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Fixed points for cyclic φ-contractions in generalized metric spaces

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Abstract

In this paper, we obtain a fixed point theorem for mappings satisfying cyclic φ-contractive conditions in complete metric spaces, which gives a positive answer to the question raised by Radenović (Fixed Point Theory Appl. 2015:189, 2015). We also find that this result and the fixed point result satisfying cyclic weak ϕ-contractions given by Karapınar (Appl. Math. Lett. 24:822-825, 2011) are independent of each other. Furthermore, when the number of cyclic sets is odd, we obtain fixed point theorems satisfying cyclic weak ϕ-contractions and cyclic φ-contractions in the setting of generalized metric spaces.

MSC: 47H10; 54H25

Keywords: fixed point; comparison function; generalized metric space; cyclic φ-contraction; cyclic weak ϕ-contraction

1 Introduction and preliminaries

The main purpose of this paper is to answer an open question raised by Radenović in [1]. In order to go further, we attempt to extend our result and the result established by Karapınar [2, 3] to the setting of generalized metric spaces. We show these results are valid in generalized metric spaces when the number of cyclic sets is odd.

Let us recall the definition of a comparison function.

Definition 1.1 [4] A function $\varphi : [0,\infty) \to [0,\infty)$ is called a comparison function if it satisfies:

$(\text{i})_\varphi$ φ is increasing;
$(\text{ii})_\varphi$ $(\varphi^n(t))_{n\in\mathbb{N}}$ converges to 0 as $n \to \infty$, for all $t \in (0,\infty)$.

If the condition $(\text{ii})_\varphi$ is replaced by

$(\text{iii})_\varphi$ $\sum_{k=0}^{\infty} \varphi^k(t) < \infty$, for all $t \in (0,\infty)$,

then φ is called a strong comparison function.

It is clear that a strong comparison function is a comparison function, but the converse is not true.

Example 1.2 Let $\varphi : [0,\infty) \to [0,\infty)$ be defined by $\varphi(t) = \frac{t}{1+t}$. Then φ is a comparison function, but it is not a strong comparison function. In fact,

$$\varphi^n(t) = \frac{t}{1+nt},$$

for all $t > 0$. Consequently, for every $t > 0$, $(\varphi^n(t))$ converges to 0 as $n \to \infty$, but $\sum_{k=0}^{\infty} \varphi^k(t) = \infty$.

Many authors considered fixed point results about cyclic φ-contractions in setting of different type of spaces; see, for example, [1–11]. Particularly, in [1], Radenović obtained a fixed point theorem for non-cyclic φ-contraction, where φ is comparison function, and raised the following question.

Question 1.3 Prove or disprove the following.

Let $\{A_i\}_i^p$ be nonempty closed subsets of a complete metric space, and suppose $f : \bigcup_{i=1}^p A_i \to \bigcup_{i=1}^p A_i$ satisfies the following conditions (where $A_{p+1} = A_1$):

(i) $f(A_i) \subset A_{i+1}$ for $1 \le i \le p$;

(ii) there exists a comparison function $\varphi : [0,\infty) \to [0,\infty)$ such that

$$d(fx,fy) \le \varphi\bigl(d(x,y)\bigr),$$

for any $x \in A_i$, $y \in A_{i+1}$, $1 \le i \le p$.

Then f has a unique fixed point $x^* \in \bigcap_{i=1}^p A_i$ and a Picard iteration $\{x_n\}_{n\ge1}$ given by $x_n = fx_{n-1}$ converging to x^* for any starting point $x_0 \in \bigcup_{i=1}^p A_i$.

In Section 2, we give an answer to Question 1.3. In Section 3, we obtain a fixed point theorem for a mapping satisfying cyclic weak ϕ-contractions and cyclic φ-contractions in complete generalized metric spaces, where the number of cyclic sets is odd.

2 Answer of Question 1.3

We start this section by presenting the notion of cyclic φ-contraction.

Definition 2.1 Let (X,d) be a metric space, $p \in \mathbb{N}$, $A_1,\dots,A_p$ nonempty subsets of X, and $Y := \bigcup_{i=1}^p A_i$. An operator $f : Y \to Y$ is called a cyclic φ-contraction if:

(i) $\bigcup_{i=1}^p A_i$ is a cyclic representation of Y with respect to f;

(ii) there exists a comparison function $\varphi : [0,\infty) \to [0,\infty)$ such that

$$d(fx,fy) \le \varphi\bigl(d(x,y)\bigr), \tag{2.1}$$

for any $x \in A_i$, $y \in A_{i+1}$, where $A_{p+1} = A_1$.

Theorem 2.2 *Let (X,d) be a complete metric space, $p \in \mathbb{N}$, $A_1,\dots,A_p$ nonempty closed subsets of X, and $Y := \bigcup_{i=1}^p A_i$. Assume that $f : Y \to Y$ is a cyclic φ-contraction. Then f has a unique fixed point $x^* \in \bigcap_{i=1}^p A_i$ and a Picard iteration $\{x_n\}_{n\ge1}$ given by $x_n = fx_{n-1}$ converging to x^* for any starting point $x_0 \in \bigcup_{i=1}^p A_i$.*

Proof Let x_0 be an arbitrary point in Y. Define the sequence $\{x_n\}$ in Y by $x_n = fx_{n-1}$, $n = 1, 2, \ldots$. If there exists n_0 such that $x_{n_0+1} = x_{n_0}$ then $fx_{n_0} = x_{n_0+1} = x_{n_0}$ and the existence of the fixed point is proved. Consequently, we always assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$.

Step 1. We will prove that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0, \qquad \lim_{n\to\infty} d(x_n, x_{n+2}) = 0, \qquad \ldots, \qquad \lim_{n\to\infty} d(x_n, x_{n+p}) = 0. \tag{2.2}$$

Using (2.1), we have

$$d(x_n, x_{n+1}) = d(fx_{n-1}, fx_n) \leq \varphi\bigl(d(x_{n-1}, x_n)\bigr), \tag{2.3}$$

for all $n \in \mathbb{N}$. From this, we deduce that

$$d(x_n, x_{n+1}) \leq \varphi\bigl(d(x_{n-1}, x_n)\bigr) \leq \varphi^2\bigl(d(x_{n-2}, x_{n-1})\bigr) \leq \cdots \leq \varphi^n\bigl(d(x_0, x_1)\bigr).$$

Using the definition of φ, we get

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0, \tag{2.4}$$

using the triangle inequality, we have

$$d(x_n, x_{n+k}) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{n+k-1}, x_{n+k}),$$

for $k = 2, 3, \ldots, p$. Combining this and (2.4), we conclude that (2.2) holds.

Step 2. We will prove the following claim.

Claim *For every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that if $n > m > N$ with $n - m \equiv 1 \bmod p$ then $d(x_n, x_m) < \varepsilon$.*

In fact, if the claim is not true, then there exists $\varepsilon_0 > 0$ such that for any $N \in \mathbb{N}$ we can find $n > m > N$ with $n - m \equiv 1 \bmod p$ satisfying $d(x_n, x_m) \geq \varepsilon_0$. By (2.2), corresponding to this ε_0, there exists n_0 such that if $n > n_0$ then

$$d(x_n, x_{n+1}) < \varepsilon_0, \qquad d(x_n, x_{n+2}) < \varepsilon_0, \qquad \ldots, \qquad d(x_n, x_{n+p}) < \varepsilon_0. \tag{2.5}$$

Taking $N = n_0$, we can find that $n_1' > m_1 > n_0$ with $n_1' - m_1 \equiv 1 \bmod p$ such that $d(x_{n_1'}, x_{m_1}) \geq \varepsilon_0$. Due to (2.5), we can choose a $n_1 \in \{m_1 + p + 1, m_1 + 2p + 1, \ldots, n_1'\}$ in such a way that it is smallest integer satisfying $d(x_{n_1}, x_{m_1}) \geq \varepsilon_0$. Then we obtain

$$d(x_{n_1}, x_{m_1}) \geq \varepsilon_0, \qquad d(x_{n_1-p}, x_{m_1}) < \varepsilon_0 \quad \text{and} \quad n_1 - m_1 \equiv 1 \bmod p.$$

Taking $N = n_1$, we can find that $n_2' > m_2 > n_1$ with $n_2' - m_2 \equiv 1 \bmod p$ such that $d(x_{n_2'}, x_{m_2}) \geq \varepsilon_0$. Similar to the choice of n_1, we can get a $n_2 \in \{m_2 + p + 1, m_2 + 2p + 1, \ldots, n_2'\}$ such that

$$d(x_{n_2}, x_{m_2}) \geq \varepsilon_0, \qquad d(x_{n_2-p}, x_{m_2}) < \varepsilon_0 \quad \text{and} \quad n_2 - m_2 \equiv 1 \bmod p.$$

Continuing the above process, by induction, we obtain two subsequences $\{x_{m_k}\}$ and $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$d(x_{n_k}, x_{m_k}) \geq \varepsilon_0, \qquad d(x_{n_k-p}, x_{m_k}) < \varepsilon_0 \quad \text{and} \quad n_k - m_k \equiv 1 \bmod p. \tag{2.6}$$

Now, using (2.6) and the triangle inequality, we have

$$\begin{aligned} \varepsilon_0 &\leq d(x_{n_k}, x_{m_k}) \leq d(x_{n_k}, x_{n_k-p}) + d(x_{n_k-p}, x_{m_k}) \\ &\leq d(x_{n_k}, x_{n_k-p}) + \varepsilon_0. \end{aligned}$$

Letting $k \to \infty$ in the above inequality, using (2.2), we obtain

$$d(x_{n_k}, x_{m_k}) \to \varepsilon_0 \quad \text{as } k \to \infty. \tag{2.7}$$

Using the triangle inequality, we get

$$d(x_{n_k-p+1}, x_{m_k+1}) \leq d(x_{n_k-p+1}, x_{n_k}) + d(x_{n_k}, x_{m_k}) + d(x_{m_k}, x_{m_k+1})$$

and

$$d(x_{n_k-p+1}, x_{m_k+1}) \geq d(x_{n_k}, x_{m_k}) - d(x_{n_k}, x_{n_k-p+1}) - d(x_{m_k+1}, x_{m_k}).$$

Letting $k \to \infty$ in the above two inequalities, using (2.2) and (2.7), we get

$$d(x_{n_k-p+1}, x_{m_k+1}) \to \varepsilon_0 \quad \text{as } k \to \infty. \tag{2.8}$$

Now, using (2.1) and (2.6), we have

$$d(x_{n_k-p+1}, x_{m_k+1}) = d(fx_{n_k-p}, fx_{m_k}) \leq \varphi\big(d(x_{n_k-p}, x_{m_k})\big) \leq \varphi(\varepsilon_0). \tag{2.9}$$

Taking the limit in (2.9) as $k \to \infty$, from (2.8), we see

$$\varepsilon_0 \leq \varphi(\varepsilon_0),$$

which is a contradiction with $\varphi(\varepsilon_0) < \varepsilon_0$. Therefore our claim is proved.

Step 3. We will prove $\{x_n\}$ is a Cauchy sequence in X.

Let $\varepsilon > 0$ be given. Using the claim, we find that $N_1 \in \mathbb{N}$ such that if $n > m > N_1$ with $n - m \equiv 1 \bmod p$ then

$$d(x_n, x_m) < \frac{\varepsilon}{p}.$$

On the other hand, using (2.4), we also find $N_2 \in \mathbb{N}$ such that, for any $n > N_2$,

$$d(x_n, x_{n+1}) < \frac{\varepsilon}{p}.$$

Let $n, m > N = \max\{N_1, N_2\}$ with $n > m$. Then we can find a $s \in \{0, 1, 2, \dots, p-1\}$ such that $n - (m+s) \equiv 1 \bmod p$. Using the triangle inequality, we obtain

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+2}) + \cdots + d(x_{m+s}, x_n) \\ &< \frac{\varepsilon}{p} + \frac{\varepsilon}{p} + \cdots + \frac{\varepsilon}{p} \\ &= (s+1) \cdot \frac{\varepsilon}{p} \leq \varepsilon. \end{aligned}$$

This proves that $\{x_n\}$ is a Cauchy sequence.

Step 4. We will prove f has a unique fixed point $x^* \in \bigcap_{i=1}^p A_i$.

As X is a complete metric space, there exists $x \in X$ such that $\lim_{n\to\infty} x_n = x$. Using the cyclic character of f, there exists a subsequence of $\{x_n\}$ for which belongs to A_i for $i \in \{1, 2, \dots, p\}$. Hence, from A_i is closed, we see that $x \in \bigcap_{i=1}^p A_i$. Now, we consider the restriction $f|_{\bigcap_{i=1}^p A_i}$ of f on $\bigcap_{i=1}^p A_i$. Since $\bigcap_{i=1}^p A_i$ is also complete, by Theorem 2.3 in [1], we see that f has a unique fixed point x^* in $\bigcap_{i=1}^p A_i$.

Step 5. We prove that the Picard iteration converges to x^* for any initial point $x_0 \in \bigcup_{i=1}^p A_i$.

Using (2.1), we have

$$d(x_n, x^*) = d(fx_{n-1}, fx^*) \leq \varphi\bigl(d(x_{n-1}, x^*)\bigr).$$

From this, we see that

$$d(x_n, x^*) \leq \varphi\bigl(d(x_{n-1}, x^*)\bigr) \leq \varphi^2\bigl(d(x_{n-2}, x^*)\bigr) \leq \cdots \leq \varphi^n\bigl(d(x_0, x^*)\bigr).$$

Using the definition of φ, we conclude that $x_n \to x^*$ as $n \to \infty$.

This completes the proof. □

Remark 2.3 From Theorem 2.2, we see that the open question raised by Radenović (that is, Question 1.3) has been answered.

Remark 2.4 Following the idea of Radenović in [1], we see that Theorem 2.3 in [1] and Theorem 2.2 are equivalent.

3 Cyclic weak ϕ-contractions and cyclic φ-contractions in generalized metric spaces

In 2000, Branciari [12] introduced the notion of generalized metric space and proved the Banach fixed point theorem in such spaces. For more information, the reader can refer to [13–17]. For some notions and facts about generalized metric spaces, one may wish to see [12].

In [3], Karapınar gave a fixed point results satisfying cyclic weak ϕ-contractions. For convenience, we rewrite his theorem (*i.e.*, [3], Theorem 2) as the following equivalent statement.

Theorem 3.1 *Let (X, d) be a complete metric space, $p \in \mathbb{N}$, $A_1, \dots, A_p$ closed nonempty subsets of X, $Y := \bigcup_{i=1}^p A_i$ and $f : Y \to Y$ an operator. Assume that*:

(i) $\bigcup_{i=1}^{p} A_i$ *is a cyclic representation of* Y *with respect to* f;
(ii) *there exists a function* $\phi : [0,\infty) \to [0,\infty)$ *with* $\phi(t) < t$ *and* $t - \phi(t)$ *is nondecreasing for* $t \in (0,\infty)$ *and* $\phi(0) = 0$ *such that*

$$d(fx,fy) \le \phi\bigl(d(x,y)\bigr),$$

for any $x \in A_i$, $y \in A_{i+1}$, *where* $A_{p+1} = A_1$.

Then f *has a unique fixed point* $x^* \in \bigcap_{i=1}^{p} A_i$.

Based on the concept of cyclic weak ϕ-contraction, we can introduce the following notion.

Definition 3.2 A function $\phi : [0,\infty) \to [0,\infty)$ is called a (w)-comparison function if it satisfies:

(i)$_\phi$ $\phi(0) = 0$;
(ii)$_\phi$ $\phi(t) < t$, for all $t \in (0,\infty)$;
(iii)$_\phi$ the function $\psi(t) := t - \phi(t)$ is increasing, *i.e.*, $t_1 \le t_2$ implies $\psi(t_1) \le \psi(t_2)$, for $t_1, t_2 \in [0,\infty)$.

Lemma 3.3 *If* $\phi : [0,\infty) \to [0,\infty)$ *is a* (w)*-comparison function, then the following hold*:
(1) $\phi(t) \le t$, *for any* $t \in [0,\infty)$;
(2) *for* $k \ge 1$, $\phi^k(t) < t$, *for any* $t \in (0,\infty)$;
(3) $(\phi^n(t))_{n\in\mathbb{N}}$ *converges to* 0 *as* $n \to \infty$, *for all* $t \in (0,\infty)$.

Proof From the definition of ϕ, it is easy to verify that (1), (2), and (3) hold. Now, we only prove that (3) holds. Let $t \in (0,\infty)$. Then we have

$$\phi^n(t) = \phi\bigl(\phi^{n-1}(t)\bigr) \le \phi^{n-1}(t), \quad \text{for all } n \in \mathbb{N}.$$

This means that $(\phi^n(t))_{n\in\mathbb{N}}$ is a decreasing sequence of non-negative real numbers. Therefore, there exists $r \ge 0$ such that $\lim_{n\to\infty} \phi^n(t) = r$. Suppose that $r > 0$. Then $\phi(r) < r$ and $r - \phi(r) > 0$. Since $r = \inf\{\phi^n(t) : n \in \mathbb{N}\}$, $0 < r \le \phi^n(t)$, for all $n \in \mathbb{N}$. By the definition of ϕ, we get

$$r - \phi(r) \le \phi^n(t) - \phi\bigl(\phi^n(t)\bigr),$$

for all $n \in \mathbb{N}$. Letting $n \to \infty$ in the above inequality, we obtain $r - \phi(r) \le r - r = 0$ and this contradicts $r - \phi(r) > 0$. □

The next are two basic examples of the comparison function and the (w)-comparison function.

Example 3.4 Let $\phi : [0,\infty) \to [0,\infty)$ be defined by

$$\phi(t) = \begin{cases} \frac{1}{4}t, & 0 \le t < 1, \\ \frac{5}{4}t - 1, & 1 \le t < 2, \\ t - \frac{1}{2}, & t \ge 2. \end{cases}$$

Then ϕ is a comparison function. But ϕ is not a (w)-comparison function because $t - \phi(t)$ is not increasing.

Example 3.5 Let $\phi : [0, \infty) \to [0, \infty)$ be defined by

$$\phi(t) = \begin{cases} \frac{3}{4}t, & 0 \le t < 1, \\ 1 - \frac{1}{4}t, & 1 \le t < 2, \\ \frac{1}{4}t, & t \ge 2. \end{cases}$$

Then ϕ is a (w)-comparison function. But ϕ is not a comparison function because $\phi(t)$ is not increasing.

Remark 3.6 From Example 3.4 and Example 3.5, we see that the comparison function and the (w)-comparison function do not imply each other. Consequently, Theorem 2 in [3] and Theorem 2.2 are independent of each other.

Now we carry over the concept of cyclic weak ϕ-contraction to generalized metric space.

Definition 3.7 Let (X, d) be a generalized metric space, $p \in \mathbb{N}$, $A_1, \ldots, A_p$ nonempty subsets of X and $Y := \bigcup_{i=1}^{p} A_i$. An operator $f : Y \to Y$ is called a cyclic weak ϕ-contraction if:

(i) $\bigcup_{i=1}^{p} A_i$ is a cyclic representation of Y with respect to f;

(ii) there exists a (w)-comparison function $\phi : [0, \infty) \to [0, \infty)$ such that

$$d(fx, fy) \le \phi\bigl(d(x, y)\bigr), \tag{3.1}$$

for any $x \in A_i$, $y \in A_{i+1}$, where $A_{p+1} = A_1$.

Theorem 3.8 *Let (X, d) be a complete generalized metric space, p an odd number, $A_1, \ldots, A_p$ nonempty closed subsets of X and $Y := \bigcup_{i=1}^{p} A_i$. Assume that $f : Y \to Y$ is a cyclic weak ϕ-contraction. Then f has a unique fixed point $x^* \in \bigcap_{i=1}^{p} A_i$ and a Picard iteration $\{x_n\}_{n \ge 1}$ given by $x_n = fx_{n-1}$ converging to x^* for any starting point $x_0 \in \bigcup_{i=1}^{p} A_i$.*

Proof Let $x_0 \in Y$, and $x_n = fx_{n-1}$, $n = 1, 2, \ldots$. If there exists n_0 such that $x_{n_0+1} = x_{n_0}$ then $fx_{n_0} = x_{n_0+1} = x_{n_0}$ and the existence of the fixed point is proved. Consequently, we will assume that $x_n \ne x_{n+1}$ for all $n \in \mathbb{N}$.

Step 1. We will prove that $x_n \ne x_m$ for all $n \ne m$.

Suppose that $x_n = x_m$ for some $n \ne m$. Without loss of generality, we may assume that $n > m + 1$. Due to the property of ϕ, we see that

$$\begin{aligned} d(x_m, x_{m+1}) &= d(x_m, fx_m) = d(x_n, fx_n) \\ &= d(fx_{n-1}, fx_n) \\ &\le \phi\bigl(d(x_{n-1}, x_n)\bigr) \\ &\le \cdots \\ &\le \phi^{n-m}\bigl(d(x_m, x_{m+1})\bigr). \end{aligned}$$

By Lemma 3.3(2), we get $\phi^{n-m}(d(x_m, x_{m+1})) < d(x_m, x_{m+1})$, which is a contradiction.

Step 2. We will prove that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0, \qquad \lim_{n\to\infty} d(x_n, x_{n+2}) = 0, \qquad \ldots, \qquad \lim_{n\to\infty} d(x_n, x_{n+p}) = 0. \tag{3.2}$$

Using (3.1), we get

$$d(x_n, x_{n+1}) = d(fx_{n-1}, fx_n) \le \phi\bigl(d(x_{n-1}, x_n)\bigr),$$

for all $n \in \mathbb{N}$. Using the definition of ϕ, we see that

$$d(x_n, x_{n+1}) < d(x_{n-1}, x_n). \tag{3.3}$$

This implies the sequence $\{d(x_n, x_{n+1})\}$ is decreasing and bounded below. Consequently, $d(x_n, x_{n+1}) \to r$ for some $r \ge 0$. Suppose that $r > 0$. Then $\phi(r) < r$. Using the definition of ϕ and $d(x_n, x_{n+1}) \ge r$, we get

$$r - \phi(r) \le d(x_n, x_{n+1}) - \phi\bigl(d(x_n, x_{n+1})\bigr),$$

for all $n \in \mathbb{N}$. From $d(x_{n+1}, x_{n+2}) \le \phi(d(x_n, x_{n-1}))$, we see that

$$r - \phi(r) \le d(x_n, x_{n+1}) - d(x_{n+1}, x_{n+2}),$$

for all $n \in \mathbb{N}$. Letting $n \to \infty$ in the above inequality, we get $r - \phi(r) \le 0$, which is a contradiction with $\phi(r) < r$. Thus, we conclude that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{3.4}$$

Using the rectangular inequality, we get

$$d(x_n, x_{n+3}) \le d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+3}).$$

From (3.4), we see that $d(x_n, x_{n+3}) \to 0$ as $n \to \infty$. By induction, we deduce that

$$\lim_{n\to\infty} d(x_n, x_{n+k}) = 0, \quad \text{for all } k \in \{1, 3, 5, \ldots, p\}. \tag{3.5}$$

Now, we prove

$$\lim_{n\to\infty} d(x_n, x_{n+p-1}) = 0. \tag{3.6}$$

Since x_n and x_{n+p-1} lie in different adjacently labeled sets A_i and A_{i+1} for certain $i \in \{1, 2, \ldots, p\}$, from (3.1) we get

$$d(x_n, x_{n+p-1}) = d(fx_{n-1}, fx_{n+p-2}) \le \phi\bigl(d(x_{n-1}, x_{n+p-2})\bigr).$$

Similar to the proof of the conclusion (3.4), we can deduce that $\{d(x_n, x_{n+p-1})\}$ is decreasing and converges to 0. This means that (3.6) holds.

For $k = 2, 4, \dots, p-3$, using the rectangular inequality, we have

$$d(x_n, x_{n+k}) \le d(x_n, x_{n+p-1}) + d(x_{n+p-1}, x_{n+p}) + d(x_{n+p}, x_{n+k}). \tag{3.7}$$

Since $p - k$ is odd, from (3.5) we get

$$\lim_{n\to\infty} d(x_{n+p}, x_{n+k}) = \lim_{n\to\infty} d(x_{n+p-k}, x_n) = 0. \tag{3.8}$$

Therefore, from (3.5), (3.6), (3.7), and (3.8) we conclude that

$$\lim_{n\to\infty} d(x_n, x_{n+k}) = 0, \quad \text{for all } k \in \{2, 4, \dots, p-1\}. \tag{3.9}$$

Combining (3.4) and (3.9), we see (3.2) is proved.

Step 3. We will prove the following claim.

Claim *For every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that if $n > m > N$ with $n - m \equiv 1 \bmod p$ then $d(x_n, x_m) < \varepsilon$.*

In fact, if this is not true, then there exists $\varepsilon_0 > 0$ such that for any $N \in \mathbb{N}$ we can find $n > m > N$ with $n - m \equiv 1 \bmod p$ satisfying $d(x_n, x_m) \ge \varepsilon_0$. By the definition of ϕ, we get

$$\varepsilon_0 - \phi(\varepsilon_0) \le d(x_n, x_m) - \phi\bigl(d(x_n, x_m)\bigr). \tag{3.10}$$

Using (3.1), we get

$$d(x_{n+1}, x_{m+1}) \le \phi\bigl(d(x_n, x_m)\bigr). \tag{3.11}$$

By (3.10), (3.11), and the rectangular inequality, we obtain

$$\begin{aligned} \varepsilon_0 - \phi(\varepsilon_0) &\le d(x_n, x_m) - d(x_{n+1}, x_{m+1}) \\ &\le d(x_n, x_{n+1}) + d(x_{n+1}, x_{m+1}) + d(x_{m+1}, x_m) - d(x_{n+1}, x_{m+1}) \\ &= d(x_n, x_{n+1}) + d(x_{m+1}, x_m). \end{aligned}$$

From (3.3), it follows that

$$\varepsilon_0 - \phi(\varepsilon_0) \le 2d(x_{m+1}, x_m) \quad \text{and} \quad d(x_{m+1}, x_m) \ge \frac{\varepsilon_0 - \phi(\varepsilon_0)}{2} > 0.$$

Therefore, $\{d(x_{m+1}, x_m)\}$ does not converge to 0 as $m \to \infty$, which contradicts (3.4).

Step 4. We will prove $\{x_n\}$ is a Cauchy sequence in X.

Let $\varepsilon > 0$ be given. Using the claim, we find that $N_1 \in \mathbb{N}$ such that if $n > m > N_1$ with $n - m \equiv 1 \bmod p$ then

$$d(x_n, x_m) < \frac{\varepsilon}{3}.$$

On the other hand, using (3.2), we also find $N_2 \in \mathbb{N}$ such that, for any $n > N_2$,

$$d(x_n, x_{n+1}) < \frac{\varepsilon}{3}, \qquad d(x_n, x_{n+2}) < \frac{\varepsilon}{3}, \qquad \dots, \qquad d(x_n, x_{n+p}) < \frac{\varepsilon}{3}.$$

Let $n, m > N = \max\{N_1, N_2\} + 1$ with $n > m$. Then we can find $s \in \{0, 1, 2, \ldots, p-1\}$ such that $n - (m+s) \equiv 1 \bmod p$.

In the case where $s = 0$, we have

$$d(x_n, x_m) < \frac{\varepsilon}{3} < \varepsilon.$$

In the other case where $s \geq 1$, using the rectangular inequality we have

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, x_{m-1}) + d(x_{m-1}, x_{m+s}) + d(x_{m+s}, x_n) \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\ &= \varepsilon. \end{aligned}$$

This proves that $\{x_n\}$ is a Cauchy sequence.

Step 5. We will prove f has a unique fixed point $x^* \in \bigcap_{i=1}^p A_i$ and the Picard iteration $\{x_n\}$ converges to x^*.

Since X is a complete generalized metric space, there exists $x^* \in X$ such that $\lim_{n\to\infty} x_n = x^*$. Using the cyclic character of f, there exists a subsequence of $\{x_n\}$ for which belongs to A_i for $i \in \{1, 2, \ldots, p\}$. Hence, from A_i is closed for $i \in \{1, 2, \ldots, p\}$, we see that $x^* \in \bigcap_{i=1}^p A_i$. Now, we will prove $d(x_n, fx^*) \to 0$ as $n \to \infty$. In fact, using (3.1), we have

$$d(x_n, fx^*) = d(fx_{n-1}, fx^*) \leq \phi(d(x_{n-1}, x^*)) \leq d(x_{n-1}, x^*) \to 0 \quad \text{as } n \to \infty,$$

which implies $d(x_n, fx^*) \to 0$ as $n \to \infty$. Using Proposition 3 of [18], we deduce that $fx^* = x^*$, *i.e.*, x^* is a fixed point of f.

In order to prove that the uniqueness of the fixed point, we take $y, z \in Y$ such that y and z are fixed points of f. The cyclic character of f implies that $y, z \in \bigcap_{i=1}^p A_i$. Using (3.1),

$$d(y, z) = d(fy, fz) \leq \phi(d(y, z)) \leq d(y, z).$$

This means $\phi(d(y,z)) = d(y,z)$. Since $\phi(t) > 0$ for $t > 0$, we get $d(y,z) = 0$ and $y = z$. This finishes the proof. □

Theorem 3.9 *Let (X, d) be a complete generalized metric space, p an odd number, $A_1, \ldots, A_p$ nonempty closed subsets of X and $Y := \bigcup_{i=1}^p A_i$. Assume that $f : Y \to Y$ is a cyclic φ-contraction. Then f has a unique fixed point $x^* \in \bigcap_{i=1}^p A_i$ and a Picard iteration $\{x_n\}_{n\geq 1}$ given by $x_n = fx_{n-1}$ converging to x^* for any starting point $x_0 \in \bigcup_{i=1}^p A_i$.*

Proof Let $x_0 \in Y$, and $x_n = fx_{n-1}$, $n = 1, 2, \ldots$.

Similar to the Step 1 and Step 2 in the proof of Theorem 3.8, we can prove

$$x_n \neq x_m, \quad \text{for all } n \neq m$$

and

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0, \qquad \lim_{n\to\infty} d(x_n, x_{n+2}) = 0, \qquad \ldots, \qquad \lim_{n\to\infty} d(x_n, x_{n+p}) = 0. \quad (3.12)$$

Now, we will prove the following claim.

Claim *For every* $\varepsilon > 0$, *there exists* $N \in \mathbb{N}$ *such that if* $n > m > N$ *with* $n - m \equiv 1 \bmod p$ *then* $d(x_n, x_m) < \varepsilon$.

In fact, in the opposite case, similar to the Step 2 in the proof of Theorem 2.2, we can find that $\varepsilon_0 > 0$ and two subsequences $\{x_{m_k}\}$ and $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$d(x_{n_k}, x_{m_k}) \geq \varepsilon_0, \qquad d(x_{n_k-p}, x_{m_k}) < \varepsilon_0 \quad \text{and} \quad n_k - m_k \equiv 1 \bmod p. \tag{3.13}$$

Next, we only prove $d(x_{n_k}, x_{m_k}) \to \varepsilon_0$ as $k \to \infty$ because the other proof is the same as in the Step 2 of Theorem 2.2. In fact, using (3.13) and the rectangular inequality, we have

$$\begin{aligned}\varepsilon_0 &\leq d(x_{n_k}, x_{m_k}) \leq d(x_{n_k}, x_{n_k-p+1}) + d(x_{n_k-p+1}, x_{n_k-p}) + d(x_{n_k-p}, x_{m_k}) \\ &\leq d(x_{n_k}, x_{n_k-p+1}) + d(x_{n_k-p+1}, x_{n_k-p}) + \varepsilon_0.\end{aligned}$$

Letting $k \to \infty$ in the above inequality, using (3.12), we obtain

$$d(x_{n_k}, x_{m_k}) \to \varepsilon_0 \quad \text{as } k \to \infty.$$

Similar to Step 4 and Step 5 in the proof of Theorem 3.8, we can finish the proof. □

Example 3.10 Let $X = \{1, 2, 3, 4, 5\}$. Define $d : X \times X \to \infty$ by

$$d(x,y) = \begin{cases} 0, & x = y, \\ 1, & |x - y| = 1, \\ 4, & x = 3, y = 4 \text{ or } x = 4, y = 3, \\ 2, & \text{otherwise.} \end{cases}$$

Then (X, d) is a generalized metric space. But d is not a metric on X because

$$d(3,4) = 4 > 3 = d(3,2) + d(2,4).$$

Now, consider $A_1 = \{1, 2, 3\}$, $A_2 = \{3\}$, $A_3 = \{3, 4, 5\}$, and $T : X \to X$ to be defined by

$$T1 = T2 = T3 = 3 \quad \text{and} \quad T4 = T5 = 2.$$

It is easy to prove that T satisfies all the conditions of Theorem 3.8 and Theorem 3.9 with $\varphi(t) = \phi(t) = \frac{1}{2}t$. Using Theorem 3.8 or Theorem 3.9, we see that T has a unique fixed point. In fact, 3 is the unique fixed point of f. But we do not apply Theorem 2.2 or Theorem 2 in [3] because d is not a metric on X.

Finally, a natural question arises.

Question 3.11 If the number of cyclic sets is even, then we may ask whether Theorem 3.8 or Theorem 3.9 is valid or not.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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