x)] d_q x}{\int_0^a w_\alpha(x)y^2(x) d_q x} = \lambda.$$

That is, $R(y) = \lambda$. Therefore, any minimum value of J is an eigenvalue and it is attained at the associated eigenfunction. Therefore the minimum value of J is the smallest eigenvalue. □

6 Conclusion and future work

This paper is the first paper that deals with variational problems of functionals defined in terms of Jackson q -integral on a finite domain and where the left-sided Caputo q -derivative appears in the integrand. We give a fractional q -analog of the Euler-Lagrange equation and a q -isoperimetric problem is defined and solved. We use these results in recasting the q FSLP under consideration as a q -isoperimetric problem, and then we solve it by a technique similar to the one used in solving regular Sturm-Liouville problems in [11] and fractional Sturm-Liouville problems in [5]. This completes the work started by the author in [34], and it generalizes the study of integer Sturm-Liouville problem introduced by Annaby and Mansour in [1, 2]. A similar study for the fractional Sturm-Liouville problem

$${}^c D_{q,\alpha} p(x) D_{q,0^+}^\alpha y(x) + r(x)y(x) = \lambda w_\alpha(x)y(x)$$

is in progress.

Competing interests

The author declares that she has no competing interests.

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