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Research Article

New Dilated LMI Characterization for the Multiobjective Full-Order Dynamic Output Feedback Synthesis Problem

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This paper introduces new dilated LMI conditions for continuous-time linear systems which not only characterize stability and H_2 performance specifications, but also, H_∞ performance specifications. These new conditions offer, in addition to new analysis tools, synthesis procedures that have the advantages of keeping the controller parameters independent of the Lyapunov matrix and offering supplementary degrees of freedom. The impact of such advantages is great on the multiobjective full-order dynamic output feedback control problem as the obtained dilated LMI conditions always encompass the standard ones. It follows that much less conservatism is possible in comparison to the currently used standard LMI based synthesis procedures. A numerical simulation, based on an empirically abridged search procedure, is presented and shows the advantage of the proposed synthesis methods.

1. Introduction

The impact of linear matrix inequalities on the systems community has been so great that it dramatically changed forever the usually utilized tools for analyzing and synthesizing control systems. The standard LMI conditions benefited greatly from breakthrough advances in convex optimization theory and offered new solutions to many analysis and synthesis problems [1–3]. When necessary and sufficient LMI conditions are not possible, as it is the case for the static output control [4, 5], the multi-objective control [6–8], or the robust control [9–12] problems, sufficient conditions were provided, but were known to be overly conservative. Some dilated versions of LMI conditions have first appeared in the literature in [13], wherein some robust dilated LMI conditions were proposed for some class of matrices. Since then, a flurry of results has been proposed in both the continuous-time [6, 7, 10, 14–17] and the discrete-time systems [5, 14, 18–20]. These conditions offer, though, no particular advantages for monoobjective and precisely known systems, but were found to greatly reduce conservatism in the multi-objective [6–8, 19] and the robust control problems [9, 10, 14–16, 18, 19]. In this respect, an interesting extension for the utilization of these dilated LMI conditions (as in, e.g., [21–23]) provided solutions to the problem of robust root-clustering analysis in some nonconnected regions with respect to polytopic and norm-bounded uncertainties. Generally, the main feature of these LMI conditions, in their dilated versions, consists in the introduction of an instrumental variable giving a suitable structure, from the synthesis viewpoint, in which the controller parameterization is completely independent from the Lyapunov matrix. A particular difficulty though with these proposed dilated versions in the continuous-time case is the absence of dilated H_∞ conditions as it is visible in [6, 15].

This paper introduces new dilated LMIs conditions for the design of full-order dynamic output feedback controllers in continuous-time linear systems, which not only characterize stability and H_2 performance specifications, but also, H_∞ performance specifications as well. Similarly to the existing dilated versions, these new dilated LMI conditions carry the same properties wherein an instrumental variable is introduced giving a suitable structure in which the controller parameterization is completely independent from the Lyapunov matrix. In addition, scalar parameters are also introduced, within these dilated LMI, to provide a supplementary degree of freedom whose impact is to further reduce, in a significant way, the conservatism in sufficient standard LMI conditions. It is also shown, in this paper, that the obtained dilated LMI conditions always encompass the standard ones. As a result, the conservatism which results whenever the standard LMI conditions are used is expected to considerably reduce in many cases. This paper focuses on the multi-objective full-order dynamic output feedback controller design in continuous-time linear systems for which the constraining necessity of using a single Lyapunov matrix to test all the objectives and all the channels, which constitutes a major source of conservatism, is no longer a necessity as a different Lyapunov matrix is separately searched for every objective and for every channel. It is shown, in this paper, that despite constraining the instrumental variable, the new dilated LMI conditions are, at worst, as good as the standard ones, and, generally, much less conservative than the standard LMI conditions. The proposed solution is quite interesting, despite an inevitable increase in the number of decision variables in the involved LMIs and a multivariable search procedure that could be abridged through empirical observations. A numerical simulation is presented and shows the advantage of the proposed synthesis method.

2. Background

Consider the linear time-invariant continuous-time and input-free system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bw(t), \\ z(t) &= Cx(t) + Dw(t), \end{aligned} \tag{2.1}$$

where the state vector $x(t) \in R^n$, the perturbation vector $w(t) \in R^m$, and the performance vector $z(t) \in R^p$. All the matrices A, B, C, and D have appropriate dimensions. Let $H_{wz}(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = C(sI - A)^{-1}B + D$ be the system transfer matrix from input w to output z. The following two lemmas are well known (see, e.g., [1, 3]) and provide necessary and sufficient conditions for System (2.1) to be asymptotically stable under an H_2 performance constraint and a H_∞ performance constraint, respectively. These standard results will be extensively used in this paper.

Lemma 2.1. *System* (2.1) *with* $D = 0$ *is asymptotically stable and* $\|H_{wz}(s)\|_2^2 < \gamma_{H2}$ *if and only if there exist symmetric matrices* $X_{H2} \in R^{n\times n}$ *and* $W \in R^{m\times m}$ *such that*

$$\begin{gathered} \operatorname{Trace}(W) < \gamma_{H2}, \\ \begin{bmatrix} X_{H2} & B \\ * & W \end{bmatrix} > 0, \\ \begin{bmatrix} \operatorname{Sym}\{AX_{H2}\} & X_{H2}C^T \\ * & -I \end{bmatrix} < 0. \end{gathered} \tag{2.2}$$

Lemma 2.2. *System* (2.1) *is asymptotically stable and* $\|H_{wz}(s)\|_\infty^2 < \gamma_{H\infty}$ *if and only if there exists a symmetric matrix* $X_{H\infty} > 0$ *in* $R^{n\times n}$ *such that*

$$\begin{bmatrix} \operatorname{Sym}\{AX_{H\infty}\} & X_{H\infty}C^T & B \\ * & -I & D \\ * & * & -\gamma_{H\infty} I \end{bmatrix} < 0. \tag{2.3}$$

3. Multiobjective Control Synthesis

Consider the continuous-time time-invariant linear system with input

$$\begin{aligned} \dot{x} &= Ax + B_w w + B_u u, \\ z &= C_z x + D_{zw} w + D_{zu} u, \\ y &= C_y x + D_{yw} w, \end{aligned} \tag{3.1}$$

where the state vector $x(t) \in R^n$, the perturbation vector $(t) \in R^m$, the input command vector $u(t) \in R^q$, the performance vector $z(t) \in R^p$, and the controlled output vector $y(t) \in R^r$, and all the matrices A, B_w, B_u, C_z, D_{zw}, D_{zu}, C_y, and D_{yw} have the appropriate dimensions. In the dynamic output feedback case, the control law is given by the state equations

$$\begin{aligned} \dot{\eta} &= \Lambda\eta + \Gamma y, \\ u &= \Phi\eta. \end{aligned} \tag{3.2}$$

As this controller is supposed to be of a full order n, $\Lambda \in R^{n\times n}$, $\Gamma \in R^{n\times r}$, and $\Phi \in R^{q\times n}$. The closed-loop system is then described by the augmented state equations

$$\begin{bmatrix} \dot{x} \\ \dot{\eta} \end{bmatrix} = A_{\mathrm{Cl}} \begin{bmatrix} x \\ \eta \end{bmatrix} + B_{\mathrm{Cl}} w, \qquad z = C_{\mathrm{Cl}} \begin{bmatrix} x \\ \eta \end{bmatrix} + D_{\mathrm{Cl}} w, \tag{3.3}$$

where

$$A_{\mathrm{Cl}} = \begin{bmatrix} A & B_u\Phi \\ \Gamma C_y & \Lambda \end{bmatrix} \in R^{2n\times 2n}, \qquad B_{\mathrm{Cl}} = \begin{bmatrix} B_w \\ \Gamma D_{yw} \end{bmatrix} \in R^{2n\times m}, \qquad C_{\mathrm{Cl}} = \begin{bmatrix} C_z & D_{zu}\Phi \end{bmatrix} \in R^{p\times 2n}, \qquad D_{\mathrm{Cl}} = D_{zw} \in R^{p\times m}. \tag{3.4}$$

The closed loop system transfer matrix from input w to output z then becomes

$$H_{wz}(s) = \left[\begin{array}{c|c} A_{\mathrm{Cl}} & B_{\mathrm{Cl}} \\ \hline C_{\mathrm{Cl}} & D_{\mathrm{Cl}} \end{array}\right] = \left[\begin{array}{cc|c} A & B_u\Phi & B_w \\ \Gamma C_y & \Lambda & \Gamma D_{yw} \\ \hline C_z & D_{zu}\Phi & D_{zw} \end{array}\right]. \tag{3.5}$$

It is supposed that this system is of a multichannel type meaning that the perturbation vector w is partitioned into a given number (say I) of block components,

$$w(t) = \left[w_1^T(t) \mid \cdots \mid w_i^T(t) \mid \cdots \mid w_I^T(t) \right]^T \in R^m; \quad w_i(t) \in R^{m_i}; \quad \sum_{i=1}^{I} m_i = m, \tag{3.6}$$

and the performance vector z is partitioned into a given number (say J) of block components,

$$z(t) = \left[z_1^T(t) \mid \cdots \mid z_j^T(t) \mid \cdots \mid z_J^T(t) \right]^T \in R^p; \quad z_j(t) \in R^{p_j}; \quad \sum_{j=1}^{J} p_j = p. \tag{3.7}$$

It is supposed that some performance specifications are defined with respect to a particular channel ij (a path relating input component w_i to output component z_j) or a combination of channels. It is also supposed that, for a given control law strategy, these performance specifications can always be expressed in terms of an H_2 and/or a H_∞ transfer matrix norm of the corresponding channel, namely, $H_{w_i z_j}(s) = E_j H_{wz}(s) F_i$, where the matrices E_j and F_i are set to select the desired input/output channel from the system closed-loop transfer matrix $H_{wz}(s)$. In fact, E_j is a J-block row matrix of dimension $p_j \times p$ such that only the jth block is nonzero and is the identity matrix in R^{p_j}. Similarly, F_i is an I-block column vector of dimension $m \times m_i$ such that only the ith block is nonzero and is the identity matrix in R^{m_i}. The problem of the multi-objective controller synthesis is to construct a controller that stabilizes the closed loop system and, simultaneously, achieves all the prescribed specifications. It is easy to see that, for each channel ij, the closed loop transfer matrix is given by

$$H_{w_i z_j}(s) = E_j \left[\begin{array}{cc|c} A & B_u\Phi & B_w \\ \Gamma C_y & \Lambda & \Gamma D_{yw} \\ \hline C_z & D_{zu}\Phi & D_{zw} \end{array}\right] F_i = \left[\begin{array}{cc|c} A & B_u\Phi & B_w F_i \\ \Gamma C_y & \Lambda & \Gamma D_{yw} F_i \\ \hline E_j C_z & E_j D_{zu}\Phi & E_j D_{zw} F_i \end{array}\right]. \tag{3.8}$$

On the channel basis, the closed-loop system is then described by

$$\begin{aligned} \begin{bmatrix} \dot{x} \\ \dot{\eta} \end{bmatrix} &= A_{\mathrm{Cl},ij} \begin{bmatrix} x \\ \eta \end{bmatrix} + B_{\mathrm{Cl},ij} w_i, \\ z_j &= C_{\mathrm{Cl},ij} \begin{bmatrix} x \\ \eta \end{bmatrix} + D_{\mathrm{Cl},ij} w_i, \end{aligned} \tag{3.9}$$

where

$$\begin{gathered} A_{\mathrm{Cl},ij} = A_{\mathrm{Cl}} = \begin{bmatrix} A & B_u\Phi \\ \Gamma C_y & \Lambda \end{bmatrix} \in R^{2n\times 2n}, \qquad B_{\mathrm{Cl},ij} = B_{\mathrm{Cl}} F_i = \begin{bmatrix} B_w F_i \\ \Gamma D_{yw} F_i \end{bmatrix} \in R^{2n\times m}, \\ C_{\mathrm{Cl},ij} = E_j C_{\mathrm{Cl}} = \begin{bmatrix} E_j C_z & E_j D_{zu}\Phi \end{bmatrix} \in R^{p\times 2n}, \qquad D_{\mathrm{Cl},ij} = E_j D_{\mathrm{Cl}} F_i = E_j D_{zw} F_i \in R^{p\times m}. \end{gathered} \tag{3.10}$$

The dynamic output feedback synthesis multi-objective problem consists of looking for a dynamic controller that stabilizes the closed loop system and, in the same time, achieves the desired H_2 and/or H_∞ performance specifications for every single system channel. More specifically, the dynamic output feedback synthesis multi-objective problem aims at making System (3.1) possess the following propriety.

Propriety P

System (3.1) is stabilizable by a dynamic output feedback law (3.2) such that, for every channel ij, either or both of the following two conditions hold:

(i) $\|H_{w_i z_j}\|_2^2 < \gamma_{H2,ij}$ with $E_j D_{zw} F_i = 0$;

(ii) $\|H_{w_i z_j}\|_\infty^2 < \gamma_{H\infty,ij}$.

Theorem 3.1 (the standard sufficient conditions). *If there exist symmetric matrices $X_1 \in R^{n\times n}$ and $X_{-1} \in R^{n\times n}$, general matrices $\Lambda_1 \in R^{n\times n}$, $\Gamma_1 \in R^{n\times r}$, and $\Phi_1 \in R^{q\times n}$ and, for every channel ij, there exists a symmetric matrix $W_{ij} \in R^{m\times m}$ such that either or both of the following two conditions* *are satisfied:*

(i) [StdH2]

$$\mathrm{Trace}(W_{ij}) < \gamma_{H2,ij},$$
$$\begin{bmatrix} X_{-1} & I & X_{-1}B_w F_i + \Gamma_1 D_{yw} F_i \\ * & X_1 & B_w F_i \\ * & * & W_{ij} \end{bmatrix} > 0,$$
$$\begin{bmatrix} \mathrm{Sym}\{X_{-1}A + \Gamma_1 C_y\} & A^T + \Lambda_1 & C_z^T E_j^T \\ * & \mathrm{Sym}\{AX_1 + B_u \Phi_1\} & X_1 C_z^T E_j^T + \Phi_1^T D_{zu}^T E_j^T \\ * & * & -I \end{bmatrix} < 0; \tag{3.11}$$

(ii) [StdH∞]

$$\begin{bmatrix} X_{-1} & I \\ I & X_1 \end{bmatrix} > 0,$$
$$\begin{bmatrix} \mathrm{Sym}\{X_{-1}A + \Gamma_1 C_y\} & A^T + \Lambda_1 & C_z^T E_j^T & X_{-1}B_w F_i + \Gamma_1 D_{yw} F_i \\ * & \mathrm{Sym}\{AX_1 + B_u \Phi_1\} & X_1 C_z^T E_j^T + \Phi_1^T D_{zu}^T E_j^T & B_w F_i \\ * & * & -I & E_j D_{zw} F_i \\ * & * & * & -\gamma_{H\infty,ij}\, I \end{bmatrix} < 0, \tag{3.12}$$

then, Propriety P holds, and furthermore, a set of the controller parameters defined in (3.2) is given by

$$\begin{aligned} \Lambda &= -X_{-2}^{-1}X_{-1}AX_1X_2^{-T} - \Gamma C_y X_1 X_2^{-T} - X_{-2}^{-1}X_{-1}B_u\Phi + X_{-2}^{-1}\Lambda_1 X_2^{-T}, \\ \Gamma &= X_{-2}^{-1}\Gamma_1, \\ \Phi &= \Phi_1 X_2^{-T}, \end{aligned} \tag{3.13}$$

where the nonsingular matrices X_2 and X_{-2} are obtained *via* the equation

$$X_1 X_{-1} + X_2 X_{-2}^T = I. \tag{3.14}$$

Proof. If either or both of conditions [StdH2] and [StdH∞] are satisfied, let $X = \begin{bmatrix} X_1 & X_2 \\ X_2^T & -X_2^T X_{-1} X_{-2}^{-T} \end{bmatrix}$ and let $T = \begin{bmatrix} X_{-1} & I \\ X_{-2}^T & 0 \end{bmatrix}$ be a nonsingular transformation matrix, with X_2 and X_{-2} selected from

(3.14) (among infinitely many possibilities) via the singular value decomposition of $I - X_1 X_{-1}$. In view of (3.13) and (3.14), the following useful identities are easily derived:

$$T^T X T = \begin{bmatrix} X_{-1} & I \\ I & X_1 \end{bmatrix},$$

$$T^T A_{\mathrm{Cl}} X T = \begin{bmatrix} X_{-1} A + \Gamma_1 C_y & \Lambda_1 \\ A & A X_1 + B_u \Phi_1 \end{bmatrix},$$

$$T^T B_{\mathrm{Cl},ij} = T^T B_{\mathrm{Cl}} F_i = \begin{bmatrix} X_{-1} B_w F_i + \Gamma_1 D_{yw} F_i \\ B_w F_i \end{bmatrix}, \tag{3.15}$$

$$C_{\mathrm{Cl},ij} X T = E_j C_{\mathrm{Cl}} X T = \begin{bmatrix} E_j C_z & E_j C_z X_1 + E_j D_{zu} \Phi_1 \end{bmatrix}.$$

As either or both of conditions [StdH2] and [StdH∞] are satisfied, by the congruence lemma applied to each LMI and in view of the identities listed just above, either or both of the following conditions are also satisfied, respectively,

(i)

$$\begin{bmatrix} T^{-T} & 0 \\ 0 & I \end{bmatrix} \left[\begin{array}{cc|c} X_{-1} & I & X_{-1} B_w F_i + \Gamma_1 D_{yw} F_i \\ I & X_1 & B_w F_i \\ \hline \multicolumn{2}{c|}{*} & W_{ij} \end{array}\right] \begin{bmatrix} T^{-1} & 0 \\ 0 & I \end{bmatrix}$$

$$= \begin{bmatrix} T^{-T} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} T^T X T & T^T B_{\mathrm{Cl},ij} \\ * & W_{ij} \end{bmatrix} \begin{bmatrix} T^{-1} & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} X & B_{\mathrm{Cl},ij} \\ * & W_{ij} \end{bmatrix} > 0,$$

$$\begin{bmatrix} T^{-T} & 0 \\ 0 & I \end{bmatrix} \left[\begin{array}{cc|c} \mathrm{Sym}\{X_{-1} A + \Gamma_1 C_y\} & A^T + \Lambda_1 & C_z^T E_j^T \\ * & \mathrm{Sym}\{A X_1 + B_u \Phi_1\} & X_1 C_z^T E_j^T + \Phi_1^T D_{zu}^T E_j^T \\ \hline \multicolumn{2}{c|}{*} & -I \end{array}\right] \begin{bmatrix} T^{-1} & 0 \\ 0 & I \end{bmatrix}$$

$$= \begin{bmatrix} T^{-T} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \mathrm{Sym}\{T^T A_{\mathrm{Cl}} X T\} & T^T X C_{\mathrm{Cl},ij}^T \\ * & -I \end{bmatrix} \begin{bmatrix} T^{-1} & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} \mathrm{Sym}\{A_{\mathrm{Cl}} X\} & X C_{\mathrm{Cl},ij}^T \\ * & -I \end{bmatrix} < 0; \tag{3.16}$$

(ii)

$$T^{-T}\begin{bmatrix} X_{-1} & I \\ I & X_1 \end{bmatrix} T^{-1} = X > 0; \tag{3.17}$$

$$\begin{bmatrix} T^{-T} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$$
$$\times \left[\begin{array}{cc|c|c} \mathrm{Sym}\{X_{-1}A + \Gamma_1 C_y\} & A^T + \Lambda_1 & C_z^T E_j^T & X_{-1}B_w F_i + \Gamma_1 D_{yw} F_i \\ * & \mathrm{Sym}\{AX_1 + B_u \Phi_1\} & X_1 C_z^T E_j^T + \Phi_1^T D_{zu}^T E_j^T & B_w F_i \\ \hline \multicolumn{2}{c|}{*} & -I & E_j D_{zw} F_i \\ \hline \multicolumn{2}{c|}{*} & * & -\gamma_{H\infty,ij} I \end{array}\right]$$
$$\times \begin{bmatrix} T^{-1} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} = \begin{bmatrix} T^{-T} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} \mathrm{Sym}\{T^T A_{\mathrm{Cl}} X T\} & T^T X C_{\mathrm{Cl},ij}^T & T^T B_{\mathrm{Cl},ij} \\ * & -I & D_{\mathrm{Cl},ij} \\ * & * & -\gamma_{H\infty,ij} I \end{bmatrix} \begin{bmatrix} T^{-1} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$$
$$= \begin{bmatrix} \mathrm{Sym}\{A_{\mathrm{Cl}} X\} & X C_{\mathrm{Cl},ij}^T & B_{\mathrm{Cl},ij} \\ * & -I & D_{\mathrm{Cl},ij} \\ * & * & -\gamma_{H\infty,ij} I \end{bmatrix} < 0. \tag{3.18}$$

According to Lemmas 2.1 and 2.2, these are precisely the sufficient standard LMI conditions, expressed on a channel basis, for Propriety P to hold. □

Theorem 3.1 provides sufficient conditions for the existence of a single multi-objective dynamic output controller in terms of LMI conditions in which common Lyapunov matrices are enforced for convexity. This is known to produce, in general, overly conservative results. The following theorem attempts at reducing the effect of this limitation.

Theorem 3.2 (the dilated sufficient conditions). *If there exist general matrices $M \in R^{n\times n}$, $G_1 \in R^{n\times n}$, $G_{-1} \in R^{n\times n}$, Λ_2, Γ_2, and Φ_2 and for every channel ij, for some scalars $\alpha_{H2,ij} > 0$ and $\alpha_{H\infty,ij} > 0$, there exist symmetric matrices $V_{ij} \in R^{m_i\times m_i}$, $N_{1,H2,ij} \in R^{n\times n}$, $Y_{1,H2,ij} \in R^{n\times n}$, $N_{1,H\infty,ij} \in R^{n\times n}$, $Y_{1,H\infty,ij} \in R^{n\times n}$, general matrices $N_{2,H2,ij} \in R^{n\times n}$ and $N_{2,H\infty,ij} \in R^{n\times n}$ such that either or both of the following two conditions are satisfied:*

(i) [DilH2]

$$\mathrm{Trace}(V_{ij}) < \gamma_{H2,ij},$$
$$\begin{bmatrix} N_{1,H2,ij} & N_{2,H2,ij} & G_{-1}^T B_w F_i + \Gamma_2 D_{yw} F_i \\ * & Y_{1,H2,ij} & B_w F_i \\ * & * & V_{ij} \end{bmatrix} > 0,$$

$$
\begin{bmatrix}
\alpha_{H2,ij}\mathrm{Sym}\left\{G_{-1}^{T}A+\Gamma_{2}C_{y}\right\} & \alpha_{H2,ij}\left(\Lambda_{2}+A^{T}\right) & \alpha_{H2,ij}C_{z}^{T}E_{j}^{T} & N_{1,H2,ij}+G_{-1}^{T}A+\Gamma_{2}C_{y}-\alpha_{H2,ij}G_{-1} & N_{2,H2,ij}+\Lambda_{2}-\alpha_{H2,ij}I \\
* & \alpha_{H2,ij}\mathrm{Sym}\left\{AG_{1}+B_{u}\Phi_{2}\right\} & \alpha_{H2,ij}\left(G_{1}^{T}C_{z}^{T}E_{j}^{T}+\Phi_{2}^{T}D_{zu}^{T}E_{j}^{T}\right) & N_{2,H2,ij}^{T}+A-\alpha_{H2,ij}\ M^{T} & \Upsilon_{1,H2,ij}+AG_{1}+B_{u}\Phi_{2}-\alpha_{H2,ij}G_{1}^{T} \\
* & * & -I & E_{j}C_{z} & E_{j}C_{z}G_{1}+E_{j}D_{zu}\Phi_{2} \\
* & * & * & -\mathrm{Sym}\left\{G_{-1}\right\} & -I-M \\
* & * & * & * & -\mathrm{Sym}\left\{G_{1}\right\}
\end{bmatrix}<0; \tag{3.19}
$$

(ii) [DilH∞]

$$
\begin{bmatrix}
\alpha_{H\infty,ij}\mathrm{Sym}\left\{G_{-1}^{T}A+\Gamma_{2}C_{y}\right\} & \alpha_{H\infty,ij}\left(\Lambda_{2}+A^{T}\right) & \alpha_{H\infty,ij}C_{z}^{T}E_{j}^{T} & G_{-1}^{T}B_{w}F_{i}+\Gamma_{2}D_{yw}F_{i} & N_{1,H\infty,ij}+G_{-1}^{T}A+\Gamma_{2}C_{y}-\alpha_{H\infty,ij}G_{-1} & N_{2,H\infty,ij}+\Lambda_{2}-\alpha_{H\infty,ij}I \\
* & \alpha_{H\infty,ij}\mathrm{Sym}\left\{AG_{1}+B_{u}\Phi_{2}\right\} & \alpha_{H\infty,ij}\left(G_{1}^{T}C_{z}^{T}E_{j}^{T}+\Phi_{2}^{T}D_{zu}^{T}E_{j}^{T}\right) & B_{w}F_{i} & N_{2,H\infty,ij}^{T}+A-\alpha_{H\infty,ij}M^{T} & \Upsilon_{1,H\infty,ij}+AG_{1}+B_{u}\Phi_{2}-\alpha_{H\infty,ij}G_{1}^{T} \\
* & * & -I & E_{j}D_{zw}F_{i} & E_{j}C_{z} & E_{j}C_{z}G_{1}+E_{j}D_{zu}\Phi_{2} \\
* & * & * & -\gamma_{H\infty,ij}I & 0 & 0 \\
* & * & * & * & -\mathrm{Sym}\left\{G_{-1}\right\} & -I-M \\
* & * & * & * & * & -\mathrm{Sym}\left\{G_{1}\right\}
\end{bmatrix}<0. \tag{3.20}
$$

Then, Propriety P holds, and furthermore, a set of the controller parameters defined in (3.2) *is given by*

$$\Lambda = -G_{-3}^{-T}G_{-1}^{T}AG_1G_3^{-1} - G_{-3}^{-T}G_{-1}^{T}B_u\Phi - \Gamma C_y G_1 G_3^{-1} + G_{-3}^{-T}\Lambda_2 G_3^{-1},$$
$$\Gamma = G_{-3}^{-T}\Gamma_2, \tag{3.21}$$
$$\Phi = \Phi_2 G_3^{-1},$$

where the nonsingular matrices G_3 and G_{-3} are obtained via the equation

$$M = G_{-1}^{T}G_1 + G_{-3}^{T}G_3. \tag{3.22}$$

Proof. If either or both of conditions [DilH2] and [DilH∞] are satisfied, let $G = \begin{bmatrix} G_1 & (I-G_1G_{-1})G_{-3}^{-1} \\ G_3 & -G_3G_{-1}G_{-3}^{-1} \end{bmatrix}$ and let $T = \begin{bmatrix} G_{-1} & I \\ G_{-3} & 0 \end{bmatrix}$ be a nonsingular transformation matrix with G_3 and G_{-3} selected from (3.22) (among infinitely many possibilities) via the singular value decomposition of $M - G_{-1}^{T}G_1$. In view of (3.21) and (3.22), the following useful identities are easily derived:

$$T^TGT = \begin{bmatrix} G_{-1}^{T} & M \\ I & G_1 \end{bmatrix},$$

$$T^TA_{\text{Cl}}GT = \begin{bmatrix} G_{-1}^{T}A + \Gamma_2 C_y & \Lambda_2 \\ A & AG_1 + B_u\Phi_2 \end{bmatrix},$$
$$\tag{3.23}$$
$$T^TB_{\text{Cl},ij} = T^TB_{\text{Cl}}F_i = \begin{bmatrix} G_{-1}^{T}A + \Gamma_2 C_y & \Lambda_2 \\ A & AG_1 + B_u\Phi_2 \end{bmatrix},$$

$$C_{\text{Cl},ij}GT = E_jC_{\text{Cl}}GT = \begin{bmatrix} E_jC_z & E_jC_zX_1 + E_jD_{zu}\Phi_2 \end{bmatrix}.$$

On the other hand, let us introduce

$$Y_{H2,ij} = T^{-T}\begin{bmatrix} N_{1,H2,ij} & N_{2,H2,ij} \\ * & Y_{1,H2,ij} \end{bmatrix}T^{-1}, \qquad Y_{H\infty,ij} = T^{-T}\begin{bmatrix} N_{1,H\infty,ij} & N_{2,H\infty,ij} \\ * & Y_{1,H\infty,ij} \end{bmatrix}T^{-1}. \tag{3.24}$$

As either or both of conditions [DilH2] and [DilH∞] are satisfied, by the congruence Lemma applied to each LMI and in view of the identities listed just above, either or both of the following conditions are also satisfied, respectively.

(i)

$$\begin{bmatrix} T^{-T} & 0 \\ 0 & I \end{bmatrix} \left[\begin{array}{cc|c} N_{1,H2,ij} & N_{2,H2,ij} & G_{-1}^{T} B_w F_i + \Gamma_2 D_{yw} F_i \\ * & Y_{1,H2,ij} & B_w F_i \\ \hline \multicolumn{2}{c|}{*} & V_{ij} \end{array}\right] \begin{bmatrix} T^{-1} & 0 \\ 0 & I \end{bmatrix}$$

$$= \begin{bmatrix} T^{-T} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} T^{T} Y_{H2,ij} T & T^{T} B_{\mathrm{Cl},ij} \\ * & V_{ij} \end{bmatrix} \begin{bmatrix} T^{-1} & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} Y_{H2,ij} & B_{\mathrm{Cl},ij} \\ * & V_{ij} \end{bmatrix} > 0,$$

$$\begin{bmatrix} T^{-T} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & T^{-T} \end{bmatrix}$$

$$\times \left[\begin{array}{cc|c|cc} \alpha_{H2,ij}\mathrm{Sym}\{G_{-1}^{T} A + \Gamma_2 C_y\} & \alpha_{H2,ij}(\Lambda_2 + A^{T}) & \alpha_{H2,ij} C_z^{T} E_j^{T} & N_{1,H2,ij} + G_{-1}^{T} A + \Gamma_2 C_y - \alpha_{H2,ij} G_{-1} & N_{2,H2,ij} + \Lambda_2 - \alpha_{H2,ij} I \\ * & \alpha_{H2,ij}\mathrm{Sym}\{A G_1 + B_u \Phi_2\} & \alpha_{H2,ij}\left(G_1^{T} C_z^{T} E_j^{T} + \Phi_2^{T} D_{zu}^{T} E_j^{T}\right) & N_{2,H2,ij}^{T} + A - \alpha_{H2,ij} M^{T} & Y_{1,H2,ij} + A G_1 + B_u \Phi_2 - \alpha_{H2,ij} G_1^{T} \\ \hline * & * & -I & E_j C_z & E_j C_z G_1 + E_j D_{zu} \Phi_2 \\ \hline * & * & * & -\mathrm{Sym}\{G_{-1}\} & -I - M \\ * & * & * & * & -\mathrm{Sym}\{G_1\} \end{array}\right]$$

$$\times \begin{bmatrix} T^{-1} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & T^{-1} \end{bmatrix}$$

$$= \begin{bmatrix} T^{-T} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & T^{-T} \end{bmatrix} \begin{bmatrix} \alpha_{H2,ij}\mathrm{Sym}\{T^{T} A_{\mathrm{Cl}} G T\} & \alpha_{H2,ij} T^{T} G^{T} C_{\mathrm{Cl},ij}^{T} & T^{T}(Y_{H2,ij} + A_{\mathrm{Cl}} G - \alpha_{H2,ij} G^{T}) T \\ 0 & -I & C_{\mathrm{Cl},ij} G T \\ 0 & 0 & -T^{T}\mathrm{Sym}\{G\} T \end{bmatrix}$$

$$\times \begin{bmatrix} T^{-1} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & T^{-1} \end{bmatrix} = \begin{bmatrix} \alpha_{H2,ij}\mathrm{Sym}\{A_{\mathrm{Cl}} G\} & \alpha_{H2,ij} G^{T} C_{\mathrm{Cl},ij}^{T} & (Y_{H2,ij} + A_{\mathrm{Cl}} G - \alpha_{H2,ij} G^{T}) \\ 0 & -I & C_{\mathrm{Cl},ij} G \\ 0 & 0 & -\mathrm{Sym}\{G\} \end{bmatrix} < 0; \tag{3.25}$$

(ii)

$$
\begin{bmatrix} T^{-T} & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & T^{-T} \end{bmatrix}
$$

$$
\times \left[\begin{array}{cc|c|c|cc}
\alpha_{H\infty,ij}\mathrm{Sym}\{G_{-1}^{T}A+\Gamma_2 C_y\} & \alpha_{H\infty,ij}(\Lambda_2+A^{T}) & \alpha_{H\infty,ij}C_z^{T}E_j^{T} & G_{-1}^{T}B_wF_i+\Gamma_2 D_{yw}F_i & N_{1,H\infty,ij}+G_{-1}^{T}A+\Gamma_2 C_y-\alpha_{H\infty,ij}G_{-1} & N_{2,H\infty,ij}+\Lambda_2-\alpha_{H\infty,ij}I \\
* & \alpha_{H\infty,ij}\mathrm{Sym}\{AG_1+B_u\Phi_2\} & \alpha_{H\infty,ij}\left(G_1^{T}C_z^{T}E_j^{T}+\Phi_2^{T}D_{zu}^{T}E_j^{T}\right) & B_wF_i & N_{2,H\infty,ij}^{T}+A-\alpha_{H\infty,ij}\ M^{T} & \Upsilon_{1,H\infty,ij}+AG_1+B_u\Phi_2-\alpha_{H\infty,ij}G_1^{T} \\
\hline
* & * & -I & E_jD_{zw}F_i & E_jC_z & E_jC_zG_1+E_jD_{zu}\Phi_2 \\
\hline
* & * & * & -\gamma_{H\infty,ij}I & 0 & 0 \\
\hline
* & * & * & * & -\mathrm{Sym}\{G_{-1}\} & -I-M \\
* & * & & * & * & -\mathrm{Sym}\{G_1\}
\end{array}\right]
$$

$$
\times \begin{bmatrix} T^{-1} & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & T^{-1} \end{bmatrix}
$$

$$
= \begin{bmatrix} T^{-T} & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & T^{-T} \end{bmatrix}
$$

$$
\times \begin{bmatrix}
\alpha_{H\infty,ij}T^{T}\mathrm{Sym}\{A_{\mathrm{Cl}}G\}T & \alpha_{H\infty,ij}T^{T}G^{T}C_{\mathrm{Cl},ij}^{T} & T^{T}B_{\mathrm{Cl},ij} & T^{T}(\Upsilon_{H\infty,ij}+A_{\mathrm{Cl}}G-\alpha_{H\infty,ij}G^{T})T \\
* & -I & D_{\mathrm{Cl},ij} & C_{\mathrm{Cl},ij}GT \\
* & * & -\gamma_{H\infty,ij}I & 0 \\
* & * & * & -T^{T}\mathrm{Sym}\{G\}T
\end{bmatrix}
$$

$$
\times \begin{bmatrix} T^{-1} & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & T^{-1} \end{bmatrix}
$$

$$= \begin{bmatrix} \alpha_{H\infty,ij}\mathrm{Sym}\{A_{\mathrm{Cl}}G\} & \alpha_{H\infty,ij}G^T C_{\mathrm{Cl},ij}^T & B_{\mathrm{Cl},ij} & \Upsilon_{H\infty,ij} + A_{\mathrm{Cl}}G - \alpha_{H\infty,ij}G^T \\ \mathrm{ef22}* & -I & D_{\mathrm{Cl},ij} & C_{\mathrm{Cl},ij}G \\ * & * & -\gamma_{H\infty,ij}I & 0 \\ * & * & * & -\mathrm{Sym}\{G\} \end{bmatrix} < 0. \tag{3.26}$$

To summarize, we have proven that if either or both conditions [DilH2] and [DilH∞] are satisfied, then either or both of the following conditions are also satisfied:

(i)

$$\mathrm{Trace}(V_{ij}) < \gamma_{H2,ij},$$

$$\begin{bmatrix} \Upsilon_{H2,ij} & B_{\mathrm{Cl},ij} \\ * & V_{ij} \end{bmatrix} > 0, \tag{3.27}$$

$$\begin{bmatrix} \alpha_{H2,ij}\mathrm{Sym}\{A_{\mathrm{Cl}}G\} & \alpha_{H2,ij}G^T C_{\mathrm{Cl},ij}^T & (\Upsilon_{H2,ij} + A_{\mathrm{Cl}}G - \alpha_{H2,ij}G^T) \\ 0 & -I & C_{\mathrm{Cl},ij}G \\ 0 & 0 & -\mathrm{Sym}\{G\} \end{bmatrix} < 0;$$

(ii)

$$\begin{bmatrix} \alpha_{H\infty,ij}\mathrm{Sym}\{A_{\mathrm{Cl}}G\} & \alpha_{H\infty,ij}G^T C_{\mathrm{Cl},ij}^T & B_{\mathrm{Cl},ij} & \Upsilon_{H\infty,ij} + A_{\mathrm{Cl}}G - \alpha_{H\infty,ij}G^T \\ * & -I & D_{\mathrm{Cl},ij} & C_{\mathrm{Cl},ij}G \\ * & * & -\gamma_{H\infty,ij}I & 0 \\ * & * & * & -\mathrm{Sym}\{G\} \end{bmatrix} < 0. \tag{3.28}$$

The third LMI of the first item condition is equivalent to

$$\begin{bmatrix} 0 & 0 & \Upsilon_{H2,ij} \\ * & -I & 0 \\ * & * & 0 \end{bmatrix} + \mathrm{Sym}\left\{ \begin{bmatrix} A_{\mathrm{Cl}} \\ C_{\mathrm{Cl},ij} \\ -I \end{bmatrix} G \begin{bmatrix} \alpha_{H2,ij}I & 0 & I \end{bmatrix} \right\} < 0 \tag{3.29}$$

which, according to the elimination lemma [3], leads to

$$\begin{bmatrix} I & 0 & A_{\mathrm{Cl}} \\ 0 & I & C_{\mathrm{Cl},ij} \end{bmatrix} \begin{bmatrix} 0 & 0 & Y_{H2,ij} \\ * & -I & 0 \\ * & * & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & I \\ A_{\mathrm{Cl}}^T & C_{\mathrm{Cl},ij}^T \end{bmatrix} < 0,$$
$$\begin{bmatrix} I & 0 & -\alpha_{H2,ij} I \\ 0 & I & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & Y_{H2,ij} \\ * & -I & 0 \\ * & * & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & I \\ -\alpha_{H2,ij} I & 0 \end{bmatrix} < 0. \tag{3.30}$$

The two previous LMIs are equivalent to $\begin{bmatrix} \mathrm{Sym}\{A_{\mathrm{Cl}} Y_{H2,ij}\} & Y_{H2,ij} C_{\mathrm{Cl},ij}^T \\ * & -I \end{bmatrix} < 0$ and $\begin{bmatrix} -2\alpha_{H2,ij} Y_{H2,ij} & 0 \\ * & -I \end{bmatrix} < 0$, that is, for any $\alpha_{H2,ij} > 0$, $Y_{H2,ij} > 0$.

Similarly, the LMI of the second item condition is equivalent to

$$\begin{bmatrix} 0 & 0 & B_{\mathrm{Cl},ij} & Y_{H\infty,ij} \\ * & -I & D_{\mathrm{Cl},ij} & 0 \\ * & * & -\gamma_{H\infty,ij} I & 0 \\ * & * & * & 0 \end{bmatrix} + \mathrm{Sym}\left\{ \begin{bmatrix} A_{\mathrm{Cl}} \\ C_{\mathrm{Cl},ij} \\ 0 \\ -I \end{bmatrix} G \left[\alpha_{H\infty,ij} I \ \ 0 \ \ 0 \ \ I \right] \right\} < 0. \tag{3.31}$$

According to the Elimination lemma, this leads to

$$\begin{bmatrix} I & 0 & 0 & A_{\mathrm{Cl}} \\ 0 & I & 0 & C_{\mathrm{Cl},ij} \\ 0 & 0 & I & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & B_{\mathrm{Cl},ij} & Y_{H\infty,ij} \\ * & -I & D_{\mathrm{Cl},ij} & 0 \\ * & * & -\gamma_{H\infty,ij} I & 0 \\ * & * & * & 0 \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \\ A_{\mathrm{Cl}}^T & C_{\mathrm{Cl},ij}^T & 0 \end{bmatrix} < 0,$$
$$\begin{bmatrix} I & 0 & 0 & -\alpha_{H\infty,ij} I \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & B_{\mathrm{Cl},ij} & Y_{H\infty,ij} \\ * & -I & D_{\mathrm{Cl},ij} & 0 \\ * & * & -\gamma_{H\infty,ij} I & 0 \\ * & * & * & 0 \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \\ -\alpha_{H\infty,ij} I & 0 & 0 \end{bmatrix} < 0. \tag{3.32}$$

The previous two matrix inequalities are equivalent to

$$\begin{bmatrix} \mathrm{Sym}\{A_{\mathrm{Cl}} Y_{H\infty,ij}\} & Y_{H\infty,ij} C_{\mathrm{Cl},ij}^T & B_{\mathrm{Cl},ij} \\ * & -I & D_{\mathrm{Cl},ij} \\ * & * & -\gamma_{H\infty,ij} I \end{bmatrix} < 0, \quad \begin{bmatrix} -2\alpha_{H\infty,ij} Y_{H\infty,ij} & 0 & B_{\mathrm{Cl},ij} \\ * & -I & D_{\mathrm{Cl},ij} \\ * & * & -\gamma_{H\infty,ij} I \end{bmatrix} < 0. \tag{3.33}$$

Table 1: Simulation results, with $G_C(s)$ representing the LMI produced full-order dynamic output feedback controller.

Problem	Synthesis method	
	Standard/controller	Dilated/controller
H_2 and H_∞	$(\gamma_{H2}, \gamma_{H\infty})=(292.27, 194.67)$ $G_C(s)=\dfrac{-16.4s^2 - 96.7s - 67.1}{s^3 + 12.3s^2 + 50.7s + 73.1}$	*Two-dimensional search procedure* $(\gamma_{H2}, \gamma_{H\infty}) = (171.7, 149.9)$ with $\alpha_{H\infty} = 6$ and $\alpha_{H2} = 11$ $G_C(s) = \dfrac{-15.4s^2 - 80.2s - 6.2}{s^3 + 11.2s^2 + 40s + 46.8}$
		One-dimensional search procedure $(\gamma_{H2}, \gamma_{H\infty}) = (199.71, 147.56)$ with $\alpha = \alpha_{H\infty} = \alpha_{H2} = 4$ $G_C(s) = \dfrac{-17s^2 - 91.5s - 23.1}{s^3 + 11.8s^2 + 44s + 51}$
	Decision variable number = 30	Decision variable number = 87

Via the Schur lemma, the latter inequality is equivalent to $Y_{H\infty,ij} > 0$ and

$$\begin{bmatrix} -I & D_{\mathrm{Cl},ij} \\ * & -\gamma_{H\infty,ij} I \end{bmatrix} + \frac{\alpha_{H\infty,ij}^{-1}}{2} \times \begin{bmatrix} 0 \\ B_{\mathrm{Cl},ij}^T \end{bmatrix} Y_{H\infty,ij}^{-1} \begin{bmatrix} 0 & B_{\mathrm{Cl},ij} \end{bmatrix} < 0. \tag{3.34}$$

Clearly, as $\begin{bmatrix} -I & D_{\mathrm{Cl},ij} \\ * & -\gamma_{H\infty,ij} I \end{bmatrix} < 0$, there always exists a sufficiently large $\alpha_{H\infty,ij} > 0$ which satisfies this LMI. According to Lemmas 2.1 and 2.2, these are precisely the sufficient standard LMI conditions, expressed on a channel basis, for Propriety P to hold. □

Theorem 3.2 also provides sufficient conditions for the existence of a single multi-objective dynamic output controller in terms of LMI conditions in which the constraint of a common Lyapunov matrix is no longer needed. Convexity is rather insured by constraining the instrumental variables G to be common. This is known to produce, in general, less conservative results than those obtained with the standard conditions of Theorem 3.1. Reducing further this conservatism is also possible through the positive scalar parameters $\alpha_{H2,ij}$ and $\alpha_{H\infty,ij}$. A simple multidimensional search procedure can be carried out in the space of these parameters in order to obtain the values of these parameters for which LMI (3.19) and/or LMI (3.20) are feasible and produce the best multi-objective dynamic output controller with optimal performance levels. This multidimensional search procedure can, however, be overly expensive if the number of channel gets larger. A solution to this rather annoying limitation will be proposed in the next section. Yet, the important results of Theorem 3.2 constitute a significant contribution to the multi-objective control problem.

Next, the important question on whether or not the standard conditions could possibly be recovered by the dilated conditions will be addressed in the following section.

4. Recovery Condition

In the following theorem, it will be shown that our proposed dilated LMI conditions for the design of multiobjective full-order dynamic output feedback controllers do indeed encompass the standard conditions. This situation will be of great importance, as it will guarantee that conservatism will be almost always reduced. Similar results do exist in the literature in both the discrete-time [19] and the continuous-time case [6, 7]. The continuous-time results were, however, strictly concerned with the multi-channel H_2 synthesis problem and only in [7] that the recovery of the standard approach is proven. In view of this, the following theorem extends the discrete-time results to the continuous-time case. This point constitutes the major contribution of this paper.

Theorem 4.1. *For, the multi-objective dynamic output feedback synthesis problem, if the standard LMI conditions of Theorem 3.1 are satisfied and achieve, with a given controller, the upper bounds* $\gamma^S_{H2,ij}$ *and* $\gamma^S_{H\infty,ij}$*, then the dilated inequality conditions of Theorem 3.2 are also satisfied with the same controller and with the upper bounds* $\gamma^D_{H2,ij} \le \gamma^S_{H2,ij}$ *and* $\gamma^D_{H\infty,ij} \le \gamma^S_{H\infty,ij}$.

Proof. If the standard LMI conditions of Theorem 3.1 are satisfied for a given controller and achieve, for every channel, the upper bounds $\gamma^S_{H2,ij}$ and $\gamma^S_{H\infty,ij}$, then there exist symmetric matrices X and W_{ij} such that

$$\operatorname{Trace}(W_{ij}) < \gamma^S_{H2,ij},$$

$$\begin{bmatrix} X & B_{\mathrm{Cl},ij} \\ * & W_{ij} \end{bmatrix} > 0, \tag{4.1}$$

$$\begin{bmatrix} \operatorname{Sym}\{A_{\mathrm{Cl}}X\} & XC^T_{\mathrm{Cl},ij} \\ * & -I \end{bmatrix} < 0$$

and/or

$$X > 0,$$

$$\begin{bmatrix} \operatorname{Sym}\{A_{\mathrm{Cl}}X\} & XC^T_{\mathrm{Cl},ij} & B_{\mathrm{Cl},ij} \\ * & -I & D_{\mathrm{Cl},ij} \\ * & * & -\gamma^S_{H\infty,ij}I \end{bmatrix} < 0. \tag{4.2}$$

Let us prove that these standard LMI conditions imply that the dilated inequality conditions of Theorem 3.2 are also satisfied with the same controller. When expressed in terms of the system closed-loop parameters, the right-hand sides of the dilated LMI conditions of Theorem 3.2 take the following form:

$$\begin{gathered}\operatorname{Trace}(V_{ij}),\\ \begin{bmatrix}\Upsilon_{H2,ij} & B_{\mathrm{Cl},ij}\\ * & V_{ij}\end{bmatrix},\\ \begin{bmatrix}\alpha_{H2,ij}\operatorname{Sym}\{A_{\mathrm{Cl}}G\} & \alpha_{H2,ij}G^{T}C_{\mathrm{Cl},ij}^{T} & \Upsilon_{H2,ij}+A_{\mathrm{Cl}}G-\alpha_{H2,ij}G^{T}\\ * & -I & C_{\mathrm{Cl},ij}G\\ * & * & -\operatorname{Sym}\{G\}\end{bmatrix}\end{gathered} \tag{4.3}$$

and/or

$$\begin{bmatrix}\alpha_{H\infty,ij}\operatorname{Sym}\{A_{\mathrm{Cl}}G\} & \alpha_{H\infty,ij}G^{T}C_{\mathrm{Cl},ij}^{T} & B_{\mathrm{Cl},ij} & \Upsilon_{H\infty,ij}+A_{\mathrm{Cl}}G-\alpha_{H\infty,ij}G^{T}\\ * & -I & D_{\mathrm{Cl},ij} & C_{\mathrm{Cl},ij}G\\ * & * & -\gamma_{H\infty,ij}I & 0\\ * & * & * & -\operatorname{Sym}\{G\}\end{bmatrix}. \tag{4.4}$$

Let, in these matrices, $\Upsilon_{H2,ij} = \Upsilon_{H\infty,ij} = X$, $V_{ij} = W_{ij}$, $\alpha_{H2,ij} = \alpha_{H\infty,ij} = \alpha$, $\gamma_{H2,ij}^{D} = \gamma_{H2,ij}^{S}$, $\gamma_{H\infty,ij}^{D} = \gamma_{H\infty,ij}^{S}$ and $G = \alpha^{-1}X$, these right-hand sides become

$$\begin{gathered}\operatorname{Trace}(W_{ij}),\\ \begin{bmatrix}X & B_{\mathrm{Cl},ij}\\ * & W_{ij}\end{bmatrix},\\ \begin{bmatrix}\operatorname{Sym}\{A_{\mathrm{Cl}}X\} & XC_{\mathrm{Cl},ij}^{T} & \alpha^{-1}A_{\mathrm{Cl}}X\\ * & -I & \alpha^{-1}C_{\mathrm{Cl},ij}X\\ * & * & -2\alpha^{-1}X\end{bmatrix}.\end{gathered} \tag{4.5}$$

and/or

$$\begin{bmatrix}\operatorname{Sym}\{A_{\mathrm{Cl}}X\} & XC_{\mathrm{Cl},ij}^{T} & B_{\mathrm{Cl},ij} & \alpha^{-1}A_{\mathrm{Cl}}X\\ * & -I & D_{\mathrm{Cl},ij} & \alpha^{-1}C_{\mathrm{Cl},ij}X\\ * & * & -\gamma_{H\infty,ij}^{S}I & 0\\ * & * & * & -2\alpha^{-1}X\end{bmatrix}. \tag{4.6}$$

Let us prove, for these four matrices above, that the second matrix is positive definite while the third and/or the fourth matrices are both negative definite. Clearly, the standard conditions imply that

$$\mathrm{Trace}(W_{ij}) < \gamma^{S}_{H2,ij}, \qquad \begin{bmatrix} X & B_{\mathrm{Cl}}F_i \\ * & W_{ij} \end{bmatrix} > 0. \tag{4.7}$$

By virtue of the Schur complement lemma, the third matrix and/or the fourth matrix will be negative definite if and only if $X > 0$,

$$\begin{bmatrix} \mathrm{Sym}\{A_{\mathrm{Cl}}X\} & XC_{\mathrm{Cl}}^{T}E_j^{T} \\ * & -I \end{bmatrix} + \frac{\alpha^{-1}}{2} \times \begin{bmatrix} A_{\mathrm{Cl}} \\ E_jC_{\mathrm{Cl}} \end{bmatrix} X \begin{bmatrix} A_{\mathrm{Cl}} \\ E_jC_{\mathrm{Cl}} \end{bmatrix}^{T} < 0, \tag{4.8}$$

and/or

$$\begin{bmatrix} \mathrm{Sym}\{A_{\mathrm{Cl}}X\} & XC_{\mathrm{Cl}}^{T}E_j^{T} & B_{\mathrm{Cl}}F_i \\ * & -I & E_jD_{\mathrm{Cl}}F_i \\ * & * & -\gamma^{S}_{H\infty,ij}I \end{bmatrix} + \frac{\alpha^{-1}}{2} \times \begin{bmatrix} A_{\mathrm{Cl}} \\ E_jC_{\mathrm{Cl}} \\ 0 \end{bmatrix} X \begin{bmatrix} A_{\mathrm{Cl}} \\ E_jC_{\mathrm{Cl}} \\ 0 \end{bmatrix}^{T} < 0. \tag{4.9}$$

As, from the standard H_2 and H_∞ conditions,

$$\begin{bmatrix} \mathrm{Sym}\{A_{\mathrm{Cl}}X\} & XC_{\mathrm{Cl}}^{T}E_j^{T} \\ * & -I \end{bmatrix} < 0, \qquad \begin{bmatrix} \mathrm{Sym}\{A_{\mathrm{Cl}}X\} & XC_{\mathrm{Cl}}^{T}E_j^{T} & B_{\mathrm{Cl}}F_i \\ * & -I & E_jD_{\mathrm{Cl}}F_i \\ * & * & -\gamma^{S}_{H\infty,ij}I \end{bmatrix} < 0, \tag{4.10}$$

there always exists an $\alpha > 0$ which achieves, simultaneously, these two conditions. As a result, the dilated inequality conditions of Theorem 3.2 are also satisfied. This proves that the dilated LMI multi-objective conditions always encompass the standard ones. Clearly, this means that the dilated-based approach yields upper bounds that are always $\gamma^{D}_{H2,ij} \le \gamma^{S}_{H2,ij}$ and $\gamma^{D}_{H\infty,ij} \le \gamma^{S}_{H\infty,ij}$. □

Theorem 4.1 has proven that the dilated LMI conditions of Theorem 3.2 do indeed encompass the standard ones of Theorem 3.1. The multidimensional search procedure carried out in the space of the scalars $\lfloor \alpha_{H2,ij}, \alpha_{H\infty,ij} \rfloor$ being exhaustive, up to a given discretization step that could be made as small as needed, does indeed cover every region, and in particular, the region where the standard conditions are recovered and which is defined by $\alpha = \alpha_{H2,ij} = \alpha_{H\infty,ij}$, where α is greater than a minimum value $\alpha_{\min}$ defined by the two LMIs just in the proof above. In practice, the value of $\alpha_{\min}$ can be easily computed through a simple one dimensional line search procedure over these two LMIs.

On the other hand, at the light of the results of Theorem 3.2, a controller which achieves the best global performance level can be obtained through the minimization of the global objective function $\sum_{i,j} \gamma_{H\infty,ij} + \gamma_{H2,ij}$. Under this setting, it appears that optimality is always achieved very close to where all the $\alpha_{H2,ij}$ and all the $\alpha_{H\infty,ij}$ coincide. This purely empirical rule, observed with many examples we have tried, fits nicely to where the recovery of the standard conditions can be proved. In order to achieve optimality, it is then reasonable to abridge the costly multi-dimensional search procedure to a much cheaper one-dimensional search in the line $\alpha_{H2,ij} = \alpha_{H\infty,ij} = \alpha$ for all channels. In this way, this proposed simple line search procedure not only provides a near optimal solution, but achieves the recovery condition which guarantees that this solution is, at least, as good as the one provided by the standard conditions.

5. An Example

Consider the LTI unstable third-order plant

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 10 & 2 \\ -1 & 1 & 0 \\ 0 & 2 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} w + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u,$$

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ u \\ \hline x_2 \\ x_3 \\ u \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ \hline 0 \\ 0 \\ 1 \end{bmatrix} u, \tag{5.1}$$

$$y = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + 2w.$$

The system is supposed to be consisting of two channels. Channel 1 connects the perturbation vector w to the performance component z_1, while Channel 2 connects the perturbation vector w to the performance component z_2. The objective here is to find a stabilizing full-order (i.e., a third order) dynamic output feedback controller which achieves simultaneously and optimally the performance specifications $\|H_{wz_2}\|_2^2 < \gamma_{H2}$ and $\|H_{wz_1}\|_\infty^2 < \gamma_{H\infty}$, relatively to Channel 2 and Channel 1, respectively. Optimality is here defined as the minimization of $\gamma_{H2}+\gamma_{H\infty}$, giving equal importance to the two channels. The use of the dilated LMI conditions can be carried out through a search procedure in the plane $\lfloor \alpha_{H2}, \alpha_{H\infty} \rfloor$. Figure 1 is a three-dimensional plot which depicts the waveform of $\gamma_{H2} + \gamma_{H\infty}$ in that plane. This figure clearly shows that optimality is achieved close to the direction where $\alpha_{H2} = \alpha_{H\infty} = \alpha$. In this example, it is found that the minimum value of α which guarantees recovery is $\alpha_{\min} = 680$. The abridged search procedure along the line $\alpha_{H2} = \alpha_{H\infty} = \alpha$ produced a near optimal global performance of $\gamma_{H2} = 199.71$ and $\gamma_{H\infty} = 147.56$ when $\alpha = \alpha_{H2} = \alpha_{H\infty} = 4$. Clearly, in this example, improvement is being made in the region below $\alpha_{\min} = 680$ where recovery is not necessarily there. Table 1 lists the simulation results obtained with the sufficient standard LMI conditions of Theorem 3.1 and with the sufficient dilated LMI conditions of Theorem 3.2.

The advantage of using the dilated rather than the standard LMI conditions is quite visible with this example. Indeed, around a 30% improvement on H_2 and a 25% improvement

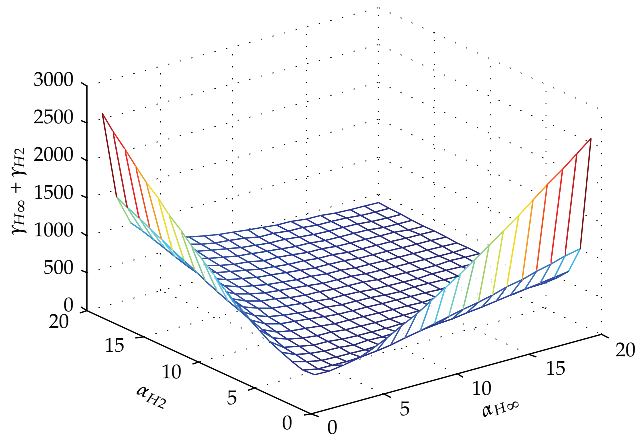

Figure 1: 3D-plot of the waveform $\gamma_{H2} + \gamma_{H\infty}$ in the plane $\lfloor \alpha_{H2}, \alpha_{H\infty} \rfloor$.

on H_∞ performance levels were possible. However, this improvement comes at the expense of almost tripling the number of decision variables involved in the proposed dilated LMI conditions (see Table 1).

6. Conclusion

This paper has presented new dilated LMI conditions for the design of multiobjective full-order dynamic output controllers in continuous-time systems that are able to cope not only with stability analysis and H_2 performance specifications, but also, with H_∞ performance specifications as well. The paper developed new controller synthesis procedures which offer no particular advantage for precisely known monoobjective systems, but significantly reduce conservatism in the multi-objective control problem, as the main property of these new dilated LMI conditions, besides the fact thatthey allow a complete independence between the standard Lyapunov matrix and the controller parametersis that they always encompass the standard ones. A numerical simulation is presented which supports these claims. The extension of these results to other control issues such as the robust controller, model predictive controller, and filter design problems is rather straightforward and yet very useful.

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