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# Spinning conformal correlators

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Abstract: We develop the embedding formalism for conformal field theories, aimed at doing computations with symmetric traceless operators of arbitrary spin. We use an index-free notation where tensors are encoded by polynomials in auxiliary polarization vectors. The efficiency of the formalism is demonstrated by computing the tensor structures allowed in $n$-point conformal correlation functions of tensors operators. Constraints due to tensor conservation also take a simple form in this formalism. Finally, we obtain a perfect match between the number of independent tensor structures of conformal correlators in $d$ dimensions and the number of independent structures in scattering amplitudes of spinning particles in $(d+1)$-dimensional Minkowski space.

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## Contents

## 1 Introduction

One hardly needs to stress the importance of Conformal Field Theories (CFT) in theoretical physics. In two dimensions, many exactly solvable models exist, thanks to the infinite dimensional extension of the global conformal group, the Virasoro algebra. Unfortunately, in three dimensions or higher, no equally efficient general approaches are known at present.

One approach which holds some promise is the 'conformal bootstrap' [1, 2], which tries to solve or constrain a higher-dimensional CFT by imposing the Operator Product Expansion (OPE) associativity. The efficiency of this method has been demonstrated in several recent applications [3–11]. However, so far this approach has been limited to the study of four-point functions of scalar operators. It is of great interest to extend this technique to other operators like the stress-energy tensor or global symmetry currents. This could provide very general constraints for any CFT or for CFTs with a given global symmetry. In this paper, we give the first step towards this goal by developing an efficient language to deal with primary tensor operators in CFT. Basically, our formalism makes CFT computations with tensor fields as easy as computations with scalars. In an upcoming paper [12], we shall use this formalism to obtain conformal blocks for four-point functions of tensor operators.

Another motivation for this work is the recently found analogy between CFT correlation functions written in the Mellin representation and scattering amplitudes [13, 14].[1] This analogy has been explored in detail in the case of CFT correlators defined holographically by Witten diagrams of scalar field theories in AdS [14, 16, 17]. It would be very interesting to find a generalization to correlators of tensor operators. The first steps towards this goal were given in [16, 17]. It is natural to expect that such a generalization could lead to recursion relations for the computation of stress-energy tensor correlators in CFTs with AdS gravity duals,[2] similar to the BCFW recursion relations for scattering amplitudes [20]. More generally, one might hope to use this analogy to translate all the powerful methods for the computation of scattering amplitudes to CFT correlation functions (at least, for CFTs with a weakly coupled AdS dual). We believe the formalism described in this paper to deal with tensor operators will also be useful in this context.

In this paper, we test the analogy between $d$-dimensional conformal correlators and $(d+1)$-dimensional scattering amplitudes at the level of counting independent coupling constants. More precisely, we show that the number of tensor structures for three point correlators of tensor operators is equal to the number of tensor structures for three particle S-matrix elements in one higher dimension. AdS/CFT provides a natural map from S-matrix elements of the bulk theory to correlators of the boundary CFT. The idea is to define the correlator by the contact Witten diagram with local interaction vertex associated with the scattering amplitude. This map can be used to obtain CFT $n$-point correlators from analytical $n$-particle S-matrix elements (contact interactions). However, for $n > 3$, the scattering amplitudes can have poles associated with particle exchange diagrams. In this case, some similarity seems to persist but it is not obvious how to define an explicit map.

[1] See also [15] for a connection between CFT anomalous dimensions and scattering amplitudes.
[2] See [18, 19] for a proposal based on momentum space correlators.



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**Structure of the paper.** The paper is built upon the embedding space formalism, which we review in section 2. In this formalism [1, 21–26], correlators in Euclidean $d$-dimensional space are uplifted to homogeneous functions on the lightcone of $(d+2)$-dimensional Minkowski spacetime, where the conformal group acts as the Lorentz group. This goes a long way towards simplifying CFT computations, but in the case of tensor fields it still falls short of our needs. In section 3, we develop a version of the embedding formalism which encodes the index structure of the tensor operators in polynomials of a 'polarization vector' in $(d+2)$-dimensions. In section 4, we use the new *index-free* formalism to compute constraints from conformal symmetry on correlators (3-, 4- and $n$-point functions) of tensor operators of arbitrary spin. We are able to rederive in a simplified and explicit way a number of known results, and to get some new ones. In section 5 we show how to implement constraints on correlation functions of conserved tensors in our language. In section 6 we discuss a rule which allows to count conformal $n$-point functions in terms of on-shell scattering amplitudes of higher spin massive fields in $(d+1)$-dimensions and, in case of conserved tensors, massless fields. For the case of three-point functions of conserved operators with spin $l_i$ in dimension $d \geq 4$, this gives the number of allowed tensor structures to be $1 + \min(l_1, l_2, l_3)$. Section 7 gives a summary of the new algorithm for dealing with CFT correlation functions and concludes.

## 2 Embedding formalism

In this paper we consider CFT in $d \geq 3$ Euclidean dimensions, so that the conformal group is $\mathrm{SO}(d+1,1)$. All of our equations can be Wick-rotated to the Minkowski signature, paying attention to the $i\epsilon$ prescription. We assume that the reader is familiar with the basics of the theory, see e.g. [27], chapter 4. As is well known, conformal symmetry imposes strong constraints on the correlation functions of primary operators in the theory. These constraints are relatively easy to work out for primary scalars, but they become less transparent for primary fields of nonzero spin. In this section we will develop the 'embedding formalism' which makes the nonzero spin case easier. The formalism has been applied on and off since the early CFT days [23, 24]. We will take as a starting point a version used recently in [25] (see also [26] for a recent discussion).[3]

The basic idea, due to Dirac [21], is that the natural habitat for the conformal group $\mathrm{SO}(d+1,1)$ is the *embedding space* $\mathbb{M}^{d+2}$, where it can be realized as the group of linear isometries. Thus, conformal symmetry constraints should become as trivial as Lorentz symmetry constraints, provided all CFT fields can somehow be lifted to $\mathbb{M}^{d+2}$. The lift is accomplished via a sort of stereographic projection; see figure 1. First, a point $x \in \mathbb{R}^d$ is put in correspondence with a null ray in $\mathbb{M}^{d+2}$ consisting of the vectors

$$P^A = \lambda \left(1, x^2, x^a\right) , \qquad \lambda \in \mathbb{R} \, , \tag{2.1}$$

where we use light cone coordinates

$$P^A = \left(P^+, P^-, P^a\right) , \tag{2.2}$$

[3] Additional work using six-dimensional field equations to describe four-dimensional theories has been done, e.g., in [28].

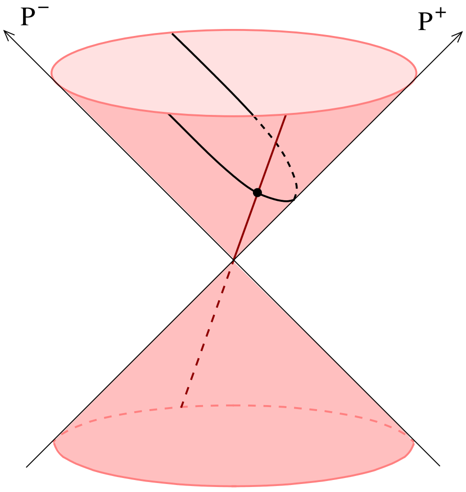


**Figure 1**. Light cone in the embedding space; light rays are in one-to-one correspondence with physical space points. The Poincaré section of the cone is also shown.

with metric given by[4]

$$P \cdot P \equiv \eta_{AB}\, P^A P^B = -P^+ P^- + \delta_{ab}\, P^a P^b \,. \tag{2.3}$$

Here and below, we use capital letters to denote embedding space ($\mathbb{M}^{d+2}$) quantities and lower case letters to denote physical space ($\mathbb{R}^d$) quantities.

Now, a linear SO$(d+1,1)$ transformation of $\mathbb{M}^{d+2}$ will map null rays into null rays, and via eq. (2.1) this defines a map of the physical space $\mathbb{R}^d$ into itself, which turns out to be a conformal transformation in the usual sense. Moreover, every conformal transformation can be realized this way [21].

Next we should establish the correspondence between fields on $\mathbb{R}^d$ and $\mathbb{M}^{d+2}$, which is done as follows. Consider a field $F_{A_1 \ldots A_l}(P)$, a tensor of SO$(d+1,1)$, with the following properties:

1. Defined on the cone $P^2 = 0$.
2. Homogeneous of degree $-\Delta$: $F_{A_1 \ldots A_l}(\lambda P) = \lambda^{-\Delta} F_{A_1 \ldots A_l}(P)$, $\lambda > 0$.
3. Symmetric and traceless.
4. Transverse: $(P \cdot F)_{A_2 \ldots A_l} \equiv P^A F_{A A_2 \ldots A_l} = 0$.

Notice that all these conditions are manifestly SO$(d+1,1)$-invariant. Because of homogeneity, $F$ is known everywhere on the cone once it is known on the *Poincaré section*,[5]

$$P_x^A = (1, x^2, x^a)\,, \qquad x \in \mathbb{R}^d\,, \tag{2.4}$$

[4]Here $\delta_{ab} \to \eta_{ab}$ when Wick-rotating to the Minkowski spacetime signature.

[5]Other sections of the cone could be useful to study CFT on curved, conformally flat, backgrounds.

whose vectors are in one-to-one correspondence with the points of $\mathbb{R}^d$. Projecting $F$ to the Poincaré section defines a symmetric tensor field on $\mathbb{R}^d$:[6]

$$f_{a_1\ldots a_l}(x) = \frac{\partial P^{A_1}}{\partial x^{a_1}} \cdots \frac{\partial P^{A_l}}{\partial x^{a_l}} F_{A_1\ldots A_l}(P_x)\,. \tag{2.5}$$

This operation has two important properties. First, any tensor proportional to $P_A$ projects to zero. We will call such $\mathrm{SO}(d+1,1)$ tensors *pure gauge* [23]. It is not difficult to show that if two symmetric transverse tensors $F$ and $F'$ project to the same $f$, then they differ by pure gauge (this is valid point by point on the Poincaré section).

Second, the projected tensor is traceless, as long as $F$ is traceless and transverse. This follows from the identity

$$K^{AB} \equiv \delta^{ab}\frac{\partial P^A}{\partial x^a}\frac{\partial P^B}{\partial x^b} = \eta^{AB} + P_x^A \bar{P}^B + P_x^B \bar{P}^A\,, \qquad \bar{P} = (0,2,0)\,, \tag{2.6}$$

which is easily verified by using the explicit form of the projection matrices:

$$\frac{\partial P^A}{\partial x^c} = (0, 2x_c, \delta^a_c)\,. \tag{2.7}$$

Given that any conformal transformation can be realized as an $\mathrm{SO}(d+1,1)$ rotation, and that $F$ transforms as a tensor of $\mathrm{SO}(d+1,1)$, it makes sense to ask how $f$ defined by (2.5) transforms under the conformal group. It can be shown [24, 26][7] that this transformation is exactly that of a spin $l$ symmetric traceless primary field of dimension $\Delta$. This is actually not surprising. Since the $f \leftrightarrow F$ correspondence is one-to-one up to pure gauge, and since pure gauge goes into pure gauge under $\mathrm{SO}(d+1,1)$, it is clear that we will have a bona fide transformation of $f$ in the sense that any ambiguity in lifting $f$ to the cone will drop out. But the Euclidean fields which transform into themselves under the conformal group are exactly the primary fields. The only question is the interpretation of the $\Delta$ parameter, and an explicit analysis shows that it has the meaning of the scaling dimension.

*To summarize*: instead of working with primary tensor fields in the physical space, we can do the computations with tensor fields in $\mathbb{M}^{d+2}$, where $\mathrm{SO}(d+1,1)$ invariance is manifest, and project the result to $\mathbb{R}^d$ using (2.5). Conformal invariance of the final result will be automatic.

### 2.1 Correlators: simplest examples

The embedding formalism provides a shortcut to solving constraints imposed by conformal symmetry on the form of CFT correlators. Consider e.g. the correlator of three primary scalars $\langle\phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\rangle$ of dimensions $\Delta_i$. It can be obtained by projecting the embedding correlator

$$\langle\Phi_1(P_1)\Phi_2(P_2)\Phi_3(P_3)\rangle = \frac{const}{(P_{12})^{\frac{\Delta_1+\Delta_2-\Delta_3}{2}}(P_{23})^{\frac{\Delta_2+\Delta_3-\Delta_1}{2}}(P_{31})^{\frac{\Delta_3+\Delta_1-\Delta_2}{2}}}\,, \tag{2.8}$$

[6] Here and below, we omit the dependence of $P_x$ on $x$ in $\partial P/\partial x$, to avoid cluttering.

[7] Ref. [24] imposes a divergence-free condition to fix the pure gauge terms in $F$, which leads to unnecessary complications.



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where we define

$$P_{ij} \equiv -2P_i \cdot P_j \,. \tag{2.9}$$

It's easy to see that the written form of the correlator is the only one consistent with the $\mathrm{SO}(d+1,1)$ invariance and the degree $-\Delta_i$ homogeneity of each $\Phi_i(P_i)$. For scalars, projection to the physical space amounts to $P_i \to P_{x_i}$. Using the identity

$$-2P_{x_i} \cdot P_{x_j} = x_{ij}^2 \qquad (x_{ij} \equiv x_i - x_j)\,, \tag{2.10}$$

we obtain the well-known result [29]

$$\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\rangle = \frac{const}{(x_{12}^2)^{\frac{\Delta_1+\Delta_2-\Delta_3}{2}}(x_{23}^2)^{\frac{\Delta_2+\Delta_3-\Delta_1}{2}}(x_{31}^2)^{\frac{\Delta_3+\Delta_1-\Delta_2}{2}}}\,. \tag{2.11}$$

As a second example, consider the two-point function $\langle v_a(x_1)v_b(x_2)\rangle$ of a dimension $\Delta$ primary vector, described in the embedding formalism by the correlator

$$G_{AB}(P_1,P_2) \equiv \langle V_A(P_1)V_B(P_2)\rangle\,. \tag{2.12}$$

$G_{AB}$ must be an $\mathrm{SO}(d+1,1)$ tensor satisfying the following properties:

$$G_{AB}(\lambda P_1,P_2) = G_{AB}(P_1,\lambda P_2) = \lambda^{-\Delta}G_{AB}(P_1,P_2)\,, \tag{2.13}$$

$$P_1^A\, G_{AB}(P_1,P_2) = 0\,, \qquad\qquad P_2^B\, G_{AB}(P_1,P_2) = 0\,, \tag{2.14}$$

following from the homogeneity and transversality conditions obeyed by $V_A(P)$. It is not difficult to convince oneself that the most general such tensor has the form

$$G_{AB}(P_1,P_2) = \frac{1}{(P_{12})^\Delta}\left[c_1\tilde{W}_{AB} + c_2\frac{P_{1A}P_{2B}}{P_1\cdot P_2}\right]\,, \tag{2.15}$$

where

$$\tilde{W}_{AB} = \eta_{AB} - \frac{P_{1B}P_{2A}}{P_1\cdot P_2}\,. \tag{2.16}$$

(The reason for the tilde in $W$ will become clear shortly.) It remains to project to the physical space, using eqs. (2.5) and (2.7). The second term in $G_{AB}$ is pure gauge and projects to zero. A short computation shows that $\tilde{W}_{AB}$ projects to

$$w_{ab} = \delta_{ab} - 2\,\frac{(x_{12})_a(x_{12})_b}{x_{12}^2}\,, \tag{2.17}$$

and we get the well-known result

$$\langle v_a(x_1)v_b(x_2)\rangle = c_1\frac{w_{ab}}{(x_{12}^2)^\Delta}\,. \tag{2.18}$$

The spin 2 case is analogous but with more indices. The embedding space two-point function is given by (up to pure gauge terms)[8]

$$G_{A_1A_2,B_1B_2}(P_1,P_2) = \frac{const}{(P_{12})^\Delta}\left[\frac{1}{2}\left(\tilde{W}_{A_1B_1}\tilde{W}_{A_2B_2} + \tilde{W}_{A_1B_2}\tilde{W}_{A_2B_1}\right) - \frac{1}{d}\,W_{A_1A_2}W_{B_1B_2}\right], \tag{2.19}$$

[8]The same expression with all $\tilde{W}$'s replaced by $W$'s would work as well, differing only by pure gauge terms. We choose the given form to facilitate comparison with projector $\Pi'$ used in eq. (3.25) below.

where we introduced the symmetric tensor

$$W_{AB} = \eta_{AB} - \frac{P_{1B}P_{2A} + P_{1A}P_{2B}}{P_1 \cdot P_2}\,, \tag{2.20}$$

differing from $\tilde{W}$ by a pure gauge term. Since both $W$ and $\tilde{W}$ are transverse, so is the above two-point function. To show that it is also traceless, notice that

$$\eta^{A_1A_2}W_{A_1A_2} = d\,, \qquad \eta^{A_1A_2}\tilde{W}_{A_1B_1}\tilde{W}_{A_2B_2} = W_{B_1B_2}\ . \tag{2.21}$$

Finally, the physical space two-point function is now obtained by projecting, which amounts to replacing $W, \tilde{W} \to w$.

The generalization to higher $l$ is, in principle, straightforward. The two-point function can always be given by a symmetrized product of $\tilde{W}_{A_iB_j}$ with trace terms subtracted using $W_{A_iA_j}$. However, the computations become increasingly cumbersome due to the proliferation of indices, particularly if we wish to compute three-point and four-point functions. It would be nice to have a more compact formalism, which for example would allow not to keep track of the trace terms. That this should be possible is intuitively clear, since these terms are not independent: they are fixed by the requirement of the overall tracelessness. In the next section we will describe such a formalism, which also has the advantage of being *index-free*.

## 3 Encoding tensors by polynomials

To begin, we will introduce a technique which allows us to represent symmetric tensors by means of polynomials obtained by contracting the tensor with a reference vector. While the basic idea is very simple, it requires some effort to develop an efficient formalism fully taking into account the tracelessness and transversality conditions. The reader may prefer to read backwards starting from the example given in section 3.3. The less essential parts (proofs) are given in smaller font and can be skipped on the first reading.

### 3.1 Tensors in the physical space

The basic idea is that any symmetric tensor can be encoded by a $d$-dimensional polynomial:

$$f_{a_1\ldots a_l} \text{ symmetric} \leftrightarrow f(z) \equiv f_{a_1\ldots a_l} z^{a_1} \cdots z^{a_l}. \tag{3.1}$$

The correspondence is clearly one-to-one: expanding the polynomial we recover the tensor.

In CFT, spin $l$ primary fields are symmetric *traceless* tensors, for which a more economical encoding is available. Such a tensor can be fully encoded by restricting the respective polynomial $f(z)$ to the submanifold $z^2 = 0$:[9]

$$f_{a_1\ldots a_l} \text{ symmetric traceless} \leftrightarrow f(z)|_{z^2=0}. \tag{3.2}$$

[9]Assuming $z$ is complex.

This fact can be formulated more fully as follows. Let $f_{a_1\ldots a_l}$ be a symmetric traceless tensor, and $\tilde{f}_{a_1\ldots a_l}$ be another symmetric tensor such that the polynomials $\tilde{f}(z)$ and $f(z)$ differ only by terms vanishing on $z^2=0$:

$$f(z) = \tilde{f}(z) + O(z^2). \tag{3.3}$$

Then $f_{a_1\ldots a_l}$ can be recovered from $\tilde{f}(z)$ (or from $\tilde{f}_{a_1\ldots a_l}$, which is the same).

Intuitively, this can be justified as follows.[10] Consider the projector onto symmetric traceless tensors:

$$\pi_{a_1\ldots a_l, b_1\ldots b_l} = \delta_{a_1(b_1}\cdots\delta_{|a_l|b_l)} - \text{traces}\,. \tag{3.4}$$

eq. (3.3) means that $f_{a_1\ldots a_l}$ and $\tilde{f}_{a_1\ldots a_l}$ can differ only by terms proportional to $\delta_{a_i a_j}$. All such terms will be subtracted away by the projector, and thus we will have:

$$f_{a_1\ldots a_l} = \pi_{a_1\ldots a_l, b_1\ldots b_l}\tilde{f}^{b_1\ldots b_l}\,. \tag{3.5}$$

*To summarize the discussion so far*: we will present results for physical-space correlators in terms of polynomials, not in terms of tensors. Moreover, we can and will drop any polynomial terms explicitly proportional to $z^2$. This gives a polynomial which encodes the original symmetric traceless tensor in the sense of eq. (3.3). The dropped terms do not create any ambiguity, as the original tensor can be recovered via (3.5).

For small values of $l$, the projector appearing in (3.5) is easy to work out explicitly, e.g.

$$\pi_{a_1 a_2, b_1 b_2} = \frac{1}{2}\left(\delta_{a_1 b_1}\delta_{a_2 b_2} + \delta_{a_1 b_2}\delta_{a_2 b_1}\right) - \frac{1}{d}\,\delta_{a_1 a_2}\delta_{b_1 b_2}\,. \tag{3.6}$$

The higher-spin projectors can be generated efficiently[11] by the differential operator of [32][12]

$$D_a = \left(h - 1 + z\cdot\frac{\partial}{\partial z}\right)\frac{\partial}{\partial z^a} - \frac{1}{2}z_a\frac{\partial^2}{\partial z\cdot\partial z}\,, \tag{3.7}$$

where we defined the shorthand $h \equiv d/2$. We then have

$$\pi_{a_1\ldots a_l, b_1\ldots b_l} = \frac{1}{l!(h-1)_l}D_{a_1}\cdots D_{a_l} z_{b_1}\cdots z_{b_l}\,, \tag{3.8}$$

where $(a)_l = \Gamma(a+l)/\Gamma(a)$ is the Pochhammer symbol. It follows that $f_{a_1\ldots a_l}$ can be recovered from a $\tilde{f}(z)$ by differentiation:

$$f_{a_1\ldots a_l} = \frac{1}{l!(h-1)_l}D_{a_1}\cdots D_{a_l}\tilde{f}(z)\,. \tag{3.9}$$

[10] A mathematician's proof that the correspondence (3.2) is one-to-one goes as follows. First, observe that symmetric traceless tensors are mapped by (3.1) onto *harmonic* polynomials. Then, use the following theorem (see [30], section 4.2): *Any d-dimensional polynomial $p(z)$ can be uniquely split as $p(z) = p_0(z) + z^2\,p_1(z)$, with $p_0(z)$ harmonic.*

[11] An alternative is to use recursion relations, see e.g. [31].

[12] See [33] for a recent use of this operator in a similar context. It was also pointed out to us by Andrew Waldron that this operator appears in the context of 'tractor calculus', where it is called the Thomas operator [34].

The $D_a$ operator is very convenient as it allows to perform operations on traceless symmetric tensors directly in terms of the polynomials that encode them. For example, consider two rank $l$ symmetric traceless tensors $f$ and $g$, encoded (in the sense of eq. (3.3)) by $\tilde{f}(z)$ and $\tilde{g}(z)$. Then their full contraction can be found by evaluating

$$f_{a_1\ldots a_l} g^{a_1\ldots a_l} = \frac{1}{l!(h-1)_l} \tilde{f}(D)\tilde{g}(z)\,. \tag{3.10}$$

If we need to free just one index but leave the rest contracted with $z$, this is done by evaluating

$$f_{aa_2\ldots a_l} z^{a_2}\cdots z^{a_l} = \frac{1}{l(h+l-2)} D_a \tilde{f}(z) + O(z^2)\,, \tag{3.11}$$

and so on.

We will just give a general idea of how these statements can be proven; see appendix A of [35] for more details. It is crucial that $D_a$ is an 'interior operator' on the cone, which means that it maps $O(z^2)$ functions to themselves:

$$h(z) = O(z^2) \implies D_a h(z) = O(z^2)\,. \tag{3.12}$$

In particular, we have

$$D_a \tilde{f}(z) = D_a f(z) + O(z^2). \tag{3.13}$$

Furthermore, tracelessness of $f$ implies that the polynomial $f(z)$ is harmonic. Thus the second term in $D_a$ does not contribute to $D_a f$, while the first term gives

$$D_a f(z) = \left(h - 1 + z\cdot\frac{\partial}{\partial z}\right)\frac{\partial}{\partial z^a} f(z) = (h+l-2)l\, f_{aa_2\ldots a_l} z^{a_2}\cdots z^{a_l}, \tag{3.14}$$

where we used the fact that $z\cdot\frac{\partial}{\partial z}$ computes the homogeneity degree, $l-1$ in this case. This proves eq. (3.11); the other properties can be shown analogously.

### 3.2 Tensors in the embedding space

Next we will extend the above discussion to the embedding space. We can similarly encode a general symmetric tensor in the embedding space by a $(d+2)$-dimensional polynomial

$$F_{A_1\ldots A_l}(P) \text{ symmetric} \leftrightarrow F(P;Z) \equiv F_{A_1\ldots A_l}(P) Z^{A_1}\ldots Z^{A_l}. \tag{3.15}$$

This notation emphasizes that the tensors will in general depend on $P$.

Now let us consider the following diagram relating embedding and physical tensors both with free indices and with encoding polynomials:

$$\begin{tikzcd} F_{A_1\ldots A_l}(P) \arrow[r, "(3.15)"] \arrow[d, "(2.5)"'] & F(P;Z) \arrow[d, dashed] \\ f_{a_1\ldots a_l}(x) \arrow[r, "(3.1)"] & f(x;z) \end{tikzcd} \tag{3.16}$$

The dashed line denotes that there is a relation between the encoding polynomial of an embedding tensor and its projection to the physical space. Using the explicit form of $\partial P/\partial x$ given in eq. (2.7), this relation takes the form

$$f(x;z) = F(P_x; Z_{z,x}), \tag{3.17}$$



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where $Z_{z,x} \equiv (0, 2x\cdot z, z)$ and has the properties

$$Z_{z,x}\cdot P_x = 0\,, \qquad Z^2_{z,x} = z^2\,. \tag{3.18}$$

Let us now specialize to tensors which are symmetric, traceless, and transverse (STT). For such tensors, we can restrict the polynomial to the subset of $Z$'s satisfying $Z^2 = 0$ and $Z\cdot P = 0$:

$$F_{A_1\ldots A_l}(P)\quad \text{STT} \qquad \leftrightarrow \qquad F(P;Z)|_{Z^2=0,Z\cdot P=0}\,. \tag{3.19}$$

More precisely, we mean the following. Let $F_{A_1\ldots A_l}(P)$ be STT and $\tilde{F}_{A_1\ldots A_l}(P)$ be any tensor whose polynomial happens to agree with $F(P;Z)$ modulo terms proportional to $Z^2$ and $Z\cdot P$:

$$F(P;Z) = \tilde{F}(P;Z) + O(Z^2, Z\cdot P). \tag{3.20}$$

Then $F_{A_1\ldots A_l}(P)$ can be recovered from $\tilde{F}_{A_1\ldots A_l}(P)$ up to pure gauge terms.

Indeed, as discussed in section 2, the tensor $F$ can be recovered up to pure gauge from its symmetric traceless projection $f$. Thus it is enough to show that $f$ can be determined from $\tilde{f}$, the projection of $\tilde{F}$. To see the latter, let us project eq. (3.20) to the physical space. Using the rule (3.17) and the properties (3.18), we obtain

$$f(x;z) = \tilde{f}(x;z) + O(z^2), \tag{3.21}$$

so $f$ can indeed be recovered from $\tilde{f}$ by one of the methods from section 3.1.

Since it will prove useful in future applications, let us give a more explicit way to recover an STT tensor $F_{A_1\ldots A_l}(P)$ from $\tilde{F}_{A_1\ldots A_l}(P)$ in the case that $\tilde{F}$ is transverse (but not necessarily traceless). In this case the projection takes the form

$$F_{A_1\ldots A_l} = \Pi_{A_1\ldots A_l,B_1\ldots B_l}\tilde{F}^{B_1\ldots B_l}\,, \tag{3.22}$$

where the projector $\Pi$ is obtained from the projector $\pi$ in eq. (3.5) by replacing

$$\delta_{a_i a_j} \to W_{A_i A_j} \equiv \eta_{A_i A_j} - \frac{P_{A_i}\bar{P}_{A_j} + P_{A_j}\bar{P}_{A_i}}{P\cdot\bar{P}}, \qquad \delta_{b_i b_j} \to \eta_{B_i B_j}, \qquad \delta_{a_i b_j} \to \eta_{A_i B_j}. \tag{3.23}$$

Here $\bar{P}$ is as in eq. (2.6). The rule may look strange, since the projector $\pi$ subtracts traces in $d$ dimensions, while $\Pi$ must do this in $d+2$ dimension. This connection between $\pi$ and $\Pi$ has to do with the assumed transversality of $\tilde{F}$.

To prove that the above formula works, notice first of all that the tensor $F$ as defined differs from $\tilde{F}$ only by terms which are proportional to $\eta_{A_i A_j}$ or $P_{A_i}$. Upon contraction with $Z$, this gives terms of $O(Z^2, Z\cdot P)$, consistent with eq. (3.20). It remains to show that $F$ is transverse and traceless. To this end, consider a different projector $\Pi'$ obtained from $\pi$ by a list of replacements which contains some extra terms compared to (3.23):

$$\delta_{a_i a_j} \to W_{A_i A_j}, \quad \delta_{b_i b_j} \to W_{B_i B_j}, \quad \delta_{a_i b_j} \to \tilde{W}_{A_i B_j} \equiv \eta_{A_i B_j} - \frac{\bar{P}_{A_i} P_{B_j}}{P\cdot\bar{P}}. \tag{3.24}$$

However, all the extra terms are proportional to $P_{B_i}$, and will vanish when contracted with $\tilde{F}$ under the assumption that it is transverse. For this reason we have an equivalent representation for $F$ as

$$F = \Pi'\tilde{F}. \tag{3.25}$$

In this form transversality and tracelessness are pretty easy to see. They just follow from the transversality of $W$ and $\tilde{W}$, and from the relations (2.21) that we already used to show that the spin-2 two-point function (2.19) was transverse and traceless. Indeed, as the reader may have noticed, that two-point function had precisely the structure of the traceless projector in $d$ dimensions, eq. (3.6).

In this paper, we will be primarily dealing with tensors which are made from metrics and from components of $\mathbb{M}^{d+2}$ vectors, such as in eq. (2.19). For such tensors, the canonical rule to get the encoding polynomial $\tilde{F}(P;Z)$ in eq. (3.20) is to simply drop all terms in $F(P;Z)$ which are proportional to $Z^2$ and $Z\cdot P$. This rule is also very convenient because it preserves the transversality condition, and even makes it stronger, in a sense that we now discuss.

In general, a transverse tensor $F_{A_1\ldots A_l}$ may contain terms which are pure gauge, and the condition $P\cdot F=0$ is only valid modulo $P^2$ terms, vanishing on the cone. We will call a tensor *identically transverse* if this condition happens to be satisfied identically, without using $P^2=0$. For example, the tensor $\tilde{W}$ from eq. (2.16) is identically transverse with respect to $P_1^A$ and $P_2^B$, while $W$ from eq. (2.20) is not. Notice that $\tilde{W}$ can be obtained from $W$ by dropping the pure gauge term. This is in fact a partial case of the following more general rule:

Take any tensor $F_{A_1\ldots A_l}(P)$ which is

1. Transverse modulo $P^2$ terms.

2. Made out of metrics and components of $P$, as well as of components of one or more vectors $Q\neq P$.

Drop any terms in the tensor which are proportional to $P^2$, $\eta_{A_iA_j}$, or $P_{A_i}$. The resulting tensor $\tilde{F}_{A_1\ldots A_l}(P)$ will be identically transverse.

To prove this, let us write $F=\tilde{F}+\hat{F}$, where $\hat{F}$ contains all terms which are to be dropped. Then $P\cdot\tilde{F}$ will contain terms proportional to $Q_{A_i}$, with coefficients which are scalar functions of $(P\cdot Q)$ and $(Q\cdot Q')$ (if there are several $Q$'s). On the other hand, $P\cdot\hat{F}$ will contain terms proportional to $P_{A_i}$ and/or $P^2$. There cannot be cancellation between these two groups of terms, and if $P\cdot F$ is to vanish on $P^2=0$, $P\cdot\tilde{F}$ must vanish identically.

Going back to the encoding polynomials, the transversality condition takes the form

$$P\cdot\frac{\partial}{\partial Z}\,F(P;Z)=0\,, \tag{3.26}$$

or equivalently

$$F(P;Z+\alpha P)=F(P;Z)\qquad(\forall\alpha)\,. \tag{3.27}$$

These conditions are satisfied modulo $P^2$ in general, and identically if the tensor is identically transverse. Translating the above discussion, the identically transverse polynomial $\tilde{F}(P;Z)$ is obtained from $F(P;Z)$ by dropping all terms proportional to $Z^2$ and $Z\cdot P$. This is precisely the 'canonical rule' introduced above.

The above discussion will prove very useful below, because the identically transverse polynomials are easy to characterize. It is not difficult to convince oneself that the following rule is true: a polynomial $\tilde{F}(P;Z)$ is identically transverse if and only if the variable $Z_A$ appears in it only via the tensor:

$$C_{AB} \equiv Z_A P_B - Z_B P_A \,. \tag{3.28}$$

To conclude this section, let us show how to compute tensor contractions using the embedding space. The problem is formulated as follows. We want to contract two symmetric traceless tensors $f_{a_1 \ldots a_l}(x)$ and $g_{a_1 \ldots a_l}(x)$. It is assumed that these tensors are projections of the embedding space STT tensors $F_{A_1 \ldots A_l}(P)$ and $G_{A_1 \ldots A_l}(P)$. The latter tensors will typically not be given in components, but in terms of their encoding polynomials $\tilde{F}(P;Z)$ and $\tilde{G}(P;Z)$ (in the sense of eq. (3.20)). Finally, we will assume that these polynomials are transverse in the sense of eq. (3.27).[13] We then have the formula (cf. eq. (3.9)):

$$f_{a_1 \ldots a_l}(x) g^{a_1 \ldots a_l}(x) = \frac{1}{l!(h-1)_l} \tilde{F}(P_x; D) \tilde{G}(P_x; Z) \,, \tag{3.29}$$

where

$$D_A = \left( h - 1 + Z \cdot \frac{\partial}{\partial Z} \right) \frac{\partial}{\partial Z^A} - \frac{1}{2} Z_A \frac{\partial^2}{\partial Z \cdot \partial Z} \tag{3.30}$$

is the same differential operator as $D_a$ made to act in the $(d+2)$-dimensional space. We stress that $h = d/2$ here as in eq. (3.7).

Let us give a quick proof. Using the notation of section 3.1, we have

$$f_{a_1 \ldots a_l} g^{a_1 \ldots a_l} = \tilde{f}_{a_1 \ldots a_l} \pi^{a_1 \ldots a_l, b_1 \ldots b_l} \tilde{g}_{b_1 \ldots b_l} = \tilde{F}_{A_1 \ldots A_l} Q^{A_1 \ldots A_l, B_1 \ldots B_l} \tilde{G}_{B_1 \ldots B_l} \,, \tag{3.31}$$

where $\tilde{f}$ and $\tilde{g}$ are the projections of $\tilde{F}$ and $\tilde{G}$ to the physical space, and $Q$ is given by

$$Q^{A_1 \ldots A_l, B_1 \ldots B_l} = \pi^{a_1 \ldots a_l, b_1 \ldots b_l} \frac{\partial P^{A_1}}{\partial x^{a_1}} \cdots \frac{\partial P^{A_l}}{\partial x^{a_l}} \frac{\partial P^{B_1}}{\partial x^{b_1}} \cdots \frac{\partial P^{B_l}}{\partial x^{b_l}} \,. \tag{3.32}$$

Remember that the projector $\pi$ is made out of $d$-dimensional metric tensors. This equation then means that the projector $Q$ can be obtained from $\pi$ by replacing each metric $\delta^{ab}$ by the effective metric $K^{AB}$ defined in eq. (2.6) (unlike in the definition of $\Pi$ above, here the replacement rule is the same whether the indices are of $a$ or $b$ type). For transverse tensors $\tilde{F}$ or $\tilde{G}$ we can replace $K^{AB}$ by $\eta^{AB}$ because the extra terms vanish identically. A moment's thought shows that this reduces (3.29) to (3.10).

### 3.3 Example

Let us now demonstrate the above formal discussion on a concrete example: the spin 2 embedding space two-point function (2.19). Since it's a double tensor, we assign to it a polynomial of two vectors $Z_1$ and $Z_2$, which defines the embedding correlation function

$$\langle F(P_1; Z_1) F(P_2; Z_2) \rangle = G(P_1, P_2; Z_1, Z_2) = Z_1^{A_1} Z_1^{A_2} Z_2^{B_1} Z_2^{B_2} \, G_{A_1 A_2, B_1 B_2}(P_1, P_2) \,. \tag{3.33}$$

[13] Although not essential here, in applications they will often be even identically transverse.

We then have the following basic contractions:

$$Z_1^A Z_2^B \tilde{W}_{AB} = (Z_1 \cdot Z_2) - \frac{(Z_1 \cdot P_2)(Z_2 \cdot P_1)}{P_1 \cdot P_2}, \tag{3.34}$$

$$Z_1^A Z_1^{A'} W_{AA'} = O(Z_1^2, Z_1 \cdot P_1), \qquad Z_2^B Z_2^{B'} W_{BB'} = O(Z_2^2, Z_2 \cdot P_2). \tag{3.35}$$

It follows that

$$\tilde{G}(P_1, P_2; Z_1, Z_2) = const \, \frac{\left((Z_1 \cdot Z_2)(P_1 \cdot P_2) - (P_1 \cdot Z_2)(P_2 \cdot Z_1)\right)^2}{(P_{12})^{\Delta+2}}, \tag{3.36}$$

where we applied the canonical rule of dropping the $O(Z_i^2, Z_i \cdot P_i)$ terms to get the encoding polynomial. Notice that $\tilde{G}$ is identically transverse, as it should be according to the discussion in section 3.2. This is already a pretty compact expression; the advantage of not having to deal with indices is starting to show.

What about the two-point function in physical space? We will write it as a polynomial contracted with $z_1$ and $z_2$. This polynomial is obtained by making the substitutions $P_i \to P_{x_i}$, $Z_i \to Z_{z_i,x_i}$ in $\tilde{G}$. Evaluating the scalar products

$$Z_1 \cdot Z_2 \to z_1 \cdot z_2, \qquad P_1 \cdot P_2 \to -\frac{1}{2} x_{12}^2, \tag{3.37}$$

$$P_1 \cdot Z_2 \to z_2 \cdot x_{12}, \qquad P_2 \cdot Z_1 \to -z_1 \cdot x_{12}, \tag{3.38}$$

we find

$$g(x_1, x_2; z_1, z_2) = const \, \frac{\left((z_1 \cdot x_{12})(z_2 \cdot x_{12}) - \frac{1}{2} x_{12}^2 (z_1 \cdot z_2)\right)^2}{(x_{12}^2)^{\Delta+2}}, \tag{3.39}$$

up to $O(z_i^2)$ terms (see eq. (3.21)). In the index-free approach that we are advocating here, this expression is *the* final answer. The indexed version can be extracted if necessary by acting with $D_a$ operators on the encoding polynomial, or in a more pedestrian way, by expanding in $z_i^a$ and acting on the coefficient tensor with the projector $\pi$. But in this paper we will not do this.

## 4 Correlation functions of spin $l$ primaries

Unitary irreducible representations of the conformal group $\mathrm{SO}(d+1,1)$ are labeled by a conformal dimension $\Delta$ and an irreducible representation of $\mathrm{SO}(d)$. In this paper, we focus on totally symmetric traceless tensors of $\mathrm{SO}(d)$. These are the spin $l$ primaries, which we will label by $\chi \equiv [l, \Delta]$. In this section, we discuss constraints imposed by conformal symmetry on the coordinate dependence of their correlators. The additional constraints appearing for conserved tensors will be discussed in the next section.

### 4.1 Two-point functions

Consider the two-point function of a spin $l$ primary in the embedding space:

$$G_{A_1 \ldots A_l, B_1 \ldots B_l}(P_1, P_2). \tag{4.1}$$



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Following the technique from the previous section, we will encode it by a function

$$G_\chi(P_1, P_2; Z_1, Z_2) = Z_1^{A_1} \cdots Z_1^{A_l} Z_2^{B_1} \cdots Z_2^{B_2}\, G_{A_1 \dots A_l, B_1 \dots B_l}(P_1, P_2)\,. \tag{4.2}$$

We have the following three conditions:

$$G_\chi(\lambda_1 P_1, \lambda_2 P_2; Z_1, Z_2) = (\lambda_1 \lambda_2)^{-\Delta} G_\chi(P_1, P_2; Z_1, Z_2)\,, \tag{4.3}$$

$$G_\chi(P_1, P_2; \beta_1 Z_1, \beta_2 Z_2) = (\beta_1 \beta_2)^l G_\chi(P_1, P_2; Z_1, Z_2)\,, \tag{4.4}$$

$$G_\chi(P_1, P_2; Z_1 + \alpha_1 P_1, Z_2 + \alpha_2 P_2) = G_\chi(P_1, P_2; Z_1, Z_2)\,. \tag{4.5}$$

The first condition follows from the fact that the embedding space fields are homogeneous of degree $-\Delta$. The second one is a fancy way of saying that $G_\chi$ is a degree $l$ polynomial in $Z_1$ and $Z_2$. The final condition encodes the transversality of the embedding space tensors; it must be satisfied modulo $O(P^2)$ terms.

As discussed in section 3.2, we may drop all the terms in $G_\chi$ proportional to $Z_i^2$ and $Z_i \cdot P_i$. The resulting function $\tilde{G}_\chi$ will be identically transverse, in the sense that it will satisfy eq. (4.5) identically, and not just modulo $O(P^2)$. The general recipe for constructing such functions says that they must be built out of the $C_{AB}$-type tensors from eq. (3.28):

$$C_{iAB} = Z_{iA} P_{iB} - Z_{iB} P_{iA} \qquad (i = 1, 2)\,. \tag{4.6}$$

Now contracting $C_i$ with itself gives terms of the kind that we dropped, and so the only possibility is to start contracting the indices of $C_1$ and $C_2$. Full contraction gives the building block

$$H_{12} \equiv -C_1 \cdot C_2 = -2\big[(Z_1 \cdot Z_2)(P_1 \cdot P_2) - (P_1 \cdot Z_2)(P_2 \cdot Z_1)\big]\,, \tag{4.7}$$

of weight one in both $Z_1$ and $Z_2$. More generally, one could try taking the trace of a string of several alternating $C_1$'s and $C_2$'s. However, one can check that

$$(C_1 C_2 C_1)_{AB} = -\frac{1}{2}\,(C_1 \cdot C_2)\, C_{1AB}\,. \tag{4.8}$$

For this reason, such iterated contractions reduce to powers of $C_1 \cdot C_2$. We conclude that the most general solution is a function of $C_1 \cdot C_2$. The spin of the operators fixes the weight in the $Z$'s, so we obtain that (cf. eq. (3.36))

$$\tilde{G}_\chi(P_1, P_2; Z_1, Z_2) = const\, \frac{H_{12}^l}{(P_{12})^{\Delta + l}}\,. \tag{4.9}$$

Thus we recover the well-known unique two-point function of spin $l$ primaries [24].

### 4.2 Three-point functions

The scalar three-point function was already given in eq. (2.8). In this section we will discuss the arbitrary spin case using the embedding formalism. It is well known that such three-point functions can be written as a linear combination of a finite number of conformally invariant building blocks [36–40]. Here, we present the explicit form of these building blocks in the embedding formalism.



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### 4.2.1 Scalar-scalar-spin $l$

Let us start with the scalar-scalar-spin $l$ case. The scalar operators of dimensions $\Delta_1$ and $\Delta_2$ are placed at points $P_1$ and $P_2$. The third operator, a symmetric traceless tensor of spin $l$ and dimension $\Delta_3$, is placed at $P_3$. In this case, the correlator is completely fixed by conformal invariance. We have ($l_3 = l$)

$$\tilde{G}_{\chi_1,\chi_2,\chi_3}(P_1, P_2, P_3; Z_3) = const \, \frac{\left((Z_3 \cdot P_1)(P_2 \cdot P_3) - (Z_3 \cdot P_2)(P_1 \cdot P_3)\right)^l}{(P_{12})^{\frac{\Delta_1+\Delta_2-\Delta_3+l}{2}} (P_{23})^{\frac{\Delta_2+\Delta_3-\Delta_1+l}{2}} (P_{31})^{\frac{\Delta_3+\Delta_1-\Delta_2+l}{2}}} \,. \tag{4.10}$$

Here we are using the same notation as in the two-point function case. The polynomial $\tilde{G}_{\chi_1,\chi_2,\chi_3}$ is obtained from the correlator polynomial $G_{\chi_1,\chi_2,\chi_3}$ by dropping all terms proportional to $Z_3^2$ and $Z_3 \cdot P_3$. This polynomial must be identically transverse, and so it must be constructed out of the tensor $C_{3AB}$. The only possibility is to contract this tensor with $P_1$ and $P_2$, which gives the structure

$$V_{3,12} \equiv \frac{P_1 \cdot C_3 \cdot P_2}{P_1 \cdot P_2} = \frac{(Z_3 \cdot P_1)(P_2 \cdot P_3) - (Z_3 \cdot P_2)(P_1 \cdot P_3)}{P_1 \cdot P_2} \tag{4.11}$$

used in (4.10). The exponents are then fixed by the homogeneity requirements.

### 4.2.2 General spins $l_1, l_2$ and $l_3$

We now proceed to the general case of the three-point function of symmetric traceless operators of spins $l_i$. We will write it as

$$\tilde{G}_{\chi_1,\chi_2,\chi_3}(\{P_i; Z_i\}) = \frac{Q_{\chi_1,\chi_2,\chi_3}(\{P_i; Z_i\})}{(P_{12})^{\frac{\tau_1+\tau_2-\tau_3}{2}} (P_{23})^{\frac{\tau_2+\tau_3-\tau_1}{2}} (P_{31})^{\frac{\tau_3+\tau_1-\tau_2}{2}}} \,, \tag{4.12}$$

where $\tau_i = \Delta_i + l_i$. The numerator $Q_{\chi_1,\chi_2,\chi_3}(\{P_i; Z_i\})$ is an identically transverse polynomial of degree $l_i$ in each $Z_i$, with coefficients which depend on $P_i$. With the above normalization, $Q$ is also homogeneous of degree $l_i$ in each $P_i$. Thus,

$$Q_{\chi_1,\chi_2,\chi_3}(\{\lambda_i P_i; \alpha_i Z_i + \beta_i P_i\}) = Q_{\chi_1,\chi_2,\chi_3}(\{P_i; Z_i\}) \prod_i (\lambda_i \alpha_i)^{l_i} \,. \tag{4.13}$$

According to the general characterization of transverse polynomials, $Q$ must be built by contracting the tensors $C_{iAB}$ among themselves and with vectors $P_i$. Not all contractions are useful, since $C_i \cdot C_i$, $C_i \cdot P_i$, $C_i \cdot Z_i$ give terms proportional to $Z_i^2$ and $Z_i \cdot P_i$ which are to be dropped.

Examples of nontrivial building blocks are given by contractions using different points, for instance $C_1 \cdot C_2$ in (4.7) and $P_1 \cdot C_3 \cdot P_2$ in (4.11). It is then clear that three-point functions can be constructed from the basic building blocks

$$V_{i,jk} \equiv \frac{P_j \cdot C_i \cdot P_k}{P_j \cdot P_k} = \frac{(Z_i \cdot P_j)(P_i \cdot P_k) - (Z_i \cdot P_k)(P_i \cdot P_j)}{(P_j \cdot P_k)} \,, \tag{4.14}$$

$$H_{ij} \equiv -C_i \cdot C_j = -2\left[(Z_i \cdot Z_j)(P_i \cdot P_j) - (Z_i \cdot P_j)(Z_j \cdot P_i)\right] , \tag{4.15}$$

which are transverse. They also satisfy the scaling conditions (4.13) with $l_i = 1$, $l_j = l_k = 0$ for $V_{i,jk}$; $l_i = l_j = 1$, $l_k = 0$ for $H_{ij}$.

However, not all $V_{i,jk}$ and $H_{ij}$ are linearly independent due to $V_{i,jk} = -V_{i,kj}$ and $H_{ij} = H_{ji}$. Hence there are three linearly independent $V$'s and three linearly independent $H$'s. Explicitly we will use the following basic structures

$$V_1 \equiv V_{1,23}\,, \quad V_2 \equiv V_{2,31}\,, \quad V_3 \equiv V_{3,12}\,, \quad H_{12}\,, \quad H_{13}\,, \quad H_{23}\,. \tag{4.16}$$

In principle, one could imagine more complicated contractions involving several $C_i$'s. However, it turns out that they will not produce any new structure. Namely, *any identically transverse polynomial $Q$ can be written as a function of $V_i$ and $H_{ij}$ only (with $P$-dependent coefficients)*. For the simplest examples, like $\mathrm{Tr}[C_1C_2C_3]$, this can be checked by an explicit computation. A general proof can be given as follows:

First, take the special case when $Q$ is identically transverse and depends only on $Z_i \cdot P_j$ but not on $Z_i \cdot Z_j$. It is easy to convince oneself that such a $Q$ must be a function of $V_i$. In the general case, let us first rewrite $Q$ by expressing all $Z_i \cdot Z_j$ products via $H_{ij}$ from eq. (4.15). This of course generates new terms, which are however all proportional to $Z_i \cdot P_j$. This shows that $Q$ can be expressed as a polynomial in $H_{ij}$ with coefficients which are functions of $Z_i \cdot P_j$. Moreover, from the way we arrived at this representation, it's clear that it is unique. In this representation, the transversality of $Q$ implies the transversality of all the coefficients (since $H_{ij}$'s are transverse by themselves). According to the special case treated first, these coefficients can be written as functions of $V_i$.

The conclusion of the above discussion is that the general solution for $Q_{\chi_1,\chi_2,\chi_3}$ can be written as a linear combination of

$$\prod_i V_i^{m_i} \prod_{i<j} H_{ij}^{n_{ij}}\,, \tag{4.17}$$

as represented schematically in figure 2. Since $Q$ must have degree $l_i$ in each $Z_i$, the exponents must satisfy the three constraints

$$m_i + \sum_{j \neq i} n_{ij} = l_i\,. \tag{4.18}$$

These equations imply as well that $Q$ has degree $l_i$ in each $P_i$, as it should. Notice that with three $P_i$'s at our disposal, we cannot construct any nontrivial functions of $P_i$ of zero homogeneity (with four $P_i$'s this would be possible; see the four-point function case below). This means that there is no further ambiguity in the coordinate dependence of $Q$.

Eq. (4.17) implies that for general spins $l_i$ there will be several inequivalent three-point function structures compatible with the conformal symmetry. Their number is equal to the number of non-negative integer points $(n_{12}, n_{13}, n_{23})$ in the three dimensional polyhedron defined by the conditions

$$n_{12} + n_{13} \leq l_1\,, \qquad n_{12} + n_{23} \leq l_2\,, \qquad n_{13} + n_{23} \leq l_3\,. \tag{4.19}$$

Counting these points, it is possible to write the number of inequivalent structures in closed form:

$$N(l_1, l_2, l_3) = \frac{(l_1+1)(l_1+2)(3l_2 - l_1 + 3)}{6} - \frac{p(p+2)(2p+5)}{24} - \frac{1-(-1)^p}{16}\,, \tag{4.20}$$

where we have ordered the spins $l_1 \leq l_2 \leq l_3$ and defined $p \equiv \max(0, l_1 + l_2 - l_3)$.

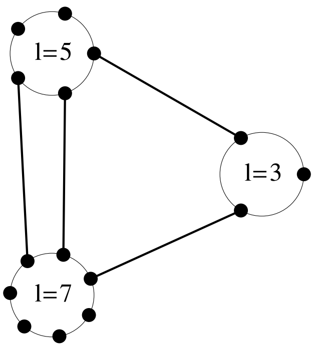


**Figure 2**. Schematic representation of one of the tensor structures appearing in the (spin 5)-(spin 3)-(spin 7) three-point function. $V_i$'s are represented as disconnected dots at the vertices and $H_{ij}$'s as lines joining the vertices.

#### 4.2.3 Parity odd three-point functions

So far we have implicitly assumed that the correlators are parity invariant. If this is not the case, then there are additional structures in the three-point function. More precisely, we can use the SO$(d+1,1)$-invariant $\epsilon$-tensor to construct new building blocks for the three-point function. Since the product of two $\epsilon$-tensors can be written in terms of metrics, it is enough to use the $\epsilon$-tensor once. The number of invariant structures that can be built from one $\epsilon$-tensor and the vectors $P_i$ and $Z_i$ depends on the dimension $d$. For $d > 4$ it is not possible to form a scalar from these ingredients. This implies that all conformally invariant three-point functions of spin $l_i$ symmetric traceless operators in $d > 4$ are necessarily parity invariant.[14]

For $d = 4$, there is a unique invariant

$$\epsilon(Z_1, Z_2, Z_3, P_1, P_2, P_3)\,, \tag{4.21}$$

where by $\epsilon(\cdots)$ we mean the contraction of the $(d+2)$-dimensional $\epsilon$-tensor with all the arguments. Thus, the number of parity odd structures of $(l_1, l_2, l_3)$ three point functions is equal to the number of parity even structures of $(l_1 - 1, l_2 - 1, l_3 - 1)$ three point functions, since (4.21) involves a single power in each $Z_i$.

For $d = 3$, there are 3 invariants

$$\epsilon(Z_i, Z_j, P_1, P_2, P_3)\,. \tag{4.22}$$

Notice that $\epsilon(Z_1, Z_2, Z_3, P_1, P_2)$ is not invariant under $Z_3 \to Z_3 + \beta P_3$ and therefore is excluded. In fact, in 3 dimensions not all conformally invariant building blocks are independent. We treat this special case separately in section 4.2.5.

[14] Sometimes the correlators containing $\epsilon$-tensors are called parity violating in the literature, which is poor terminology. The theory may be perfectly parity preserving even though some correlators are parity odd, provided that the fields themselves are assigned negative parity. A notable exception is the stress tensor, which must be assigned positive parity by its very meaning as the generator of spacetime transformations, and also more formally since the correlator $\langle TTT\rangle$ necessarily contains a parity even term (due to the Ward identity) [38]. In this case, any admixture of a parity odd structure [39, 40] would imply parity violation.

### 4.2.4 Relation to leading OPE coefficient

Mack [36] and Osborn and Petkou [38] give a prescription to uplift the leading OPE coefficient into a conformally invariant three point function. Here we wish to make direct contact with this work, starting from the embedding formalism.

Let us rewrite eq. (2.31) of [38] as follows:

$$\phi_1(x;z_1)\phi_2(0;z_2) \sim \phi_3(0;\partial_{z_3})\, t(x;z_1,z_2,z_3)\, x^{-(\Delta_1+\Delta_2-\Delta_3+\sum l_i)}, \tag{4.23}$$

where $x^\alpha$ stands for $(x^2)^{\frac{\alpha}{2}}$, and

$$\phi(x;z) = z^{\mu_1}\cdots z^{\mu_l}\phi_{\mu_1\ldots\mu_l}(x)\,. \tag{4.24}$$

The choice of a rotationally invariant tensor structure for the leading OPE coefficient is the choice of rotationally invariant polynomial $t$ such that

$$t(\lambda x,\lambda_1 z_1,\lambda_2 z_2,\lambda_3 z_3) = t(x;z_1,z_2,z_3)\prod_{i=1}^{3}(\lambda\lambda_i)^{l_i}\,. \tag{4.25}$$

Equation (2.36) of [38] then becomes

$$\langle\phi_1(x_1;z_1)\phi_2(x_2;z_2)\phi_3(x_3;z_3)\rangle = \frac{t\left(X_{12};\tilde z_1,\tilde z_2,z_3\right)}{x_{13}^{2\Delta_1}x_{23}^{2\Delta_2}X_{12}^{\Delta_1+\Delta_2-\Delta_3+\sum l_i}}\,, \tag{4.26}$$

where

$$X_{12} = \frac{x_{13}}{x_{13}^2} - \frac{x_{23}}{x_{23}^2}\,, \qquad \tilde z_1 = R(x_{13})z_1\,, \qquad \tilde z_2 = R(x_{23})z_2\,, \tag{4.27}$$

where $R(x)$ is a linear transformation acting on $z_i$ as

$$R(x)_{\mu\nu} = \delta_{\mu\nu} - \frac{2x_\mu x_\nu}{x^2}\,. \tag{4.28}$$

Using $X_{12}^2 = x_{12}^2/(x_{13}^2 x_{23}^2)$ and the scaling properties of $t$ we can write

$$\langle\phi_1(x_1;z_1)\phi_2(x_2;z_2)\phi_3(x_3;z_3)\rangle = \frac{t\left(\tilde x_{12};\tilde z_1,\tilde z_2,z_3\right)}{x_{13}^{\Delta_1+\Delta_3-\Delta_2+\sum l_i}x_{23}^{\Delta_2+\Delta_3-\Delta_1+\sum l_i}x_{12}^{\Delta_1+\Delta_2-\Delta_3+\sum l_i}}\,, \tag{4.29}$$

where

$$\tilde x_{12} = x_{13}^2 x_{23}^2 X_{12} = x_{13}\,x_{23}^2 - x_{23}\,x_{13}^2\,. \tag{4.30}$$

This expression treats the operator $\phi_3$ differently from the other two operators. However, if needed, one can easily rewrite it, so that the role of $\phi_3$ is taken by, say, $\phi_1$. To do this, one needs to re-express the numerator as

$$t\left(\tilde x_{12};R(x_{13})z_1,R(x_{23})z_2,z_3\right) = t'\left(\tilde x_{23};z_1,R(x_{12})z_2,R(x_{13})z_3\right)\,, \tag{4.31}$$

where $t'$ is some other polynomial. To find $t'$, notice first of all that the transformation $R(x)$ is orthogonal.[15] Since $t$ is a rotationally invariant polynomial it will not change if every argument is multiplied by $R(x_{13})$. Using the relations

$$R(x_{13})R(x_{23}) = R(\tilde x_{23})R(x_{12}), \qquad R(x_{13})\tilde x_{12} = \tilde x_{23}\,, \tag{4.32}$$

[15] It's actually a very trivial orthogonal transformation; it just flips the sign of the component in the direction of $x$.

we see that this transformation accomplishes the needed rewriting, and that

$$t'(x; z_1, z_2, z_3) = t(x; z_1, R(x)z_2, z_3) . \tag{4.33}$$

Now, it is clear that if $\phi_1 = \phi_2$, then the polynomial $t$ obeys

$$t(x; z_1, z_2, z_3) = t(-x; z_2, z_1, z_3) . \tag{4.34}$$

On the other hand, if $\phi_2 = \phi_3$ it is the $t'$ which satisfies the simple condition, while for $t$ the condition is less transparent:

$$t\,(x; z_1, z_2, z_3) = t\,(-x; z_1, R(x)z_3, R(x)z_2) \ . \tag{4.35}$$

We now wish to compare with eq. (4.12). In order to do that, we should project the embedding correlator onto the Poincaré section, using

$$P_i = (1, x_i^2, x_i)\,, \qquad\qquad Z_i = (0, 2x_i \cdot z_i, z_i) \ . \tag{4.36}$$

One can then check that

$$\begin{aligned} P_{23}V_1 &= -\tilde{z}_1 \cdot \tilde{x}_{12}\,, \qquad & P_{13}V_2 &= -\tilde{z}_2 \cdot \tilde{x}_{12}\,, \qquad P_{12}V_3 = z_3 \cdot \tilde{x}_{12}\,, \\ P_{12}P_{23}H_{13} &= (\tilde{z}_1 \cdot z_3)\tilde{x}_{12}^2\,, \qquad & P_{12}P_{13}H_{23} &= (\tilde{z}_2 \cdot z_3)\tilde{x}_{12}^2\,, \\ P_{13}P_{23}H_{12} &= (\tilde{z}_1 \cdot \tilde{z}_2)\tilde{x}_{12}^2 - 2(\tilde{z}_1 \cdot \tilde{x}_{12})(\tilde{z}_2 \cdot \tilde{x}_{12})\,. \end{aligned} \tag{4.37}$$

Therefore, the structure (4.17) corresponds to $t(x; z_1, z_2, z_3)$ given by

$$(x^2 z_1 \cdot z_3)^{n_{13}} (x^2 z_2 \cdot z_3)^{n_{23}} (x^2 z_1 \cdot z_2 - 2x \cdot z_1\, x \cdot z_2)^{n_{12}}\, (-x \cdot z_1)^{m_1} (-x \cdot z_2)^{m_2} (x \cdot z_3)^{m_3}\,, \tag{4.38}$$

modulo terms $O(z_i^2)$ which are not independent but fixed by tracelessness of $\phi$'s. It is also clear that this is a basis for the most general rotational and parity invariant polynomial $t(x; z_1, z_2, z_3)$.

Parity odd structures are dimension specific. In order to form a scalar from the $d$-dimensional $\epsilon$-tensor we need at least $d$ linearly independent vectors. Therefore, for $d > 4$ the polynomial $t(x; z_1, z_2, z_3)$ is necessarily parity invariant, as stated in the previous section. In four dimensions, we can make parity odd three-point functions using $\epsilon(x, z_1, z_2, z_3)$. This corresponds to the use of (4.21) in the embedding language. To see that, we just need to project onto the Poincaré section,

$$\epsilon(Z_1, Z_2, Z_3, P_1, P_2, P_3) = \begin{vmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 2z_1 \cdot x_1 & 2z_2 \cdot x_2 & 2z_3 \cdot x_3 & x_1^2 & x_2^2 & x_3^2 \\ z_1 & z_2 & z_3 & x_1 & x_2 & x_3 \end{vmatrix} . \tag{4.39}$$

Using translation invariance we can write

$$\epsilon(Z_1, Z_2, Z_3, P_1, P_2, P_3) = \begin{vmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 2z_1 \cdot x_{13} & 2z_2 \cdot x_{23} & 0 & x_{13}^2 & x_{23}^2 & 0 \\ z_1 & z_2 & z_3 & x_{13} & x_{23} & 0 \end{vmatrix} , \tag{4.40}$$

and expanding in the last column, we find

$$\epsilon(Z_1, Z_2, Z_3, P_1, P_2, P_3) = \epsilon(\tilde{x}_{12}, \tilde{z}_1, \tilde{z}_2, z_3) \ . \tag{4.41}$$

The problem in three dimensions is special so we treat it separately in the next subsection.

### 4.2.5 Three dimensions

The problem of constructing conformally invariant three-point functions in three dimensional CFTs has been recently addressed in [40]. In this subsection we shall explain how their results fit into the formalism of this paper.

Using the group theoretic approach of [36] it is easy to count how many independent structures exist for a three-point function of operators with spin $l_1 \leq l_2 \leq l_3$. We just need to count how many irreducible representations of SO(3) appear in the tensor product $l_1 \otimes l_2 \otimes l_3$ (notice that all irreducible representations of SO(3) are totally symmetric and traceless representations). This gives

$$N_{3d}(l_1, l_2, l_3) = \sum_{l=l_3-l_2}^{l_3+l_2} \sum_{m=|l-l_1|}^{l+l_1} 1 = (2l_1+1)(2l_2+1) - p(1+p)\,, \tag{4.42}$$

where $p = \max(0, l_1 + l_2 - l_3)$. Of these, there are

$$N^{+}_{3d}(l_1, l_2, l_3) = 2l_1 l_2 + l_1 + l_2 + 1 - \frac{p(p+1)}{2} \tag{4.43}$$

parity even structures and

$$N^{-}_{3d}(l_1, l_2, l_3) = 2l_1 l_2 + l_1 + l_2 - \frac{p(p+1)}{2} \tag{4.44}$$

parity odd structures. The split between parity even and parity odd structures follows from the fact that in the product of two SO(3) tensors with spin $l_1$ and $l_2$, the tensors with spin $l_1+l_2, l_1+l_2-2, \dots, |l_1-l_2|$ are parity even, and the tensors with spin $l_1+l_2-1, l_1+l_2-3, \dots, |l_1-l_2|+1$ are parity odd because they contain one $\epsilon$-tensor.

The number of parity even structures $N^{+}_{3d}$ is smaller than the general result (4.20). To explain this mismatch we need to notice that, in three dimensions, there are identities relating some of the general tensor structures. The easiest way to derive these relations is to consider the expression for the leading OPE coefficient $t(x; z_1, z_2, z_3)$. As in section 3, we can restrict the polynomial to $z_i^2 = 0$, which translates to an $O(Z_i^2, Z_i \cdot P_i)$ term in the embedding space.

In three dimensions, the four arguments of $t$ cannot be linearly independent vectors:

$$x = \sum_{i=1}^{3} \alpha_i\, z_i\,. \tag{4.45}$$

For $z_i^2 = 0$, the coefficients $\alpha_i$ can be given explicitly as

$$\alpha_i = \frac{(z_j \cdot x)(z_k \cdot z_i) + (z_k \cdot x)(z_j \cdot z_i) - (z_i \cdot x)(z_j \cdot z_k)}{2(z_i \cdot z_j)(z_i \cdot z_k)} \qquad (j \neq k \neq i)\,. \tag{4.46}$$

Another way to express the linear dependence is as

$$\det_{1\leq i,j\leq 4} (z_i \cdot z_j)\Big|_{z_4=x} = 0\,. \tag{4.47}$$

Using the rules in eq. (4.37), this last identity corresponds to the relation

$$(V_1H_{23}+V_2H_{13}+V_3H_{12}+2V_1V_2V_3)^2 \approx -2H_{12}H_{13}H_{23} \tag{4.48}$$

between the conformally invariant structures. Here $\approx$ means modulo $O(Z_i^2, Z_i\cdot P_i)$. This identity is (the square of) the identity (2.14) of [40]. The identity (4.48) can also be obtained directly from the $(3+2)$-dimensional embedding space by noting that the 6 vectors $Z_i$ and $P_i$ can not be linearly independent. Equation (4.48) then follows from $\det_{1\leq i,j\leq 6}(Z_i\cdot Z_j)=0$ where $Z_{i+3}\to P_i$ for $i=1,2,3$. The existence of this identity means that one does not need to use the substructure $H_{12}H_{13}H_{23}$ to write the most general three-point function. It is then simple to correct the overcounting of the general analysis for parity even structures, by subtracting all structures containing the factor $H_{12}H_{13}H_{23}$. This gives

$$N_{3d}^{+}(l_1,l_2,l_3)=N(l_1,l_2,l_3)-N(l_1-2,l_2-2,l_3-2)\,, \tag{4.49}$$

which agrees with the counting (4.43) from group theory.

We can also find relations between the parity odd structures by expanding the following determinant along the first line,

$$\begin{vmatrix} A_1 & A_2 & A_3 & \sum\alpha_i A_i \\ z_1 & z_2 & z_3 & x \end{vmatrix}=0\,, \tag{4.50}$$

where we recall that $z_i$ and $x$ are three dimensional vectors here represented as columns. The simplest identity follows from choosing $A_i=x\cdot z_i$:

$$(x\cdot z_1)\,\epsilon(z_2,z_3,x)+(x\cdot z_2)\,\epsilon(z_3,z_1,x)+(x\cdot z_3)\,\epsilon(z_1,z_2,x)-x^2\,\epsilon(z_1,z_2,z_3)=0\,. \tag{4.51}$$

This tells us that we never need to use the substructure $\epsilon(z_1,z_2,z_3)$, since it can be obtained as a linear combination of $\epsilon(z_i,z_j,x)$. Furthermore, choosing

$$A_1=-(x\cdot z_1)^2\,,\quad A_2=x^2\,(z_1\cdot z_2)-(x\cdot z_1)(x\cdot z_2)\,,\quad A_3=x^2\,(z_1\cdot z_3)-(x\cdot z_1)(x\cdot z_3)\,, \tag{4.52}$$

we obtain

$$A_1\,\epsilon(z_2,z_3,x)-A_2\,\epsilon(z_1,z_3,x)+A_3\,\epsilon(z_1,z_2,x)=0\,, \tag{4.53}$$

where we have used that $\sum\alpha_iA_i=0$ (as one can check from eq. (4.47)). This identity is invariant under the permutation $z_2\leftrightarrow z_3$, but one can generate two more identities by permuting $z_1\leftrightarrow z_2$ and $z_1\leftrightarrow z_3$. In terms of our conformally invariant structures, these identities read

$$\begin{aligned}
0 &\approx V_1^2\,\epsilon_{23}+(H_{12}+V_1V_2)\,\epsilon_{13}-(H_{13}+V_1V_3)\,\epsilon_{12}\,,\\
0 &\approx V_2^2\,\epsilon_{13}+(H_{23}+V_2V_3)\,\epsilon_{12}+(H_{12}+V_1V_2)\,\epsilon_{23}\,,\\
0 &\approx V_3^2\,\epsilon_{12}-(H_{13}+V_1V_3)\,\epsilon_{23}+(H_{23}+V_2V_3)\,\epsilon_{13}\,,
\end{aligned} \tag{4.54}$$

where

$$\epsilon_{ij}\equiv P_{ij}\,\epsilon(Z_i,Z_j,P_1,P_2,P_3)\,. \tag{4.55}$$

This follows from the projections to the Poincaré section (4.37) and

$$\epsilon_{12} = -x_{12}^2\, \epsilon(\tilde{z}_1, \tilde{z}_2, \tilde{x}_{12})\,, \quad \epsilon_{13} = -x_{13}^2\, \epsilon(\tilde{z}_1, z_3, \tilde{x}_{12})\,, \quad \epsilon_{23} = -x_{23}^2\, \epsilon(\tilde{z}_2, z_3, \tilde{x}_{12})\,. \tag{4.56}$$

The identities (4.54) are equivalent to the eqs. (2.19) given in [40]. The identity (4.48) follows from the compatibility of these three equations. The easiest way to count all parity odd three-point functions is to take these three identities as the only independent relations between the building blocks. Then we have

$$\begin{aligned} N_{3d}^-(l_1,l_2,l_3) = &\, N(l_1-1,l_2-1,l_3) + N(l_1-1,l_2,l_3-1) + N(l_1,l_2-1,l_3-1) \\ &- N(l_1-2,l_2-1,l_3-1) - N(l_1-1,l_2-2,l_3-1) \\ &- N(l_1-1,l_2-1,l_3-2), \end{aligned} \tag{4.57}$$

where the first line corresponds to all parity even structures times $\epsilon_{12}$, $\epsilon_{13}$ and $\epsilon_{23}$, respectively. The second and third lines corresponds to the subtraction of the identities (4.54), multiplied by parity even structures to avoid overcounting. This expression agrees with the explicit formula given in eq. (4.44).

### 4.3 Four-point functions

Now let us move on to discuss the possible structures that can appear in CFT four-point functions. The simplest case is when all four operators are scalar primaries. A correlation function $\langle\phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4)\rangle$ containing primaries of dimension $\Delta_i$ can be obtained from the projection of the embedding correlator

$$\langle\Phi_1(P_1)\Phi_2(P_2)\Phi_3(P_3)\Phi_4(P_4)\rangle = \left(\frac{P_{24}}{P_{14}}\right)^{\frac{\Delta_1-\Delta_2}{2}} \left(\frac{P_{14}}{P_{13}}\right)^{\frac{\Delta_3-\Delta_4}{2}} \frac{f(u,v)}{(P_{12})^{\frac{\Delta_1+\Delta_2}{2}}(P_{34})^{\frac{\Delta_3+\Delta_4}{2}}}\,, \tag{4.58}$$

where $u$ and $v$ are the conformally invariant cross-ratios

$$u = \frac{P_{12}P_{34}}{P_{13}P_{24}}\,, \qquad v = \frac{P_{14}P_{23}}{P_{13}P_{24}}\,. \tag{4.59}$$

Thus, in this very simple case, the correlation function depends on a single function of the cross ratios.

The generalization to operators with spin is clear and follows the same logic explained in section 4.2. In this case, however, the correlation function will be a linear combination of tensor structures that are polynomial in the $Z$'s, with coefficients given by undetermined functions of the cross ratios. Thus, for a generic four-point function we write

$$\tilde{G}_{\chi_1,\chi_2,\chi_3,\chi_4} = \frac{\left(\frac{P_{24}}{P_{14}}\right)^{\frac{\tau_1-\tau_2}{2}} \left(\frac{P_{14}}{P_{13}}\right)^{\frac{\tau_3-\tau_4}{2}}}{(P_{12})^{\frac{\tau_1+\tau_2}{2}}(P_{34})^{\frac{\tau_3+\tau_4}{2}}} \sum_k f_k(u,v)\, Q^{(k)}_{\chi_1,\chi_2,\chi_3,\chi_4}(\{P_i;Z_i\})\,, \tag{4.60}$$

where $\tau_i = \Delta_i + l_i$. With this choice of pre-factor, the $Q^{(k)}$ have weight $l_i$ in each point $P_i$. Conformal invariance is equivalent to the following condition for each linearly independent $Q^{(k)}$ polynomial:

$$Q^{(k)}_{\chi_1,\chi_2,\chi_3,\chi_4}(\{\lambda_i P_i;\alpha_i Z_i + \beta_i P_i\}) = Q^{(k)}_{\chi_1,\chi_2,\chi_3,\chi_4}(\{P_i;Z_i\}) \prod_i (\lambda_i\alpha_i)^{l_i}\,. \tag{4.61}$$



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Similar to the three-point function case, these polynomials are constructed from the basic building blocks $V_{i,jk}$ and $H_{ij}$ introduced in section 4.2. However, not all $V_{i,jk}$ are linearly independent. In addition to $V_{i,jk} = -V_{i,kj}$ we have, in the case of four points,

$$(P_2 \cdot P_3)(P_1 \cdot P_4)V_{1,23} + (P_2 \cdot P_4)(P_1 \cdot P_3)V_{1,42} + (P_3 \cdot P_4)(P_1 \cdot P_2)V_{1,34} = 0 \,. \tag{4.62}$$

This shows that there are only 2 independent $V_{i,jk}$ for each $i$. A convenient choice for the example given below is to use linear combinations that are even and odd under the interchange $3 \leftrightarrow 4$,

$$W_1 \equiv V_{1,23} + V_{1,24}\,, \qquad \bar{W}_1 \equiv V_{1,23} - V_{1,24}\,, \tag{4.63}$$
$$W_2 \equiv V_{2,13} + V_{2,14}\,, \qquad \bar{W}_2 \equiv V_{2,13} - V_{2,14}\,. \tag{4.64}$$

Similarly, we may define $W_3, W_4$ and $\bar{W}_3, \bar{W}_4$ to be, respectively, even and odd under the interchange $1 \leftrightarrow 2$. Then, all solutions $Q^{(k)}$ of (4.61) have the form

$$\prod_i W_i^{m_i} \prod_i \bar{W}_i^{\bar{m}_i} \prod_{i<j} H_{ij}^{n_{ij}}\,, \tag{4.65}$$

such that

$$m_i + \bar{m}_i + \sum_{j \neq i} n_{ij} = l_i\,. \tag{4.66}$$

The problem of finding the number of structures for the four-point function is given by counting the 6-tuples $(n_{12}, n_{13}, n_{14}, n_{23}, n_{24}, n_{34})$ of non-negative integers such that

$$\begin{aligned}
n_{12} + n_{13} + n_{14} &= a_1 \leq l_1\,, \\
n_{12} + n_{23} + n_{24} &= a_2 \leq l_2\,, \\
n_{13} + n_{23} + n_{34} &= a_3 \leq l_3\,, \\
n_{14} + n_{24} + n_{34} &= a_4 \leq l_4\,.
\end{aligned} \tag{4.67}$$

Then, for each of these 6-tuples with a given set $\{a_i\}$, there are

$$\prod_{i=1}^{4} (l_i - a_i + 1)\,, \tag{4.68}$$

possible ways of distributing the $W_i$ and $\bar{W}_i$ structures (counting number of integers $m_i$ and $\bar{m}_i$ such that $m_i + \bar{m}_i = l_i - a_i$). We will not attempt here to count the number of general structures allowed for a generic four-point function. The whole point of this analysis was to make it clear how to construct such structures in any given particular case that one may wish to consider.

### 4.3.1 Example: vector-vector-scalar-scalar

As an example of the previous general formalism let us consider the case of a four-point function between two vectors and two scalars $\langle v_a(x_1) v_b(x_2) \phi(x_3) \phi(x_4) \rangle$, even under the exchange of both vectors and of both scalars. To start there are five possible independent structures, namely

$$W_1 W_2\,, \quad \bar{W}_1 \bar{W}_2\,, \quad W_1 \bar{W}_2, \quad \bar{W}_1 W_2, \quad H_{12}\,. \tag{4.69}$$

Noticing that under $P_1 \leftrightarrow P_2$ or $P_3 \leftrightarrow P_4$ the cross ratios transform as $u \leftrightarrow w \equiv u/v$, it is clear that in this case the linear combination of the $Q^{(k)}$ entering (4.60) is given by

$$f_1(u,w) W_1 W_2 + f_2(u,w) \bar{W}_1 \bar{W}_2 + f_3(u,w) H_{12} + f_4(u,w) \left( W_1 \bar{W}_2 - \bar{W}_1 W_2 \right) , \tag{4.70}$$

with

$$f_4(u,w) = -f_4(w,u)\,, \qquad f_k(u,w) = f_k(w,u)\,, \quad k = 1,2,3\,. \tag{4.71}$$

Hence we recover the counting already presented in [25].

### 4.4 $n$-point functions

We will finish this section with some general remarks on the case of $n$-point functions, for which there are $n(n-3)/2$ independent conformally invariant cross-ratios $u_a$ (actually, for $n$ high enough they are not all independent, but this fact will not be important here).

A generic $n$-point function can be written as

$$\tilde{G}_{\chi_1,\ldots,\chi_n} = \prod_{i<j}^{n} P_{ij}^{-\alpha_{ij}} \sum_k f_k(u_a)\, Q^{(k)}_{\chi_1,\ldots,\chi_n}(\{P_i; Z_i\})\,, \tag{4.72}$$

where

$$\alpha_{ij} = \frac{\tau_i + \tau_j}{n-2} - \frac{1}{(n-1)(n-2)} \sum_{k=1}^{n} \tau_k\,. \tag{4.73}$$

With the chosen pre-factor, the $Q^{(k)}$ have weight $l_i$ in each point $P_i$. They are also identically transverse:

$$Q^{(k)}_{\chi_1,\ldots,\chi_n}(\{\lambda_i P_i; \alpha_i Z_i + \beta_i P_i\}) = Q^{(k)}_{\chi_1,\ldots,\chi_n}(\{P_i; Z_i\}) \prod_i (\lambda_i \alpha_i)^{l_i}\,. \tag{4.74}$$

These polynomials can then be constructed from the basic building blocks $V_{i,jk}$ and $H_{ij}$ given in (4.14) and (4.15); see figure 3. For each $i$, since only $n-2$ of the (anti-symmetric) $V_{i,jk}$ are linearly independent, we can choose to work with

$$\mathcal{V}_{ij} \equiv V_{i,(i+1)j} \qquad\qquad (j = 1, \cdots, \hat{i}, \widehat{i+1}, \cdots, n)\,, \tag{4.75}$$

where hatted integers are excluded. Then, all solutions $Q^{(k)}$ have the form

$$\left( \prod_{i=1}^{n} \prod_{j\neq i,i+1}^{n} \mathcal{V}_{ij}^{m_{ij}} \right) \prod_{i<j}^{n} H_{ij}^{n_{ij}}\,, \tag{4.76}$$

such that

$$\sum_{j\neq i,i+1}^{n} m_{ij} + \sum_{j\neq i}^{n} n_{ij} = l_i\,. \tag{4.77}$$

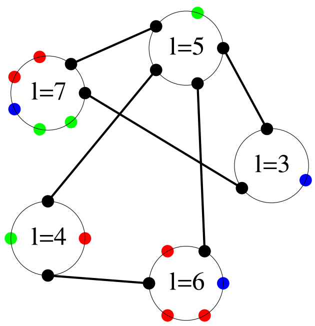


**Figure 3**. Same as figure 2 but for a five-point function. The isolated dots representing $V$'s appear in several colors because for an $n$-point function there are several possible $V$'s per vertex.

Thus, the problem of finding the number of structures of the $n$-point function separates again in finding the $(n(n-1)/2)$-tuples, $\{n_{ij}\}$ with $i < j$, such that

$$\sum_{j \neq i}^{n} n_{ij} = a_i \leq l_i \,. \tag{4.78}$$

For each set of non-negative integers $a_i$, a moment's thought shows that there are

$$\prod_{i=1}^{n} \frac{(l_i - a_i + n - 3)!}{(l_i - a_i)!(n-3)!} \tag{4.79}$$

possible ways of distributing the $\mathcal{V}_{ij}$ structures.

In the above counting we neglected identities following from the finite dimensionality of spacetime. The $2n$ vectors $Z_i$ and $P_i$ can not be linearly independent in the $(d+2)$-dimensional embedding space if $n > \frac{d}{2} + 1$. In a given dimension, one can obtain identities between the above tensor structures by expanding $\det(Z_i \cdot Z_j) = 0$, where the matrix is of size $(d+3) \times (d+3)$ or larger and some of the $Z$'s can be $P$'s.

## 5 Conserved tensors

In unitary CFTs, the dimensions of spin $l$ primaries must satisfy the unitarity bound [41–44]:

$$\Delta \geq l + d - 2 \quad (l \geq 1) \,. \tag{5.1}$$

When $\Delta$ takes the lowest value allowed by this bound for a given $l$, the corresponding primary field is conserved. Physically important examples of such fields are the stress tensor ($l = 2$) and global symmetry currents ($l = 1$).[16] The conservation condition then leads to additional constraints on the form of three and higher point functions. In this section we will discuss these constraints and show how to impose them directly in the embedding space.

[16] Note that it is not as interesting to consider scalars, since only a free field can saturate the scalar unitarity bound $\Delta \geq (d-2)/2$.

### 5.1 Conservation condition and conformal invariance

Let us begin by considering the conservation condition for a spin $l$ dimension $\Delta$ primary:

$$\partial \cdot f = 0\,, \tag{5.2}$$

$$(\partial \cdot f)^{a_2 \dots a_l} \equiv \frac{\partial}{\partial x^{a_1}} f^{a_1 a_2 \dots a_l}(x)\,. \tag{5.3}$$

We would like to learn how to impose this condition in terms of the embedding space tensor $F$ which projects to $f$. Differentiating eq. (2.5), there will be two types of terms depending whether the derivative falls on $\partial P/\partial x$ or on $F$. These terms can be simplified using

$$\frac{\partial}{\partial x^a}\left(\frac{\partial P^A}{\partial x^b}\right) = \delta_{ab}\bar{P}^A, \tag{5.4}$$

$$\frac{\partial P^{A_1}}{\partial x_{a_1}}\frac{\partial F_{A_1 \dots A_l}}{\partial x^{a_1}} = \frac{\partial P^{A_1}}{\partial x_{a_1}}\frac{\partial P^B}{\partial x^{a_1}}\frac{\partial F_{A_1 \dots A_l}}{\partial P^B} \equiv K^{A_1 B}\frac{\partial F_{A_1 \dots A_l}}{\partial P^B}\,, \tag{5.5}$$

where the metric $K^{AB}$ and the vector $\bar{P}^A$ were given in eq. (2.6). Commuting $P$ with $\partial/\partial P$ and using the property that $F$ is homogeneous of degree $-\Delta$, the end result can be put in the form

$$(\partial \cdot f)_{a_2 \dots a_l}(x) = \frac{\partial P^{A_2}}{\partial x^{a_2}} \dots \frac{\partial P^{A_l}}{\partial x^{a_l}} R_{A_2 \dots A_l}(P_x)\,, \tag{5.6}$$

with

$$R_{A_2 \dots A_l}(P) = \left[\frac{\partial}{\partial P_{A_1}} - \frac{1}{P \cdot \bar{P}}(\bar{P} \cdot \frac{\partial}{\partial P})P^{A_1} - (l + d - 2 - \Delta)\frac{\bar{P}^{A_1}}{P \cdot \bar{P}}\right] F_{A_1 \dots A_l}(P)\,. \tag{5.7}$$

Note that the $1/(P \cdot \bar{P})$ prefactors are needed to ensure that all terms in $R$ have the same homogeneity in $P$.

The tensor $F$ is originally defined on the cone $P^2 = 0$, while the derivatives $\partial/\partial P$ appearing in the definition of $R$ are unrestricted. To compute the derivatives along the non-tangent directions, the tensor $F$ has to be extended away from the cone. It is easy to see that different extensions of $F$ change $R$ by terms which project to zero. This is a sanity check, since the l.h.s. of the formula does not allow for any ambiguity. The same is true about pure gauge modifications of $F$.

The terms in $R$ involving $\bar{P}$ may seem problematic from the point of view of SO$(d+1,1)$ invariance. The last term clearly breaks it unless its coefficient vanishes. On the other hand, the second term *is* SO$(d+1,1)$ invariant, though not manifestly. To see this, one should use the condition that $P \cdot F$ vanishes on the cone. Writing this as $P \cdot F = O(P^2)$, we see that $P \cdot \bar{P}$ cancels out and $\bar{P}$ drops out from the second term.

Now we see what is special about $\Delta = l + d - 2$: precisely for this dimension $R$ becomes an SO$(d+1,1)$ invariant tensor. This tensor is also traceless (obvious) and transverse (straightforward to show by using the tracelessness and transversality of $F$). We conclude that its projection to the physical space, $\partial \cdot f$, will transform as a primary under the conformal group. In particular, the transformation of $\partial \cdot f$ will be homogeneous: $\partial \cdot f(x)$ is proportional to $\partial \cdot f(x')$. This is to be contrasted with the usual transformation rule for the derivative of a primary, which contains a term proportional to the primary itself.

One consequence of the above discussion is that for $\Delta = l + d - 2$, and only for this dimension, the conservation condition $\partial \cdot f = 0$ can be imposed in a way that is consistent with the conformal symmetry.

But one can say more. The fact that for $\Delta = l + d - 2$ the divergence $\partial \cdot f$ is both a primary and a descendant implies, using the argument familiar from 2D CFT, that it is a *null state*. In particular, the two-point function of $\partial f$ with itself, as with any other primary, will vanish:

$$\langle \partial \cdot f(x)\ \partial \cdot f(0) \rangle = 0\,. \tag{5.8}$$

The latter equality can be also checked using the two-point function of spin $l$ primaries discussed in section 4.1.

Now, in a unitary theory eq. (5.8) implies that $\partial \cdot f = 0$ as an operator equation. Thus imposing the conservation condition for $\Delta = l + d - 2$ is not only consistent, but also mandatory.

In practice, we will have to impose that three-point functions of $f$ with any other fields should be conserved. However, unlike for the two-point functions, this will not happen automatically. Rather, we will find constraints beyond those discussed in section 4.2. On the other hand, once all the three-point function constraints are satisfied, higher point functions will be automatically conserved as a consequence of the OPE.

### 5.2 Conservation condition for polynomials

Since the conservation constraint must be imposed in addition to the constraints discussed in section 4, we should write it in a form compatible with the index-free notation that we developed there. In particular, we will work with the encoding polynomial $\tilde{F}(P;Z)$ introduced in section 3.2, which is identically transverse and agrees with $F(P;Z)$ modulo $O(Z^2, Z \cdot P)$. Similarly, we will also encode the tensor $R$ via the identically transverse function $\tilde{R}(P;Z)$.

The result of this section will be that $\tilde{R}(P;Z)$ can be computed from $\tilde{F}(P;Z)$ by the following simple formula:

$$\tilde{R}(P;Z) = \frac{1}{l(h+l-2)} (\partial \cdot D) \tilde{F}(P;Z) - O(Z^2, Z \cdot P), \tag{5.9}$$

where

$$\partial \cdot D \equiv \frac{\partial}{\partial P_M} D_M, \tag{5.10}$$

and $D_M$ is the differential operator in $Z$ defined in eq. (3.30). "$-O(\cdots)$" means that the corresponding terms must be dropped.

Let us prove formula (5.9). First we need to recover $F$ from $\tilde{F}$. According to the result from section 3.2, the necessary projector can be obtained from a $d$-dimensional traceless symmetric projector:

$$\pi_{a_1 \ldots a_l, b_1 \ldots b_l} = \delta_{a_1 b_1} \cdots \delta_{a_l b_l} - c_l \sum_{i<j} \delta_{a_i a_j} \delta_{b_i b_j} \prod_{k \neq i,j} \delta_{a_k b_k} + O(\delta_{a_i a_j} \delta_{a_k a_n})\,. \tag{5.11}$$

Here we are not symmetrizing in $b$'s, assuming that $\pi$ is contracted with a symmetric tensor. The second term in the formula subtracts single traces, which fixes its coefficient $c_l = 1/(d + 2l - 4)$.

The $O(\delta_{a_ia_j}\delta_{a_ka_n})$ stands for terms which subtract multiple traces; we will not need to know them explicitly. Performing the replacements from eq. (3.23), we obtain the representation

$$F_{A_1\ldots A_l} = \tilde{F}_{A_1\ldots A_l} - c_l \sum_{i<j} W_{A_iA_j} \tilde{F}^B{}_{BA_1\ldots\hat{A}_i\ldots\hat{A}_j\ldots A_l} + O(W_{A_iA_j}W_{A_kA_n}), \tag{5.12}$$

where the hatted indices are skipped.

Now we can start computing $\tilde{R}$. Assuming that $\Delta = l + d - 2$, eq. (5.7) gives

$$\tilde{R}_{A_2\ldots A_l} = \left[\frac{\partial}{\partial P_{A_1}} - \frac{1}{P\cdot\bar{P}}(\bar{P}\cdot\frac{\partial}{\partial P})P^{A_1}\right] F_{A_1\ldots A_l} - O(\eta_{A_iA_j}, P_{A_i}), \tag{5.13}$$

where $-O(\cdots)$ again indicates the terms which will be dropped when passing from $R$ to $\tilde{R}$. In fact, it is easy to see that the $O(W_{A_iA_j}W_{A_kA_n})$ part of $F$ only leads to such terms. Similarly, all of the terms in $F$ proportional to $\eta_{A_iA_j}$ with $i,j \neq 1$ will also be dropped.

The remaining terms are

$$F_{A_1\ldots A_l} = \Big(\tilde{F}_{A_1\ldots A_l} - c_l \sum_{j\geq 2} \eta_{A_1A_j} \tilde{F}^B{}_{BA_2\ldots\hat{A}_j\ldots A_l}\Big) + c_l \sum_i P_{A_i} S_{A_1\ldots\hat{A}_i\ldots A_l} + \cdots, \tag{5.14}$$

where

$$S_{A_2\ldots A_l} \equiv \frac{1}{P\cdot\bar{P}} \sum_{j=2}^{l} \bar{P}_{A_j} \tilde{F}^B{}_{BA_2\ldots\hat{A}_j\ldots A_l}. \tag{5.15}$$

Now let us apply the differential operator. Using the fact that $\tilde{F}$ is transverse, the action on the first term of eq. (5.14) gives

$$\tilde{R}_{A_2\ldots A_l} = \frac{\partial}{\partial P_{A_1}}\Big(\tilde{F}_{A_1\ldots A_l} - c_l \sum_{j\geq 2} \eta_{A_1A_j} \tilde{F}^B{}_{BA_2\ldots\hat{A}_j\ldots A_l}\Big) + c_l S_{A_2\ldots A_l} - O(\eta_{A_iA_j}, P_{A_i}) + \cdots, \tag{5.16}$$

where the $-O(\cdots)$ reminds us that some of the terms generated by $\partial/\partial P_{A_1}$ will have to be dropped.

To compute the action on the second term, we use a formula valid for any $S$ of homogeneity $-\Delta_S$:

$$\left[\frac{\partial}{\partial P_{A_1}} - \frac{1}{P\cdot\bar{P}}(\bar{P}\cdot\frac{\partial}{\partial P})P^{A_1}\right] \sum_i P_{A_i} S_{A_1\ldots\hat{A}_i\ldots A_l} = \tag{5.17}$$

$$= (d + l - 1 - \Delta_S) S_{A_2\ldots A_l} - \frac{1}{P\cdot\bar{P}} \sum_i \bar{P}_{A_i} (P\cdot S)_{A_2\ldots\hat{A}_i\ldots A_l} + O(\eta_{A_iA_j}, P_{A_i}). \tag{5.18}$$

Specializing to the $S$ in eq. (5.15), $\Delta_S = d + l - 1$ and the first term vanishes. Using the contraction

$$(P\cdot S)_{A_3\ldots A_l} = \tilde{F}^B{}_{BA_3\ldots A_l} \tag{5.19}$$

(for $\tilde{F}$ transverse), we see that the contribution to $\tilde{R}_{A_2\ldots A_l}$ is simply $-c_l S_{A_2\ldots A_l}$, canceling the second term in eq. (5.16). Thus we obtain the final result

$$\tilde{R}_{A_2\ldots A_l} = \frac{\partial}{\partial P_{A_1}}\Big(\tilde{F}_{A_1\ldots A_l} - c_l \sum_{j\geq 2} \eta_{A_1A_j} \tilde{F}^B{}_{BA_2\ldots\hat{A}_j\ldots A_l}\Big) - O(\eta_{A_iA_j}, P_{A_i}). \tag{5.20}$$

It remains to convert this equation to the polynomial notation by contracting with $Z$'s. Using the definition of the operator $D_M$, it is straightforward to show that the resulting formula is identical to eq. (5.9).

### 5.3 Examples

Now we will give some simple examples of how to apply the above formalism, focusing on three-point functions where two of the three operators are conserved currents. We will then show how the conservation condition restricts possible structures that appear in these three-point functions. Conservation constraints on the structure of three-point functions have been studied previously by Osborn and Petkou [38], directly in the physical space. Where comparison is possible, we have verified explicitly that our methods reproduce their results. We consider only the parity even case in $d \geq 4$.

Let us consider the simplest nontrivial example of a three-point function between two vector currents at points $x_1$ and $x_2$ and a scalar operator at $x_3$,

$$\langle v_a^1(x_1) v_b^2(x_2) \phi(x_3) \rangle \,. \tag{5.21}$$

Here we assume that $\phi$ has dimension $\Delta$, while $v$'s necessarily have dimension $d-1$. The currents do not necessarily belong to the same nonabelian current multiplet, so we can consider both symmetry possibilities under the exchange of $v$'s.

First we consider the symmetric case (e.g. if the currents are identical). According to the results of section 4.2, the embedding function encoding this three-point function has the form

$$\tilde{G}(P_1, P_2, P_3; Z_1, Z_2) = \frac{\alpha V_1 V_2 + \beta H_{12}}{(P_{12})^{d-\frac{\Delta}{2}} (P_{13})^{\frac{\Delta}{2}} (P_{23})^{\frac{\Delta}{2}}} \,, \tag{5.22}$$

with a priori independent constants $\alpha$ and $\beta$. The conservation condition can be imposed by using eq. (5.9). Computing the divergence at $P_1$ and dropping the terms of $O(Z_1^2, Z_1 \cdot P_1)$, we find the result

$$(\partial_{P_1} \cdot D_{Z_1}) \tilde{G} \to \left(\frac{d}{2} - 1\right) (\alpha(d-1-\Delta) + \beta\Delta) \frac{V_2}{(P_{12})^{d-\frac{\Delta}{2}} (P_{13})^{\frac{\Delta}{2}} (P_{23})^{\frac{\Delta}{2}}} \,. \tag{5.23}$$

For any $\alpha$ and $\beta$, this embedding function is identically transverse and has the correct structure to represent a three-point function between a scalar $\partial^a v_a^1(x_1)$, a vector $v_b^2(x_2)$ and another scalar $\phi(x_3)$. This is exactly how it should be, since taking divergence is consistent with conformal symmetry for the canonical field dimensions. Moreover, current conservation demands that the result should actually vanish, which implies that $\alpha$ and $\beta$ must be related by

$$\alpha(d-1-\Delta) + \beta\Delta = 0. \tag{5.24}$$

This example demonstrates how the conservation condition can be simply imposed directly in the embedding space. Note that the computations in this formalism are completely mechanical and easily lend themselves to automatization, e.g. in Mathematica.

Let us now generalize to the three-point function when the scalar is replaced by a spin $l$, dimension $\Delta$ operator:

$$\langle v_a^1(x_1) v_b^2(x_2) \phi_{c_1 \cdots c_l}(x_3) \rangle, \tag{5.25}$$

still symmetric in $1 \leftrightarrow 2$. When $l \geq 2$ is even this three-point function has an embedding function that a priori depends on the four constants $\alpha$, $\beta$, $\gamma$ and $\eta$:

$$\tilde{G}(\{P_i; Z_i\}) = \frac{\alpha V_1 V_2 V_3^l + \beta\, (H_{13} V_2 + H_{23} V_1)\, V_3^{l-1} + \gamma H_{12} V_3^l + \eta H_{13} H_{23} V_3^{l-2}}{(P_{12})^{d-\frac{\Delta+l}{2}} (P_{13})^{\frac{\Delta+l}{2}} (P_{23})^{\frac{\Delta+l}{2}}}. \tag{5.26}$$

| | symmetric | anti-symmetric |
|---|---|---|
| $\langle v^1 v^2 O^{(l)} \rangle$ | $l = 0 : 2 \to \mathbf{1}$ | $l = 0 : 0$ |
| | $l \geq 1$ odd $: 1 \to \mathbf{0}$ | $l = 1$ conserved $: 3 \to \mathbf{2}$ [45] |
| | $l \geq 2$ even $: 4 \to \mathbf{2}$ | $l = 1$ non-conserved $: 3 \to \mathbf{1}$ |
| | | $l \geq 2$ even $: 1 \to \mathbf{0}$ |
| | | $l \geq 3$ odd $: 4 \to \mathbf{2}$ |
| $\langle TTO^{(l)} \rangle$ | $l = 0 : 3 \to \mathbf{1}$ [38] | |
| | $l \geq 1$ odd $: 4 \to \mathbf{0}$ | |
| | $l = 2$ conserved $: 8 \to \mathbf{3}$ | |
| | $l = 2$ non-conserved $: 8 \to \mathbf{2}$ | |
| | $l \geq 4$ even $: 10 \to \mathbf{3}$ | |

**Table 1**. The number of parity even structures in the three-point function of two conserved spin $j$ currents ($j = 1, 2$) with an arbitrary spin $l$ primary in $d \geq 4$. We consider symmetric and anti-symmetric structures with respect to exchanging spin 1 currents, while only symmetric structures are relevant for the stress tensor correlators. "$n \to \boldsymbol{m}$" means that $n$ conformal structures compatible with the assumed exchange symmetry are reduced to $\boldsymbol{m}$ when the conservation condition is imposed.

The particular combinations of elementary tensor structures are fixed by the requirement that the function be even under the exchange of points $P_1$ and $P_2$. Computing the divergence at $P_1$ and dropping the usual terms, we find the following result

$$(\partial_{P_1} \cdot D_{Z_1}) \tilde{G} \to \left( \frac{d}{2} - 1 \right) \frac{a V_2 V_3^l + b H_{23} V_3^{l-1}}{(P_{12})^{d - \frac{\Delta + l}{2}} (P_{13})^{\frac{\Delta + l}{2}} (P_{23})^{\frac{\Delta + l}{2}}} \,, \tag{5.27}$$

with

$$a = \alpha(d - 1 - \Delta) + \beta(2 - 2d - l + \Delta) + \gamma(l + \Delta) \,, \tag{5.28}$$

$$b = \beta(d - 2 - \Delta) + \gamma l + \eta(4 - 2d - l + \Delta) \,. \tag{5.29}$$

Current conservation then forces $a = b = 0$, reducing the number of independent tensor structures in this three-point function from four to two.

For odd $l \geq 1$ there is a single tensor structure invariant under the exchange of points $P_1$ and $P_2$, given by

$$V(\{P_i; Z_i\}) = \frac{\alpha \left( H_{13} V_2 - H_{23} V_1 \right) V_3^{l-1}}{(P_{12})^{d - \frac{\Delta + l}{2}} (P_{13})^{\frac{\Delta + l}{2}} (P_{23})^{\frac{\Delta + l}{2}}} \,. \tag{5.30}$$

However, imposing conservation as above, we find $\alpha = 0$. This means that an odd $l$ field cannot appear in the OPE of two identical conserved currents.

The three-point functions anti-symmetric under current exchanges are straightforward to consider by the same method. One can also consider (spin 2)-(spin 2)-(spin $l$) three-point function, imposing stress tensor conservation (appendix A). The results are summarized in table 1.

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We would like to comment about the case when the spin $l$ operator is also conserved. One could naïvely expect that imposing spin $l$ conservation would lead to a further reduction of structures, but that's not what happens. For $l$ unequal to the spin $j$ of the other two conserved currents, spin $l$ conservation turns out to be satisfied automatically as a consequence of the spin $j$ conservation and setting the spin $l$ dimension to the canonical value $\Delta = l + d - 2$. Furthermore, for $l = j$ we actually get one more structure by going to the canonical spin $l$ dimension, as the table shows. What happens is that for this dimension some of the constraints for the coefficients of elementary structures become linearly dependent.

## 6 S-matrix rule for counting structures

In the previous sections we have rigorously derived a number of results related to counting CFT three-point function structures, with or without conservation constraints. We will now present a rule which allows us to intuitively explain all of the found results. The first appearance of this rule was the observation by Hofman and Maldacena [46] that the number of conformally invariant structures in the stress tensor three-point function in $d \geq 4$, computed to be 3 by Osborn and Petkou [38], coincides with the number of on-shell three-graviton vertices in $\mathbb{M}^{d+1}$, computed to be 3 by Metsaev and Tseytlin [47].

We propose the following generalization of this rule, which covers both the conserved and non-conserved case: *The number of independent structures in a three-point function containing operators of spins $\{l_1, l_2, l_3\}$ is equal to the number of independent on-shell scattering amplitudes for particles of spins $\{l_1, l_2, l_3\}$ in $d+1$ flat Minkowski dimensions. The particles should be taken massless or massive depending on whether or not the corresponding operator is conserved.*

To demonstrate how this works, let us first consider the case of a scattering amplitude between 3 massive particles of arbitrary spin. It is a Lorentz invariant function of the momentum $p_i$ and polarization tensor $\zeta_i$ of each particle. Since the spin $l_i$ polarization tensors $\zeta_i$ are symmetric and traceless, we can trade them for a polynomial of degree $l_i$ in the null vector $z_i$. Moreover, the transversality condition $(p_i)_{\mu_1}\zeta_i^{\mu_1 \ldots \mu_{l_1}} = 0$ translates to $z_i \cdot p_i = 0$.[17] Therefore, we must count polynomials such that

$$S(p_1, p_2, p_3; \lambda_1 z_1, \lambda_2 z_2, \lambda_3 z_3) = \lambda_1^{l_1} \lambda_2^{l_2} \lambda_3^{l_3} S(p_1, p_2, p_3; z_1, z_2, z_3)\,, \tag{6.1}$$

where $z_i \cdot p_i = 0$ and

$$p_1 + p_2 + p_3 = 0\,, \qquad p_i^2 = -M_i^2\,. \tag{6.2}$$

On-shellness and momentum conservation tell us that the contractions $p_i \cdot p_j$ can be written in terms of the particle masses and can therefore be dropped. Further, momentum conservation and transversality imply that $z_1 \cdot p_2 = -z_1 \cdot p_3$. Therefore, the general solution is a linear combination of

$$S(n_{12}, n_{13}, n_{23}) = (z_1 \cdot z_2)^{n_{12}} (z_1 \cdot z_3)^{n_{13}} (z_2 \cdot z_3)^{n_{23}} (z_1 \cdot p_2)^{m_1} (z_2 \cdot p_3)^{m_2} (z_3 \cdot p_1)^{m_3}\,, \tag{6.3}$$

[17] That this is the right condition to recover the tensor is clear in the rest frame of the particle, where the polarization tensor is purely spatial.

where

$$m_i = l_i - \sum_{j\neq i} n_{ij} \geq 0\,. \tag{6.4}$$

Since this is the same condition as eq. (4.18), the number of solutions is given by exactly the same combinatorial problem that we solved for CFT three-point functions. It is clear that there are no parity odd structures available in dimension bigger than 5 and in 5 dimensions we have the unique structure

$$\epsilon(z_1, z_2, z_3, p_1, p_2) = -\epsilon(z_1, z_2, z_3, p_1, p_3) = \epsilon(z_1, z_2, z_3, p_2, p_3)\,, \tag{6.5}$$

in perfect agreement with the results of section 4.2.3 for parity odd correlators.

Actually, the rule seems to work even beyond the three-point function level. Indeed, the most general $n$-particle scattering amplitude is a linear combination of

$$\left(\prod_{i=1}^{n}\prod_{j\neq i,i+1}^{n}(z_i\cdot p_j)^{m_{ij}}\right)\prod_{i<j}^{n}(z_i\cdot z_j)^{n_{ij}}\,, \tag{6.6}$$

where

$$\sum_{j\neq i,i+1}^{n} m_{ij} + \sum_{j\neq i}^{n} n_{ij} = l_i\,. \tag{6.7}$$

This is identical to the condition (4.77) for counting general tensor structures in an $n$-point conformal correlator. Moreover, the coefficients in the linear combination of structures for the S-matrix can be arbitrary functions of the Mandelstam invariants, in direct analogy with the functions $f_k$ of the cross-ratios in the $n$-point conformal correlators (4.72). This match strongly suggests that there is a one-to-one correspondence between $n$-particle scattering amplitudes and $n$-point conformal correlators.

### 6.1 Massless particles

Let us now study massless particles. In this case, the scattering amplitude must be invariant under the infinitesimal gauge transformation

$$\zeta_{\mu_1\ldots\mu_l} \to \zeta_{\mu_1\ldots\mu_l} + p_{(\mu_1}\Lambda_{\mu_2\ldots\mu_l)}\,. \tag{6.8}$$

This corresponds to invariance under

$$z_\mu \to z_\mu + \epsilon\, p_\mu \tag{6.9}$$

to first order in $\epsilon$. The problem of finding gauge invariant 3-particle scattering amplitudes is then reduced to finding linear combinations of the structures (6.3) that are invariant under (6.9) to first order in $\epsilon$. Recalling that $p_i^2 = 0$, it is easy to see that

$$\delta_1 S(n_{12}, n_{13}, n_{23}) = \epsilon_1\Big[n_{13}\, S_1(n_{12}, n_{13}-1, n_{23}) - n_{12}\, S_1(n_{12}-1, n_{13}, n_{23})\Big]\,, \tag{6.10}$$

$$\delta_2 S(n_{12}, n_{13}, n_{23}) = \epsilon_2\Big[n_{12}\, S_2(n_{12}-1, n_{13}, n_{23}) - n_{23}\, S_2(n_{12}, n_{13}, n_{23}-1)\Big]\,, \tag{6.11}$$

$$\delta_3 S(n_{12}, n_{13}, n_{23}) = \epsilon_3\Big[n_{23}\, S_3(n_{12}, n_{13}, n_{23}-1) - n_{13}\, S_3(n_{12}, n_{13}-1, n_{23})\Big]\,, \tag{6.12}$$

where $S_i$ is given by the same expression as $S$ but with $l_i \to l_i - 1$. This suggests starting with the ansatz

$$\sum_{i=0}^{k} a_i \, S(i, k-i, n_{23}) \tag{6.13}$$

to impose gauge invariance for particle 1. We then find that

$$\begin{aligned} 0 &= \sum_{i=0}^{k} \left( a_i \, i \, S_1(i-1, k-i, n_{23}) - a_i \, (k-i) \, S_1(i, k-i-1, n_{23}) \right) \\ &= \sum_{i=1}^{k} \left( a_i \, i - a_{i-1} \, (k-i+1) \right) \, S_1(i-1, k-i, n_{23}) \, , \end{aligned} \tag{6.14}$$

which fixes all the coefficients up to an overall normalization,

$$a_i = \frac{k-i+1}{i} a_{i-1} = \frac{k!}{i!(k-i)!} a_0 \, . \tag{6.15}$$

Notice that this solution only exists for $k \leq l_1$.

Imposing gauge invariance also on particle 2, we find the amplitude

$$T_k = \sum_{i=0}^{k} \sum_{j=0}^{k-i} \frac{k!}{i! j! (k-i-j)!} \, S(i, j, k-i-j) \, . \tag{6.16}$$

Gauge invariance of particle 3 is automatic. Note that this solution only exists for $k$ smaller (or equal) than all the spins $l_i$. Therefore, the number of possible scattering amplitudes between 3 massless higher spin particles is

$$1 + \min(l_1, l_2, l_3) \, .$$

This matches the counting of conformal three-point functions of conserved tensors in $d \geq 4$ (see table 1).

It is also interesting to notice the permutation symmetry properties

$$T_k(1,2,3) = T_k(2,3,1) = T_k(3,1,2) = (-1)^{\sum l_i} T_k(2,1,3) \, . \tag{6.17}$$

In particular, this means that photons don't interact; one needs a non-abelian gauge symmetry to have a three-point function of spin 1 massless particles.

To make further contact with the results of section 5.3, we can consider the case when one of the three particles is massive. In this case the analysis is simplified by going to the rest frame of the massive particle, so that we are dealing with a decay amplitude. It is also helpful to completely fix the gauge symmetry. The amplitude has to be constructed by contracting the purely spatial polarization tensors $\varepsilon_{1,2,3}$ with the spatial momentum of the decay products $\boldsymbol{p}$. We will assume that the decaying particle 3 has arbitrary spin $l$, while the massless decay products have the same spin $j$, focusing on the case $j = 1, 2$. Since $\varepsilon_{1,2}$ are transverse to $\boldsymbol{p}$, it's easy to construct the amplitudes:

$$\begin{aligned} j = 1: \quad & (\varepsilon_1 \cdot \varepsilon_2)(\varepsilon_3 \cdot \boldsymbol{p}^l) \, , \quad \varepsilon_{1\mu_1} \varepsilon_{2\mu_2} (\varepsilon_3 \cdot \boldsymbol{p}^{l-2})^{\mu_1 \mu_2} \, , \\ j = 2: \quad & (\varepsilon_1 \cdot \varepsilon_2)(\varepsilon_3 \cdot \boldsymbol{p}^l) \, , \quad (\varepsilon_1 \cdot \varepsilon_2)_{\mu_1 \mu_2} (\varepsilon_3 \cdot \boldsymbol{p}^{l-2})^{\mu_1 \mu_2} \, , \quad \varepsilon_{1\mu_1\mu_2} \varepsilon_{2\mu_3\mu_4} (\varepsilon_3 \cdot \boldsymbol{p}^{l-4})^{\mu_1 \mu_2 \mu_3 \mu_4} \, . \end{aligned} \tag{6.18}$$

This matches the number of structures found in table 1, including the symmetry/anti-symmetry of the current correlators, corresponding to parity under $\boldsymbol{p} \to -\boldsymbol{p}$. Notice that for low $l$ one runs out of indices to contract with $\boldsymbol{p}$ and the number of amplitudes is reduced, again in agreement with table 1.

### 6.2 Four dimensions

In four dimensions, the 5 vectors $z_1, z_2, z_3, p_1, p_2$ can not be linearly independent. Therefore, the determinant

$$\det_{1\le i,j\le 5}(z_i \cdot z_j)\Big|_{\substack{z_4=p_1\\ z_5=p_2}} \tag{6.19}$$

must vanish. This gives the following identity:

$$\begin{aligned}&\left(\frac{1}{2}\sum_{i}^{3} M_i^4 - \sum_{i<j}^{3} M_i^2 M_j^2\right)(z_1\cdot z_2)(z_1\cdot z_3)(z_2\cdot z_3)\\ &= 2(z_1\cdot z_2)(z_1\cdot p_2)(z_2\cdot p_3)(z_3\cdot p_1)^2 - M_1^2(z_1\cdot p_2)(z_2\cdot z_3)\\ &\quad + (M_1^2 - M_2^2 - M_3^2)(z_2\cdot p_3)(z_3\cdot p_1)(z_1\cdot z_2)(z_1\cdot z_3) + \text{cyclic}\,.\end{aligned} \tag{6.20}$$

Thus, we do not need to use the structure $(z_1\cdot z_2)(z_1\cdot z_3)(z_2\cdot z_3)$, and we recover precisely the counting of CFT three-point functions in three dimensions.

In the massless case, there is an even simpler relation

$$\begin{aligned}0 &= (z_2\cdot p_3)(z_3\cdot p_1)z_1 + (z_1\cdot p_2)(z_3\cdot p_1)z_2 + (z_2\cdot p_3)(z_1\cdot p_2)z_3\\ &\quad - (z_1\cdot z_3)(z_2\cdot p_3)p_1 + (z_2\cdot z_3)(z_1\cdot p_2)p_2\,.\end{aligned} \tag{6.21}$$

Taking the inner product with $z_1$ we re-obtain the identity (6.20) in the massless case

$$(z_1\cdot z_2)(z_3\cdot p_1) + (z_1\cdot z_3)(z_2\cdot p_3) + (z_2\cdot z_3)(z_1\cdot p_2) = 0\,, \tag{6.22}$$

which relates the basic structures as

$$S(n_{12}+1, n_{13}, n_{23}) + S(n_{12}, n_{13}+1, n_{23}) + S(n_{12}, n_{13}, n_{23}+1) = 0\,, \tag{6.23}$$

assuming that all $m_i = l_i - \sum_j n_{ij}$ are non-zero. Therefore, we can write all structures in terms of structures with $n_{23} = 0$:

$$S(n_{12}, n_{13}, n_{23}) = (-1)^{n_{23}} \sum_{i=0}^{n_{23}} \frac{n_{23}!}{i!(n_{23}-i)!} S(n_{12}+i, n_{13}+n_{23}-i, 0)\,. \tag{6.24}$$

This reduces the gauge invariant amplitude to

$$\begin{aligned}T_k &= \sum_{i=0}^{k}\sum_{j=0}^{k-i} \frac{k!(-1)^{k-i-j}}{i!j!(k-i-j)!} \sum_{t=0}^{k-i-j} \frac{(k-i-j)!}{t!(k-i-j-t)!} S(i+t, k-i-t, 0)\\ &= \sum_{i+j+t+u=k} \frac{k!(-1)^{t+u}}{i!j!t!u!} S(i+t, j+u, 0)\\ &= \sum_{r+s=k} S(r,s,0)\frac{k!}{r!s!} \sum_{i+t=r} (-1)^t \frac{r!}{i!t!} \sum_{j+u=s} (-1)^u \frac{s!}{j!u!}\\ &= 0\end{aligned} \tag{6.25}$$

in general. However, there are two special cases: $k = 0$ and $k = \min(l_1, l_2, l_3)$. It is clear that $T_0$ does not vanish identically and is gauge invariant. When $k$ takes its maximal value, equal to the smaller spin (which we choose to be $l_1$), the identity (6.24) can not be used. In particular, $S(i, l_1 - i - 1, 1)$ with $i = 0, 1, \ldots, l_1 - 1$ can not be written solely in terms of structures with $n_{23} = 0$. The best we can do is to reduce it down to structures with $n_{23} = 0$ and $n_{23} = 1$. We conclude that in 4 dimensions there are only 2 parity even structures for the scattering amplitude of 3 massless higher spin fields. This result agrees with the conjecture of [40] that there are only 2 independent structures for the three-point function of conserved tensors in $\mathrm{CFT}_3$.

Here we only considered parity even structures. It should be possible to give an analogous discussion for parity odd structures, where we expect to find one amplitude if the spins $\{l_1, l_2, l_3\}$ satisfy the triangle inequality and zero otherwise, to match the conjecture of [40] in the parity odd case.

### 6.3 Relation to AdS/CFT duality

In the case of polynomial scattering amplitudes, we can use AdS/CFT to provide an explicit map from scattering amplitudes in $\mathbb{M}^{d+1}$ to $\mathrm{CFT}_d$ correlators. We simply construct a contact Witten diagram that connects $n$ bulk-to-boundary propagators to the local interaction vertex corresponding to the $n$-particle S-matrix element. This map was already explored in the case of four-point functions of scalar operators in [9, 48]. Above, we saw that it should also extend to $n$-point functions of tensor operators. However, when the scattering amplitude has poles describing a mediated interaction, the situation is more complicated. It would be very interesting to construct an explicit map from S-matrix elements to conformal correlators that is also valid in this case. The Mellin representation of conformal correlators [14, 16, 17, 36] may be useful in this context, given its close structural analogy to scattering amplitudes.

Let us now give this map explicitly in the simplest case of three particle scattering. To each S-matrix element $S(n_{12}, n_{13}, n_{23})$ given in (6.3) we can associate a cubic local interaction vertex in the Lagrangian for AdS fields given by

$$\mathcal{V}(n_{12}, n_{13}, n_{23}) = \left((\nabla_\nu)^{m_2}\, \phi_1^{\mu_1 \ldots \mu_{l_1}}\right)\left((\nabla_\rho)^{m_3}\, \phi_2^{\nu_1 \ldots \nu_{l_2}}\right) \times \left((\nabla_\mu)^{m_1}\, \phi_3^{\rho_1 \ldots \rho_{l_3}}\right)(g_{\mu\nu})^{n_{12}}(g_{\mu\rho})^{n_{13}}(g_{\nu\rho})^{n_{23}}\,, \tag{6.26}$$

where Greek indices denote AdS indices. We use a schematic notation where, for example, $(\nabla_\nu)^{m_2}$ is the covariant derivative acting $m_2$ times on the field $\phi_1$, with indices contracted with the $\nu$ indices of the field $\phi_2$. The notation used in $(g_{\mu\nu})^{n_{12}}$ tells us that there are $n_{12}$ contractions of the indices of the fields $\phi_1$ and $\phi_2$. We recall that the integers $m_i$ are determined by the $n_{ij}$'s through the constraint $m_i + \sum_j n_{ij} = l_i$.

The AdS/CFT duality gives an explicit rule on how to map the above interaction vertex to a correlation function of operators dual to the fields $\phi_i$: one simply computes the Witten diagram by replacing in (6.26) the fields by their bulk-to-boundary propagators, and then integrates over the AdS interaction point. We shall denote the bulk-to-boundary propagator from an AdS point $y$ to a boundary point $x$ by

$$\Pi^{\mu_1\ldots\mu_l,a_1\ldots a_l}(y,x)\,. \tag{6.27}$$

This propagator obeys the bulk equation

$$\nabla^\nu\nabla_\nu\Pi^{\mu_1\ldots\mu_l,a_1\ldots a_l}=(\Delta(\Delta-d)-l)\,\Pi^{\mu_1\ldots\mu_l,a_1\ldots a_l}\,, \tag{6.28}$$

and has vanishing divergence

$$\nabla_\mu\Pi^{\mu\mu_2\ldots\mu_l,a_1\ldots a_l}=0\,. \tag{6.29}$$

From AdS/CFT one expects that all three-point functions can be written as a linear combination of this set of Witten diagrams. Of course the basis of three-point functions obtained this way is not the same basis of section 4. In particular, Witten diagrams give a basis of tensor structures where the constraints arising from operator conservation are simpler to formulate.

Let us then analyze in more detail the case of conserved spin $l$ operators. We wish to understand the constraints imposed on the bulk interaction vertices $\mathcal{V}(n_{12},n_{13},n_{23})$ that arise from current conservation in the CFT side. The boundary divergence acting on the bulk-to-boundary propagator of dimension $\Delta=d-l+2$ is pure gauge, i.e.

$$\partial_a\,\Pi^{\mu_1\ldots\mu_{l_1},aa_2\ldots a_l}=\nabla^{(\mu_1}\Lambda^{\mu_2\ldots\mu_{l_1}),a_2\ldots a_l}\,, \tag{6.30}$$

where $\Lambda$ satisfies the bulk equation (6.28). Therefore, as expected, current conservation in the boundary becomes gauge invariance in the bulk.[18]

Let us then look for gauge invariant linear combinations of vertices of the type

$$\mathcal{V}=\sum_{\{n_{ij}\}}a(n_{ij})\mathcal{V}(n_{ij})\,. \tag{6.31}$$

Suppose that we consider the field $\phi_1^{\mu_1\ldots\mu_l}=\nabla^{(\mu_1}\Lambda^{\mu_2\ldots\mu_l)}$ to be pure gauge. After some integrations by parts, and using the equations of motion, the vertex $\mathcal{V}(n_{12},n_{13},n_{23})$ transforms to

$$\delta_1\mathcal{V}(n_{ij})=\gamma\,(l_1-n_{12}-n_{13})\,\tilde{\mathcal{V}}(n_{ij})+n_{12}\tilde{\mathcal{V}}(n_{12}-1,n_{13},n_{23})-n_{13}\tilde{\mathcal{V}}(n_{12},n_{13}-1,n_{23})\,, \tag{6.32}$$

where

$$\gamma=\frac{1}{2}\left(\mu_2^2-\mu_1^2-\mu_3^2\right)\,,\qquad\qquad \mu_i^2=\Delta_i(\Delta_i-d)-l_i\,. \tag{6.33}$$

Note that here $\tilde{\mathcal{V}}$ denotes the vertex introduced in (6.26) with $\phi_1$ replaced by the gauge tensor $\Lambda$ of spin $l_1-1$. This equation is the direct analogue of (6.10) in flat space. The only difference is the appearance of an extra term proportional to the mass squared of the higher spin gauge fields in AdS. We conclude that gauge invariance imposes the constraint

$$\gamma\,(l_1-n_{12}-n_{13})\,a(n_{ij})+(n_{12}+1)\,a(n_{12}+1,n_{13},n_{23})-(n_{13}+1)\,a(n_{12},n_{13}+1,n_{23})=0\,, \tag{6.34}$$

on the coefficients of the expansion (6.31). Imposing gauge invariance on $\phi_2$ and $\phi_3$ produces similar equations.

[18] In the original three graviton case of Hofman and Maldacena [46] this gauge invariance was general covariance and the vertices were extracted from a generally covariant Lagrangian including the Einstein-Hilbert term and contractions of the Weyl tensor.

## 7 Summary and conclusions

With the formalism developed in this paper, the kinematical constraints arising from conformal invariance can be implemented for symmetric traceless operators of arbitrary spin almost as easily as for scalar operators. Bellow, we briefly summarize the basic rules for the more pragmatic reader.

### Summary

- **Embedding space**

  The natural habitat for conformal field theories is the light cone of the origin of $\mathbb{M}^{d+2}$. SO$(d+1,1)$ Lorentz transformations of the light rays generate conformal transformations. The usual flat physical space $\mathbb{R}^d$ can be obtained by projecting into the Poincaré section of the light cone

$$P_x = (P^+, P^-, P^a) = (1, x^2, x^a)\,. \tag{7.1}$$

- **Primary fields**

  Primary fields of dimension $\Delta$ and spin $l$ are encoded into a field $F(P;Z)$, polynomial in the polarization vector $Z$, such that

$$F(\lambda P; \alpha Z + \beta P) = \lambda^{-\Delta}\alpha^l F(P;Z)\,. \tag{7.2}$$

  The usual tensor form of the operator on $\mathbb{R}^d$ is obtained from

$$f_{a_1\ldots a_l}(x) = \frac{1}{l!(h-1)_l} D_{a_1}\cdots D_{a_l} F(P_x; Z_{z,x})\,, \tag{7.3}$$

  where $D_a$ is the differential operator defined in (3.7) and $Z_{z,x} = (0, 2x\cdot z, z^a)$.

- **Correlators**

  The most general form of the correlator

$$\langle F_1(P_1;Z_1)\cdots F_n(P_n;Z_n)\rangle \tag{7.4}$$

  compatible with conformal invariance is a linear combination of homogeneous polynomials of degree $l_i$ in each $Z_i$, each constructed by multiplying the basic building blocks $V_{i,jk}$ and $H_{ij}$ given in (4.14) and (4.15). The $P_i$ dependence is then constrained by the scaling in (7.2).

- **Conserved fields**

  A spin $l$ primary field of dimension $\Delta = d-2+l$ obeys the conservation equation

$$(\partial_P \cdot D)\, F(P;Z) = 0\,, \tag{7.5}$$

  where $D$ is the differential operator defined in (3.30). This condition generates additional constraints on the correlators of conserved fields that can be easily implemented.

The focus of this paper was to develop the formalism, postponing applications to the near future [12]. We were careful to establish connections with previous work and exemplify the strength of the method by rederiving many known results. For example, we have established a one-to-one correspondence with the general three-point function analysis of Mack [36] and of Osborn and Petkou [38], as well as with recent work on the three dimensional case in [40].

We have also presented several new results, interesting in their own right. For example, we reduced the problem of counting conformal three-point functions of operators with spin to the simple combinatorial problem depicted in figure 2, which we solved in closed form in eq. (4.20). For spin 1 currents and the stress tensor, we studied how conservation leads to a reduction in the number of three-point functions with an arbitrary spin $l$ primary. We have also discussed a general rule for counting the number of three-point functions in terms of flat space S-matrices in $d+1$ dimensions. Using this rule, we conjecture that the number of independent tensor structures for three-point functions of conserved tensors in $d \geq 4$ is given by $1 + \min(l_1, l_2, l_3)$. In three dimensions, the number of structures is reduced to 2 as claimed in [40].

In this paper we have been dealing with correlators of bosonic fields, but it should be pointed out that the embedding formalism can be also developed for fermion correlators [26]. Finally, although we have limited the discussion to the symmetric traceless primaries, it should not be too difficult to extend the formalism to anti-symmetric fields or fields of mixed symmetry, using polynomials in Grassmann variables.

**Note added.** When this paper was being finalized, ref. [17] appeared which among other things points out that conformal structures corresponding to operators with spin can be constructed from a smaller set of elementary structures. Our structures $V_{i,jk}$ and $H_{ij}$ are index-free equivalents of the structures $X_{ij}^{M_k}$ and $I^{M_i M_j}$ appearing in [17]. We believe that our index-free formalism is cleaner and more versatile, especially when various degeneracies among basic structures need to be taken into account, as in several situations discussed above, and also when considering traceless tensors.

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## A Three-point function for (spin 2)-(spin 2)-(spin $l$)

In this appendix we will apply the formalism developed in section 5 to the case of a three-point function between the spin 2 stress tensor $T_{ab}$ at $x_1$ and $x_2$ and a dimension $\Delta$ operator of spin $l$ at $x_3$,

$$\langle T_{ab}(x_1) T_{cd}(x_2) \phi_{e_1 \cdots e_l}(x_3) \rangle. \tag{A.1}$$

When $l$ is even, the embedding function (prior to imposing the conservation constraints) has 10 possible structures with coefficients $\alpha_a$:

$$\tilde{G}(\{P_i; Z_i\}) = \frac{\sum_{a=1}^{10} \alpha_a A_a(V_i, H_{ij})}{(P_{12})^{d+2-\frac{\Delta+l}{2}} (P_{13})^{\frac{\Delta+l}{2}} (P_{23})^{\frac{\Delta+l}{2}}}, \tag{A.2}$$

where the structures symmetric under exchanging $\{P_1, Z_1\}$ with $\{P_2, Z_2\}$ are given by

$$A_a(V_i, H_{ij}) = \begin{pmatrix} V_1^2 V_2^2 V_3^l \\ (H_{13} V_2^2 V_1 + H_{23} V_1^2 V_2) V_3^{l-1} \\ H_{12} V_1 V_2 V_3^l \\ (H_{13} V_2 + H_{23} V_1) H_{12} V_3^{l-1} \\ H_{13} H_{23} V_1 V_2 V_3^{l-2} \\ H_{12}^2 V_3^l \\ (H_{13}^2 V_2^2 + H_{23}^2 V_1^2) V_3^{l-2} \\ H_{12} H_{23} H_{13} V_3^{l-2} \\ (H_{13} H_{23}^2 V_1 + H_{23} H_{13}^2 V_2) V_3^{l-3} \\ H_{13}^2 H_{23}^2 V_3^{l-4} \end{pmatrix}. \tag{A.3}$$

We can then compute the divergence at $P_1$ and drop terms of $O(Z_1^2, Z_1 \cdot P_1)$ to obtain

$$(\partial_{P_1} \cdot D_{Z_1}) \tilde{G} \to \frac{\sum_{a=1}^{8} \beta_a B_a(V_i, H_{ij})}{(P_{12})^{d+2-\frac{\Delta+l}{2}} (P_{13})^{\frac{\Delta+l}{2}} (P_{23})^{\frac{\Delta+l}{2}}}, \tag{A.4}$$

where we have chosen the basis of structures

$$B_a(V_i, H_{ij}) = \begin{pmatrix} V_1 V_2^2 V_3^l \\ H_{13} V_2^2 V_3^{l-1} \\ H_{23} V_1 V_2 V_3^{l-1} \\ H_{12} V_2 V_3^l \\ H_{13} H_{23} V_2 V_3^{l-2} \\ H_{12} H_{23} V_3^{l-1} \\ H_{23}^2 V_1 V_3^{l-2} \\ H_{13} H_{23}^2 V_3^{l-3} \end{pmatrix}, \tag{A.5}$$

and the coefficients $\beta_a$ are given by

$$\begin{aligned}
\beta_1 &= \alpha_1 \left(2 - l + \Delta - d(1 - d + \Delta)\right) + \alpha_2 \left(-2 + l - \Delta + \frac{1}{2}d(2 - 2d - l + \Delta)\right) \\
&\quad + \alpha_3 \left(-2 + l - \Delta + \frac{1}{2}d(2 + l + \Delta)\right) + 2\alpha_4 \left(2 - d - l + \Delta\right), \qquad \text{(A.6)} \\
\beta_2 &= -\alpha_1 l + \frac{1}{2}\alpha_2 \left(d^2 + 2l - d\Delta\right) + \alpha_3 l + \frac{1}{2}\alpha_4 \left((d-4)l + d\Delta\right) + \alpha_7 d(-2d - l + \Delta), \qquad \text{(A.7)} \\
\beta_3 &= \alpha_2 \left(2 + d^2 - l + \Delta - d(2 + \Delta)\right) + \frac{1}{2}\alpha_3 dl + \frac{1}{2}\alpha_4 d(-2 + l + \Delta) \\
&\quad + \alpha_5 \left(-2 + l - \Delta + \frac{1}{2}d(4 - 2d - l + \Delta)\right) - 2\alpha_8(-2 + d + l - \Delta), \qquad \text{(A.8)} \\
\beta_4 &= 2\alpha_1 - 2\alpha_2 + \frac{1}{2}\alpha_3 \left(-4 + d^2 - d\Delta\right) + \alpha_4 \left(4 - \frac{1}{2}d(2d + l - \Delta)\right) + \alpha_6 d(1 + \Delta), \qquad \text{(A.9)} \\
\beta_5 &= -\alpha_2(l-1) + \frac{1}{2}\alpha_4 d(l-1) + \frac{1}{2}\alpha_5 \left(-2 + d^2 + 2l - d\Delta\right) - 2\alpha_7 d \\
&\quad + \frac{1}{2}\alpha_8 \left(4 - 4l + d(-2 + l + \Delta)\right) + \alpha_9 d(2 - 2d - l + \Delta), \qquad \text{(A.10)} \\
\beta_6 &= \alpha_2 + \frac{1}{2}\alpha_4 d(-1 + d - \Delta) - \alpha_5 + \alpha_6 dl + \alpha_8 \left(2 + \frac{1}{2}d(2 - 2d - l + \Delta)\right), \qquad \text{(A.11)} \\
\beta_7 &= \frac{1}{2}\alpha_4 d(l-1) - \frac{1}{2}\alpha_5 d + \alpha_7 \left(2 + d^2 - l + \Delta - d(1 + \Delta)\right) + \alpha_8 \left(2 - d - l + \Delta\right) \\
&\quad - \frac{1}{2}\alpha_9 (d-2) \left(-2 + 2d + l - \Delta\right), \qquad \text{(A.12)} \\
\beta_8 &= -\alpha_7(l-2) + \frac{1}{2}\alpha_8(d-2)(l-2) + \frac{1}{2}\alpha_9 \left(d^2 + 2(l-2) - d(2 + \Delta)\right) \\
&\quad + \alpha_{10} d \left(4 - 2d - l + \Delta\right). \qquad \text{(A.13)}
\end{aligned}$$

Setting each of these coefficients to zero would naïvely reduce the number of structures from 10 down to 2. However, one of the equations is linearly dependent due to the relation

$$\begin{aligned}
0 &= 2\beta_1 l + 2\beta_2(d^2 - l + \Delta - d\Delta) - \beta_3(2l + d(2 + d - \Delta)) + \beta_4(d-2)l \\
&\quad - \beta_5(d-2)(2d + l - \Delta) - \beta_6(d-2)(l + \Delta) + 2\beta_7 d(2d + l - \Delta), \qquad \text{(A.14)}
\end{aligned}$$

so the number of independent structures is actually reduced from 10 down to 3.

Next let us consider the case that $l$ is odd. In this case there are initially 4 possible structures invariant under exchanging $\{P_1, Z_1\}$ with $\{P_2, Z_2\}$:

$$\tilde{G}(\{P_i; Z_i\}) = \frac{\sum_{a=1}^{4} \gamma_i G_a(V_i, H_{ij})}{(P_{12})^{d+2-\frac{\Delta+l}{2}} (P_{13})^{\frac{\Delta+l}{2}} (P_{23})^{\frac{\Delta+l}{2}}}, \qquad \text{(A.15)}$$

with

$$G_a(V_i, H_{ij}) = \begin{pmatrix} (H_{13}V_2^2V_1 - H_{23}V_1^2V_2)V_3^{l-1} \\ (H_{13}V_2 - H_{23}V_1)H_{12}V_3^{l-1} \\ (H_{13}^2V_2^2 - H_{23}^2V_1^2)V_3^{l-2} \\ (H_{23}^2H_{13}V_1 - H_{13}^2H_{23}V_2)V_3^{l-3} \end{pmatrix}. \qquad \text{(A.16)}$$

Then computing the divergence at $P_1$ and dropping the usual terms gives

$$(\partial_{P_1} \cdot D_{Z_1})\, \tilde{G} \to \frac{\sum_{a=1}^{8} \delta_a B_a(V_i, H_{ij})}{(P_{12})^{d+2-\frac{\Delta+l}{2}} (P_{13})^{\frac{\Delta+l}{2}} (P_{23})^{\frac{\Delta+l}{2}}}, \tag{A.17}$$

where the coefficients $\delta_i$ are given by

$$\delta_1 = \frac{1}{2}\gamma_1 \left((d-2)(2-l+\Delta) - 2d^2\right) + 2\gamma_2 (2-d-l+\Delta)\,, \tag{A.18}$$

$$\delta_2 = \frac{1}{2}\gamma_1 \left(d^2 + 2l - d\Delta\right) + \frac{1}{2}\gamma_2 ((d-4)l + d\Delta) + \gamma_3 d(-2d-l+\Delta)\,, \tag{A.19}$$

$$\delta_3 = \gamma_1 (-2+l-\Delta-d(d-\Delta)) + \gamma_2 \left(2(2-l+\Delta) - \frac{1}{2}d(2+l+\Delta)\right), \tag{A.20}$$

$$\delta_4 = -2\gamma_1 + \gamma_2 \left(4 - \frac{1}{2}d(2d+l-\Delta)\right), \tag{A.21}$$

$$\delta_5 = \gamma_1(l-1) + \frac{1}{2}\gamma_2(d-4)(l-1) - 2\gamma_3 d + \gamma_4 d(-2+2d+l-\Delta)\,, \tag{A.22}$$

$$\delta_6 = -\gamma_1 + \gamma_2 \left(2 - \frac{1}{2}d(1+d-\Delta)\right), \tag{A.23}$$

$$\begin{aligned} \delta_7 = &-\frac{1}{2}\gamma_2 d(l-1) + \gamma_3 (-2+l-\Delta+d(1-d+\Delta)) \\ &+ \frac{1}{2}\gamma_4 (d-2)(2-2d-l+\Delta), \end{aligned} \tag{A.24}$$

$$\delta_8 = \gamma_3(l-2) + \gamma_4 \left(-2+l+\frac{1}{2}d(2+d-\Delta)\right). \tag{A.25}$$

Setting each of these coefficients to zero, it is straightforward to verify that there are precisely 4 linearly independent constraints, forcing $\gamma_1 = \gamma_2 = \gamma_3 = \gamma_4 = 0$. Thus, an odd $l$ operator cannot appear in the OPE of the stress tensor with itself.



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