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Existence and multiplicity of positive solutions for a nonlocal differential equation

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Abstract
In this paper, the existence and multiplicity results of positive solutions for a nonlocal differential equation are mainly considered.

Keywords: Nonlocal boundary value problems, Cone, Fixed point theorem

Introduction

In this paper, we are concerned with the existence and multiplicity of positive solutions for the following nonlinear differential equation with nonlocal boundary value condition

$$\begin{cases} -\Phi\left(\int\limits_0^1 |u(s)|^q \mathrm{d}\varphi(s)\right) u''(t) = h(t)f(u(t)), & \text{in } 0 < t < 1, \\ \alpha u(0) - \beta u'(0) = 0,\ \gamma u(1) + \delta u'(1) = g\left(\int\limits_0^1 u(s)\mathrm{d}\varphi(s)\right), \end{cases} \tag{1}$$

where α, β, γ, δ are nonnegative constants, $\rho = \alpha\gamma + \alpha\delta + \beta\gamma > 0$, $q \geq 1$; $\int_0^1 |u(s)|^q \mathrm{d}\varphi(s)$,$\int_0^1 |u(s)|^q \mathrm{d}\varphi(s)$ denote the Riemann-Stieltjes integrals.

Many authors consider the problem

$$-\Delta u = M\frac{f(u)^{\alpha}}{\left(\int_{\Omega} f(u)\right)^{\beta}},\ \text{in } \Omega \subset R^n,\ u = 0, \quad \text{on } \partial\Omega, \tag{2}$$

because of the importance in numerous physical models: system of particles in thermodynamical equilibrium interacting via gravitational potential, 2-D fully turbulent behavior of a real flow, one-dimensional fluid flows with rate of strain proportional to a power of stress multiplied by a function of temperature, etc. In [1,2], the authors use the Kras-noselskii fixed point theorem to obtain one positive solution for the following nonlocal equation with zero Dirichlet boundary condition

$$-a\left(\int\limits_0^1 |u(s)|^q\right) u''(t) = h(t)f(u(t)),$$

when the nonlinearity f is a sublinear or superlinear function in a sense to be established when necessary. Nonlocal BVPs of ordinary differential equations or system arise in a variety of areas of applied mathematics and physics. In recent years, more and more papers were devoted to deal with the existence of positive solutions of nonlocal BVPs (see [3-9] and references therein). Inspired by the above references, our aim in the present paper is to investigate the existence and multiplicity of positive solutions to Equation 1 using the Krasnosel'skii fixed point theorem and Leggett-Williams fixed point theorem.

This paper is organized as follows: In Section 2, some preliminaries are given; In Section 3, we give the existence results.

Preliminaries

Lemma 2.1 [3]. Let $y(t) \in C([0, 1])$, then the problem

$$\begin{cases} -u''(t) = y(t), \text{ in } 0 < t < 1, \\ \alpha u(0) - \beta u'(0) = 0, \ \gamma u(1) + \delta u'(1) = g\left(\int_0^1 u(s) \mathrm{d}\varphi(s)\right), \end{cases}$$

has a unique solution

$$u(t) = \frac{\beta + \alpha t}{\rho} g\left(\int_0^1 u(s) \mathrm{d}\varphi(s)\right) + \int_0^1 G(t, s) y(s) \mathrm{d}s,$$

where the Green function $G(t, s)$ is

$$G(t,s) = \frac{1}{\rho}, \begin{cases} (\beta + \alpha s)(\delta + \gamma - \gamma t), & \text{in } 0 \le s \le t \le 1, \\ (\beta + \alpha t)(\delta + \gamma - \gamma s), & \text{in } 0 \le t \le s \le 1. \end{cases}$$

It is easy to see that

$$G(t,s) > 0, \ 0 < t, s < 1; \ G(t,s) \le G(s, s), \quad 0 \le t, s \le 1,$$

and there exists a $\theta \in (0, \frac{1}{2})$ such that $G(t, s) \ge \theta G(s, s)$, $\theta \le t \le 1 - \theta$, $0 \le s \le 1$.

For convenience, we assume the following conditions hold throughout this paper:

(H1) f, g, Φ: $R^+ \to R^+$ are continuous and nondecreasing functions, and $\Phi(0) > 0$;

(H2) $\phi(t)$ is an increasing nonconstant function defined on [0, 1] with $\phi(0) = 0$;

(H3) $h(t)$ does not vanish identically on any subinterval of (0, 1) and satisfies

$$0 < \int_\theta^{1-\theta} G(t, s) h(s) \mathrm{d}s < +\infty.$$

Obviously, $u \in C^2(0, 1)$ is a solution of Equation 1 if and only if $u \in C(0, 1)$ satisfies the following nonlinear integral equation

$$u(t) = \frac{\beta + \alpha t}{\rho} g\left(\int_0^1 u(s) \mathrm{d}\varphi(s)\right) + \int_0^1 G(t, s) \frac{h(s) f(u(s))}{\Phi\left(\int_0^1 |u|^q \mathrm{d}\varphi\right)} ds.$$

At the end of this section, we state the fixed point theorems, which will be used in Section 3.

Let E be a real Banach space with norm $||\cdot||$ and $P \subset E$ be a cone in E, $P_r = \{x \in P : ||x|| < r\} (r > 0)$. Then, $\overline{P_r} = \{x \in P : ||x|| \le r\}$. A map α is said to be a nonnegative continuous concave functional on P if α: $P \to [0, +\infty)$ is continuous and

$$\alpha(tx + (1 - t)y) \ge t\alpha(x) + (1 - t)\alpha(y)$$

for all x, $y \in P$ and $t \in [0, 1]$. For numbers a, b such that $0 < a < b$ and α is a nonnegative continuous concave functional on P, we define the convex set

$$P(\alpha, a, b) = \{x \in P : a \le \alpha(x), ||x|| \le b\}.$$

Lemma 2.2 [10]. Let $A : \overline{P_c} \to \overline{P_c}$ be completely continuous and α be a nonnegative continuous concave functional on P such that $\alpha(x) = ||x||$ for all $x \in \overline{P_c}$. Suppose there exists $0 < d < a < b = c$ such that

(i) $\{x \in P(\alpha, a, b): \alpha(x) > a\} \neq \emptyset$ and $\alpha(Ax) > a$ for $x \in P(\alpha, a, b)$;

(ii) $||Ax|| < d$ for $||x|| \le d$;

(iii) $\alpha(Ax) > a$ for $x \in P(\alpha, a, c)$ with $||Ax|| > b$.

Then, A has at least three fixed points x_1, x_2, x_3 satisfying

$$||x_1|| < d, \quad a < \alpha(x_2),$$
$$||x_3|| > d \quad \text{and} \quad \alpha(x_3) < a.$$

Lemma 2.3 [10]. Let E be a Banach space, and let $P \subset E$ be a closed, convex cone in E, assume Ω_1, Ω_2 are bounded open subsets of E with $0 \in \Omega_1, \bar{\Omega}_1 \subset \Omega_2$, and $A : P \cap (\bar{\Omega}_2 \backslash \Omega_1) \to P$ be a completely continuous operator such that either

(i) $||Au|| \le ||u||$, $u \in P \cap \partial\Omega_1$ and $||Au|| \ge ||u||$, $u \in P \cap \partial\Omega_2$; or

(ii) $||Au|| \ge ||u||$, $u \in P \cap \partial\Omega_1$ and $||Au|| \le ||u||$, $u \in P \cap \partial\Omega_2$.

Then, A has a fixed point in $P \cap (\bar{\Omega}_2 \backslash \Omega_1)$.

Main result

Let $E = C[0, 1]$ endowed norm $||u|| = \max_{0 \le t \le 1} |u|$, and define the cone $P \subseteq E$ by

$$P = \left\{u \in E : u(t) \ge 0, \min_{\theta \le t \le 1-\theta} u(t) \ge \theta ||u||\right\}.$$

Then, it is easy to prove that E is a Banach space and P is a cone in E.

Define the operator T: $E \to E$ by

$$T(u)(t) = \frac{\beta + \alpha t}{\rho} g\left(\int_0^1 u(s) \mathrm{d}\varphi(s)\right) + \int_0^1 G(t, s) \frac{h(s) f(u(s))}{\Phi\left(\int_0^1 |u|^q \mathrm{d}\varphi\right)} \mathrm{d}s.$$

Lemma 3.1. T: $E \to E$ is completely continuous, and *Te now prove that*$P \subseteq P$.

Proof. For any $u \in P$, then from properties of $G(t, s)$, $T(u)(t) \ge 0$, $t \in [0, 1]$, and it follows from the definition of T that

$$||T(u)|| \le \frac{\alpha + \beta}{\rho} g\left(\int_0^1 u(s) \mathrm{d}\varphi(s)\right) + \int_0^1 G(s, s) \frac{h(s) f(u(s))}{\Phi\left(\int_0^1 |u|^q \mathrm{d}\varphi\right)} \mathrm{d}s.$$

Thus, it follows from above that

$$\begin{aligned} \min_{\theta \le t \le 1-\theta} T(u)(t) &= \min_{\theta \le t \le 1-\theta} \left[\frac{\beta + \alpha t}{\rho} g\left(\int_0^1 u(s) \mathrm{d}\varphi(s)\right) + \int_0^1 G(t, s) \frac{h(s) f(u(s))}{\Phi\left(\int_0^1 |u|^q \mathrm{d}\varphi\right)} \mathrm{d}s\right] \\ &\ge \theta \frac{\alpha + \beta}{\rho} g\left(\int_0^1 u(s) \mathrm{d}\varphi(s)\right) + \theta \int_0^1 G(s, s) \frac{h(s) f(u(s))}{\Phi\left(\int_0^1 |u|^q \mathrm{d}\varphi\right)} \mathrm{d}s \\ &\ge \theta ||T(u)|| \end{aligned}$$

From the above, we conclude that $TP \subseteq P$. Also, one can verify that T is completely continuous by the Arzela-Ascoli theorem. □

Let

$$l = \min_{0 \le t \le 1} \int_{\theta}^{1-\theta} G(t,s)h(s)\mathrm{d}s, \quad L = \min_{\theta \le t \le 1-\theta} \int_{\theta}^{1-\theta} G(t,s)h(s)\mathrm{d}s,$$

$$Ł = \min_{0 \le t \le 1} \int_{0}^{1} G(t,s)h(s)\mathrm{d}s.$$

Then, it is clear to see that $0 < l \le L < Ł$.

Theorem 3.2. Assume (H1) to (H3) hold. In addition,

(H4)

$$\lim_{r \to 0^+} \inf \frac{f(\theta r)}{r\Phi(r^q \varphi(1))} \ge \frac{1}{l};$$

(H5) There exists a constant $2 \le p_1$ such that

$$\lim_{r \to \infty} \sup \frac{f(r)}{r\Phi((\varphi(1-\theta) - \varphi(\theta))\theta^q r^q)} \le \frac{1}{p_1 Ł};$$

(H6) There exists a constant p_2 with $\frac{1}{p_1} + \frac{1}{p_2} = 1$ such that

$$\lim_{r \to \infty} \sup \frac{g(r)}{r} \le \frac{\rho}{p_2 \varphi(1)(\beta + \alpha)}$$

Then, problem (Equation 1) has one positive solution.

Proof. From (H4), there exists a $0 < \eta < \infty$ such that

$$\frac{f(\theta r)}{r\Phi(r^q \varphi(1))} \ge \frac{1}{l}, \quad \forall 0 < r \le \eta. \tag{3}$$

Choosing $R_1 \in (0, \eta)$, set $\Omega_1 = \{u \in E : ||u|| < R_1\}$. We now prove that

$$||Tu|| \ge ||u||, \quad \forall u \in P \cap \partial\Omega_1. \tag{4}$$

Let $u \in P \cap \partial\Omega_1$. Since $\min_{\theta \le t \le 1-\theta} u(t) \ge \theta\, ||u||$ and $||u|| = R_1$, from Equation 3, (H1) and (H3), it follows that

$$\begin{aligned}
Tu(t) &= \frac{\beta + \alpha t}{\rho} g\left(\int_0^1 u(s)\mathrm{d}\varphi(s)\right) + \int_0^1 G(t,s)\frac{h(s)f(u(s))}{\Phi\left(\int_0^1 |u|^q \mathrm{d}\varphi\right)}\mathrm{d}s \\
&\ge \int_0^1 G(t,s)\frac{h(s)f(u(s))}{\Phi\left(\int_0^1 |u|^q \mathrm{d}\varphi\right)}\mathrm{d}s \\
&\ge \int_{\theta}^{1-\theta} G(t,s)\frac{h(s)f(u(s))}{\Phi\left(\int_0^1 |u|^q \mathrm{d}\varphi\right)}\mathrm{d}s \\
&\ge \frac{f(\theta R_1)}{\Phi(R_1^q \varphi(1))} \int_{\theta}^{1-\theta} G(t,s)h(s)\mathrm{d}s \\
&\ge \frac{f(\theta R_1)}{\Phi(R_1^q \varphi(1))} l \\
&\ge R_1 = ||u||.
\end{aligned}$$

Then, Equation 4 holds.

On the other hand, from (H5), there exists $\overline{R_1} > 0$ such that

$$\frac{f(r)}{r\Phi((\varphi(1-\theta)-\varphi(\theta))\theta^q r^q)} \le \frac{1}{p_1 Ł}, \quad \forall r \ge \overline{R_1}. \tag{5}$$

From (H6), there exists $\overline{R_2} > 0$ such that

$$\frac{g(r)}{r} \le \frac{\rho}{p_2\varphi(1)(\beta+\alpha)}, \quad \forall r \ge \overline{R_2}. \tag{6}$$

Choosing $R_2 = \max\left\{R_1, \overline{R_1}, \frac{\overline{R_2}}{\theta(\varphi(1-\theta)-\varphi(\theta))}\right\} + 1$, set $\Omega_2 = \{u \in E : ||u|| < R_2\}$. We now prove that

$$||Tu|| \le ||u||, \quad \forall u \in P \cap \partial\Omega_2. \tag{7}$$

If $u \in P \cap \partial\Omega_2$, we have

$$\int_1^0 u(s)\mathrm{d}\varphi(s) \ge \int_\theta^{1-\theta} u(s)\mathrm{d}\varphi(s) \ge \theta R_2(\varphi(1-\theta)-\varphi(\theta)) \ge \overline{R_2}.$$

From Equations 5, 6, we can prove

$$\begin{aligned}
Tu(t) &= \frac{\beta+\alpha t}{\rho} g\left(\int_0^1 u(s)\mathrm{d}\varphi(s)\right) + \int_0^1 G(t,s)\frac{h(s)f(u(s))}{\Phi\left(\int_0^1 |u|^q \mathrm{d}\varphi\right)}\mathrm{d}s \\
&\le \frac{\beta+\alpha}{\rho} g\left(\int_0^1 u(s)\mathrm{d}\varphi(s)\right) + \int_0^1 G(t,s)\frac{h(s)f(u(s))}{\Phi\left(\int_0^1 |u|^q \mathrm{d}\varphi\right)}\mathrm{d}s \\
&\le \frac{\beta+\alpha}{\rho}\frac{\rho}{p_2\varphi(1)(\beta+\alpha)}\int_0^1 u(s)\mathrm{d}\varphi(s) + f(||u||)\int_0^1 G(t,s)\frac{h(s)}{\Phi\left(\int_\theta^{1-\theta} |u|^q \mathrm{d}\varphi\right)}\mathrm{d}s \\
&\le \frac{\beta+\alpha}{\rho}\frac{\rho}{p_2\varphi(1)(\beta+\alpha)}||u||\varphi(1) + \frac{f(||u||)}{\Phi((\varphi(1-\theta)-\varphi(\theta))\theta^q||u||^q)}\int_0^1 G(t,s)h(s)\mathrm{d}s \\
&\le \frac{R_2}{p_1} + \frac{R_2}{p_2} \\
&= R_2 = ||u||.
\end{aligned}$$

Then, Equation 7 holds.

Therefore, by Equations 4 and 7 and the second part of Lemma 2.3, T has a fixed point in $P \cap (\bar{\Omega}_2 \backslash \Omega_1)$, which is a positive solution of Equation 1. □

Example. Let $q = 2$, $h(t) = 1$, $\Phi(s) = 2 + s$, $\phi(t) = 2t$, $f(u) = \frac{\theta^2(1-2\theta)}{4Ł}(u^{\frac{1}{3}} + u^3)$ and $g(s) = s^{\frac{1}{2}}$, namely,

$$\begin{cases} -\left(2 + \int_0^1 |u(s)|^2 \mathrm{d}(2s)\right) u''(t) = \frac{\theta^2(1-2\theta)}{4Ł}(u^{\frac{1}{3}} + u^3), \quad \text{in } 0 < t < 1, \\ \alpha u(0) - \beta u'(0) = 0, \ \gamma u(1) + \delta u'(1) = \left[\int_0^1 u(s)\mathrm{d}(2s)\right]^{\frac{1}{2}}. \end{cases}$$

It is easy to see that (H1) to (H3) hold. We also can have

$$\lim_{r\to 0+}\ \inf\frac{f(\theta r)}{r\Phi(r^q\varphi(1))}=\lim_{r\to 0+}\ \inf\frac{\dfrac{\theta^2(1-2\theta)}{4Ł}((\theta r)^{\frac{1}{3}}+(\theta r)^3)}{r(2+2r^2)}=\infty,$$

$$\lim_{r\to\infty}\sup\frac{f(r)}{r\Phi((\varphi(1-\theta)-\varphi(\theta))\theta^q r^q)}=\lim_{r\to\infty}\sup\frac{\dfrac{\theta^2(1-2\theta)}{4Ł}(r^{\frac{1}{3}}+r^3)}{r(2+2(1-2\theta)\theta^2 r^2)}=\frac{1}{8Ł}.$$

Take $p_1 = 2$, then it is clear to see that (H4) and (H5) hold. Since

$$\lim_{r\to\infty}\sup\frac{g(r)}{r}=\lim_{r\to\infty}\sup\frac{r^{\frac{1}{2}}}{r}=0,$$

then (H6) hold.

Theorem 3.3. Assume (H1) to (H3) hold. In addition,

(H7) There exists a constant $2 \le p_1$ such that

$$\lim_{r\to 0}\sup\frac{f(r)}{r\Phi((\varphi(1-\theta)-\varphi(\theta))\theta^q r^q)}\le\frac{1}{p_1 Ł};$$

(H8) There exists a constant p_2 with $\dfrac{1}{p_1}+\dfrac{1}{p_2}=1$ such that

$$\lim_{r\to 0}\sup\frac{g(r)}{r}\le\frac{\rho}{p_2\varphi(1)(\beta+\alpha)};$$

(H9)

$$\lim_{r\to\infty}\ \inf\frac{f(\theta r)}{r\Phi(r^q\varphi(1))}\ge\frac{1}{l}.$$

Then, problem (Equation 1) has one positive solution.

Proof. From (H7), there exists $\eta_1 > 0$ such that

$$\frac{f(r)}{r\Phi((\varphi(1-\theta)-\varphi(\theta))\theta^q r^q)}\le\frac{1}{p_1 Ł},\quad \forall 0<r<\eta_1. \tag{8}$$

From (H8), there exists $\eta_2 > 0$ such that

$$\frac{g(r)}{r}\le\frac{\rho}{p_2\varphi(1)(\beta+\alpha)},\quad \forall 0<r<\eta_2. \tag{9}$$

Choosing $R_1 = \min\{\eta_1, \frac{\eta_2}{\varphi(1)}\}$, set $\Omega_1 = \{u \in E : ||u|| < R_1\}$. We now prove that

$$||Tu|| \le ||u||,\quad \forall u \in P\cap\partial\Omega_1. \tag{10}$$

If $u \in P \cap \partial\Omega_1$, we have

$$\int_0^1 u(s)\mathrm{d}\varphi(s)\le\int_0^1 R_1\mathrm{d}\varphi(s)\le R_1\varphi(1)\le\eta_2.$$

From Equations 8, 9, we can prove

$$\begin{aligned}
Tu(t) &= \frac{\beta+\alpha t}{\rho} g\left(\int_0^1 u(s)\mathrm{d}\varphi(s)\right) + \int_0^1 G(t,s)\frac{h(s)f(u(s))}{\Phi\left(\int_0^1 |u|^q \mathrm{d}\varphi\right)}\mathrm{d}s \\
&\le \frac{\beta+\alpha}{\rho} g\left(\int_0^1 u(s)\mathrm{d}\varphi(s)\right) + \int_0^1 G(t,s)\frac{h(s)f(u(s))}{\Phi\left(\int_0^1 |u|^q \mathrm{d}\varphi\right)}\mathrm{d}s \\
&\le \frac{\beta+\alpha}{\rho}\frac{\rho}{p_2\varphi(1)(\beta+\alpha)}\int_0^1 u(s)\mathrm{d}\varphi(s) + f(||u||)\int_0^1 G(t,s)\frac{h(s)}{\Phi\left(\int_\theta^{1-\theta} |u|^q \mathrm{d}\varphi\right)}\mathrm{d}s \\
&\le \frac{\beta+\alpha}{\rho}\frac{\rho}{p_2\varphi(1)(\beta+\alpha)}||u||\varphi(1) + \frac{f(||u||)}{\Phi((\varphi(1-\theta)-\varphi(\theta))\theta^q||u||^q)}\int_0^1 G(t,s)h(s)\mathrm{d}s \\
&\le \frac{R_1}{p_1} + \frac{R_1}{p_2} \\
&= R_1 = ||u||.
\end{aligned}$$

Then, Equation 10 holds.

On the other hand, from (H7), there exists $\overline{R_1} > 0$ such that

$$\frac{f(\theta r)}{r\Phi(r^q\varphi(1))} \ge \frac{1}{l}, \quad \forall r \ge \overline{R_1}. \tag{11}$$

Choosing $R_2 = \max\{R_1, (\frac{\overline{R_1}}{\theta^q(\varphi(1-\theta)-\varphi(\theta))})^{\frac{1}{q}}\} + 1$, set $\Omega_2 = \{u \in E : ||u|| < R_2\}$. We now prove that

$$||Tu|| \ \ge \ ||u||, \quad \forall u \in P \cap \partial\Omega_2. \tag{12}$$

If $u \in P \cap \partial\Omega_2$, Since $\min_{\theta \le t \le 1-\theta} u(t) \ge \theta\, ||u||$ and $||u|| = R_2$, we have

$$\int_0^1 |u|^q \mathrm{d}\varphi(s) \ge \int_\theta^{1-\theta} |u|^q \mathrm{d}\varphi \ge \theta^q R_2^q(\varphi(1-\theta)-\varphi(\theta)) \ge \overline{R_1}. \tag{13}$$

By Equation 11, (H1) and (H3), it follows that

$$\begin{aligned}
Tu(t) &= \frac{\beta+\alpha t}{\rho} g\left(\int_0^1 u(s)\mathrm{d}\varphi(s)\right) + \int_0^1 G(t,s)\frac{h(s)f(u(s))}{\Phi\left(\int_0^1 |u|^q \mathrm{d}\varphi\right)}\mathrm{d}s \\
&\ge \int_0^1 G(t,s)\frac{h(s)f(u(s))}{\Phi\left(\int_0^1 |u|^q \mathrm{d}\varphi\right)}\mathrm{d}s \\
&\ge \int_\theta^{1-\theta} G(t,s)\frac{h(s)f(u(s))}{\Phi\left(\int_0^1 |u|^q \mathrm{d}\varphi\right)}\mathrm{d}s \\
&\ge \frac{f(\theta R_2)}{\Phi(R_2^q\varphi(1))}\int_\theta^{1-\theta} G(t,s)h(s)\mathrm{d}s \\
&\ge \frac{f(\theta R_2)}{\Phi(R_2^q\varphi(1))} l \\
&\ge R_2 = ||u||.
\end{aligned}$$

Then, Equation 12 holds.

Therefore, by Equations 10 and 12 and the first part of Lemma 2.3, T has a fixed point in $P \cap (\bar{\Omega}_2 \backslash \Omega_1)$, which is a positive solution of Equation 1. □

Example. Let $q = 2$, $h(t) = t$, $\Phi(s) = 2 + s$, $\phi(t) = 2t$, $f(u) = \frac{2}{l\theta^3}u^2$ and $g(s) = s^2$.

Theorem 3.4. Assume that (H1) to (H3) hold. In addition, $\phi(1) \geq 1$, and the functions f, g satisfy the following growth conditions:

(H10)

$$\limsup_{r\to\infty} \frac{f(r)}{\Phi((\varphi(1-\theta)-\varphi(\theta))\theta^q r^q)r} < \frac{1}{4Ł},$$
$$\limsup_{r\to\infty} \frac{g(r)}{r} < \frac{\rho}{4(\beta+\alpha)\varphi(1)};$$

(H11)

$$\limsup_{r\to 0} \frac{f(r)}{\Phi((\varphi(1-\theta)-\varphi(\theta))\theta^q r^q)r} < \frac{1}{2Ł},$$
$$\limsup_{r\to 0} \frac{g(r)}{r} < \frac{\rho}{2(\beta+\alpha)\varphi(1)};$$

(H12) There exists a constant $a > 0$ such that

$$f(u) > \frac{\Phi((\frac{a}{\theta})^q\varphi(1))a}{L}, \quad \text{for } u \in [a, \frac{a}{\theta}].$$

Then, BVP (Equation 1) has at least three positive solutions.

Proof. For the sake of applying the Leggett-Williams fixed point theorem, define a functional $\sigma(u)$ on cone P by

$$\sigma(u) = \min_{\theta\leq t\leq 1-\theta} u(t), \quad \forall u \in P.$$

Evidently, $\sigma: P \to R^+$ is a nonnegative continuous and concave. Moreover, $\sigma(u) \leq ||u||$ for each $u \in P$.

Now, we verify that the assumption of Lemma 2.2 is satisfied.

Firstly, it can verify that there exists a positive number c with $c \geq b = \frac{a}{\theta}$ such that $T : \overline{P_c} \to P_c$.

By (H10), it is easy to see that there exists $\tau > 0$ such that

$$\frac{f(r)}{\Phi((\varphi(1-\theta)-\varphi(\theta))\theta^q r^q)r} < \frac{1}{4Ł}, \quad \forall r \geq \tau,$$
$$\frac{g(r)}{r} < \frac{\rho}{4(\beta+\alpha)\varphi(1)}, \quad \forall r \geq \tau,$$

Set

$$M_1 = \frac{f(\tau)}{\Phi(0)}, \quad M_2 = g(\tau).$$

Taking

$$c > \max\{b, 4ŁM_1, \frac{4M_2(\beta+\alpha)}{\rho}\}.$$

If $u \in \overline{P_c}$, then

$$\begin{aligned}
||Tu(t)|| &= \max_{t\in[0,1]} |Tu(t)| \\
&= \max_{t\in[0,1]} \frac{\beta+\alpha t}{\rho} g\left(\int_0^1 u(s)\mathrm{d}\varphi(s)\right) + \max_{t\in[0,1]} \int_0^1 G(t,s)\frac{h(s)f(u(s))}{\Phi(\int_0^1 |u|^q \mathrm{d}\varphi)}\mathrm{d}s \\
&\le \frac{\beta+\alpha}{\rho} g\left(\int_0^1 u(s)\mathrm{d}\varphi(s)\right) + \max_{t\in[0,1]} \int_0^1 G(t,s)\frac{h(s)f(u(s))}{\Phi(\int_0^1 |u|^q \mathrm{d}\varphi)}\mathrm{d}s \\
&\le \frac{\beta+\alpha}{\rho} g(\varphi(1)||u||) + \max_{t\in[0,1]} \frac{f(||u||)}{\Phi((\varphi(1-\theta)-\varphi(\theta))\theta^q||u||^q)} \int_0^1 G(t,s)h(s)\mathrm{d}s \\
&\le \frac{\beta+\alpha}{\rho}\left(\frac{\rho}{4(\beta+\alpha)\varphi(1)}\varphi(1)||u|| + M_2\right) + Ł\left(\frac{||u||}{4Ł} + M_1\right) \\
&< c.
\end{aligned}$$

by (H1) to (H3) and (H10).

Next, from (H11), there exists $d' \in (0, a)$ such that

$$\begin{aligned}
\frac{f(r)}{\Phi((\varphi(1-\theta)-\varphi(\theta))\theta^q r^q)r} &< \frac{1}{2Ł}, \quad \forall r \in [0,\ d'], \\
\frac{g(r)}{r} &< \frac{\rho}{2(\beta+\alpha)\varphi(1)}, \quad \forall r \in [0,\ d'].
\end{aligned}$$

Take $d = \frac{d'}{\varphi(1)}$. Then, for each $u \in \overline{P_d}$, we have

$$\begin{aligned}
||Tu(t)|| &= \max_{t\in[0,1]} |Tu(t)| \\
&= \max_{t\in[0,1]} \frac{\beta+\alpha t}{\rho} g\left(\int_0^1 u(s)\mathrm{d}\varphi(s)\right) + \max_{t\in[0,1]} \int_0^1 G(t,s)\frac{h(s)f(u(s))}{\Phi(\int_0^1 |u|^q \mathrm{d}\varphi)}\mathrm{d}s \\
&\le \frac{\beta+\alpha}{\rho} g\left(\int_0^1 u(s)\mathrm{d}\varphi(s)\right) + \max_{t\in[0,1]} \int_0^1 G(t,s)\frac{h(s)f(u(s))}{\Phi(\int_0^1 |u|^q \mathrm{d}\varphi)}\mathrm{d}s \\
&\le \frac{\beta+\alpha}{\rho} g(\varphi(1)||u||) + \max_{t\in[0,1]} \frac{f(||u||)}{\Phi((\varphi(1-\theta)-\varphi(\theta))\theta^q||u||^q)} \int_0^1 G(t,s)h(s)\mathrm{d}s \\
&\le \frac{\beta+\alpha}{\rho}\left(\frac{\rho}{2(\beta+\alpha)\varphi(1)}\varphi(1)||u||\right) + Ł\left(\frac{||u||}{2Ł}\right) \\
&< d.
\end{aligned}$$

Finally, we will show that $\{u \in P(\sigma, a, b): \sigma(u) > a\} \neq \emptyset$ and $\sigma(Tu) > a$ for all $u \in P(\sigma, a, b)$.

In fact,

$$u(t) = \frac{a+b}{2} \in \{u \in P(\sigma, a, b) : \sigma(u) > a\}.$$

For $u \in P(\sigma, a, b)$, we have

$$b \ge ||u|| \ge u \ge \min_{t\in[\theta,1-\theta]} u(t) \ge a,$$

for all $t \in [\theta, 1 - \theta]$. Then, we have

$$\begin{aligned}
\min_{t\in[\theta,1-\theta]} Tu(t) &= \min_{t\in[\theta,1-\theta]} \frac{\beta+\alpha t}{\rho} g\left(\int_0^1 u(s)\mathrm{d}\varphi(s)\right) + \min_{t\in[\theta,1-\theta]} \int_0^1 G(t,s)\frac{h(s)f(u(s))}{\Phi\left(\int_0^1 |u|^q \mathrm{d}\varphi\right)}\mathrm{d}s \\
&\geq \min_{t\in[\theta,1-\theta]} \int_0^1 G(t,s)\frac{h(s)f(u(s))}{\Phi\left(\int_0^1 |u|^q \mathrm{d}\varphi\right)}\mathrm{d}s \\
&\geq \frac{1}{\Phi(\varphi(1)b^q)} \min_{t\in[\theta,1-\theta]} \int_\theta^{1-\theta} G(t,s)h(s)f(u(s))\mathrm{d}s \\
&> \frac{1}{\Phi(\varphi(1)b^q)} \frac{\Phi(b^q\varphi(1))a}{L} \min_{t\in[\theta,1-\theta]} \int_\theta^{1-\theta} G(t,s)h(s))\mathrm{d}s \\
&= a
\end{aligned}$$

by (H1) to (H3), (H12). In addition, for each $u \in P(\theta, a, c)$ with $||Tu|| > b$, we have

$$\min_{t\in[\theta,1-\theta]} (Tu)(t) \geq \theta||Tu|| > \theta b \geq a.$$

Above all, we know that the conditions of Lemma 2.2 are satisfied. By Lemma 2.2, the operator T has at least three fixed points $u_i (i = 1, 2, 3)$ such that

$$\begin{aligned}
&||u_1|| < d, \\
&a < \min_{t\in[\theta,1-\theta]} u_2(t) \\
&||u_3|| > d \text{ with } \min_{t\in[\theta,1-\theta]} u_3(t) < a.
\end{aligned}$$

The proof is complete. □

Example. Let $q = 2$, $h(t) = t$, $\Phi(s) = 2 + s$, $\phi(t) = 2t$, $f(u) = 4\frac{1+\theta^2}{L\theta^2}u^2$ and, $g(s) = \frac{\rho}{16(\beta+\alpha)}\frac{s^2}{2+s}$, namely,

$$\begin{cases}
-\left(2 + \int_0^1 |u(s)|^2 \mathrm{d}(2s)\right)u''(t) = t4\frac{1+\theta^2}{l\theta^2}u^2, \text{ in } 0 < t < 1, \\
\alpha u(0) - \beta u'(0) = 0, \ \gamma u(1) + \delta u'(1) = \frac{\rho}{16(\beta+\alpha)} \frac{\left(\int_0^1 u(s)\mathrm{d}(2s)\right)^2}{2+\int_0^1 u(s)\mathrm{d}(2s)}.
\end{cases}$$

From a simple computation, we have

$$\begin{aligned}
\limsup_{r\to\infty} \frac{f(r)}{\Phi((\varphi(1-\theta)-\varphi(\theta))\theta^2 r^2)r} &= \limsup_{r\to\infty} \frac{4\frac{1+\theta^2}{L\theta^2}r^2}{(2+2(1-2\theta)\theta^2 r^2)r} = 0, \\
\limsup_{r\to\infty} \frac{g(r)}{r} = \limsup_{r\to\infty} \frac{\frac{\rho}{16(\beta+\alpha)}\frac{r^2}{2+r}}{r} &= \frac{\rho}{16(\beta+\alpha)} < \frac{\rho}{4(\beta+\alpha)\varphi(1)}, \\
\limsup_{r\to 0} \frac{f(r)}{\Phi((\varphi(1-\theta)-\varphi(\theta))\theta^q r^q)r} &= \limsup_{r\to 0} \frac{4\frac{1+\theta^2}{L\theta^2}r^2}{(2+2(1-2\theta)\theta^2 r^2)r} = 0, \\
\limsup_{r\to 0} \frac{g(r)}{r} &= \limsup_{r\to 0} \frac{\frac{\rho}{16(\beta+\alpha)}\frac{r^2}{2+r}}{r} = 0,
\end{aligned}$$

Then, it is easy to see that (H1) to (H3) and (H10) to (H11) hold. Especially, take $a = 1$, by $f(a) = f(1) = 4\frac{1+\theta^2}{L\theta^2} > 2\frac{1+\theta^2}{L\theta^2} = \frac{\Phi((\frac{a}{\theta})^q\varphi(1))a}{L}$ and (H1), then (H12) holds.

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Authors' contributions
In this manuscript the authors studied the existence and multiplicity of positive solutions for an interesting nonlocal differential equation using the Cone-Compression and Cone-Expansion Theorem due to M. Krasnosel'skii for the existence result and Leggett-Williams fixed point Theorem for the multiplicity result. Moreover, in this work, the authors supplements the studies done in [12], because here they consider the case nonlocal boundary value condition. All authors typed, read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

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