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On mirror symmetry for Calabi-Yau fourfolds with three-form cohomology

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Abstract: We study the action of mirror symmetry on two-dimensional $\mathcal{N} = (2, 2)$ effective theories obtained by compactifying Type IIA string theory on Calabi-Yau fourfolds. Our focus is on fourfold geometries with non-trivial three-form cohomology. The couplings of the massless zero-modes arising by expanding in these forms depend both on the complex structure deformations and the Kähler structure deformations of the Calabi-Yau fourfold. We argue that two holomorphic functions of the deformation moduli capture this information. These are exchanged under mirror symmetry, which allows us to derive them at the large complex structure and large volume point. We discuss the application of the resulting explicit expression to F-theory compactifications and their weak string coupling limit. In the latter orientifold settings we demonstrate compatibility with mirror symmetry of Calabi-Yau threefolds at large complex structure. As a byproduct we find an interesting relation of no-scale like conditions on Kähler potentials to the existence of chiral and twisted-chiral descriptions in two dimensions.

Keywords: F-Theory, M-Theory, Supersymmetry and Duality, Supergravity Models

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Contents

1 Introduction

The derivation of four-dimensional low-energy effective actions arising from string theory requires a detailed understanding of the geometries used as compactification spaces. Since the early days of string theory much research has focused on the study of Calabi-Yau manifolds of complex dimension three. These threefolds were identified as valid compactification backgrounds to four space-time dimensions and can yield to, for example when used in the heterotic string theories, potentially phenomenologically interesting four-dimensional effective theories with the minimal amount of supersymmetry. In contrast, there is significantly less known about the geometry of Calabi-Yau manifolds of complex dimension four. With the advent of F-theory [1–3] it became clear that these fourfolds are relevant in obtaining four-dimensional effective theories with the minimal amount of supersymmetry from Type IIB string theory. It is therefore crucial to further our understanding of the geometry of Calabi-Yau fourfolds and investigate the relation to couplings in the effective theories.

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In contrast to Calabi-Yau threefolds one finds that Calabi-Yau fourfolds admit three non-trivial independent Hodge numbers that count the number of harmonic forms of different degree. In full analogy to threefolds, two of them encode the number of complex structure and Kähler structure deformations of the geometry. When compactifying string theory or M-theory on a Calabi-Yau fourfold, these deformations will appear as massless fields, so-called moduli, in the effective action. The moduli space geometry has been studied in various works [4–11]. The additional Hodge number on Calabi-Yau fourfolds is associated to the number of harmonic three-forms. In this work we will discuss in detail how the presence of these three-forms affects the effective theory when compactifying Type IIA string theory and M-theory on Calabi-Yau fourfolds. The effective theory will then admit new scalars N_l with couplings non-trivially varying over both the complex structure and Kähler structure moduli spaces. This was already observed for M-theory compactifications in [12, 13], for Type IIA compactifications in [14], and for F-theory compactifications in [15]. We will show in this work that the moduli dependence at certain points in the moduli space can actually be computed explicitly by using mirror symmetry for Calabi-Yau fourfolds.

Our first focus is on a refined understanding of the moduli variations of three-forms. Therefore, we begin by introducing a basis of $(2,1)$-forms on the fourfold that is convenient when performing the dimensional reduction. The so-defined set of forms is adapted to the underlying complex structure and it was pointed out in [15] that their variation with the complex structure moduli can be captured by a holomorphic function f_{kl}, with indices ranging over the number of harmonic $(2,1)$-forms. This holomorphic function can be used in the compactification of Type IIA string theory, accessed via its low-energy supergravity theory, on a Calabi-Yau fourfold. Such dimensional reductions of the Type IIA theory have already been investigated in [14, 16, 17]. They are expected to yield two-dimensional effective theories with $\mathcal{N}=(2,2)$ supersymmetry describing the dynamics of chiral and twisted-chiral multiplets. Without including three-forms the supersymmetry properties of such Type IIA effective theories were already discussed in [17] by extending earlier results [18–20]. We will consider the generalization of this result including the three-form scalars N_l and suggest that it leads to a more general class of supersymmetric dilaton supergravities.

The two-dimensional $\mathcal{N}=(2,2)$ effective action is expected to be invariant under the action of mirror symmetry. More precisely, considering Type IIA string theory on two Calabi-Yau fourfolds that are mirror manifolds to each other, the resulting two effective actions should admit an identification under an appropriate mirror map. This map identifies complex coordinates and couplings at special points in moduli space. Mirror symmetry exchanges complex structure and Kähler structure deformations, but preserves the number of non-trivial three-forms and thus the number of three-form scalars N_l. It also maps chiral to twisted-chiral multiplets. Therefore, we are forced to perform an appropriate duality transformation for the three-form scalars appearing in pairs of effective actions arising from mirror manifolds. In both effective actions the dynamics of the three-forms are described by two holomorphic functions f_{kl} and h^k_l. The former is holomorphic in the complex structure moduli, while the latter is holomorphic in the complexified Kähler moduli. These functions are exchanged by mirror symmetry and we are able to derive the complex structure dependence of f_{kl} in the large complex structure limit by using the results of a large volume compactification on the mirror geometry.

Our results have several interesting applications, in particular, when using the Calabi-Yau fourfolds with non-trivial three-forms as F-theory backgrounds. To determine the four-dimensional F-theory effective actions for such configurations one uses the duality with M-theory [1, 2, 15]. It was shown in [15] that the function f_{lm} can either lift to a gauge coupling function of R-R vector fields or to the metric of a special set of complex scalars. In both cases it is desirable to explicitly compute the moduli dependence of their coupling function. For example, the three-form scalars lifting to four-dimensional scalars naturally admit real shift symmetries or even a generalized Heisenberg symmetry and might be of profound phenomenological interest (see, for example, [21–23]). Furthermore, considering the weak string coupling limit of the F-theory setting following [24, 25] the resulting effective theory should match with the orientifold effective actions [26, 27]. In the case that such a limit exists one can associate a Calabi-Yau threefold to the F-theory Calabi-Yau fourfold. We are then able to show that our result for f_{kl} obtained by fourfold mirror symmetry is consistent with the weak coupling analog obtained from threefold mirror symmetry.

This paper is organized as follows. In section 2 we recall some basics about Calabi-Yau fourfolds and discuss a convenient basis of $(2,1)$-forms parametrized by a holomorphic function f_{lk}. We dimensionally reduce Type IIA supergravity on a Calabi-Yau fourfold in section 3. This allows us to investigate the $\mathcal{N} = (2,2)$ supersymmetric structure of the effective theory and perform a set of important dualizations to interchange chiral multiplets and twisted-chiral multiplets. In section 4 we discuss mirror symmetry with a focus on the $(2,1)$-form sector. This allows us to determine f_{lm} in the large complex structure limit. We use these results in an F-theory compactification on an elliptically fibered Calabi-Yau fourfold in section 5. Moving to the weak string coupling limit, we find compatibility of our result for f_{lm} with the answers predicted by mirror symmetry of Calabi-Yau threefolds. This work has two appendices with useful computational results. In appendix A we perform the circle reduction of a general three-dimensional un-gauged $\mathcal{N} = 2$ supergravity theory with focus on the bosonic action. We find interesting conditions on the kinetic potential to match the proposed $\mathcal{N} = (2,2)$ action in two dimensions. The dualization of chiral to twisted-chiral multiplets in the bosonic sector is performed in appendix B. We again find conditions on the kinetic potential in order that this dualization can be performed. The results of both appendices are immediately applicable to Calabi-Yau fourfold effective actions of Type IIA string theory and M-theory.

2 On the geometry of Calabi-Yau fourfolds with three-form cohomology

In this section we introduce important facts about the geometry of Calabi-Yau fourfolds Y_4. A brief summary of some basics about their differential structure and topology will be given in section 2.1. The focus of section 2.2 will be to introduce relevant properties of the three-form cohomology of Y_4. We argue that an appropriate definition of three-forms of Hodge-type (2,1) can be given in terms of a function f_{mn} holomorphic in the complex structure moduli. This function will be of key interest throughout this work.

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2.1 Some basic properties of Calabi-Yau fourfolds

We define a compact real eight-dimensional manifold Y_4 to be a Calabi-Yau fourfold if its holonomy group is exactly $SU(4)$. Such manifolds are Kähler, i.e. admit a closed Kähler two-form J, and possess a unique Ricci-flat metric within the class of J. Furthermore, one can introduce a non-trivial closed $(4,0)$-form Ω on Y_4 that is unique up to constant rescalings. Note that J and Ω can be used to form a top-form on Y_4 and one has

$$\frac{1}{4!} J \wedge J \wedge J \wedge J = |\Omega|^{-2}\, \Omega \wedge \bar{\Omega}\,, \qquad\qquad |\Omega|^2 = \frac{1}{\mathcal{V}} \int_{Y_4} \Omega \wedge \bar{\Omega}\,, \tag{2.1}$$

where $\mathcal{V}$ is the total volume of Y_4. The $SU(4)$ holonomy also allows one to introduce one complex covariantly constant and no-where vanishing spinor of definite chirality. The forms J and Ω are obtained as bilinear contractions using this spinor.

With our definition of a Calabi-Yau fourfold, we can also constrain the Hodge numbers $h^{p,q}(Y_4) = \dim(H^{p,q}(Y_4, \mathbb{C}))$. There are three independent Hodge numbers on Y_4: $h^{1,1}(Y_4)$, $h^{3,1}(Y_4)$, and $h^{2,1}(Y_4)$. The significance of $h^{1,1}(Y_4)$ and $h^{3,1}(Y_4)$ in the dimensional reduction are very similar to the case of a Calabi-Yau threefold (see e.g. [28]). On the one hand, the number $h^{1,1}(Y_4)$ counts the allowed Kähler structure deformations, which we will denote by v^A. On the other hand, the number $h^{3,1}(Y_4)$ counts the complex structure deformations denoted by z^K. Both turn out to become moduli fields in the effective theory obtained by dimensional reduction on Y_4 and will be discussed in more detail in section 3.1. The Hodge number $h^{2,1}$ has no threefold analog and understanding the geometries with $h^{2,1}(Y_4) > 0$ will be the main focus of this work. Having three independent Hodge numbers, the Hodge diamond takes the form

$$\begin{array}{ccccccccc}
 & & & & h^{0,0} & & & & \\
 & & & h^{1,0} & & h^{0,1} & & & \\
 & & h^{2,0} & & h^{1,1} & & h^{0,2} & & \\
 & h^{3,0} & & h^{2,1} & & h^{1,2} & & h^{0,3} & \\
h^{4,0} & & h^{3,1} & & h^{2,2} & & h^{1,3} & & h^{0,4} \\
 & h^{4,1} & & h^{3,2} & & h^{2,3} & & h^{1,4} & \\
 & & h^{4,2} & & h^{3,3} & & h^{2,4} & & \\
 & & & h^{4,3} & & h^{3,4} & & & \\
 & & & & h^{4,4} & & & &
\end{array}
\quad = \quad
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 0 & & 0 & & & \\
 & & 0 & & h^{1,1} & & 0 & & \\
 & 0 & & h^{2,1} & & h^{2,1} & & 0 & \\
1 & & h^{3,1} & & h^{2,2} & & h^{3,1} & & 1 \\
 & 0 & & h^{2,1} & & h^{2,1} & & 0 & \\
 & & 0 & & h^{1,1} & & 0 & & \\
 & & & 0 & & 0 & & & \\
 & & & & 1 & & & &
\end{array}\,,$$

where we have indicated for later use the action of mirror symmetry on the Hodge numbers. More precisely, mirror symmetry identifies two Calabi-Yau geometries with Hodge numbers mirrored along the dashed line. A more detailed discussion of mirror symmetry will be presented in section 4. In addition, one finds the formulas [6]

$$h^{2,2}(Y_4) = 2(22 + 2h^{1,1} + 2h^{3,1} - h^{2,1})\,, \qquad\qquad \chi(Y_4) = 6(8 + h^{1,1} + h^{3,1} - h^{2,1}) \tag{2.2}$$

where $\chi = \sum_{p,q} (-1)^{p+q} h^{p,q}$ is the Euler characteristic of Y_4.

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2.2 Non-trivial three-forms on Calabi-Yau fourfolds

Of key importance in this work is the inclusion of non-trivial three-forms in the dimensional reduction and discussion of mirror symmetry. In this subsection we summarize some basic properties of such three-forms that will be useful throughout the later sections.

To begin with, we comment on the moduli dependence of three-forms when choosing them to represent elements of $H^{2,1}(Y_4)$. In order to do that, recall that the Hodge filtration of the three-cohomology $H^3(Y_4,\mathbb{C})$ is given by the holomorphic bundles $F^p(Y_4) = \bigoplus_{j=p}^3 H^{j,3-j}$ over the complex structure moduli space. Since $H^{3,0}(Y_4)$ is trivial, this enables us to find a basis ψ_l of $F^2(Y_4) = H^{2,1}(Y_4)$, which varies holomorphically with the complex structure moduli z^K, i.e. one has

$$\frac{\partial}{\partial \bar{z}^K}\psi_l = 0, \qquad l = 1,\dots,h^{2,1}(Y_4)\,, \tag{2.3}$$

where $K = 1,\dots,h^{3,1}(Y_4)$ labels the complex structure moduli. Note that in the dimensional reduction we can think of ψ_l to be the harmonic representative in each class of $H^{2,1}(Y_4)$. At a given point in the complex structure space we can write this basis in the form

$$\psi_l = \alpha_l + i f_{lm}(z)\beta^m \qquad \in H^{2,1}(Y_4)\,, \tag{2.4}$$

where (α_l, β^m) comprise a real moduli-independent basis of $H^3(Y_4,\mathbb{R})$.[1] Holomorphicity of the forms ψ_l translates to the fact that $f_{lm}(z)$ is a holomorphic function of the complex structure moduli z^K. Furthermore, we assume that (α_l,β^m) is chosen such that $\text{Re}f_{lm}$ is a positive definite and invertible matrix.[2]

After performing the dimensional reduction on Y_4 in section 3, we aim to find the proper complex fields that are compatible with two-dimensional supersymmetry. For the zero-modes arising from the ψ_l it turns out that a further normalization is useful, i.e. we introduce the (1,2)-forms

$$\Psi^l = \frac{1}{2}(\text{Re}f)^{lm}\bar{\psi}_m = \frac{1}{2}(\text{Re}f)^{lm}(\alpha_m - i\bar{f}_{mn}(\bar{z})\beta^n) \qquad \in H^{1,2}(Y_4)\ . \tag{2.5}$$

In this expression we have multiplied by the *inverse* $(\text{Re}f)^{lm}$ of the real part of f_{lm}, i.e. $(\text{Re}f)^{lm}\,(\text{Re}f)_{mk} = \delta^l_k$. This definition allows to give a simple expressions for $\text{Im}\,\Psi^l$ and the derivative of Ψ^l with respect to the complex structure moduli:

$$\bar{\Psi}^l - \Psi^l = i\beta^l\,, \qquad \partial_{z^K}\Psi^l = -\Psi^k\,(\text{Re}f)^{lm}\,\partial_{z^K}(\text{Re}f)_{mk}\,, \tag{2.6}$$

and accordingly $\partial_{z^K}\Psi^l = \partial_{z^K}\bar{\Psi}^l$. Note that Ψ^l is a $(1,2)$-form and therefore satisfies

$$*\,\Psi^l = -iJ\wedge\Psi^l\,, \tag{2.7}$$

where $*$ is the Hodge-star for the Calabi-Yau metric on Y_4.

[1] It might be natural to chose (α_l,β^m) to be a basis of $H^3(Y_4,\mathbb{Z})$, but quantization of the coefficients will not be important in this work.

[2] While we have no complete proof that this is always possible, we note that $H^{2,1}(Y_4)/H^3(Y_4,\mathbb{Z})$ is actually a complex torus. f_{lm} sets the complex structure on this torus.

To evaluate the integrals appearing in the dimensional reduction we impose one further condition on the basis (α_l, β^m). More precisely, we demand

$$\beta^l \wedge \beta^m = 0\,, \qquad \forall\; l, m = 1, \ldots, h^{2,1}(Y_4)\,, \tag{2.8}$$

which is supposed to hold in cohomology.[3] Introducing a basis ω_A of $H^{1,1}(Y_4)$ we thus find that

$$\int_{Y_4} \omega_A \wedge \beta^l \wedge \beta^m = 0\,, \qquad \forall\; A = 1, \ldots, h^{1,1}(Y_4)\;. \tag{2.9}$$

The remaining integrals are in general non-trivial and denoted by

$$C_{Am}{}^k = \int_{Y_4} \omega_A \wedge \alpha_m \wedge \beta^k\,, \qquad C_{Amk} = \int_{Y_4} \omega_A \wedge \alpha_m \wedge \alpha_k\;. \tag{2.10}$$

Using a basis (α_m, β^k) satisfying (2.9) one checks that the metric $\int \Psi^l \wedge *\bar{\Psi}^k$ is symmetric in the indices l, k and real. This property will be crucial in determining a kinetic potential for this metric.

3 Dimensional reduction of Type IIA supergravity

In this section we perform the dimensional reduction of Type IIA supergravity on a Calabi-Yau fourfold Y_4. Such reductions have already been performed in [12–14, 17]. Our analysis follows [12–14], but we will apply in addition the improved understanding about the three-form cohomology of section 2.

3.1 The effective action from a Calabi-Yau fourfold reduction

The Kaluza-Klein reduction of Type IIA supergravity can be trusted in the limit in which the typical length scale of the physical volumes of submanifolds of Y_4 are sufficiently large compared to the string scale. This limit is referred to as the large volume limit. Furthermore, these typical length scales set the Kaluza-Klein scale which we assume to be sufficiently above the energy scale of the effective action. We therefore keep only the massless Kaluza-Klein modes in the following reduction.

Our starting point will be the bosonic part of the ten-dimensional Type IIA action in string-frame given by[4]

$$\begin{aligned} S^{(10)}_{\mathrm{IIA}} = & \int e^{-2\check{\phi}_{\mathrm{IIA}}} \left(\frac{1}{2} \check{R}\, \check{*}1 + 2 d\check{\phi}_{\mathrm{IIA}} \wedge \check{*} d\check{\phi}_{\mathrm{IIA}} - \frac{1}{4} \check{H}_3 \wedge \check{*}\check{H}_3 \right) \\ & - \frac{1}{4} \int \left(\check{F}_2 \wedge \check{*}\check{F}_2 + \check{\mathbf{F}}_4 \wedge \check{*}\check{\mathbf{F}}_4 + \check{B}_2 \wedge \check{F}_4 \wedge \check{F}_4 \right)\;. \end{aligned} \tag{3.1}$$

where $\check{\phi}_{\mathrm{IIA}}$ is the ten-dimensional dilaton, $\check{H}_3 = d\check{B}_2$ is the field strength of the NS-NS two-form $\check{B}_2$, and $\check{F}_p = d\check{C}_p$ are the field strengths of the R-R p-forms $\check{C}_1$ and $\check{C}_3$. We also

[3] Considering hypersurfaces in toric varieties, this condition can be satisfied for non-trivial three-forms arising from singular Riemann surfaces. This allows us to choose a symplectic basis with respect to a certain divisor for the three-forms.

[4] Note that for convenience we have set $\kappa^2 = 1$.

have used the modified field strength $\check{\mathbf{F}}_4 = \check{F}_4 - \check{C}_1 \wedge \check{H}_3$. Here and in the following we will use a check to indicate ten-dimensional fields.

The background solution around which we want to consider the effective theory is taken to be of the form $\mathbb{M}_{1,1} \times Y_4$, where $\mathbb{M}_{1,1}$ is the two-dimensional Minkowski space-time, and Y_4 is a Calabi-Yau fourfold with properties introduced in section 2. As pointed out there such a manifold admits one complex covariantly constant spinor of definite chirality. This spinor can be used to dimensionally reduce the $\mathcal{N} = (1,1)$ supersymmetry of Type IIA supergravity to obtain a two-dimensional $\mathcal{N} = (2,2)$ supergravity theory. In particular, the two ten-dimensional gravitinos of opposite chirality reduce to two pairs of two-dimensional Majorana-Weyl gravitinos with opposite chirality. We will have more to say about the supersymmetry properties of the two-dimensional action in section 3.2. Furthermore, recall that Y_4 admits a Ricci-flat metric $g^{(8)}_{mn}$ and one can thus check that a metric of the form

$$d\check{s}^2 = \eta_{\mu\nu} dx^\mu dx^\nu + g^{(8)}_{mn} dy^m dy^n \,, \tag{3.2}$$

solves the ten-dimensional equations of motion in the absence of background fluxes.[5] Note that in (3.2) we denote by x^μ the two-dimensional coordinates of the space-time $\mathbb{M}_{1,1}$, whereas the eight-dimensional real coordinates on the Calabi-Yau fourfold Y_4 are denoted by y^m.

The massless perturbations around this background both consist of fluctuations of the internal metric $g^{(8)}_{mn}$ that preserve the Calabi-Yau condition as well as the fluctuations of the form fields $\check{B}_2$, $\check{C}_1, \check{C}_3$ and the dilaton $\check{\phi}_{\mathrm{IIA}}$. The metric fluctuations give rise to the real Kähler structure moduli v^A, $A = 1, \ldots, h^{1,1}(Y_4)$ that preserve the complex structure and are given by

$$g_{i\bar{\jmath}} + \delta g_{i\bar{\jmath}} = -i J_{i\bar{\jmath}} = -i v^A \, (\omega_A)_{i\bar{\jmath}}, \tag{3.3}$$

where J is the Kähler form on Y_4 and ω_A comprises a real basis of harmonic $(1,1)$-forms spanning $H^{1,1}(Y_4)$. The Kähler structure moduli appear also in the expression of the total string-frame volume $\mathcal{V}$ of Y_4 given by

$$\mathcal{V} \equiv \int_{Y_4} *1 = \frac{1}{4!} \int_{Y_4} J \wedge J \wedge J \wedge J \ . \tag{3.4}$$

In addition to the Kähler structure moduli one finds a set of complex structure moduli z^K, $K = 1, \ldots, h^{3,1}(Y_4)$. These fields parameterize the change in the complex structure of Y_4 preserving the class of its Kähler form J. Infinitesimally they are given by the fluctuations δz^K as

$$\delta g_{\bar{\imath}\bar{\jmath}} = -\frac{1}{3|\Omega|^2} \overline{\Omega}_{\bar{\imath}}{}^{lmn} (\chi_K)_{lmn\bar{\jmath}} \, \delta z^K \,, \tag{3.5}$$

where Ω is the $(4,0)$-form, the χ_K form a basis of harmonic $(3,1)$-forms spanning $H^{3,1}(Y_4)$, and $|\Omega|^2$ was already given in (2.1).

[5] The inclusion of background fluxes complicates the reduction further. In particular, it requires to introduce a warp-factor. The M-theory reduction with warp-factor was recently performed in [29–31].

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The Kaluza-Klein ansatz for the remaining fields takes the form

$$\begin{aligned}\check{B}_2 &= b^A\omega_A\,, \qquad \check{C}_1 = A\,, \\ \check{C}_3 &= V^A\wedge\omega_A + N_l\Psi^l + \bar{N}_l\bar{\Psi}^l\,,\end{aligned} \tag{3.6}$$

where Ψ^l is a basis of harmonic $(1,2)$-forms spanning $H^{1,2}(Y_4)$ as introduced in (2.5). A discussion of the properties of Ψ^l was already given in section 2. Finally, we dimensionally reduce the Type IIA dilaton by dropping its dependence on the internal manifold Y_4. It turns out to be convenient to define a two-dimensional dilaton $\phi_{\rm IIA}$ in terms of the ten-dimensional dilaton $\check{\phi}_{\rm IIA}$ as

$$e^{2\phi_{\rm IIA}} \equiv \frac{e^{2\check{\phi}_{\rm IIA}}}{\mathcal{V}}\,. \tag{3.7}$$

In summary, we find in the two-dimensional $\mathcal{N}=(2,2)$ supergravity theory the $2h^{1,1}(Y_4)+1$ real scalar fields $v^A(x)$, $b^A(x)$, $\phi_{\rm IIA}(x)$ as well as the $h^{3,1}(Y_4)+h^{2,1}(Y_4)$ complex scalar fields z^K, N_l. In addition there are $h^{1,1}(Y_4)+1$ vectors A, V^A, which carry, however, no physical degrees of freedom in a two-dimensional theory if they are not involved in any gauging. Since the effective action considered here contains no gaugings, we will drop these in the following analysis.

To perform the dimensional reduction one inserts the expansions (3.3), (3.5), (3.6), and (3.7) into the Type IIA action (3.1). It reduces to the two-dimensional action

$$\begin{aligned}S^{(2)} = \int & e^{-2\phi_{\rm IIA}}\Big(\frac{1}{2}R*1 + 2d\phi_{\rm IIA}\wedge *\phi_{\rm IIA} - G_{K\bar{L}}\,dz^K\wedge *d\bar{z}^L - G_{AB}\,dt^A\wedge *d\bar{t}^B\Big) \\ & -\frac{1}{2}v^A d_A{}^{lk} DN_l\wedge *\overline{DN}_k - \frac{i}{4}d_A{}^{lk}db^A\wedge(N_l\overline{DN}_k - DN_l\bar{N}_k)\,.\end{aligned} \tag{3.8}$$

We note that the NS-NS part, which is summarized in the first line of (3.1), reduces to the first line of (3.8), while the R-R part, i.e. the second line of (3.1), reduces to the second line of (3.8).

Let us introduce the various objects appearing in the action (3.8). First, we have defined the complex coordinates

$$t^A \equiv b^A + iv^A\,, \tag{3.9}$$

which combine the Kähler structure moduli with the B-field moduli. Furthermore, we have introduced the metric[6]

$$G_{AB} = \frac{1}{4\mathcal{V}}\int_{Y_4}\omega_A\wedge *\omega_B = -\frac{1}{8\mathcal{V}}\Big(\mathcal{K}_{AB} - \frac{1}{18\mathcal{V}}\mathcal{K}_A\mathcal{K}_B\Big)\,, \tag{3.10}$$

where $\mathcal{V}$, $\mathcal{K}_A$ and $\mathcal{K}_{AB}$ are given in terms of the quadruple intersection numbers $\mathcal{K}_{ABCD}$ as

$$\begin{aligned}\mathcal{K}_{ABCD} &= \int_{Y_4}\omega_A\wedge\omega_B\wedge\omega_C\wedge\omega_D\,, \\ \mathcal{V} &= \frac{1}{4!}\mathcal{K}_{ABCD}v^Av^Bv^Cv^D\,, \quad \mathcal{K}_A = \mathcal{K}_{ABCD}v^Bv^Cv^D\,, \quad \mathcal{K}_{AB} = \mathcal{K}_{ABCD}v^Cv^D\,.\end{aligned} \tag{3.11}$$

[6]The second equality follows from the cohomological identity $*\omega_A = -\frac{1}{2}\omega_A\wedge J\wedge J + \frac{1}{36}\mathcal{V}^{-1}\mathcal{K}_A J\wedge J\wedge J$.

With these definitions at hand, we can further evaluate the metric G_{AB} and show that it can be obtained from a Kähler potential as

$$G_{AB} = -\partial_{t^A}\partial_{\bar{t}^B} \log \mathcal{V} \ . \tag{3.12}$$

Also the metric $G_{K\bar{L}}$ is actually a Kähler metric. It only depends on the complex structure moduli z^K and takes the form

$$G_{K\bar{L}} = -\frac{\int_{Y_4} \chi_K \wedge \bar{\chi}_L}{\int_{Y_4} \Omega \wedge \bar{\Omega}} = -\partial_{z^K}\partial_{\bar{z}^L} \log \int_{Y_4} \Omega \wedge \bar{\Omega} \ . \tag{3.13}$$

Note that both G_{AB} and $G_{K\bar{L}}$ are actually positive definite and therefore define physical kinetic terms in (3.8). Both terms scale with the dilaton ϕ_{IIA} and it is easy to check that this dependence cannot be removed using a Weyl-rescaling of the two-dimensional metric. We will show in section 3.2 that this is consistent with the form of the $\mathcal{N} = (2,2)$ dilaton supergravity.

Let us now turn to the R-R part of the action (3.1) and discuss the couplings appearing in the second line of (3.8). First, we introduce the coupling function

$$d_A{}^{lm} \equiv i \int_{Y_4} \omega_A \wedge \Psi^l \wedge \bar{\Psi}^m = -\int_{Y_4} \omega_A \wedge \Psi^l \wedge \beta^m = -\frac{1}{2}(\text{Re} f)^{ln} C_{An}{}^m \,, \tag{3.14}$$

where we have used (2.6) to evaluate the second equality, and (2.9), (2.10) to show the third equality. One also checks the relation

$$H^{lm} \equiv \int_{Y_4} \Psi^l \wedge * \bar{\Psi}^m = i \int_{Y_4} J \wedge \Psi^l \wedge \bar{\Psi}^m = v^A d_A{}^{lm} \,, \tag{3.15}$$

where we have used (2.7) for the (1,2)-forms Ψ^l. This contraction gives precisely the positive definite metric of the complex scalars N_l in (3.8). It turns out to be convenient to write

$$H^{lm} = v^A d_A{}^{lm} = -\frac{1}{2}(\text{Re} f)^{ln} v^A C_{An}{}^m \equiv -\frac{1}{2}(\text{Re} f)^{ln} \ \text{Re}\, h_n^m \,, \tag{3.16}$$

where $h_n^m = -it^A C_{An}{}^m$. Note that H^{lm} thus depends non-trivially on the complex structure moduli z^K through the holomorphic functions f_{kl} and on the Kähler structure moduli t^A through the holomorphic function h_n^m. Second, we note that the modified derivative DN_l appearing in (3.8) is a shorthand for

$$DN_l = dN_l - 2\text{Re} N_m (\text{Re} f)^{mn} \partial_{z^K} (\text{Re} f_{nl}) dz^K \ . \tag{3.17}$$

Using this expression one easily reads off the coefficient function in front of $dN_l \wedge * dz^K$ and checks that it can be obtained by taking derivatives of a real function. In the next subsection we show that this is true for all terms in (3.8) and discuss the connection with two-dimensional supersymmetry.

3.2 Comments on two-dimensional $\mathcal{N} = (2, 2)$ supergravity

Having performed the dimensional reduction we next want to comment on the supersymmetry properties of the action (3.8). As pointed out already in the previous subsection the counting of covariantly constant spinors on the Calabi-Yau fourfold suggests that the two-dimensional effective theory admits $\mathcal{N} = (2, 2)$ supersymmetry. It was pointed out in [17] that, at least in the case of $h^{2,1}(Y_4) = 0$ one expects to be able to bring the action (3.8) into the standard form of an two-dimensional $\mathcal{N} = (2, 2)$ dilaton supergravity. In this work the dilaton supergravity action was constructed using superspace techniques. Earlier works in this direction include [18–20]. In the following we comment on this matching for $h^{2,1}(Y_4) = 0$ and then discuss the general case in which $h^{2,1}(Y_4) > 0$.

In order to display the supergravity actions we first have to introduce two sets of multiplets containing scalars in two-dimensions: (1) a set of chiral multiplets with complex scalars ϕ^κ, (2) a set of twisted-chiral multiplets with complex scalars σ^A. In a superspace description these multiplets obey the two inequivalent linear spinor derivative constraints leading to irreducible representations.

To discuss the actions we first focus on the case $h^{2,1}(Y_4) = 0$ and follow the constructions of [17]. For simplicity we will not include gaugings or a scalar potential. The superspace action used in [17] is given by

$$S^{(2)}_{\rm dil} = \int d^2x d^4\theta E^{-1} e^{-2V-\mathcal{K}} \, . \tag{3.18}$$

Here E^{-1} is the superspace measure, V is a real superfield with $V| = \varphi$ as lowest component, and $\mathcal{K}$ is a function of the chiral and twisted-chiral multiplets with lowest components ϕ^κ and σ^A, respectively. To display the bosonic part of the action (3.18) we first set

$$e^{-2\tilde{\varphi}} = e^{-2\varphi-\mathcal{K}} \, , \tag{3.19}$$

where $\mathcal{K}(\phi^\kappa, \bar{\phi}^\kappa, \sigma^A, \bar{\sigma}^A)$ is evaluated as a function of the bosonic scalars. With this definition at hand one finds the bosonic action

$$\begin{aligned} S^{(2)}_{\rm dil} = \int e^{-2\tilde{\varphi}} \Big(& \frac{1}{2} R * 1 + 2 d\tilde{\varphi} \wedge * d\tilde{\varphi} - \mathcal{K}_{\phi^\kappa \bar{\phi}^\lambda} \, d\phi^\kappa \wedge * d\bar{\phi}^\lambda + \mathcal{K}_{\sigma^A \bar{\sigma}^B} \, d\sigma^A \wedge * d\bar{\sigma}^B \\ & - \mathcal{K}_{\phi^\kappa \bar{\sigma}^B} \, d\phi^\kappa \wedge d\bar{\sigma}^B - \mathcal{K}_{\sigma^A \bar{\phi}^\lambda} \, d\bar{\phi}^\lambda \wedge d\sigma^A \Big) \, , \end{aligned} \tag{3.20}$$

where $\mathcal{K}_{\phi^\kappa \bar{\phi}^\lambda} = \partial_{\phi^\kappa} \partial_{\bar{\phi}^\lambda} \mathcal{K}$, $\mathcal{K}_{\phi^\kappa \bar{\sigma}^A} = \partial_{\phi^\kappa} \partial_{\bar{\sigma}^A} \mathcal{K}$ with a similar notation for the other coefficients. It is now straightforward to compare (3.20) with the action (3.8) for the case $h^{2,1}(Y_4) = 0$, i.e. in the absence of any complex scalars N_l. One first identifies

$$\tilde{\varphi} = \phi_{\rm IIA} \, , \qquad \phi^K = z^K \, , \qquad \sigma^A = t^A \, , \tag{3.21}$$

and then infers that

$$\mathcal{K} = -\log \int_{Y_4} \Omega \wedge \bar{\Omega} + \log \mathcal{V} \, . \tag{3.22}$$

Note that we find here a positive sign in front of the logarithm of $\mathcal{V}$. This is related to the fact that there is an extra minus sign in the kinetic terms of the twisted-chiral fields σ^A in (3.20). Clearly, the kinetic terms of the complex structure deformations z^K and complexified Kähler structure deformations t^A in the action (3.8) have both positive definite kinetic terms.[7]

Let us now include the complex scalars N_l. It is important to note that the action (3.8) cannot be brought into the form (3.20). In fact, we see in (3.8) that the terms independent of the two-dimensional metric do not contain an $\phi_{\rm IIA}$-dependent pre-factor, while the terms of this type in (3.20) do admit an $\tilde\varphi$-dependence. Any field redefinition in (3.8) involving the dilaton seems to introduce new undesired mixed terms that cannot be matched with (3.20) either. However, we note that the action (3.8) actually can be brought to the form

$$\begin{aligned} S^{(2)} = \int e^{-2\tilde\varphi}\Big(\frac{1}{2}R*1 + 2d\tilde\varphi\wedge *d\tilde\varphi - \tilde K_{\phi^\kappa\bar\phi^\lambda}\,d\phi^\kappa\wedge *d\bar\phi^\lambda + \tilde K_{\sigma^A\bar\sigma^B}\,d\sigma^A\wedge *d\bar\sigma^B \\ - \tilde K_{\phi^\kappa\bar\sigma^B}\,d\phi^\kappa\wedge d\bar\sigma^B - \tilde K_{\sigma^A\bar\phi^\lambda}\,d\bar\phi^\kappa\wedge d\sigma^A\Big)\,, \end{aligned} \tag{3.23}$$

where $\tilde K$ is now allowed to be dependent on $\tilde\varphi$ and given by

$$\tilde K = \mathcal{K} + e^{2\tilde\varphi}\mathcal{S}\,, \tag{3.24}$$

Similar to $\mathcal{K}$, the new function $\mathcal{S}$ is allowed to depend on the chiral and twisted-chiral scalars ϕ^κ, σ^A, but is taken to be independent of $\tilde\varphi$. The action (3.23) trivially reduces to (3.20) for $\mathcal{S} = 0$. Note that the new terms induced by $\mathcal{S}$ do not scale with $e^{-2\tilde\varphi}$. Comparison with (3.8) reveals that one can identify

$$\tilde\varphi = \phi_{\rm IIA}\,, \qquad \phi^\kappa = (z^K, N_l)\,, \qquad \sigma^A = t^A\,, \tag{3.25}$$

and introduce the generating functions

$$\begin{aligned} \mathcal{K} &= -\log\int_{Y_4}\Omega\wedge\bar\Omega + \log\mathcal{V}\,, \\ \mathcal{S} &= H^{lk}\,\mathrm{Re}\,N_l\,\mathrm{Re}N_k\,, \qquad H^{lk}\equiv v^A d_A{}^{lk}\,. \end{aligned} \tag{3.26}$$

To show this, it is useful to note that $d_A{}^{lk}$ can be evaluated as in (3.14) and depends on the complex structure moduli through the holomorphic function $f_{mn}(z)$ only.

Let us close this subsection with two remarks. First, note that (3.23) is expected to be compatible with $\mathcal{N} = (2,2)$ supersymmetry and gives an extension of the two-dimensional dilaton supergravity action (3.18). A suggestive form of the extended superspace action is

$$S^{(2)} = \int d^2x d^4\theta E^{-1}\big(e^{-2V-\mathcal{K}} + \mathcal{S}\big)\,, \tag{3.27}$$

where $\mathcal{S}$ is now evaluated as a function of the chiral and twisted-chiral superfields. It would be interesting to check that (3.27) indeed correctly reproduces the bosonic action (3.23) with $\tilde K$ as in (3.24).

Second, the action (3.23) with the identification (3.25) can also be straightforwardly obtained by dimensionally reducing M-theory, or rather eleven-dimensional supergravity,

[7]Our discussion differs here from the one in [17], where the sign in front of $\log\mathcal{V}$ was claimed to be negative.

first on Y_4 and then on an extra circle of radius r. The reduction of M-theory on Y_4 was carried out in [12, 13]. We give the resulting three-dimensional action in (5.5) and briefly recall this reduction in section 5.1 when considering applications to F-theory. Using the standard relation of eleven-dimensional supergravity on a circle and Type IIA supergravity, one straightforwardly identifies

$$r = e^{-2\phi_{\text{IIA}}}\,, \qquad e^{2\phi_{\text{IIA}}} v^A = \frac{v^A_{\text{M}}}{\mathcal{V}_{\text{M}}} \equiv L^A\,, \tag{3.28}$$

where v^A_{M} and $\mathcal{V}_{\text{M}}$ are the analogs of v^A and $\mathcal{V}$ used in the M-theory reduction. Note that the scalars L^A are the appropriate fields to appear in three-dimensional vector multiplets. Inserting the identification (3.28) into (3.24) together with (3.25), (3.26) one finds

$$\tilde{K}^{\text{M}} = -\log \int_{Y_4} \Omega \wedge \bar{\Omega} + \log\left(\frac{1}{4!}\mathcal{K}_{ABCD} L^A L^B L^C L^D\right) + L^A d_A{}^{lk}\, \text{Re}N_l\, \text{Re}N_k\,, \tag{3.29}$$

where we have dropped the logarithm containing the circle radius. Indeed $\tilde{K}^{\text{M}}$ agrees precisely with the result found in [12, 13, 15] from the M-theory reduction. The general discussion of the circle reduction of a three-dimensional un-gauged $\mathcal{N} = 2$ supergravity theory to a two-dimensional $\mathcal{N} = (2, 2)$ supergravity theory can be found in appendix A.

3.3 Legendre transforms from chiral and twisted-chiral scalars

In this subsection we want to introduce an operation that allows to translate the dynamics of certain chiral multiplets to twisted-chiral multiplets and vice versa. More precisely, we will assume that some of the scalars, say the scalars λ_l, in the $\mathcal{N} = (2, 2)$ supergravity action have continuous shift symmetries, i.e. $\lambda_l \to \lambda_l + c_l$ for constant c_l. These scalars therefore only appear with derivatives $d\lambda^l$ in the action. By the standard duality of massless p-forms to $(D - p - 2)$-forms in D dimensions, one can then replace the scalars λ_l by dual scalars λ'^l. Accordingly, one has to adjust the complex structure on the scalar field space by performing a Legendre transform. In the following we will give representative examples of how this works in detail. We will see that this duality, in particular as described in the first example, becomes crucial in the discussion of mirror symmetry of section 4.

As a first example, let us consider the above theory with complex scalars z^K, N_l in chiral multiplets and complex scalars t^A in twisted-chiral multiplets. The kinetic potential for these fields $\tilde{K}$ was given in (3.24) with (3.26). Two facts about this example are crucial for the following discussion. First, the fields N_l admit a shift symmetry $N_l \to N_l + ic_l$ in the action, i.e. the kinetic potential $\tilde{K}$ given in (3.26) is independent of $N_l - \bar{N}_l$. Second, the N_l only appear in the term $\mathcal{S}$ of the kinetic potential and thus carry no dilaton pre-factor in the action. One can thus straightforwardly dualize $N_l - \bar{N}_l$ into real scalars λ'^l. The new complex scalars N'^l are then given by

$$N'^l = \frac{1}{2}\frac{\partial \mathcal{S}}{\partial \text{Re}\, N_l} + i\lambda'^l\,, \tag{3.30}$$

where we have included a factor of $1/2$ for later convenience. Furthermore, the new kinetic potential $\tilde{K}'$ is now a function of z^K, t^A, N'^l and given by the Legendre transform

$$\tilde{K}' = \tilde{K} - 2\, e^{2\tilde{\varphi}} \text{Re}\, N'^l \text{Re}\, N_l\,, \tag{3.31}$$

where $\mathrm{Re}\, N_l$ has to be evaluated as a function of $\mathrm{Re}\, N'^l$ and the other complex fields by solving (3.30) for $\mathrm{Re}\, N_l$. One now checks that the scalars N'^l actually reside in twisted-chiral multiplets. Using the transformation (3.30) and (3.31) in the action (3.23) simply yields a dual description in which certain chiral multiplets are consistently replaced by twisted-chiral multiplets. It is simple to evaluate (3.30), (3.31) for $\mathcal{S}$ given in (3.26) to find

$$N'^l = H^{lm}\, \mathrm{Re}\, N_m + i\lambda'^l\,, \tag{3.32}$$

$$\tilde{K}' = \mathcal{K} - e^{2\phi_{\mathrm{IIA}}} H_{kl}\, \mathrm{Re}\, N'^k\, \mathrm{Re}\, N'^l\,, \tag{3.33}$$

where H^{lm} is the inverse of the matrix H_{lm} introduced in (3.15), (3.16). It is interesting to realize that upon inserting (3.32) into (3.33) one finds that $\tilde{K}'$ evaluated as a function of N_k only differs by a minus sign in front of the term linear in $e^{2\phi_{\mathrm{IIA}}}$ from the original $\tilde{K}$. This simple transformation arises from the fact that $\tilde{K}$ is only quadratic in the N_k. This observation will be crucial again in the discussion of mirror symmetry in section 4.

As a second example, we briefly want to discuss a dualization that transforms all multiplets containing scalars to become chiral. The detailed computation for a general $\mathcal{N} = (2,2)$ setting can be found in appendix B. For the example of section 3.2 we focus on the twisted-chiral multiplets with complex scalars t^A. These admit a shift symmetry $t^A \to t^A + c^A$ for constant c^A, such that $\mathrm{Re}\, t^A$ only appears with derivatives in the action. Accordingly, the kinetic potential $\tilde{K}$ is independent of $t^A + \bar{t}^A$ as seen in (3.24) with (3.26). Due to the shift symmetry we can dualize the scalars $t^A + \bar{t}^A$ to scalars ρ_A. However, note that by using the kinetic potential (3.24), (3.26) there are couplings of t^A in (3.23) that have a dilaton factor $e^{\tilde{\varphi}}$, and others that are independent of $e^{\tilde{\varphi}}$. This seemingly prevents us from performing a straightforward Legendre transform to bring the resulting action to the form (3.23) with only chiral multiplets. Remarkably, the special properties of the kinetic potential (3.24), (3.26), however, allow us to nevertheless achieve this goal as we will see in the following.

The action (3.23) for a setting with only chiral multiplets with complex scalars M^I takes the form

$$S^{(2)} = \int e^{-2\tilde{\varphi}} \left(\frac{1}{2} R * 1 + 2 d\tilde{\varphi} \wedge * d\tilde{\varphi} - \mathbf{K}_{M^I \bar{M}^J}\, dM^I \wedge * d\bar{M}^J \right)\,, \tag{3.34}$$

where $\mathbf{K}_{M^I \bar{M}^J} = \partial_{M^I} \partial_{\bar{M}^J} \mathbf{K}$. In other words, the potential $\mathbf{K}$ is in this case actually a Kähler potential on the field space spanned by the complex coordinates M^I. For our example (3.24), (3.26) the scalars M^I consist of z^K, N_l, and T_A, where T_A are the duals of the complex fields t^A. We make the following Ansatz for the dual coordinates T_A

$$T_A = e^{-2\tilde{\varphi}} \frac{\partial \tilde{K}}{\partial \mathrm{Im}\, t^A} + i\rho_A = e^{-2\tilde{\varphi}} \frac{\partial \mathcal{K}}{\partial \mathrm{Im}\, t^A} + \frac{\partial \mathcal{S}}{\partial \mathrm{Im}\, t^A} + i\rho_A\,, \tag{3.35}$$

and the dual potential $\mathbf{K}$

$$\mathbf{K} = \tilde{K} - e^{2\tilde{\varphi}} \mathrm{Re}\, T_A \mathrm{Im}\, t^A\;. \tag{3.36}$$

These expressions describe the standard Legendre transform for $\mathrm{Im}\, t^A$, but crucially contain dilaton factors $e^{2\tilde{\varphi}}$. This latter fact allows to factor out $e^{-2\tilde{\varphi}}$ as required in (3.34), but requires to perform a two-dimensional Weyl rescaling as we will discuss below. Using (3.24) with (3.26) one straightforwardly evaluates

$$T_A = e^{-2\phi_{\rm IIA}} \frac{1}{3!} \frac{\mathcal{K}_A}{\mathcal{V}} + d_A{}^{kl}\, \mathrm{Re}\, N_l\, \mathrm{Re}\, N_k + i\rho_A\,, \tag{3.37}$$

$$\mathbf{K} = -\log \int_{Y_4} \Omega\wedge\bar\Omega + \log\mathcal{V}\ . \tag{3.38}$$

Clearly, upon using the map (3.28) this result is familiar from the study of M-theory compactifications on Calabi-Yau fourfolds [12, 13, 15]. Also note that the contribution $\mathcal{S}$ present in the kinetic potential (3.24) is removed by the Legendre transform in $\mathbf{K}$ and reappears in a more involved definition of the coordinates T_A.

At first it appears that (3.35) induces new mixed terms involving one $d\tilde\varphi$ due to the dilaton dependence in front of the derivatives of $\mathcal{K}$. Interestingly, these can be removed by a two-dimensional Weyl rescaling if $\mathcal{K}$ satisfies the conditions

$$\mathcal{K}_{t^A}\, \mathcal{K}^{t^A\bar t^B}\, \mathcal{K}_{\bar t^B} = k\,, \qquad \mathcal{K}_{v^A}\, d\mathrm{Im}t^A = df\,, \tag{3.39}$$

for some constant k and some real field dependent function f. In this expression $\mathcal{K}^{t^A\bar t^B}$ is the inverse of $\mathcal{K}_{t^A\bar t^B}$ and $\mathcal{K}_{v^A} \equiv \partial_{\mathrm{Im}\, t^A}\mathcal{K}$. In fact, one can perform the rescaling $\tilde g_{\mu\nu} = e^{2\omega} g_{\mu\nu}$, which transforms the Einstein-Hilbert action as

$$\int e^{-2\tilde\varphi}\frac{1}{2}\tilde R\ \tilde{*}1 = \int e^{-2\tilde\varphi}\left(\frac{1}{2}R\ *1 - 2d\omega\wedge *d\tilde\varphi\right)\,, \tag{3.40}$$

while leaving all other terms invariant. Using (3.40) to absorb the mixed terms one needs to chose

$$\omega = -\frac{k}{2}\tilde\varphi - \frac{f}{2}\ . \tag{3.41}$$

The details of this computation can be found in appendix B. Indeed, for the example (3.26) one finds $f = \log\mathcal{V}$ and $k = -4$. Remarkably, the condition (3.39) essentially states that $\mathcal{K}$ has to satisfy a no-scale like condition. A recent discussion and further references on the subject of studying four-dimensional supergravities satisfying such conditions can be found in [32].

4 Mirror symmetry at large volume/large complex structure

In section 3 we have determined the two-dimensional action obtained from Type IIA supergravity compactified on a Calabi-Yau fourfold. We commented on its $\mathcal{N} = (2, 2)$ supersymmetry structure which relies on the proper identification of chiral and twisted-chiral multiplets in two dimensions. In this section we are exploring the action of mirror symmetry. More precisely, we consider pairs of geometries Y_4 and $\hat Y_4$ that are mirror manifolds [4–6]. From a string theory world-sheet perspective one expects the two theories obtained from string theory on Y_4 and $\hat Y_4$ to be dual. This implies that after finding the appropriate identification of coordinates the two-dimensional effective theories should be identical when considered at dual points in moduli space. We will make this more precise for the large volume and large complex structure point in this section. Note that in contrast to mirror symmetry for Calabi-Yau threefolds the mirror theories encountered here are both arising in Type IIA string theory.[8]

4.1 Mirror symmetry for complex and Kähler structure

Mirror symmetry arises from the observation that the conformal field theories associated with Y_4 and $\hat{Y}_4$ are equivalent. It describes the identification of Calabi-Yau fourfolds Y_4, $\hat{Y}_4$ with Hodge numbers

$$h^{p,q}(Y_4) = h^{4-p,q}(\hat{Y}_4) \ . \tag{4.1}$$

Note that this particularly includes the non-trivial conditions

$$h^{1,1}(Y_4) = h^{3,1}(\hat{Y}_4) \, , \qquad h^{3,1}(Y_4) = h^{1,1}(\hat{Y}_4) \, , \tag{4.2}$$

$$h^{2,1}(Y_4) = h^{2,1}(\hat{Y}_4) \ . \tag{4.3}$$

The first identification (4.2) together with the observations made in section 3 implies that mirror symmetry exchanges Kähler structure deformations of Y_4 ($\hat{Y}_4$) with complex structure deformations of $\hat{Y}_4$ (Y_4). Accordingly one needs to exchange chiral multiplets and twisted-chiral multiplets in the effective $\mathcal{N} = (2, 2)$ supergravity theory. The second identification (4.3) seems to suggest that for the fields N_l the mirror map is trivial. However, as we will see in section 4.2 this is not the case and one has to equally change from a chiral to a twisted-chiral description.

To present a more in-depth discussion of mirror symmetry we first need to introduce some notation. All fields and couplings obtained by compactification on Y_4 are denoted as in section 3. To destinguish them from the quantities obtained in the $\hat{Y}_4$ reduction we will dress the latter with a hat. In particular for the fields we write

$$\begin{aligned} Y_4 : \quad & \phi_{\rm IIA} \, , \ t^A \, , \ z^K \, , \ N_l \, , \\ \hat{Y}_4 : \quad & \hat{\phi}_{\rm IIA} \, , \ \hat{t}^K \, , \ \hat{z}^A \, , \ \hat{N}_l \ . \end{aligned} \tag{4.4}$$

Note that we have exchanged the indices on $\hat{t}^K$ and $\hat{z}^A$ in accordance with the fact that complex structure and Kähler structure deformations are interchanged by mirror symmetry. In other words, $K = 1, \ldots, h^{1,1}(\hat{Y}_4)$ and $A = 1, \ldots, h^{3,1}(\hat{Y}_4)$ is compatible with the previous notation due to (4.2). Similarly we will adjust the notation for the couplings. For example, the functions introduced in (3.26) and (2.4), (2.5) are

$$Y_4 : \quad f_{mn}(z) \, , \ H^{mn}(v, z) \, , \tag{4.5}$$

$$\hat{Y}_4 : \quad \hat{f}_{mn}(\hat{z}) \, , \ \hat{H}^{mn}(\hat{v}, \hat{z}) \, . \tag{4.6}$$

The functional form of the various couplings will in general differ for Y_4 and $\hat{Y}_4$. A match of the two mirror-symmetric effective theories should, however, be possible when identifying the mirror map, which we denote formally by $\mathcal{M}[\cdot]$.

[8]This can be seen immediately when employing the SYZ-understanding of mirror symmetry as T-duality [33]. Mirror symmetry is thereby understood as T-duality along half of the compactified dimensions, i.e. Y_4 is argued to contain real four-dimensional tori along which T-duality can be performed. Clearly, this inverts an even number of dimensions for Calabi-Yau fourfolds.

We want to focus on the sector of the theory independent of the three-forms. Recall that in the two-dimensional effective theory obtained from Y_4 the kinetic terms of the complex structure moduli z^K and Kähler structure moduli t^A are obtained from the kinetic potential (3.22), (3.26) as

$$\mathcal{K}(Y_4) = \log\left(\frac{1}{4!}\mathcal{K}_{ABCD}\,\mathrm{Im}\,t^A\,\mathrm{Im}\,t^B\,\mathrm{Im}\,t^C\,\mathrm{Im}\,t^D\right) - \log\int_{Y_4}\Omega\wedge\overline{\Omega} \tag{4.7}$$

when used in the action (3.20). Mirror symmetry exchanges the Kähler moduli t^K of Y_4 with the complex structure moduli $\hat{z}^K$ of $\hat{Y}_4$. The expression (4.7) was computed at the large volume point in Kähler moduli space, i.e. with the assumption that $\mathrm{Im}\,t^A \gg 1$ in string units. Accordingly one has to evaluate $\mathcal{K}(\hat{Y}_4)$ at the large complex structure point as

$$\int_{\hat{Y}_4}\hat{\Omega}\wedge\overline{\hat{\Omega}} = \frac{1}{4!}\mathcal{K}_{ABCD}\,\mathrm{Im}\,\hat{z}^A\,\mathrm{Im}\,\hat{z}^B\,\mathrm{Im}\,\hat{z}^C\,\mathrm{Im}\,\hat{z}^D\,, \tag{4.8}$$

where now $\mathrm{Im}\,\hat{z}^A \gg 1$. Similarly, one has to proceed for the Kähler moduli part of the kinetic potential $\mathcal{K}(\hat{Y}_4)$ and evaluate $\mathcal{K}(Y_4)$ at the large complex structure point

$$\int_{Y_4}\Omega\wedge\overline{\Omega} = \frac{1}{4!}\hat{\mathcal{K}}_{KLMN}\,\mathrm{Im}\,z^K\,\mathrm{Im}\,z^L\,\mathrm{Im}\,z^M\,\mathrm{Im}\,z^N\,, \tag{4.9}$$

where $\hat{\mathcal{K}}_{KLMN}$ are now the quadruple intersection numbers on the geometry $\hat{Y}_4$. Therefore, at the large volume and large complex structure point the two effective theories obtained from Y_4 and $\hat{Y}_4$ are identified under the mirror map

$$\mathcal{M}\big[t^A\big] = \hat{z}^A\,,\quad \mathcal{M}\big[z^K\big] = \hat{t}^K\,, \tag{4.10}$$

and

$$\mathcal{M}\big[\mathcal{K}(Y_4)\big] = -\mathcal{K}(\hat{Y}_4)\,,\qquad \mathcal{M}\big[\phi_{\mathrm{IIA}}\big] = \hat{\phi}_{\mathrm{IIA}}\,. \tag{4.11}$$

It is important to stress that a sign change occurs when applying the mirror map to $\mathcal{K}$. This can be traced back to the fact that scalars in chiral and twisted-chiral multiplets have different sign kinetic terms in the actions (3.20), (3.23). The quantum corrections to $\mathcal{K}$ were discussed using mirror symmetry in [4–7] and localization in [8, 9] (using and extending the results of [34–36]).

4.2 Mirror symmetry for non-trivial three-forms

Let us next include the moduli N_l arising for Calabi-Yau fourfolds Y_4 with non-vanishing $h^{2,1}(Y_4)$. In section 3 we have seen that these complex scalars are part of chiral multiplets. Their dynamics was described by the real function $\mathcal{S}$ in the kinetic potential $\tilde{K}$ given in (3.24) and (3.26). For completeness we recall that

$$\mathcal{S}(Y_4) = H^{lk}\,\mathrm{Re}\,N_l\,\mathrm{Re}\,N_k\,,\qquad H^{lk} \equiv v^A d_A{}^{lk}\,, \tag{4.12}$$

where $d_A{}^{lk}$ is a function of the complex structure moduli of Y_4. Mirror symmetry should map the fields N_l to scalars $\hat{N}_l$ arising in the reduction on the mirror Calabi-Yau fourfold $\hat{Y}_4$, i.e. one should have

$$\mathcal{M}\big[N_l\big] = Q_l(\hat{N},\hat{z},\hat{t})\,, \tag{4.13}$$

where we have allowed the image of N_l to be a non-trivial function that will be determined in the following. In fact, note that the map cannot be as simple as $\mathcal{M}(N_l) = \hat{N}_l$. As already pointed out in [17] the mirror duals $\mathcal{M}(N_l)$ need to be, in contrast to the N_l, parts of *twisted*-chiral multiplets. To achieve this we need to use the results of section 3.3.

Let us therefore consider the reduction on $\hat{Y}_4$ using the same notation as in section 3 but with hatted symbols. The two-dimensional theory will contain a set of complex scalars $\hat{N}_l$ that reside in chiral multiplets. We can transform them to scalars in twisted-chiral multiplets using (3.32) and (3.33). In other words, we find a dual description with scalars $\hat{N}'^l$ defined as

$$\hat{N}'^{\,l} = \hat{H}^{lm}\,\mathrm{Re}\,\hat{N}_m + i\hat{\lambda}'^{\,l}\,, \tag{4.14}$$

where $\hat{H}^{lm}$ is a function of the mirror complex structure moduli $\hat{z}^A$ and Kähler moduli $\hat{v}^K$. The dual kinetic potential takes the form

$$\tilde{K}'(\hat{Y}_4) = \mathcal{K}(\hat{Y}_4) - e^{2\hat{\phi}_{\mathrm{IIA}}}\hat{H}_{kl}\;\mathrm{Re}\,\hat{N}'^{\,k}\;\mathrm{Re}\,\hat{N}'^{\,l}\,. \tag{4.15}$$

The mirror map (4.10), (4.11) and (4.13) exchanges chiral and twisted-chiral states and therefore has to take the form

$$\mathcal{M}\big[N_l\big] = \hat{N}'^{\,l}(\hat{N},\hat{z},\hat{t})\,, \qquad \mathcal{M}\big[t^A\big] = \hat{z}^A\,, \qquad \mathcal{M}\big[z^K\big] = \hat{t}^K\,, \tag{4.16}$$

$$\mathcal{M}\big[\tilde{K}(Y_4)\big] = -\tilde{K}'(\hat{Y}_4)\,, \qquad \mathcal{M}\big[\phi_{\mathrm{IIA}}\big] = \hat{\phi}_{\mathrm{IIA}}\,. \tag{4.17}$$

and is evaluated as a function of $\hat{N}_l$, $\hat{z}^A$ and $\hat{t}^K$ by using (4.14).

Using these insights we are now able to infer the mirror image of the function H_{mn} appearing in $\tilde{K}(Y_4)$. To do that, we apply the mirror map to the kinetic potential $\tilde{K}$. Note that

$$\mathcal{M}\big[\tilde{K}(Y_4)\big] = -\mathcal{K}(\hat{Y}_4) + e^{2\hat{\phi}_{\mathrm{IIA}}}\mathcal{M}\big[\mathcal{S}(Y_4)\big]\,, \tag{4.18}$$

where we have used (4.11). Furthermore, we insert (4.17) into (4.12) to find

$$\mathcal{M}\big[\mathcal{S}(Y_4)\big] = \sum_{k,l}\mathcal{M}\big[H^{kl}\big]\mathrm{Re}\,\hat{N}'^k\,\mathrm{Re}\,\hat{N}'^l\,. \tag{4.19}$$

We next apply (4.17) together with (4.15) which requires

$$\sum_{k,l}\mathcal{M}\big[H^{kl}\big]\mathrm{Re}\,\hat{N}'^k\,\mathrm{Re}\,\hat{N}'^l \overset{!}{=} \hat{H}_{kl}\;\mathrm{Re}\,\hat{N}'^{\,k}\;\mathrm{Re}\,\hat{N}'^{\,l}\,, \tag{4.20}$$

and thus enforces

$$\mathcal{M}\big[H^{kl}\big] \overset{!}{=} \hat{H}_{kl}\,. \tag{4.21}$$

We therefore find that the mirror map actually identifies H^{kl} with the *inverse* $\hat{H}_{kl}$ of $\hat{H}^{kl}$. This inversion is crucial and stems from the exchange of chiral an twisted-chiral multiplets under mirror symmetry. In the final part of this section we evaluate the condition (4.21) at the large complex structure point, since H^{kl} given in (4.12) was computed at large volume.

Using the mirror map we are now able to determine the holomorphic function f_{kl} appearing in the definition of H_{kl} at the large complex structure point. Note that (3.16) translates on Y_4 and $\hat{Y}_4$ to

$$\begin{aligned} H^{lm} &= -\frac{1}{2}(\mathrm{Re} f)^{ln}\,\mathrm{Re}\, h_n^m\,, \qquad h_n^m = -it^A C_{An}{}^m\,, \\ \hat{H}^{lm} &= -\frac{1}{2}(\mathrm{Re}\hat{f})^{ln}\,\mathrm{Re}\, \hat{h}_n^m\,, \qquad \hat{h}_n^m = -i\hat{t}^K \hat{C}_{Kn}{}^m\,, \end{aligned} \tag{4.22}$$

where on the mirror geometry we introduced the intersection numbers

$$\hat{C}_{Kn}{}^m = \int_{\hat{Y}_4} \hat{\omega}_K \wedge \hat{\alpha}_n \wedge \hat{\beta}^m\,. \tag{4.23}$$

Using (4.16), (4.17), (4.21), and (4.22) in the mirror map one infers that a possible identification is[9]

$$\mathrm{Re} f_{nm} = \mathrm{Im} z^K \hat{C}_{Kn}{}^m\,. \tag{4.24}$$

By holomorphicity of f_{nm} we finally conclude

$$f_{nm} = -iz^K \hat{C}_{Kn}{}^m \tag{4.25}$$

Having determined the function f_{mn} at the large complex structure point we have established a complete match of the two two-dimensional effective theories obtained from Y_4 and $\hat{Y}_4$ under the mirror map $\mathcal{M}[\cdot]$. The result (4.25) is not unexpected. In fact, from the variation of Hodge-structures one could have expected a leading linear dependence on z^K. Furthermore, we will find agreement with a dual Calabi-Yau threefold result when using the geometry Y_4 as F-theory background and performing the orientifold limit. This will be the task of the final section of this work.

5 Applications for F-theory and Type IIB orientifolds

In this section we want to apply the result obtained by using mirror symmetry to compactifications of F-theory and their orientifold limit. The F-theory effective action is studied via the M-theory to F-theory limit. Therefore, we will briefly review in section 5.1 the dimensional reduction of M-theory on a smooth Calabi-Yau fourfold including three-form moduli. This reduction was already performed in [13], but we will use the insights we have gained in the previous sections to include the three-form moduli more conveniently. In section 5.2 we will then restrict to a certain class of elliptically fibered Calabi-Yau fourfolds and perform the M-theory to F-theory limit. This allows us to identify the characteristic data determining the four-dimensional $\mathcal{N} = 1$ F-theory effective action in terms of the geometric quantities of the internal space [15]. We note that for certain fourfolds the holomorphic function f_{kl} lifts to a four-dimensional gauge coupling function. Starting from these F-theroy settings we will then perform the weak string coupling limit in section 5.3. In this limit f_{kl} can be partially computed by using mirror symmetry for Calabi-Yau threefolds and we show compatibility with the fourfold result of section 4.

[9]Note that in general the basis (α_l, β^k) might not directly map to $(\hat{\alpha}_l, \hat{\beta}^k)$ on the mirror geometry $\hat{Y}_4$. In this expression we have assumed that there is no non-trivial base change under mirror symmetry.

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5.1 M-theory on Calabi-Yau fourfolds

In this subsection we review the dimensional reduction of M-theory on a Calabi-Yau fourfold Y_4 in the large volume limit without fluxes. The ansatz here is similar to the one used for Type IIA supergravity in section 3.1.

We start with eleven-dimensional supergravity as the low-energy limit of M-theory. Its bosonic two-derivative action is given by

$$S^{(11)} = \int \frac{1}{2}\check{R}\,\check{*}1 - \frac{1}{4}\check{F}_4 \wedge \check{*}\check{F}_4 - \frac{1}{12}\check{C}_3 \wedge \check{F}_4 \wedge \check{F}_4\,, \tag{5.1}$$

with $\check{F}_4 = d\check{C}_3$ the eleven-dimensional three-form field strength. This will be dimensionally reduced on the background

$$d\check{s}^2 = \eta^{(3)}_{\mu\nu}dx^\mu dx^\nu + g^{(8)}_{mn}dy^m dy^n\,, \tag{5.2}$$

where $\eta^{(3)}$ is the metric of three-dimensional Minkowski space-time $\mathbb{M}_{2,1}$ and $g^{(8)}$ the metric of the Calabi-Yau fourfold Y_4. This is the analog to (3.2) and, as we briefly discussed at the end of section 3.2, the Type IIA supergravity vacuum can be obtained by a circle-reduction of this Ansatz.

To perform the dimensional reduction one inserts similar expansions of (3.3), (3.5) and (3.6) into the eleven-dimensional action (5.1). For the metric deformations consisting of Kähler and complex structure deformations, this is exactly the same as (3.3) and (3.5), hence we obtain $h^{1,1}(Y_4)$ real scalars v^A_{M} by expanding the M-theory Kähler form J_{M} as

$$J_{\mathrm{M}} = v^A_{\mathrm{M}}\omega_A \tag{5.3}$$

and $h^{3,1}(Y_4)$ complex scalars z^K in three dimensions. Since the eleven-dimensional three-form $\check{C}_3$ is the common origin of the Type IIA fields $\check{B}_2$, $\check{C}_3$, we expand

$$\check{C}_3 = V^A \wedge \omega_A + N_l\Psi^l + \bar{N}_l\bar{\Psi}^l\,. \tag{5.4}$$

This yields $h^{2,1}(Y_4)$ three-dimensional complex scalars N_l and $h^{1,1}(Y_4)$ vectors V^A. The latter combine with the real scalars v^A_{M} into three-dimensional vector multiplets, whereas z^K, N_l give rise to three-dimensional chiral multiplets. Combining the expansions (3.3), (3.5) and (5.4) with the action (5.1) by using the notation of section 3.1 and section 3.2 we thus obtain the three-dimensional effective action[10]

$$\begin{aligned}S^{(3)} = \int \frac{1}{2}R*1 - G_{K\overline{L}}dz^K \wedge *d\bar{z}^L - \frac{1}{2}d\log\mathcal{V}_{\mathrm{M}} \wedge *d\log\mathcal{V}_{\mathrm{M}} - G^{\mathrm{M}}_{AB}dv^A_{\mathrm{M}} \wedge *dv^B_{\mathrm{M}}\\ - \frac{1}{2}v^A_{\mathrm{M}}\,d_A{}^{lk}DN_l \wedge *D\overline{N}_k - \mathcal{V}^2_{\mathrm{M}}G^{\mathrm{M}}_{AB}dV^A \wedge *dV^B\\ + \frac{i}{4}d_A{}^{lk}dV^A \wedge \left(N_l D\overline{N}_k - \overline{N}_k DN_l\right)\,.\end{aligned} \tag{5.5}$$

Note the G^{M}_{AB} takes the same functional form as (3.10), but uses the M-theory Kähler structure deformations v^A_{M}.

[10] The action has been Weyl-rescaled to the three-dimensional Einstein frame by introducing $g^{\mathrm{new}}_{\mu\nu} = \mathcal{V}^{-2}g^{\mathrm{old}}_{\mu\nu}$.

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The three-dimensional action given in (5.5) is an $\mathcal{N}=2$ supergravity theory. The proper scalars in the vector multiplets are denoted by L^A and are expressed in terms of the $v^A_{\rm M}$ as $L^A=\frac{v^A_{\rm M}}{\mathcal{V}_{\rm M}}$, as already given in (3.28). The complex scalars in the chiral multiplets are collectively denoted by $\phi^\kappa=(z^K,N_l)$. The action (5.5) can then be written using a kinetic potential $\tilde{K}^{\rm M}$ as

$$
\begin{aligned}
S^{(3)}=\int \frac{1}{2}R^{(3)}*1&+\frac{1}{4}\tilde{K}^{\rm M}_{L^AL^B}dL^A\wedge *dL^B+\frac{1}{4}\tilde{K}^{\rm M}_{L^AL^B}dV^A\wedge *dV^B\\
&-\tilde{K}^{\rm M}_{\phi^\kappa\bar{\phi}^\lambda}\,d\phi^\kappa\wedge *d\bar{\phi}^\lambda+dV^A\wedge \mathrm{Im}(\tilde{K}^{\rm M}_{L^A\phi^\kappa}d\phi^\kappa)\,,
\end{aligned}
\tag{5.6}
$$

where $\tilde{K}^{\rm M}_{L^AL^B}=\partial_{L^A}\partial_{L^B}\tilde{K}$, $\tilde{K}^{\rm M}_{\phi^\kappa\bar{\phi}^\lambda}=\partial_{\phi^\kappa}\partial_{\bar{\phi}^\lambda}\tilde{K}^{\rm M}$, and $\tilde{K}^{\rm M}_{L^A\phi^\kappa}=\partial_{L^A}\partial_{\phi^\kappa}\tilde{K}^{\rm M}$. Comparing (5.5) with (5.6) the kinetic potential obtained for this M-theory reduction therefore reads

$$
\tilde{K}^{\rm M}=-\log\int_{Y_4}\Omega\wedge\bar{\Omega}+\log\left(\frac{1}{4!}\mathcal{K}_{ABCD}L^AL^BL^CL^D\right)+L^Ad_A{}^{lk}\,\mathrm{Re}\,N_l\,\mathrm{Re}\,N_k\,,
\tag{5.7}
$$

and was already given in (3.29). Recalling the discussion at the end of section 3.2 it is not hard to check that (5.5) reduces to the Type IIA result found in section 3.1 upon a circle compactification. The detailed circle reduction is performed for a general three-dimensional un-gauged $\mathcal{N}=2$ theory in appendix A.

5.2 M-theory to F-theory lift

Let us now lift the result (5.6) of the M-theory reduction on a general smooth Calabi-Yau fourfold Y_4 to a four-dimensional effective F-theory compactification. To do so, we need to restrict Y_4 to be an elliptic fibration $\pi: Y_4\to B_3$ over a base manifold B_3 which is a three-dimensional complex Kähler manifold. This four-dimensional theory exhibits $\mathcal{N}=1$ supersymmetry. In the following we will not need to consider the full four-dimensional theory, but will rather focus on the kinetic terms of the complex scalars and vectors without including gaugings or a scalar potential. Supersymmetry ensures that these can be written in the form [37]

$$
S^{(4)}=\int\frac{1}{2}R*1-K^{\rm F}_{M^I\bar{M}^J}\,dM^I\wedge *d\bar{M}^J-\frac{1}{2}\mathrm{Re}\,\mathbf{f}_{\Lambda\Sigma}F^\Lambda\wedge *F^\Sigma-\frac{1}{2}\mathrm{Im}\,\mathbf{f}_{\Lambda\Sigma}F^\Lambda\wedge F^\Sigma\,. \tag{5.8}
$$

In this expression we denoted by M^I the bosonic degrees of freedom in chiral multiplets, and by F^Λ the field strengths of vectors A^Λ. The metric $K^{\rm F}_{M^I\bar{M}^J}$ is Kähler and thus can be obtained from a Kähler potential $K^{\rm F}$ via $K^{\rm F}_{M^I\bar{M}^J}=\partial_{M^I}\partial_{\bar{M}^J}K^{\rm F}$. The gauge-kinetic coupling function $\mathbf{f}_{\Lambda\Sigma}$ is holomorphic in the complex scalars M^I.

In order to determine the Kähler potential $K^{\rm F}$ and the gauge coupling function $\mathbf{f}_{\Lambda\Sigma}$ via M-theory one next would have to compactify (5.8) on a circle. The resulting three-dimensional theory then has to be pushed to the Coulomb branch and all massive modes, including the excited Kaluza-Klein modes of all four-dimensional fields, have to be integrated out. The resulting three-dimensional effective theory can then, after a number of dualizations, be compared with the M-theory effective action (5.5). Performing all these steps is in general complicated. However, a relevant special case has been considered in [15] and will be the focus in the following discussion. Despite the fact that we could refer to [15] we will try to keep the derivation of K^{F} and $\mathbf{f}_{\Lambda\Sigma}$ in this subsection self-contained.

Let us therefore assume that Y_4 is an elliptically fibered Calabi-Yau fourfold that satisfies the conditions

$$h^{2,1}(Y_4) = h^{2,1}(B_3)\,, \qquad h^{1,1}(Y_4) = h^{1,1}(B_3) + 1\,. \tag{5.9}$$

It is not hard to use toric geometry to construct examples that satisfy these conditions (see, for example, refs. [38, 39]). From the point of view of F-theory, or Type IIB string theory, the first condition in (5.9) implies that all scalars N_l in (5.5) lift to R-R vectors A^l in four dimensions. In other words, one can compactify Type IIB on the base B_3 and obtain vectors A^l by expanding the R-R four-form as

$$C_4 = A^l \wedge \alpha_I - \tilde{A}_l \wedge \beta^l + \ldots\,. \tag{5.10}$$

The vectors $\tilde{A}_l$ are the magnetic duals of the A^l and can be eliminated by using the self-duality of the field-strength of C_4.

The second condition in (5.9) implies that there are no further vectors in the four-dimensional theory, i.e. there are no massless vector degrees of freedom arising from seven-branes. The two-forms used in (5.3) and (5.4) split simply as

$$\omega_A = (\omega_0, \omega_\alpha)\,, \tag{5.11}$$

where ω_0 is the Poincaré-dual of the base divisor B_3 and ω_α is the Poincaré-dual of the vertical divisors $D^\alpha = \pi^{-1}(D^\alpha_{\mathrm{b}})$ stemming from divisors D^α_{b} of B_3. Accordingly one splits the three-dimensional vector multiplets in (5.6) as

$$L^A = (R, L^\alpha)\,, \qquad V^A = (A^0, A^\alpha)\,. \tag{5.12}$$

One can now evaluate the kinetic potential (5.7) for the special case (5.9). The only relevant non-vanishing quadruple intersection numbers are given by

$$\mathcal{K}_{0\alpha\beta\gamma} = \int_{Y_4} \omega_0 \wedge \omega_\alpha \wedge \omega_\beta \wedge \omega_\gamma \equiv \mathcal{K}_{\alpha\beta\gamma}\,, \tag{5.13}$$

which are simply the triple intersections $\mathcal{K}_{\alpha\beta\gamma}$ of the base B_3. Crucially, for an elliptic fibration one has $\mathcal{K}_{\alpha\beta\gamma\delta} = 0$. Furthermore, note that due to (5.9) all non-trivial three-forms come from the base B_3 and we can chose the basis (α_I, β^l) such that

$$C_{0m}{}^k = \int_{Y_4} \omega_0 \wedge \alpha_l \wedge \beta^k = \delta^k_l\,, \qquad C_{\alpha m}{}^k = C_{Amk} = 0\,, \tag{5.14}$$

with $C_{Am}{}^k$ and C_{Amk} introduced in (2.10). Inserting (5.13) and (5.14) into (5.7) one finds

$$\tilde{K}^{\mathrm{M}} = -\log \int_{Y_4} \Omega \wedge \bar{\Omega} + \log\left(\frac{1}{3!}\mathcal{K}_{\alpha\beta\gamma} L^\alpha L^\beta L^\gamma\right) + \log(R) - \frac{1}{2} R\, \mathrm{Re} f^{lk}\, \mathrm{Re}\, N_l\, \mathrm{Re}\, N_k\,, \tag{5.15}$$

where we have used that $L^A d_A{}^{lk} = -\frac{1}{2} L^A C^l_{Am} \mathrm{Re} f^{mk} = -\frac{1}{2} R\, \mathrm{Re} f^{lk}$, and we have dropped terms in the logarithm that are higher order in R.

In order to compare this kinetic potential with the result of the circle reduction of (5.8) we next have to dualize (L^α, A^α) into three-dimensional complex scalars T_α, and N_k into three-dimensional vectors (ξ^k, A^k). Due to our assumption (5.9) leading to (5.14) we can perform these dualizations independently. The change from (L^α, A^α) to $\text{Re}T_\alpha = \partial_{L^\alpha}\tilde{K}^{\text{M}}$ is similar to (3.37). It is conveniently parameterized by the base Kähler deformations v_{b}^α and the base volume $\mathcal{V}_{\text{b}}$ defined as [15, 26]

$$L^\alpha = \frac{v_{\text{b}}^\alpha}{\mathcal{V}_{\text{b}}}\,, \qquad \mathcal{V}_{\text{b}} = \frac{1}{3!}\mathcal{K}_{\alpha\beta\gamma}v_{\text{b}}^\alpha v_{\text{b}}^\beta v_{\text{b}}^\gamma\;. \tag{5.16}$$

The dualization of the complex scalars N_k into three-dimensional vectors is similar to the dualization yielding (3.30), (3.31) and (3.32), (3.33). First, one introduces

$$\xi^k = \partial_{\text{Re}N_k}\tilde{K}^{\text{M}}\,, \qquad \tilde{K}^{\text{M}\to\text{F}} = \tilde{K}^{\text{M}} - \xi^k \text{Re}\,N_l\,, \tag{5.17}$$

and then dualizes the field $\text{Im}N_k$ with a shift symmetry into the vector A^k. Together both Legendre transforms yield

$$\tilde{K}^{\text{M}\to\text{F}} = -\log\int_{Y_4}\Omega\wedge\bar{\Omega} - 2\log\,\mathcal{V}_{\text{b}}\, + \log\,R + \frac{1}{2R}\text{Re}f_{lk}\;\xi^l\xi^k\,, \tag{5.18}$$

which has to be evaluated as a function of z^K, ξ^k and

$$T_\alpha = \partial_{L^\alpha}\tilde{K}^{\text{M}} + i\rho_\alpha = \frac{1}{2!}\mathcal{K}_{\alpha\beta\gamma}v_{\text{b}}^\beta v_{\text{b}}^\gamma + i\rho_\alpha\;. \tag{5.19}$$

The kinetic potential (5.18) is now in the correct frame to be lifted to four space-time dimensions.

To derive K^{F}, $\mathbf{f}_{kl}$ one reduces (5.8) on a circle of radius r with the usual Kaluza-Klein ansatz the four-dimensional metric and vectors as

$$g^{(4)}_{\mu\nu} = \begin{pmatrix} g^{(3)}_{pq} + r^2 A^0_p A^0_q & r^2 A^0_q \\ r^2 A^0_p & r^2 \end{pmatrix}, \qquad A^k_\mu = (A^k_p + A^0_p\zeta^k, \zeta^k)\,, \tag{5.20}$$

where we introduced the three-dimensional indices $p, q = 0, 1, 2$ and the Kaluza-Klein vector A^0. Note that we use for three-dimensional vectors the same symbol A^k as in four dimensions. Furthermore, we introduced the new three-dimensional real scalars r, ζ^k into the theory. We next define

$$R = r^{-2}\,, \quad \xi^{\hat{k}} = (R, R\zeta^k)\,, \quad A^{\hat{k}} = (A^0, A^k)\;. \tag{5.21}$$

The three-dimensional theory obtained by reducing (5.8) has thus the field content: chiral multiplets with complex scalars M^I and vector multiplets $(\xi^{\hat{k}}, A^{\hat{k}})$. Its action can be written in the form (5.6) with a kinetic potential

$$\tilde{K}(M,\bar{M},\xi) = K^F(M,\bar{M}) + \log(R) - \frac{1}{R}\text{Re}\,\mathbf{f}_{kl}(M)\xi^k\xi^l\;, \tag{5.22}$$

when replacing $L^A \to \xi^{\hat{k}}$, $V^A \to A^{\hat{k}}$, and $\phi^\kappa \to M^I$. Finally, comparing (5.22) with (5.18) implies that one finds $M^I = \{T_\alpha, z^K\}$

$$K^{\mathrm{F}} = -\log\left(\int_{Y_4} \Omega \wedge \overline{\Omega}\right) - 2\log \mathcal{V}_b\,, \tag{5.23}$$

$$\mathbf{f}_{kl} = \frac{1}{2} f_{kl}\,. \tag{5.24}$$

In the next section, we want to derive the orientifold limit of this result relating the data of F-theory on Y_4 to Type IIB supergravity with $O7/O3$-planes on the closely related Calabi-Yau three-fold Y_3, a double cover of B_3.

5.3 Orientifold limit of F-theory and mirror symmetry

In this final subsection we investigate the orientifold limit of the F-theory effective action introduced above. More precisely, we assume that the F-theory compactification on the elliptically fibered geometry Y_4 admits a weak string coupling limit as introduced by Sen [24, 25]. This limit takes one to a special region in the complex structure moduli space of Y_4 in which the dilaton-axion $\tau = C_0 + ie^{-\phi_{\mathrm{IIB}}}$, given by the complex structure of the two-torus fiber of Y_4, is almost everywhere constant along the base B_3. The locations where τ is not constant are precisely the orientifold seven-planes (O7-planes). In the weak string coupling limit the geometry Y_4 can be approximated by

$$Y_4 \cong (Y_3 \times T^2)/\tilde{\sigma} \tag{5.25}$$

where we introduced the involution $\tilde{\sigma} = (\sigma, -1, -1)$ with σ being a holomorphic and isometric orientifold involution such that $Y_3/\sigma = B_3$. The two one-cycles of the torus are both odd under the involution, but its volume form is even. It was shown in [24, 25] that the double cover Y_3 of B_3 is actually a Calabi-Yau threefold. The location of the O7-planes in Y_3 is simply the fixed-point set of σ.

In the limit (5.25) we can check compatibility of the mirror symmetry results of section 4 with the mirror symmetry of the Calabi-Yau threefold Y_3. By using the mirror fourfold $\hat{Y}_4$ of Y_4 we have found that the function f_{lk} is linear in the large complex structure limit of Y_4. Here we recall that the weak string coupling expression gives a compatible result. Using the mirror $\hat{Y}_3$ of Y_3 one shows that the function f_{lk} is linear in the large complex structure limit of Y_3. This can be depicted as

$$\begin{array}{ccc} \text{F-theory on } Y_4 & \xrightarrow{\text{weak coupling}} & \text{Type IIB orientifolds } Y_3/\sigma \\ & & \updownarrow \quad \text{physical mirror duality} \\ & & \text{Type IIA orientifolds } \hat{Y}_3/\hat{\sigma} \end{array} \tag{5.26}$$

Note that mirror symmetry of Y_3 and $\hat{Y}_3$ gives a physical map between Type IIB and Type IIA orientifolds. The mirror map between Y_4 and $\hat{Y}_4$ has no apparent physical meaning in F-theory. Nevertheless, using the geometry Y_4 in Type IIA compactifications it can be used to calculate f_{lk} as we explained in section 4.

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Let us now introduce the function f_{lm} for the geometry (5.25). In the orientifold setting one splits the cohomologies of Y_3 as $H^{p,q}(Y_3) = H^{p,q}_+(Y_3) \oplus H^{p,q}_-(Y_3)$, which are the two eigenspaces of σ^*. We denote their dimensions as $h^{p,q}_\pm(Y_3)$. As reviewed, for example, in [2] the complex structure moduli z^K of Y_4 split into three sets of fields at weak string coupling. First, there is the dilaton-axion τ, which is now a modulus of the effective theory. Second, there are $h^{2,1}_-$ complex structure moduli z^κ of the quotient Y_3/σ. Third, the remaining number of complex structure deformations of Y_4 correspond to D7-brane position moduli. The last set are open string degrees of freedom and are not captured by the geometry of Y_3. For simplicity, we will not include them in the following discussion. With this simplifying assumption one finds that the pure complex structure part of the F-theory Kähler potential (5.23) splits as

$$-\log\left(\int_{Y_4}\Omega\wedge\overline{\Omega}\right) = -\log\big[-i(\tau-\bar\tau)\big] - \log\left[i\int_{Y_3}\Omega_3\wedge\bar\Omega_3\right] + \dots\,, \tag{5.27}$$

where Ω_3 is the $(3,0)$-form on Y_3 that varies holomorphically in the complex structure moduli z^κ. The dots indicate that further corrections arise that are suppressed at weak string coupling $-i(\tau-\bar\tau)\gg 1$. Taking the weak coupling limit for the Kähler potential (5.23) of the Kähler structure deformations is more straightforward. The deformations are counted by $h^{1,1}_+(Y_3)$ and identified with the Kähler structure deformations $v^\alpha_{\rm b}$ of the base B_3 introduced in (5.16). The orientifold Kähler potential for this set of deformations is then simply the second term in (5.23) and the Kähler coordinates are given by (5.19).

Turning to the gauge theory sector, we note that the number of R-R vectors A^l arising from C_4 as in (5.10) are counted by $h^{2,1}_+(Y_3)$ in the orientifold setting. The gauge coupling function for these vectors is determined as function of the complex structure moduli z^κ of Y_3 in [26].[11] It is given by

$$f_{kl}(z^\kappa) = -i\mathcal{F}_{kl}|(z^\kappa) \equiv \partial_{z^k}\partial_{z^l}\mathcal{F}|(z^\kappa)\,, \tag{5.28}$$

where $\mathcal{F}$ is the pre-potential determining the moduli-dependence of the Ω_3 of the geometry Y_3. To evaluate (5.28) one first splits the complex structure moduli of Y_3 into $h^{2,1}_-(Y_3)$ fields z^κ and $h^{2,1}_+(Y_3)$ fields z^k. The pre-potential $\mathcal{F}(z^\kappa, z^k)$ of Y_3 at first depends on both sets of fields. Then one has to take derivatives of $\mathcal{F}$ with respect to z^k and afterwards set these fields to constant background values compatible with the orientifold involution σ. This freezing of the z^k is indicated by the symbol $|$ in (5.28). Using mirror symmetry for Calabi-Yau threefolds it is well-known that the pre-potential at the large complex structure point of Y_3 is a cubic function of the complex structure moduli z^κ and z^k. Taking derivatives and evaluating the expression on the orientifold moduli space one thus finds

$$f_{kl}(z^\kappa) = -iz^\kappa\hat{\mathcal{K}}_{\kappa kl}\,, \tag{5.29}$$

where $\hat{\mathcal{K}}_{\kappa kl} = \int_{\hat Y_3}\hat\omega_\kappa\wedge\hat\omega_k\wedge\hat\omega_l$ are the triple intersection numbers of the mirror threefold $\hat Y_3$. This result agrees with the one for Type IIA orientifolds, which have been studied at large

[11] Note that we have slightly changed the index conventions with respect to [26] in order to match the F-theory discussion.

volume in [27]. Hence, we find consistency with the F-theory result (4.25) obtained by using mirror symmetry for Y_4 at the large complex structure point. To obtain a complete match of the results the intersection matrix $\hat{C}_{\kappa k}{}^{l}$ of $\hat{Y}_4$ is identified with the triple intersection $\hat{\mathcal{K}}_{\kappa kl}$ of $\hat{Y}_3$.

To close this section we stress again that we have only discussed the matching with the orientifold limit for special geometries satisfying (5.9). Furthermore, we have not included the open string degrees of freedom on the orientifold side. Clearly, our result for f_{lk} obtained in section 4 can be more generally applied. For example, a simple generalization is the inclusion of $h^{1,1}_{-}(Y_3)$ moduli G^a into the orientifold setting, which arise in the expansion of the complex two-form $C_2 - \tau B_2$. In F-theory the same degrees of freedom appear from the expansion (5.4) into non-trivial three-forms Ψ_a that have two legs in the base B_3 and one leg in the torus fiber, i.e. are not present in the geometries satisfying (5.9). In the orientifold setting one finds that the fields G^a correct the complex coordinates (5.19). We read off the result from [26] to find[12]

$$T_\alpha = \frac{1}{2!}\mathcal{K}_{\alpha\beta\gamma} v_{\mathrm{b}}^{\beta} v_{\mathrm{b}}^{\gamma} + \frac{1}{2\,\mathrm{Im}\tau}\,\mathcal{K}_{\alpha ab}\,\mathrm{Im}\,G^a \mathrm{Im}\,G^b + i\rho_\alpha\,. \tag{5.30}$$

Comparing this expression with (3.37) we read off that

$$N^a = iG^a\,, \qquad d_{\alpha ab} = \frac{1}{2}\frac{1}{\mathrm{Im}\tau}\mathcal{K}_{\alpha ab}\,, \qquad f_{ab}(\tau) = i\tau\delta_{ab}\,, \tag{5.31}$$

in order to match the F-theory result as already done in [40]. Again we find that the result is linear in one of the complex structure moduli, namely the field τ, of the Calabi-Yau fourfold Y_4 in the orientifold limit (5.25). It would be interesting to generalize these results even further and also include the open string moduli into the orientifold setting.

6 Conclusions

In this paper we first studied the two-dimensional low-energy effective action obtained from Type IIA string theory on a Calabi-Yau fourfold with non-trivial three-form cohomology. The couplings of the three-forms were shown to be encoded by two holomorphic functions f_{kl} and h^l_k, where the former depends on the complex structure moduli and the latter on the complexified Kähler structure moduli. Performing a large volume dimensional reduction of Type IIA supergravity, we were able to derive h^l_k explicitly as a linear function. We argued that f_{kl} and h^l_k computed on mirror pairs of Calabi-Yau manifolds will be exchanged, at least, if one considers the theories at large volume and large complex structure. In order to show this, we investigated the non-trivial map between the three-form moduli arising from mirror geometries and argued that it involves a scalar field dualization together with a Legendre transformation. This can be also motivated by the fact that chiral and twisted-chiral multiplets are expected to be exchanged by mirror symmetry. We thus established a linear dependence of the function f_{lk} on the complex structure moduli near the large complex structure point and determined the constant topological pre-factor.

[12] Note that compared with [26] we have redefined ρ_α to make the terms in T_α involving the G^a real.

In this work we also included a discussion of the superymmetry properties of the two-dimensional low-energy effective action. This action is expected to be an $\mathcal{N} = (2, 2)$ supergravity theory, which we showed to extend the dilaton supergravity action of [17]. The bosonic action was brought to an elegant form with all kinetic and topological terms determined by derivatives of a single function $\tilde{K} = \mathcal{K} + e^{2\tilde{\varphi}}\mathcal{S}$, where $\mathcal{K}$ and $\mathcal{S}$ can depend on the scalars in chiral and twisted-chiral multiplets, but are independent of the two-dimensional dilaton $\tilde{\varphi}$. In the Type IIA supergravity reduction the three-form scalars only appeared in the function $\mathcal{S}$ and are thus suppressed by $e^{2\tilde{\varphi}} = e^{2\phi_{\text{IIA}}}$. In this analysis the complex structure moduli and the three-form moduli were argued to fall into chiral multiplets, while the complexified Kähler moduli are in twisted-chiral multiplets. However, due to apparent shift symmetries of the three-form moduli and complexified Kähler moduli a scalar dualization accompanied by a Legendre transformation can be performed in two dimensions. This lead to dual descriptions in which certain chiral multiplets are replaced by twisted-chiral multiplets and vice versa. Remarkably, if one dualizes a subset of scalars appearing in $\mathcal{K}$, we found that the requirement to bring the dual action back to the standard $\mathcal{N} = (2, 2)$ dilaton supergravity form imposes conditions on viable $\mathcal{K}$. These constraints include a no-scale type condition on $\mathcal{K}$. The emergence of such restrictions arose from general arguments about two-dimensional theories coupled to an overall $e^{-2\tilde{\varphi}}$ factor. For Calabi-Yau fourfold reductions we checked that these conditions are indeed satisfied. It would be interesting to investigate this further and to get a deeper understanding of this result.

Having shown that in the large complex structure limit the function f_{kl} is linear in the complex structure moduli, we discussed the application of this result in an F-theory compactification. By assuming that the Calabi-Yau fourfold is elliptically fibered and that the three-forms exclusively arise from the base of this fibration, we recalled that f_{kl} is actually the gauge-coupling function of four-dimensional R-R vector fields. This gauge-coupling function was already evaluated in the weak string coupling limit in the orientifold literature. In this orientifold limit one can double-cover the base with a Calabi-Yau threefold. We found compatibility of the fourfold result with the expectation from mirror symmetry for Calabi-Yau threefold orientifolds. In this analysis we only included closed string moduli in the orientifold setting. Clearly, the results obtained from the Calabi-Yau fourfold analysis are more powerful and it would be interesting to further investigate the open string dependence in orientifolds using our results. Additionally we commented briefly on the case in which the three-forms have legs in the fiber of the elliptic fibration. In this situation the inverse of $\text{Re} f_{lk}$ sets the value of decay constants of four-dimensional axions [22]. Again we found compatibility in the closed string sector at weak string coupling in which $f_{lk} \propto i\tau$. It would be interesting to include the open string moduli in the orientifold setting and derive corrections to f_{lk} without restricting to the weak string coupling limit. The latter task requires to compute f_{lk} away from the large complex structure limit for elliptically fibered Calabi-Yau fourfolds.

In order to derive the complete moduli dependence of f_{lk} at various points in complex structure moduli space it would be desirable to obtain differential equations obeyed by the $(2, 1)$-forms. This should be possible by investigating the variations of Hodge structures

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and is expected to yield equations of second order in derivatives. The linear solutions found in this work can then provide the boundary conditions for the complete solutions. It would be important to develop the necessary tools for such an analysis and we hope to return to this issue in a future publication.

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A Three-dimensional $\mathcal{N} = 2$ supergravity on a circle

In this appendix we consider $\mathcal{N} = 2$ supergravity compactified on a circle of radius r. Our goal is to derive the resulting $\mathcal{N} = (2, 2)$ action. We also briefly discuss the dualization of vector multiplets in three dimensions and point out the relation to appendix B.

We start with a three-dimensional $\mathcal{N} = 2$ supergravity theory coupled to chiral multiplets with complex scalars ϕ^κ and vector multiplets with bosonic fields (L^A, A^A). HereL^A is a real scalar and A^A a vector of an U(1) gauge theory. The bosonic part of the ungauged $\mathcal{N} = 2$ action takes the form

$$S^{(3)} = \int \frac{1}{2} R^{(3)} * 1 - \tilde{K}_{\phi^\kappa \bar{\phi}^\lambda} d\phi^\kappa \wedge * d\bar{\phi}^\lambda + \frac{1}{4} \tilde{K}_{L^A L^B} dL^A \wedge * dL^B \\ + \frac{1}{4} \tilde{K}_{L^A L^B}\, dA^A \wedge * dA^B + dA^A \wedge \mathrm{Im}(\tilde{K}_{L^A \phi^\kappa} d\phi^\kappa) \tag{A.1}$$

where the kinetic terms of the vectors and scalars are determined by the single real kinetic potential $\tilde{K}$.

We want to put this on a circle of radius r and period one, i.e. the background metric is of the form

$$ds^2_{(3)} = g_{\mu\nu} dx^\mu dx^\nu + r^2 dz^2 \tag{A.2}$$

where we already drop vectors, since in an un-gauged theory they do not carry degrees of freedom in two dimensions. Similarly, the vectors A^A are only reduced to real scalars $dA^A = db^A \wedge dz$. The resulting two-dimensional action thus reads

$$S^{(2)} = \int \frac{1}{2} r R * 1 - r \tilde{K}_{\phi^\kappa \bar{\phi}^\lambda} d\phi^\kappa \wedge * d\bar{\phi}^\lambda + \frac{1}{4} r \tilde{K}_{L^A L^B} dL^A \wedge * dL^B \\ + \frac{1}{4r} \tilde{K}_{L^A L^B} db^A \wedge * db^B - db^A \wedge \mathrm{Im}(\tilde{K}_{L^A \phi^\kappa} d\phi^\kappa)\,, \tag{A.3}$$

with a two-dimensional R and Hodge star $*$. Note that the last term is topological and does not couple to the radius r of the circle. We can perform Weyl rescaling of the two-dimensional metric setting $\tilde{g}_{\mu\nu} = e^{2\omega} g_{\mu\nu}$. This transforms the Einstein-Hilbert term as

$$\int \frac{1}{2} r \tilde{R}\, \tilde{*} 1 = \int \frac{1}{2} r\, R\ \ * 1 + d\omega \wedge * dr\,, \tag{A.4}$$

while leaving all other terms in the action (A.3) invariant. We then find the action

$$S^{(2)} = \int r\left(\frac{1}{2}R*1 + d\log r\wedge *d\omega - \tilde{K}_{\phi^\kappa\bar{\phi}^\lambda}d\phi^\kappa\wedge *d\bar{\phi}^\lambda + \frac{1}{4}\tilde{K}_{L^AL^B}dL^A\wedge *dL^B \right. \\ \left. + \frac{1}{4r^2}\tilde{K}_{L^AL^B}db^A\wedge *db^B\right) - db^A\wedge \mathrm{Im}(\tilde{K}_{L^A\phi^\kappa}d\phi^\kappa) \tag{A.5}$$

To make contact with the $\mathcal{N}=(2,2)$ dilaton supergravity action (3.23) we set

$$L^A = r^{-1}v^A\,, \qquad r = e^{-2\tilde{\varphi}}\,, \tag{A.6}$$

$$\sigma^A \equiv b^A + iv^A\,. \tag{A.7}$$

Inserted into (A.5) we then obtain

$$S^{(2)} = \int e^{-2\tilde{\varphi}}\left(\frac{1}{2}R*1 - 2d\tilde{\varphi}\wedge *\Big(d\omega - \frac{1}{2}\tilde{K}_{v^Av^B}v^Adv^B - \frac{1}{2}\tilde{K}_{v^Av^B}v^Av^Bd\tilde{\varphi}\Big) \right. \\ \left. - \tilde{K}_{\phi^\kappa\bar{\phi}^\lambda}d\phi^\kappa\wedge *d\bar{\phi}^\lambda + \tilde{K}_{\sigma^A\bar{\sigma}^B}d\sigma^A\wedge *d\bar{\sigma}^B - d\mathrm{Re}\,\sigma^A\wedge\mathrm{Im}(\tilde{K}_{v^A\phi^\kappa}d\phi^\kappa)\right)\,. \tag{A.8}$$

In order to match the action (3.23) one therefore has to find an ω such that

$$d\omega = -d\tilde{\varphi} + \frac{1}{2}\tilde{K}_{v^Av^B}v^Adv^B + \frac{1}{2}\tilde{K}_{v^Av^B}v^Av^Bd\tilde{\varphi} \tag{A.9}$$

To solve this condition, we first notice that any term in $\tilde{K}$ that is linear v^A drops out from this relation, i.e. $\tilde{K}$ can take the form

$$\tilde{K} = \mathcal{K} + v^A\mathcal{S}_A\,, \tag{A.10}$$

with an arbitrary function $\mathcal{S}_A(\phi,\bar{\phi})$. Furthermore, we can solve (A.9) by assuming that $\mathcal{K}=\mathcal{K}_1+\mathcal{K}_2$ splits into a v^A-independent term $\mathcal{K}_1(\phi,\bar{\phi})$ and a term $\mathcal{K}_2(v)$ that only depends on v^A. Then (A.9) is satisfied if

$$v^A\mathcal{K}_{v^A} = -k\,, \qquad \omega = -\tilde{\varphi} + \frac{k}{2}\tilde{\varphi} - \frac{\mathcal{K}_2(v)}{2}\,, \tag{A.11}$$

It is easy to check that the conditions (A.10) and (A.11) are actually satisfied for the M-theory example (3.29) of $\tilde{K}$. One finds

$$\mathcal{K}_1(z) = -\log\int_{Y_4}\Omega\wedge\bar{\Omega}\,, \qquad \mathcal{K}_2(v) = \log\mathcal{V}\,, \qquad \mathcal{S}_A = e^{2\varphi}d_A{}^{l\bar{k}}\,\mathrm{Re}N_l\,\mathrm{Re}N_k\,, \tag{A.12}$$

such that $k=-4$. Finally, in order to show that (A.8) is indeed identical to the action (3.23), we still have to complete the last term in (A.8) to $\mathrm{Im}(d\sigma^A\wedge\tilde{K}_{v^A\phi^\kappa}d\phi^\kappa)$. In order to do that we use

$$d\mathrm{Im}\,\sigma^A\wedge\mathrm{Re}(\tilde{K}_{v^A\phi^\kappa}d\phi^\kappa) = \frac{1}{2}d\mathrm{Im}\,\sigma^A\wedge d\tilde{K}_{v^A}\,, \tag{A.13}$$

which follows from the fact that $d\tilde{K}_{v^A} = 2\mathrm{Re}(\tilde{K}_{v^A\phi^\kappa}d\phi^\kappa) + \tilde{K}_{v^Av^B}dv^B$. This implies that these terms simply yield a total derivative and shows that the reduction of $\mathcal{N}=2$ supergravity of the form (A.1) indeed yields the extended form of $\mathcal{N}=(2,2)$ dilaton supergravity suggested in (3.23) coupled to the chiral multiplets with scalars ϕ^κ and twisted-chiral multiplets with scalars σ^A. Interestingly, we had to employ the conditions (A.10) and (A.11), which hints to the fact that the action (3.23) might admit further interesting extensions.

Let us end this appendix by pointing out that we could also have first dualized the vectors A^A to real scalars in three dimensions and then performed the circle reduction. The dual multiplets to the vector multiplets (L^A, A^A) are three-dimensional chiral multiplets with bosonic parts being complex scalars T_A given by

$$T_A = \partial_{L^A} \tilde{K} + i\rho_A \ . \tag{A.14}$$

The metric is determined now from a proper Kähler potential given by

$$\mathbf{K}(T + \bar{T}, M) = K - \mathrm{Re}\, T_A\, L^A \, , \tag{A.15}$$

such that the final action reads

$$S^{(3)} = \int \frac{1}{2} R^{(3)} * 1 - \mathbf{K}_{M^I \bar{M}^J} dM^I \wedge * d\bar{M}^J \tag{A.16}$$

with $M^I = (T_A, \phi^\kappa)$. We can again reduce this theory on a circle (A.2) and perform a Weyl-rescaling (A.4) to find

$$S^{(2)} = \int \frac{1}{2} r R * 1 + dr \wedge * d\omega - r \mathbf{K}_{M^I \bar{M}^J} dM^I \wedge * d\bar{M}^J \ . \tag{A.17}$$

With the choices $r = e^{-2\tilde{\varphi}}$ and $\omega = -\tilde{\varphi}$ this reads

$$S^{(2)} = \int e^{-2\tilde{\varphi}} \left(\frac{1}{2} R * 1 + 2 d\tilde{\varphi} \wedge * d\tilde{\varphi} - \mathbf{K}_{M^I \bar{M}^J} dM^I \wedge * d\bar{M}^J \right) \ . \tag{A.18}$$

This result should also be obtainable from (A.8) by dualizing the chiral multiplets with scalars σ^A. This is possible since b^A appears only with its field-strength db^A. The details of this dualization in two dimensions will be discussed in appendix B.

B Twisted-chiral to chiral dualization in two dimensions

In this appendix we present the details of the dualization discussed in section 3.3 of a twisted-chiral multiplet to a chiral multiplet in two dimensions. The starting point is the action

$$\begin{aligned} S^{(2)}_{\text{C-TC}} = \int e^{-2\tilde{\varphi}} \Big(& \frac{1}{2} R * 1 + 2 d\tilde{\varphi} \wedge * d\tilde{\varphi} - \tilde{K}_{\phi^\kappa \bar{\phi}^\lambda}\, d\phi^\kappa \wedge * d\bar{\phi}^\lambda + \tilde{K}_{\sigma^A \bar{\sigma}^B}\, d\sigma^A \wedge * d\bar{\sigma}^B \\ & - \tilde{K}_{\phi^\kappa \bar{\sigma}^B}\, d\phi^\kappa \wedge d\bar{\sigma}^B - \tilde{K}_{\sigma^A \bar{\phi}^\lambda}\, d\bar{\phi}^\lambda \wedge d\sigma^A \Big) \, , \end{aligned} \tag{B.1}$$

where $\tilde{K}$ is given by

$$\tilde{K} = \mathcal{K} + e^{2\tilde{\varphi}} \mathcal{S} \ . \tag{B.2}$$

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In the following we use sub-scripts to indicate derivatives with respect to fields, e.g. $\tilde{K}_{\phi^\kappa} \equiv \partial_{\phi^\kappa}\tilde{K}$. $\tilde{K}$ depends on a number of chiral multiplets with complex scalars ϕ^κ and a number of twisted-chiral multiplets with complex scalars σ^A.

In order to perform a dualization, we assume that $\mathrm{Re}\,\sigma^A$ has a shift symmetry and only appears via $d\,\mathrm{Re}\,\sigma^A$ in (B.1). This implies that $\mathrm{Re}\,\sigma^A$ can be dualized into a scalar ρ_A by the standard procedure. One first replaces $d\mathrm{Re}\,\sigma^A \to F^A$ in (B.1) and then adds a Lagrange multiplier term promotional to $F^A \wedge d\rho_A$. Then F^A can be consistently eliminated from (B.1). Denoting the imaginary part of σ^A by $v^A = \mathrm{Im}\,\sigma^A$ the resulting action reads

$$S_{\mathrm{C}}^{(2)} = \int e^{-2\tilde{\varphi}}\Big(\frac{1}{2}R*1 + 2d\tilde{\varphi}\wedge *d\tilde{\varphi} - \tilde{K}_{\phi^\kappa\bar{\phi}^\lambda}\, d\phi^\kappa \wedge *d\bar{\phi}^\lambda + \frac{1}{4}\tilde{K}_{v^A v^B}\, dv^A \wedge *dv^B \\ + \tilde{K}^{v^A v^B}\big(e^{2\tilde{\varphi}}d\rho_A - \mathrm{Im}\,(\tilde{K}_{v^A\phi^\kappa}d\phi^\kappa)\big)\wedge *\big(e^{2\tilde{\varphi}}d\rho_B - \mathrm{Im}\,(\tilde{K}_{v^B\phi^\lambda}d\phi^\lambda)\big)\Big) \tag{B.3}$$

To compute the dualized action we make the following ansatz for the Legendre transformed variables T_A

$$T_A = e^{-2\tilde{\varphi}}\frac{\partial\tilde{K}}{\partial v^A} + i\rho_A = e^{-2\tilde{\varphi}}\frac{\partial\mathcal{K}}{\partial v^A} + \frac{\partial\mathcal{S}}{\partial v^A} + i\rho_A\,, \tag{B.4}$$

and the dual potential $\mathbf{K}$

$$\mathbf{K} = \tilde{K} - e^{2\tilde{\varphi}}\,\mathrm{Re}\,T_A\, v^A\,. \tag{B.5}$$

We want to derive the conditions on $\tilde{K}$ under which the action (B.3) can be brought to the form

$$S_{\mathrm{C}}^{(2)} = \int e^{-2\tilde{\varphi}}\Big(\frac{1}{2}R*1 + 2d\tilde{\varphi}\wedge *d\tilde{\varphi} - \mathbf{K}_{M^I\bar{M}^J}\, dM^I \wedge *d\bar{M}^J\Big)\,, \tag{B.6}$$

with $M^I = (\phi^\kappa, T_A)$.

We first determine from (B.4) and (B.5) that

$$\begin{aligned} \frac{\partial v^A}{\partial T_B} &= \frac{1}{2}e^{2\tilde{\varphi}}\tilde{K}^{v^A v^B}\,, & \frac{\partial v^A}{\partial\phi^\kappa} &= -\tilde{K}^{v^A v^B}\tilde{K}_{v^B\phi^\kappa}\,, \\ \mathbf{K}_{T_A} &= -\frac{1}{2}e^{2\tilde{\varphi}}v^A\,, & \mathbf{K}_{\phi^\kappa} &= \tilde{K}_{\phi^\kappa}\,, \end{aligned} \tag{B.7}$$

where $\tilde{K}^{v^A v^B}$ is the inverse of $\tilde{K}_{v^A v^B} \equiv \partial_{v^A}\partial_{v^B}\tilde{K} = 4\tilde{K}_{\sigma^A\bar{\sigma}^B}$. Crucially, one also derives from (B.4) that

$$d\mathrm{Re}T_A = e^{-2\tilde{\varphi}}\big(\tilde{K}_{v^A v^B}dv^A + 2\mathrm{Re}(\tilde{K}_{v^A\phi^\kappa}d\phi^\kappa) - 2\mathcal{K}_{v^A}d\tilde{\varphi}\big)\,. \tag{B.8}$$

Note that there is the additional $d\tilde{\varphi}$-term, which is absent in the standard dualization procedure. The conditions on $\tilde{K}$ arise from demanding that the dual action can be brought to the form (B.1) and no additional mixed terms involving $d\tilde{\varphi}$ appear. To evaluate (B.1) one uses (B.7) to derive the identities

$$\begin{aligned} \mathbf{K}_{T_A\bar{T}_B} &= -\frac{1}{4}e^{4\tilde{\varphi}}\tilde{K}^{v^A v^B}\,, \qquad \mathbf{K}_{T_A\bar{\phi}^\kappa} = \frac{1}{2}e^{2\tilde{\varphi}}\tilde{K}^{v^A v^B}\tilde{K}_{v^B\bar{\phi}^\kappa}\,, \\ \mathbf{K}_{\phi^\kappa\bar{\phi}^\lambda} &= \tilde{K}_{\phi^\kappa\bar{\phi}^\lambda} - \tilde{K}_{\phi^\kappa v^A}\tilde{K}^{v^A v^B}\tilde{K}_{v^B\bar{\phi}^\lambda}\,. \end{aligned} \tag{B.9}$$

Inserting (B.8), (B.9) into (B.6) one finds the following terms involving $d\tilde{\varphi}$

$$S^{(2)}_{d\tilde{\varphi}} = \int e^{-2\tilde{\varphi}}\Big(\big(2 + \mathcal{K}_{v^A}\mathcal{K}^{v^A v^B}\mathcal{K}_{v^B}\big)d\tilde{\varphi}\wedge *d\tilde{\varphi} + \mathcal{K}_{v^A}dv^A \wedge *d\tilde{\varphi}\Big)\ . \tag{B.10}$$

These terms can be removed by a Weyl rescaling of the three-dimensional metric if certain conditions on $\mathcal{K}$ are satisfied. To see this, we perform a Weil rescaling

$$\tilde{g}_{\mu\nu} = e^{2\omega}g_{\mu\nu} \tag{B.11}$$

which transforms the Einstein-Hilbert term as

$$\int e^{-2\tilde{\varphi}}\frac{1}{2}\tilde{R}\ \tilde{*}1 = \int e^{-2\tilde{\varphi}}\left(\frac{1}{2}R\ *1 - 2d\omega\wedge *d\tilde{\varphi}\right)\ , \tag{B.12}$$

while leaving all other terms invariant. Hence we can absorb the extra terms in (B.10) by a Weyl rescaling iff

$$-2d\omega = \mathcal{K}_{v^A}\mathcal{K}^{v^A v^B}\mathcal{K}_{v^B}d\tilde{\varphi} + \mathcal{K}_{v^A}dv^A\ . \tag{B.13}$$

Clearly, a simple solution to this equation is found if $\mathcal{K}$ satisfies

$$\mathcal{K}_{v^A}\mathcal{K}^{v^A v^B}\mathcal{K}_{v^B} = k\,, \qquad \mathcal{K} = \mathcal{K}_1(\phi,\bar{\phi}) + \mathcal{K}_2(v)\,, \tag{B.14}$$

for a constant k, a function $\mathcal{K}_1(\phi,\bar{\phi})$ independent of v^A, and a function $\mathcal{K}_2(v)$ independent of ϕ^κ. In this case one can chose

$$\omega = -\frac{k}{2}\tilde{\varphi} - \frac{1}{2}\mathcal{K}_2(v)\ . \tag{B.15}$$

Note that (B.14) is satisfied for the result found in a Calabi-Yau fourfold reduction (3.26), i.e. $k=-4$ and $\mathcal{K}_2 = \log\mathcal{V}$.

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