Fixed Point Theory and Applications
a SpringerOpen Journal

RESEARCH Open Access

Hybrid iterative method for finding common solutions of generalized mixed equilibrium and fixed point problems

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Abstract

Recently, Colao et al. (J Math Anal Appl 344:340-352, 2008) introduced a hybrid viscosity approximation method for finding a common element of the set of solutions of an equilibrium problem and the set of fixed points of a finite family of nonexpansive mappings in a real Hilbert space. In this paper, by combining Colao, Marino and Xu's hybrid viscosity approximation method and Yamada's hybrid steepest-descent method, we propose a hybrid iterative method for finding a common element of the set *GMEP* of solutions of a generalized mixed equilibrium problem and the set $\bigcap_{i=1}^{N} \mathrm{Fix}\,(S_i)$ of fixed points of a finite family of nonexpansive mappings $\{S_i\}_{i=1}^{N}$ in a real Hilbert space. We prove the strong convergence of the proposed iterative algorithm to an element of $\bigcap_{i=1}^{N} \mathrm{Fix}\,(S_i) \cap GMEP$, which is the unique solution of a variational inequality.

AMS subject classifications: 49J40; 47J20; 47H09.

Keywords: Generalized mixed equilibrium problem, fixed point, nonexpansive mapping, variational inequality, hybrid iterative method.

1 Introduction

The theory of equilibrium problems has played an important role in the study of a wide class of problems arising in economics, finance, transportation, network and structural analysis, elasticity and optimization, and has numerous applications, including but not limited to problems in economics, game theory, finance, traffic analysis, circuit network analysis and mechanics. The ideas and techniques of this theory are being used in a variety of diverse areas and proved to be productive and innovative. It is remarkable that the variational inequalities and mathematical programming problems can be viewed as a special realization of the abstract equilibrium problems [1,2].

Let H be a real Hilbert space. Throughout this paper, we write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x. The $x_n \to x$ indicates that $\{x_n\}$ converges strongly to x. Let C be a nonempty closed convex subset of H and Θ be a bifunction of $C \times C$ into $\mathbf{R}$, where $\mathbf{R}$ is the set of real numbers. The equilibrium problem for Θ: $C \times C \to \mathbf{R}$ is to find $\bar{x} \in C$ such that

$$\Theta(\bar{x}, y) \geq 0, \quad \forall y \in C. \tag{1.1}$$

The set of solutions of problem (1.1) is denoted by $EP(\Theta)$. Given a mapping $T: C \to H$, let $\Theta(x, y) = \langle Tx, y - x\rangle$ for all $x, y \in C$. Then, $z \in EP(\Theta)$ if and only if $\langle Tz, y - z\rangle \geq 0$ for all $y \in C$. Numerous problems in physics, optimization, and economics reduce to finding a solution of problem (1.1). Equilibrium problems have been studied extensively [2-18]. Combettes and Hirstoaga [3] introduced an iterative scheme for finding the best approximation to the initial data when $EP(\Theta)$ is nonempty and derived a strong convergence theorem. Very recently, Peng and Yao [4] introduced the following generalized mixed equilibrium problem of finding $\bar{x} \in C$ such that

$$\Theta(\bar{x}, y) + \varphi(y) - \varphi(\bar{x}) + \langle A\bar{x}, y - \bar{x}\rangle \geq 0, \quad \forall y \in C, \tag{1.2}$$

where $A: H \to H$ is a nonlinear mapping, $\phi: C \to \mathbf{R}$ is a function and $\Theta: C \times C \to \mathbf{R}$ is a bifunction. The set of solutions of problem (1.2) is denoted by $GMEP$.

In particular, whenever $A = 0$, problem (1.2) reduces to the following mixed equilibrium problem of finding $\bar{x} \in C$ such that

$$\Theta(\bar{x}, y) + \varphi(y) - \varphi(\bar{x}) \geq 0, \quad \forall y \in C,$$

which was considered by Ceng and Yao [5]. The set of solutions of this problem is denoted by MEP.

Whenever $\phi = 0$, problem (1.2) reduces to the following generalized equilibrium problem of finding $\bar{x} \in C$ such that

$$\Theta(\bar{x}, y) + \langle A\bar{x}, y - \bar{x}\rangle \geq 0, \quad \forall y \in C, \tag{1.3}$$

which was introduced and studied by Takahashi and Takahashi [13]. The set of solutions of problem (1.3) is denoted by GEP. Obviously, the generalized equilibrium problem covers the equilibrium problem as a special case. It is assumed in [4] that $\Theta: C \times C \to \mathbf{R}$ is a bifunction satisfying conditions (H1)-(H4) and $\phi: C \to \mathbf{R}$ is a lower semicontinuous and convex function with restriction (A1) or (A2), where

(H1) $\Theta(x, x) = 0$, $\forall x \in C$;

(H2) Θ is monotone, i.e., $\Theta(x, y) + \Theta(y, x) \leq 0$, $\forall x, y \in C$;

(H3) for each $y \in C$, $x \mapsto \Theta(x, y)$ is weakly upper semicontinuous;

(H4) for each $x \in C$, $y \mapsto \Theta(x, y)$ is convex and lower semicontinuous;

(A1) for each $x \in H$ and $r > 0$, there exist a bounded subset $D_x \subset C$ and $y_x \in C$ such that for any $z \in C \setminus D_x$,

$$\Theta(z, y_x) + \varphi(y_x) - \varphi(z) + \frac{1}{r}\langle y_x - z, z - x\rangle < 0;$$

(A2) C is a bounded set.

It is worth pointing out that, related iterative methods for solving fixed point problems, variational inequalities and optimization problems can be found in [19-35].

Recall that a ρ-Lipschitzian mapping $T: C \to H$ is a mapping on C such that

$$\| Tx - Ty \| \leq \rho \| x - y \|, \quad \forall x, y \in C,$$

where $\rho \geq 0$ is a constant. In particular, if $\rho \in [0, 1)$ then T is called a contraction on C; if $\rho = 1$ then T is called a nonexpansive mapping on C. Denote the set of fixed points of T by Fix(T). It is well known that if C is a nonempty bounded closed convex subset of H and $S: C \to C$ is nonexpansive, then Fix(S) ≠ Ø. Let P_C be the metric projection of H onto C, that is, for every point $x \in H$, there exists a unique nearest point of C, denoted by $P_C x$, such that $\| x - P_C x \| \leq \| x - y \|$ for all $y \in C$. Recall also that a mapping A of C into H is called

(i) monotone if

$$\langle Ax - Ay, x - y\rangle \geq 0, \ \forall x, y \in C;$$

(ii) η-strongly monotone if there exists a constant $\eta > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \eta \|x - y\|^2, \ \forall x, y \in C;$$

(iii) δ-inverse strongly monotone if there exists a constant $\delta > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \delta \|Ax - Ay\|^2, \ \forall x, y \in C.$$

Furthermore, let A be a strongly positive bounded linear operator on H, that is, there exists a constant $\bar{\gamma} > 0$ such that

$$\langle Ax, x\rangle \geq \bar{\gamma}\|x\|^2, \quad \forall x \in H. \tag{1.4}$$

1.1 The W-mappings

The concept of W-mappings was introduced in Atsushiba and Takahashi [22]. It is very useful in establishing the convergence of iterative methods for computing a common fixed point of nonlinear mappings (see, for instance, [23,25,27]).

Let $\lambda_{n,1}, \lambda_{n,2} \ldots, \lambda_{n,N} \in (0, 1]$, $n \geq 1$. Given the nonexpansive mappings $S_1, S_2, \ldots, S_N$ on H, Atsushiba and Takahashi defines, for each $n \geq 1$, mappings $U_{n,1}, U_{n,2}, \ldots, U_{n,N}$ by

$$\begin{aligned}
U_{n,1} &= \lambda_{n,1}S_1 + (1 - \lambda_{n,1})I,\\
U_{n,2} &= \lambda_{n,2}S_2U_{n,1} + (1 - \lambda_{n,2})I,\\
&\vdots\\
U_{n,N-1} &= \lambda_{n,N-1}S_{N-1}U_{n,N-2} + (1 - \lambda_{n,N-1})I,\\
W_n := U_{n,N} &= \lambda_{n,N}S_NU_{n,N-1} + (1 - \lambda_{n,N})I.
\end{aligned} \tag{1.5}$$

The W_n is called the W-mapping generated by $S_1, \ldots, S_N$ and $\lambda_{n,1}, \lambda_{n,2}, \ldots, \lambda_{n,N}$. Note that Nonexpansivity of S_i implies the nonexpansivity of W_n.

Colao et al. [14] introduced an iterative method for finding a common element of the set of solutions of an equilibrium problem and the set of fixed points of a finite family of nonexpansive mappings in a real Hilbert space H. Moreover, they proved the strong convergence of the proposed iterative algorithm.

1.2 Theorem CMX

(See [[14], Theorem 3.1]). Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{S_i\}_{i=1}^N$ be a finite family of nonexpansive mappings on H, A a strongly positive bounded linear operator on H with coefficient $\bar{\gamma}$ and f an α-contraction on H for some $\alpha \in (0, 1)$. Moreover, let $\{\alpha_n\}$ be a sequence in (0, 1), $\{\lambda_{n,i}\}_{i=1}^N$ a sequence in $[a, b]$ with $0 < a \leq b < 1$, $\{r_n\}$ a sequence in $(0, \infty)$ and γ and β two real numbers such that $0 < \beta < 1$ and $0 < \gamma < \bar{\gamma}/\alpha$. Let Θ: $C \times C \to \mathbf{R}$ be a bifunction satisfying assumptions (H1)-(H4) and $\cap_{i=1}^{N} \mathrm{Fix}(S_i) \cap EP(\Theta) \neq \emptyset$. For every $n \geq 1$, let W_n be the W-mapping generated by $S_1,\ldots, S_N$ and $\lambda_{n,1}, \lambda_{n,2},\ldots, \lambda_{n,N}$. Given $x_1 \in H$ arbitrarily, suppose the sequences $\{x_n\}$ and $\{u_n\}$ are generated iteratively by

$$\begin{aligned} &\Theta(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ &x_{n+1} = \alpha_n \gamma f(x_n) + \beta x_n + ((1-\beta)I - \alpha_n A)W_n u_n, \quad \forall n \geq 1, \end{aligned} \tag{1.6}$$

where the sequences $\{\alpha_n\}$, $\{r_n\}$ and the finite family of sequences $\{\lambda_{n,i}\}_{i=1}^{N}$ satisfy the conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(ii) $\liminf_{n\to\infty} r_n > 0$ and $\lim_{n\to\infty} r_n/r_{n+1} = 1$ (or $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$);
(iii) $\lim_{n\to\infty} |\lambda_{n,i} - \lambda_{n-1,i}| = 0$ for every $i \in \{1,\ldots, N\}$.

Then both $\{x_n\}$ and $\{u_n\}$ converge strongly to $x^* \in \cap_{i=1}^{N} \mathrm{Fix}(S_i) \cap EP(\Theta)$, which is the unique fixed point of the composite mapping $P_{\cap_{i=1}^{N} \mathrm{Fix}(S_i) \cap EP(\Theta)}(I - A + \gamma f)$, i.e.,

$$x^* = P_{\cap_{i=1}^{N} \mathrm{Fix}(S_i) \cap EP(\Theta)}(I - A + \gamma f)x^*.$$

Very recently, Yao et al. [10] relaxed the β in Colao, Marino and Xu's iterative scheme (1.6) by a sequence of $\{\beta_n\}$. They showed that if with additional condition $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$ holds, then the sequences $\{x_n\}$ and $\{u_n\}$ generated by (1.6) (but now with β_n in the place of β) still converge strongly to $x^* \in \cap_{i=1}^{N} \mathrm{Fix}(S_i) \cap EP(\Theta)$, which is the unique fixed point of the composite mapping $P_{\cap_{i=1}^{N} \mathrm{Fix}(S_i) \cap EP(\Theta)}(I - A + \gamma f)$, i.e.,

$$x^* = P_{\cap_{i=1}^{N} \mathrm{Fix}(S_i) \cap EP(\Theta)}(I - A + \gamma f)x^*.$$

1.3 Hybrid steepest-descent method

Let F: $H \to H$ be a κ-Lipschitzian and η-strongly monotone operator with constants κ, $\eta > 0$, and let T: $H \to H$ be nonexpansive such that $\mathrm{Fix}(T) \neq \emptyset$. Yamada [20] introduced the so-called hybrid steepest-descent method for solving the variational inequality problem: finding $\tilde{x} \in \mathrm{Fix}(T)$ such that

$$\langle F\tilde{x}, x - \tilde{x}\rangle \geq 0, \ \forall x \in \mathrm{Fix}(T).$$

This method generates a sequence $\{x_n\}$ via the following iterative scheme:

$$x_{n+1} = Tx_n - \lambda_{n+1}\mu F(Tx_n), \ \forall n \geq 0, \tag{1.7}$$

where $0 < \mu < 2\eta/\kappa^2$, the initial guess $x_0 \in H$ is arbitrary and the sequence $\{\lambda_n\}$ in (0, 1) satisfies the conditions:

$$\lambda_n \to 0, \ \sum_{n=0}^{\infty} \lambda_n = \infty \text{ and } \sum_{n=0}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty.$$

A key fact in Yamada's argument is that, for small enough $\lambda > 0$, the mapping

$$T^{\lambda}x := Tx - \lambda\mu F(Tx), \ \forall x \in H$$

is a contraction, due to the κ-Lipschitz continuity and η-strong monotonicity of F.

1.4 Our hybrid model

In this paper, assume Θ: $C \times C \to \mathbf{R}$ is a bifunction satisfying assumptions (H1)-(H4) and ϕ: $C \to \mathbf{R}$ is a lower semicontinuous and convex function with restriction (A1) or (A2). Let the mapping $A: H \to H$ be δ-inverse strongly monotone, and $\{S_i\}_{i=1}^N$ be a finite family of nonexpansive mappings on H such that $\cap_{i=1}^N \mathrm{Fix}(S_i) \cap GMEP \neq \emptyset$. Let F: $H \to H$ be a κ-Lipschitzian and η-strongly monotone operator with constants κ, $\eta > 0$ and f: $H \to H$ a ρ-Lipschitzian mapping with constant $\rho \geq 0$. Let $0 < \mu < 2\eta/\kappa^2$ and $0 \leq \gamma\rho < \tau$, where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$. By combining Yamada's hybrid steepest-descent method [20] and Colao, Marino and Xu's hybrid viscosity approximation method [14] (see also [10]), we propose the following hybrid iterative method for finding a common element of the set of solutions of generalized mixed equilibrium problem (1.2) and the set of fixed points of finitely many nonexpansive mappings $\{S_i\}_{i=1}^N$, that is, for given $x_1 \in H$ arbitrarily, let $\{x_n\}$ and $\{u_n\}$ be generated iteratively by

$$\begin{cases} \Theta(u_n, y) + \varphi(y) - \varphi(u_n) + \langle Ax_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, & \forall y \in C, \\ x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n \mu F)W_n u_n, & \forall n \geq 1, \end{cases} \tag{1.8}$$

where $\{a_n\}$, $\{\beta_n\} \subset (0, 1)$, $\{r_n\} \subset (0, 2\delta]$, $\{\lambda_{n,i}\}_{i=1}^N \subset [a, b]$ with $0 < a \leq b < 1$, and W_n is the W-mapping generated by $S_1,\ldots, S_N$ and $\lambda_{n,1}, \lambda_{n,2},\ldots, \lambda_{n,N}$. We shall prove that under quite mild hypotheses, both sequences $\{x_n\}$ and $\{u_n\}$ converge strongly to $x^* \in \cap_{i=1}^N \mathrm{Fix}(S_i) \cap GMEP$, where $x^* = P_{\cap_{i=1}^N \mathrm{Fix}(S_i)\cap GMEP}(I - \mu F + \gamma f)x^*$ is a unique solution of the variational inequality:

$$\langle(\mu F - \gamma f)x^*, x^* - x\rangle \leq 0, \forall x \in \bigcap_{i=1}^N \mathrm{Fix}(S_i) \cap GMEP. \tag{1.9}$$

Compared with Theorem 3.2 of Yao et al. [10], our Theorem 3.1 improves and extends their Theorem 3.2 [10] in the following aspects:

(i) The contraction f: $H \to H$ with coefficient $\rho \in (0, 1)$ in [[10], Theorem 3.2] is extended to the case of general Lipschitzian mapping f on H with constant $\rho \geq 0$.

(ii) The strongly positive bounded linear operator A: $H \to H$ with coefficient $\tilde{\gamma} > 0$ in [[10], Theorem 3.2] is extended to the case of general κ-Lipschitzian and η-strongly monotone operator F: $H \to H$ with constants κ, $\eta > 0$.

(iii) The equilibrium problem in [[10], Theorem 3.2] is extended to the case of generalized mixed equilibrium problem (1.2). Obviously, the problem (1.2) is more complicated than their problem (1.1).

(iv) The hybrid viscosity approximation method in [[10], Theorem 3.2] (see also [[14], Theorem 3.1]) is extended to develop our iterative method by virtue of Yamada's hybrid steepest-descent method [20].

2 Preliminaries

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$, and norm $\|\cdot\|$. Let C be a nonempty closed convex subset of H. Recall that the metric (or nearest point) projection from H onto C is the mapping P_C: $H \to C$ which assigns to each point $x \in H$ the unique point $P_C x \in C$ satisfying the property

$$\| x - P_C x \| = \inf_{y \in C} \| x - y \| =: d(x, C).$$

In order to prove our main results in the next section, we need the following lemmas and propositions.

Lemma 2.1 (See [36]). Let C be a nonempty closed convex subset of a real Hilbert space H. Given $x \in H$ and $z \in C$, we then have

(i) $z = P_C x$ if and only if $\langle x - z, y - z \rangle \leq 0$, $\forall y \in C$.

(ii) $z = P_C x$ if and only if $|| x - z ||^2 \leq || x - y ||^2 - || y - z ||^2$, $\forall_y \in C$.

(iii) $\langle P_C x - P_C y, x - y \rangle \geq || P_C x - P_C y ||^2$, $\forall x, y \in H$.

Consequently, P_C is nonexpansive and monotone.

Lemma 2.2 (See [5]). Let C be a nonempty closed convex subset of H. Let Θ: $C \times C \to \mathbf{R}$ be a bifunction satisfying conditions (H1)-(H4) and let ϕ: $C \to \mathbf{R}$ be a lower semicontinuous and convex function. For $r > 0$ and $x \in H$, define a mapping $T_r^{(\Theta,\varphi)} : H \to C$ as follows:

$$T_r^{(\Theta,\varphi)}(x) = \{z \in C : \Theta(z, y) + \varphi(y) - \varphi(z) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C\}$$

for all $x \in H$. Assume that either (A1) or (A2) holds. Then the following assertions hold:

(i) $T_r^{(\Theta,\varphi)}(x) \neq \emptyset$ for each $x \in H$ and $T_r^{(\Theta,\varphi)}$ is single-valued;

(ii) $T_r^{(\Theta,\varphi)}$ is firmly nonexpansive, i.e., for any $x, y \in H$,

$$\left\| T_r^{(\Theta,\varphi)}x - T_r^{(\Theta,\varphi)}y \right\|^2 \leq \langle T_r^{(\Theta,\varphi)}x - T_r^{(\Theta,\varphi)}y, x - y\rangle;$$

(iii) $\mathrm{Fix}(T_r^{(\Theta,\varphi)}) = MEP(\Theta, \varphi)$;

(iv) $MEP(\Theta, \phi)$is closed and convex.

Remark 2.1. If $\phi = 0$, then $T_r^{(\Theta,\varphi)}$ is rewritten as T_r^{Θ}; if $\Theta = 0$ additionally, then $T_r^{\Theta} = P_C$.

Lemma 2.3 (See [21]). Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in [0, 1] with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose $x_{n+1} = (1 - \beta_n)y_n + \beta_n x_n$ for all integers $n \geq 0$ and $\limsup_{n\to\infty}(||y_{n+1} - y_n|| - ||x_{n+1} - x_n||) \leq 0$. Then, $\lim_{n\to\infty} ||y_n - x_n|| = 0$.

Proposition 2.1 (See [[6], Proposition 2.1]). Let C, H, Θ, ϕ and $T_r^{(\Theta,\varphi)}$ be as in Lemma 2.2. Then the following inequality holds:

$$\left\| T_s^{(\Theta,\varphi)}x - T_t^{(\Theta,\varphi)}x \right\|^2 \leq \frac{s-t}{s}\langle T_s^{(\Theta,\varphi)}x - T_t^{(\Theta,\varphi)}x, T_s^{(\Theta,\varphi)}x - x\rangle$$

for all $s, t > 0$ and $x \in H$.

Lemma 2.4 (See [19]). Let $\{a_n\}$ be a sequence of nonnegative numbers satisfying the condition

$$a_{n+1} \leq (1 - \delta_n)a_n + \delta_n\sigma_n, \ \forall n \geq 1,$$

where $\{\delta_n\}$, $\{\sigma_n\}$ are sequences of real numbers such that

(i) $\{\delta_n\} \subset [0, 1]$ and $\sum_{n=1}^{\infty} \delta_n = \infty$, or equivalently,

$$\prod_{n=1}^{\infty}(1-\delta_n) := \lim_{n\to\infty}\prod_{k=1}^{n}(1-\delta_k) = 0;$$

(ii) $\limsup_{n\to\infty} \sigma_n \le 0$, or

(ii)' $\sum_{n=1}^{\infty} \delta_n\sigma_n$ is convergent.

Then $\lim_{n\to\infty} a_n = 0$.

We will need the following result concerning the W-mapping W_n generated by $S_1,\ldots, S_N$ and $\lambda_{n,1}, \lambda_{n,2},\ldots, \lambda_{n,N}$ in (1.5).

Proposition 2.2 (See [23]). Let C be a nonempty closed convex subset of a Banach space X. Let $S_1, S_2,\ldots, S_N$ be a finite family of nonexpansive mappings of C into itself such that $\cap_{i=1}^{N}\mathrm{Fix}(S_i) \neq \emptyset$, and let $\lambda_{n,1}, \lambda_{n,2},\ldots, \lambda_{n,N}$ be real numbers such that $0 < \lambda_{n,i} \le b < 1$ for $i = 1, 2,\ldots, N$. For any $n \ge 1$, let W_n be the W-mapping of C into itself generated by $S_1,\ldots, S_N$ and $\lambda_{n,1},\ldots, \lambda_{n,N}$. If X is strictly convex, then $\mathrm{Fix}(W_n) = \cap_{i=1}^{N}\mathrm{Fix}(S_i)$.

Proposition 2.3 (See [[14], Lemma 2.8]). Let C be a nonempty convex subset of a Banach space. Let $\{S_i\}_{i=1}^{N}$ be a finite family of nonexpansive mappings of C into itself and $\{\lambda_{n,i}\}_{i=1}^{N}$ be sequences in [0, 1] such that $\lambda_{n,i} \to \lambda_i$ ($i = 1,\ldots, N$). Moreover for every integer $n \ge 1$, let W and W_n be the W-mappings generated by $S_1,\ldots, S_N$ and $\lambda_1,\ldots, \lambda_N$ and $S_1,\ldots, S_N$ and $\lambda_{n,1},\ldots, \lambda_{n,N}$ respectively. Then for every $x \in C$, it follows that

$$\lim_{n\to\infty} \| W_n x - Wx \| = 0.$$

The following two lemmas are the immediate consequences of the inner product on H.

Lemma 2.5. For all $x, y \in H$, there holds the inequality

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, x+y\rangle.$$

Lemma 2.6 (See [36]). For all $x, y, z \in H$ and $\alpha, \beta, \gamma \in [0, 1]$ with $\alpha + \beta + \gamma = 1$, there holds the equality

$$\|\alpha x+\beta y+\gamma z\|^2 = \alpha\|x\|^2+\beta\|y\|^2+\gamma\|z\|^2-\alpha\beta\|x-y\|^2-\beta\gamma\|y-z\|^2-\gamma\alpha\|z-x\|^2.$$

The following lemma plays a crucial role in proving strong convergence of our iterative schemes.

Lemma 2.7 (See [[19], Lemma 3.1]). Let λ be a number in (0, 1] and let $\mu > 0$. Let $F: H \to H$ be an operator on H such that, for some constants $\kappa, \eta > 0$, F is κ-Lipschitzian and η-strongly monotone. Associating with a nonexpansive mapping $T: H \to H$, define the mapping $T^\lambda: H \to H$ by

$$T^\lambda x := Tx - \lambda\mu F(Tx),\ \forall x \in H.$$

Then T^λ is a contraction provided $\mu < 2\eta/\kappa^2$, that is,

$$\| T^\lambda x - T^\lambda y \| \le (1-\lambda\tau)\| x-y \|,\ \forall x, y \in H,$$

where $\tau = 1 - \sqrt{1-\mu(2\eta-\mu\kappa^2)} \in (0, 1]$.

Remark 2.2. Put $F = \frac{1}{2}I$, where I is the identity operator of H. Then we have $\mu < 2\eta/\kappa^2 = 4$. Also, put $\mu = 2$. Then it is easy to see that $\kappa = \eta = \frac{1}{2}$ and

$$\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)} = 1 - \sqrt{1 - 2(2 \cdot \frac{1}{2} - 2(\frac{1}{2})^2)} = 1.$$

In particular, whenever $\lambda > 0$, we have $T^{\lambda}x := Tx - \lambda\mu F(Tx) = (1 - \lambda)\, Tx$.

3 Iterative scheme and strong convergence

In this section, based on Yamada's hybrid steepest-descent method [20] and Colao, Marino and Xu's hybrid viscosity approximation method [14] (see also [10]), we introduce a hybrid iterative method for finding a common element of the set of solutions of generalized mixed equilibrium problem (1.2) and the set of fixed points of finitely many nonexpansive mappings in a real Hilbert space. Moreover, we derive the strong convergence of the proposed iterative algorithm to a common solution of problem (1.2) and the fixed point problem of finitely many nonexpansive mappings.

Theorem 3.1. Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\Theta: C \times C \to \mathbf{R}$ be a bifunction satisfying assumptions (H1)-(H4) and $\phi: C \to \mathbf{R}$ be a lower semicontinuous and convex function with restriction (A1) or (A2). Let the mapping $A: H \to H$ be δ-inverse strongly monotone, and $\{S_i\}_{i=1}^N$ be a finite family of nonexpansive mappings on H such that $\cap_{i=1}^N \mathrm{Fix}(S_i) \cap GMEP \neq \emptyset$. Let $F: H \to H$ be a κ-Lipschitzian and η-strongly monotone operator with constants κ, $\eta > 0$ and $f: H \to H$ a ρ-Lipschitzian mapping with constant $\rho \geq 0$. Let $0 < \mu < 2\eta/\kappa^2$ and $0 \leq \gamma\rho < \tau$, where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$. Suppose $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in (0, 1), $\{r_n\}$ is a sequence in $(0, 2\delta)$ and $\{\lambda_{n,i}\}_{i=1}^N$ is a sequence in $[a, b]$ with $0 < a \leq b < 1$. For every $n \geq 1$, let W_n be the W-mapping generated by $S_1,\ldots, S_N$ and $\lambda_{n,1}, \lambda_{n,2},\ldots, \lambda_{n,N}$. Given $x_1 \in H$ arbitrarily, suppose the sequences $\{x_n\}$ and $\{u_n\}$ are generated iteratively by

$$\begin{cases} \Theta(u_n, y) + \varphi(y) - \varphi(u_n) + \langle Ax_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, & \forall y \in C, \\ x_{n+1} = \alpha_n\gamma f(x_n) + \beta_n x_n + ((1 - \beta_n)I - \alpha_n\mu F)W_n u_n, & \forall n \geq 1, \end{cases} \tag{3.1}$$

where the sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{r_n\}$ and the finite family of sequences $\{\lambda_{n,i}\}_{i=1}^N$ satisfy the conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;
(iii) $0 < \liminf_{n\to\infty} r_n \leq \limsup_{n\to\infty} r_n < 2\delta$ and $\lim_{n\to\infty} (r_{n+1} - r_n) = 0$;
(iv) $\lim_{n\to\infty} (\lambda_{n+1,i} - \lambda_{n,i}) = 0$ for all $i = 1, 2,\ldots, N$.

Then both $\{x_n\}$ and $\{u_n\}$ converge strongly to $x^* \in \bigcap_{i=1}^N \mathrm{Fix}(S_i) \cap GMEP$, where $x^* = P_{\bigcap_{i=1}^N \mathrm{Fix}(S_i)\cap GMEP}(I - \mu F + \gamma f)x^*$ is a unique solution of the variational inequality:

$$\langle(\mu F - \gamma f)x^*, x^* - x\rangle \leq 0,\ \forall x \in \bigcap_{i=1}^N \mathrm{Fix}(S_i) \cap GMEP. \tag{3.2}$$

Proof. Let $Q = P_{\bigcap_{i=1}^{N}\mathrm{Fix}(S_i)\cap GMEP}$. Note that $F: H \to H$ is a κ-Lipschitzian and η-strongly monotone operator with constants κ, $\eta > 0$ and $f: H \to H$ is a ρ-Lipschitzian mapping with constant $\rho \geq 0$. Then, we have

$$\begin{aligned}\|(I-\mu F)x-(I-\mu F)y\|^2 &= \|x-y\|^2 - 2\mu\langle x-y,\ Fx-Fy\rangle + \mu^2\|Fx-Fy\|^2\\ &\leq (1-2\mu\eta+\mu^2\kappa^2)\|x-y\|^2\\ &= (1-\tau)^2\|x-y\|^2,\end{aligned}$$

where $\tau = 1-\sqrt{1-\mu(2\eta-\mu\kappa^2)}$, and hence

$$\begin{aligned}\| Q(I-\mu F+\gamma f)(x) - Q(I-\mu F+\gamma f)(y) \| &\leq \| (I-\mu F+\gamma f)(x) - (I-\mu F+\gamma f)(y) \|\\ &\leq \| (I-\mu F)x-(I-\mu F)y \| + \gamma \| f(x)-f(y) \|\\ &\leq (1-\tau) \| x-y \| + \gamma\rho \| x-y \|\\ &= (1-(\tau-\gamma\rho)) \| x-y \|,\end{aligned}$$

for all $x, y \in H$. Since $0 \leq \gamma\rho < \tau \leq 1$, it is known that $1 - (\tau - \gamma\rho) \in [0, 1)$. Therefore, $Q(I - \mu F + \gamma f)$ is a contraction of H into itself, which implies that there exists a unique element $x^* \in H$ such that $x^* = Q(I-\mu F+\gamma f)x^* = P_{\cap_{i=1}^{N}\mathrm{Fix}(S_i)\cap GMEP}(I-\mu F+\gamma f)x^*$.

From the definition of $T_r^{(\Theta,\varphi)}$, we know that $u_n = T_{r_n}^{(\Theta,\varphi)}(x_n - r_nAx_n)$. Take $p \in \bigcap_{i=1}^{N}\mathrm{Fix}(S_i)\cap GMEP$ arbitrarily. Since $p = T_{r_n}^{(\Theta,\varphi)}(p-r_nAp) = S_ip$, A is δ-inverse strongly monotone and $0 < r_n \leq 2\delta$, we deduce that, for any $n \geq 1$,

$$\begin{aligned}\|u_n-p\|^2 &= \left\|T_{r_n}^{(\Theta,\varphi)}(x_n-r_nAx_n) - T_{r_n}^{(\Theta,\varphi)}(p-r_nAp)\right\|^2\\ &\leq \|(x_n-r_nAx_n)-(p-r_nAp)\|^2\\ &= \|x_n-p-r_n(Ax_n-Ap)\|^2\\ &= \|x_n-p\|^2 - 2r_n\langle x_n-p,\ Ax_n-Ap\rangle + r_n^2\|Ax_n-Ap\|^2\\ &\leq \|x_n-p\|^2 + r_n(r_n-2\delta)\|Ax_n-Ap\|^2\\ &\leq \|x_n-p\|^2.\end{aligned} \tag{3.3}$$

First we will prove that both $\{x_n\}$ and $\{u_n\}$ are bounded.

Indeed, taking into account the control conditions (i) and (ii), we may assume, without loss of generality, that $\alpha_n \leq 1 - \beta_n$ for all $n \geq 1$. Now, by Proposition 2.2 we have $p \in \mathrm{Fix}(W_n)$.

Then utilizing Lemma 2.7, from (3.1) and (3.3) we obtain

$$\begin{aligned}&\| x_{n+1}-p \|\\ &= \| \alpha_n(\gamma f(x_n)-\mu Fp) + \beta_n(x_n-p) + ((1-\beta_n)I-\alpha_n\mu F)W_nu_n - ((1-\beta_n)I-\alpha_n\mu F)W_np \|\\ &\leq \alpha_n \| \gamma f(x_n)-\mu Fp \| + \beta_n \| x_n-p \| + \| ((1-\beta_n)I-\alpha_n\mu F)W_nu_n - ((1-\beta_n)I-\alpha_n\mu F)W_np \|\\ &= \alpha_n \| \gamma f(x_n)-\mu Fp \| + \beta_n \| x_n-p \| + (1-\beta_n) \| (I-\tfrac{\alpha_n}{1-\beta_n}\mu F)W_nu_n - (I-\tfrac{\alpha_n}{1-\beta_n}\mu F)W_np \|\\ &\leq (1-\beta_n)(1-\tfrac{\alpha_n\tau}{1-\beta_n}) \| u_n-p \| + \beta_n \| x_n-p \| + \alpha_n \| \gamma f(x_n)-\mu Fp \|\\ &\leq (1-\beta_n-\alpha_n\tau) \| u_n-p \| + \beta_n \| x_n-p \| + \alpha_n \| \gamma f(x_n)-\mu Fp \|\\ &\leq (1-\alpha_n\tau) \| x_n-p \| + \alpha_n\gamma \| f(x_n)-f(p) \| + \alpha_n \| \gamma f(p)-\mu Fp \|\\ &\leq (1-\alpha_n\tau) \| x_n-p \| + \alpha_n\gamma\rho \| x_n-p \| + \alpha_n \| \gamma f(p)-\mu Fp \|\\ &= (1-(\tau-\gamma\rho)\alpha_n) \| x_n-p \| + \alpha_n \| \gamma f(p)-\mu Fp \|\\ &= (1-(\tau-\gamma\rho)\alpha_n) \| x_n-p \| + (\tau-\gamma\rho)\alpha_n\tfrac{\|\gamma f(p)-\mu Fp\|}{\tau-\gamma\rho}\\ &\leq \max\left\{\| x_n-p \|,\ \tfrac{\|\gamma f(p)-\mu Fp\|}{\tau-\gamma\rho}\right\}.\end{aligned} \tag{3.4}$$

It follows from (3.4) and induction that

$$\| x_n - p \| \le \max\{\| x_0 - p \|, \frac{\| \gamma f(p) - \mu Fp \|}{\tau - \gamma\rho}\},\ \forall n \ge 1.$$

Therefore $\{x_n\}$ is bounded. We also obtain that $\{u_n\}$, $\{Ax_n\}$, $\{W_n u_n\}$ and $\{f(x_n)\}$ are all bounded. We shall use M to denote the possible different constants appearing in the following reasoning.

Next, we show that $\| x_{n+1} - x_n \| \to 0$.

Indeed, set $x_{n+1} = \beta_n x_n + (1 - \beta_n)z_n$ for all $n \ge 1$. Then from the definition of z_n we obtain

$$\begin{aligned} z_{n+1} - z_n &= \tfrac{x_{n+2} - \beta_{n+1}x_{n+1}}{1-\beta_{n+1}} - \tfrac{x_{n+1} - \beta_n x_n}{1-\beta_n} \\ &= \tfrac{\alpha_{n+1}\gamma f(x_{n+1}) + ((1-\beta_{n+1})I - \alpha_{n+1}\mu F)W_{n+1}u_{n+1}}{1-\beta_{n+1}} - \tfrac{\alpha_n \gamma f(x_n) + ((1-\beta_n)I - \alpha_n \mu F)W_n u_n}{1-\beta_n} \\ &= \tfrac{\alpha_{n+1}}{1-\beta_{n+1}}\gamma f(x_{n+1}) - \tfrac{\alpha_n}{1-\beta_n}\gamma f(x_n) + W_{n+1}u_{n+1} \\ &\quad - W_n u_n + \tfrac{\alpha_n}{1-\beta_n}\mu F W_n u_n - \tfrac{\alpha_{n+1}}{1-\beta_{n+1}}\mu F W_{n+1}u_{n+1} \\ &= \tfrac{\alpha_{n+1}}{1-\beta_{n+1}}[\gamma f(x_{n+1}) - \mu F W_{n+1}u_{n+1}] + \tfrac{\alpha_n}{1-\beta_n}[\mu F W_n u_n - \gamma f(x_n)] \\ &\quad + W_{n+1}u_{n+1} - W_{n+1}u_n + W_{n+1}u_n - W_n u_n. \end{aligned}$$

It follows that

$$\begin{aligned} \| z_{n+1} - z_n \| - \| x_{n+1} - x_n \| &\le \tfrac{\alpha_{n+1}}{1-\beta_{n+1}}(\gamma \| f(x_{n+1}) \| + \mu \| F W_{n+1}u_{n+1} \|) \\ &\quad + \tfrac{\alpha_n}{1-\beta_n}(\mu \| F W_n u_n \| + \gamma \| f(x_n) \|) + \| W_{n+1}u_{n+1} \\ &\quad - W_{n+1}u_n \| + \| W_{n+1}u_n - W_n u_n \| - \| x_{n+1} - x_n \| \\ &\le \tfrac{\alpha_{n+1}}{1-\beta_{n+1}}(\gamma \| f(x_{n+1}) \| + \mu \| F W_{n+1}u_{n+1} \|) \\ &\quad + \tfrac{\alpha_n}{1-\beta_n}(\mu \| F W_n u_n \| + \gamma \| f(x_n) \|) + \| W_{n+1}u_n - W_n u_n \| \\ &\quad + \| u_{n+1} - u_n \| - \| x_{n+1} - x_n \|. \end{aligned} \tag{3.5}$$

From (1.5), since S_i and $U_{n,\ i}$ for all $i = 1, 2,..., N$ are nonexpansive,

$$\begin{aligned} &\| W_{n+1}u_n - W_n u_n \| \\ &= \| \lambda_{n+1,N}S_N U_{n+1,N-1}u_n + (1 - \lambda_{n+1,N})u_n - \lambda_{n,N}S_N U_{n,N-1}u_n - (1 - \lambda_{n,N})u_n \| \\ &\le |\lambda_{n+1,N} - \lambda_{n,N}| \| u_n \| + \| \lambda_{n+1,N}S_N U_{n+1,N-1}u_n - \lambda_{n,N}S_N U_{n,N-1}u_n \| \\ &\le |\lambda_{n+1,N} - \lambda_{n,N}| \| u_n \| + \| \lambda_{n+1,N}(S_N U_{n+1,N-1}u_n - S_N U_{n,N-1}u_n) \| \\ &\quad + |\lambda_{n+1,N} - \lambda_{n,N}| \| S_N U_{n,N-1}u_n \| \\ &\le 2M|\lambda_{n+1,N} - \lambda_{n,N}| + \lambda_{n+1,N} \| U_{n+1,N-1}u_n - U_{n,N-1}u_n \|. \end{aligned} \tag{3.6}$$

Again, from (1.5),

$$\begin{aligned} &\| U_{n+1,N-1}u_n - U_{n,N-1}u_n \| \\ &= \| \lambda_{n+1,N-1}S_{N-1}U_{n+1,N-2}u_n + (1 - \lambda_{n+1,N-1})u_n - \lambda_{n,N-1}S_{N-1}U_{n,N-2}u_n - (1 - \lambda_{n,N-1})u_n \| \\ &\le |\lambda_{n+1,N-1} - \lambda_{n,N-1}| \| u_n \| + \| \lambda_{n+1,N-1}S_{N-1}U_{n+1,N-2}u_n - \lambda_{n,N-1}S_{N-1}U_{n,N-2}u_n \| \\ &\le |\lambda_{n+1,N-1} - \lambda_{n,N-1}| \| u_n \| + \lambda_{n+1,N-1} \| S_{N-1}U_{n+1,N-2}u_n - S_{N-1}U_{n,N-2}u_n \| \\ &\quad + |\lambda_{n+1,N-1} - \lambda_{n,N-1}|M \\ &\le 2M|\lambda_{n+1,N-1} - \lambda_{n,N-1}| + \lambda_{n+1,N-1} \| U_{n+1,N-2}u_n - U_{n,N-2}u_n \| \\ &\le 2M|\lambda_{n+1,N-1} - \lambda_{n,N-1}| + \| U_{n+1,N-2}u_n - U_{n,N-2}u_n \|. \end{aligned} \tag{3.7}$$

Therefore, we have

$$\begin{aligned}&\| U_{n+1,N-1}u_n - U_{n,N-1}u_n \| \\ &\le 2M|\lambda_{n+1,N-1} - \lambda_{n,N-1}| + 2M|\lambda_{n+1,N-2} - \lambda_{n,N-2}| + \| U_{n+1,N-3}u_n - U_{n,N-3}u_n \| \\ &\le 2M\sum_{i=2}^{N-1}|\lambda_{n+1,i} - \lambda_{n,i}| + \| U_{n+1,1}u_n - U_{n,1}u_n \| \\ &= \| \lambda_{n+1,1}S_1u_n + (1-\lambda_{n+1,1})u_n - \lambda_{n,1}S_1u_n - (1-\lambda_{n,1})u_n \| + 2M\sum_{i=2}^{N-1}|\lambda_{n+1,i} - \lambda_{n,i}|,\end{aligned}$$

and then

$$\begin{aligned}&\| U_{n+1,N-1}u_n - U_{n,N-1}u_n \| \\ &\le |\lambda_{n+1,1} - \lambda_{n,1}| \| u_n \| + \| \lambda_{n+1,1}S_1u_n - \lambda_{n,1}S_1u_n \| + 2M\sum_{i=2}^{N-1}|\lambda_{n+1,i} - \lambda_{n,i}| \\ &\le 2M\sum_{i=1}^{N-1}|\lambda_{n+1,i} - \lambda_{n,i}|.\end{aligned} \tag{3.8}$$

Substituting (3.8) into (3.6), we have

$$\begin{aligned}\| W_{n+1}u_n - W_nu_n \| &\le 2M|\lambda_{n+1,N} - \lambda_{n,N}| + 2\lambda_{n+1,N}M\sum_{i=1}^{N-1}|\lambda_{n+1,i} - \lambda_{n,i}| \\ &\le 2M\sum_{i=1}^{N}|\lambda_{n+1,i} - \lambda_{n,i}|.\end{aligned} \tag{3.9}$$

On the other hand, utilizing the δ-inverse strongly monotonicity of A we have

$$\begin{aligned}\| (x_{n+1} - r_{n+1}Ax_{n+1}) - (x_n - r_nAx_n) \| &= \| x_{n+1} - x_n - r_{n+1}(Ax_{n+1} - Ax_n) + (r_n - r_{n+1})Ax_n \| \\ &\le \| x_{n+1} - x_n - r_{n+1}(Ax_{n+1} - Ax_n) \| + |r_{n+1} - r_n| \| Ax_n \| \\ &\le \| x_{n+1} - x_n \| + |r_{n+1} - r_n| \| Ax_n \|,\end{aligned} \tag{3.10}$$

Since $u_n = T_{r_n}^{(\Theta,\varphi)}(x_n - r_nAx_n)$ and $u_{n+1} = T_{r_{n+1}}^{(\Theta,\varphi)}(x_{n+1} - r_{n+1}Ax_{n+1})$, we get

$$\begin{aligned}&\| u_{n+1} - u_n \| \\ &= \| T_{r_{n+1}}^{(\Theta,\varphi)}(x_{n+1} - r_{n+1}Ax_{n+1}) - T_{r_n}^{(\Theta,\varphi)}(x_n - r_nAx_n) \| \\ &= \| T_{r_{n+1}}^{(\Theta,\varphi)}(x_{n+1} - r_{n+1}Ax_{n+1}) - T_{r_{n+1}}^{(\Theta,\varphi)}(x_n - r_nAx_n) + T_{r_{n+1}}^{(\Theta,\varphi)}(x_n - r_nAx_n) - T_{r_n}^{(\Theta,\varphi)}(x_n - r_nAx_n) \| \\ &\le \| T_{r_{n+1}}^{(\Theta,\varphi)}(x_{n+1} - r_{n+1}Ax_{n+1}) - T_{r_{n+1}}^{(\Theta,\varphi)}(x_n - r_nAx_n) \| + \| T_{r_{n+1}}^{(\Theta,\varphi)}(x_n - r_nAx_n) - T_{r_n}^{(\Theta,\varphi)}(x_n - r_nAx_n) \| \\ &\le \| (x_{n+1} - r_{n+1}Ax_{n+1}) - (x_n - r_nAx_n) \| + \| T_{r_{n+1}}^{(\Theta,\varphi)}(x_n - r_nAx_n) - T_{r_n}^{(\Theta,\varphi)}(x_n - r_nAx_n) \| \\ &\le \| x_{n+1} - x_n \| + |r_{n+1} - r_n| \| Ax_n \| + \| T_{r_{n+1}}^{(\Theta,\varphi)}(x_n - r_nAx_n) - T_{r_n}^{(\Theta,\varphi)}(x_n - r_nAx_n) \|,\end{aligned} \tag{3.11}$$

Using (3.9) and (3.11) in (3.5), we get

$$\begin{aligned}&\| z_{n+1} - z_n \| - \| x_{n+1} - x_n \| \\ &\le \tfrac{\alpha_{n+1}}{1-\beta_{n+1}}(\gamma \| f(x_{n+1}) \| + \mu \| FW_{n+1}u_{n+1} \|) + \tfrac{\alpha_n}{1-\beta_n}(\mu \| FW_nu_n \| + \gamma \| f(x_n) \|) \\ &\quad + 2M\sum_{i=1}^{N}|\lambda_{n+1,i} - \lambda_{n,i}| + \| x_{n+1} - x_n \| + |r_{n+1} - r_n| \| Ax_n \| \\ &\quad + \| T_{r_{n+1}}^{(\Theta,\varphi)}(x_n - r_nAx_n) - T_{r_n}^{(\Theta,\varphi)}(x_n - r_nAx_n) \| - \| x_{n+1} - x_n \| \\ &= \tfrac{\alpha_{n+1}}{1-\beta_{n+1}}(\gamma \| f(x_{n+1}) \| + \mu \| FW_{n+1}u_{n+1} \|) + \tfrac{\alpha_n}{1-\beta_n}(\mu \| FW_nu_n \| + \gamma \| f(x_n) \|) \\ &\quad + 2M\sum_{i=1}^{N}|\lambda_{n+1,i} - \lambda_{n,i}| + |r_{n+1} - r_n| \| Ax_n \| \\ &\quad + \| T_{r_{n+1}}^{(\Theta,\varphi)}(x_n - r_nAx_n) - T_{r_n}^{(\Theta,\varphi)}(x_n - r_nAx_n) \| .\end{aligned} \tag{3.12}$$

Note that $0 < \liminf_{n\to\infty} r_n \le \limsup_{n\to\infty} r_n < 2\delta$ and $\lim_{n\to\infty} (r_{n+1} - r_n) = 0$. Then utilizing Proposition 2.1 we have

$$\lim_{n\to\infty} \| T_{r_{n+1}}^{(\Theta,\varphi)}(x_n - r_nAx_n) - T_{r_n}^{(\Theta,\varphi)}(x_n - r_nAx_n) \| = 0. \tag{3.13}$$

Consequently, it follows from (3.13) and conditions (i), (iii), (iv) that

$$\begin{aligned}
&\limsup_{n\to\infty}(\| z_{n+1} - z_n \| - \| x_{n+1} - x_n \|) \\
&\le \limsup_{n\to\infty}\{\frac{\alpha_{n+1}}{1-\beta_{n+1}}(\gamma \| f(x_{n+1}) \| + \mu \| FW_{n+1}u_{n+1} \|) + \frac{\alpha_n}{1-\beta_n}(\mu \| FW_nu_n \| + \gamma \| f(x_n) \|) \\
&\quad +2M\sum_{i=1}^{N} |\lambda_{n+1,i} - \lambda_{n,i}| + |r_{n+1} - r_n| \| Ax_n \| + \| T_{r_{n+1}}^{(\Theta,\varphi)}(x_n - r_nAx_n) - T_{r_n}^{(\Theta,\varphi)}(x_n - r_nAx_n) \|\} \\
&= 0.
\end{aligned}$$

Hence by Lemma 2.3 we have

$$\lim_{n\to\infty} \| z_n - x_n \| = 0.$$

Consequently

$$\lim_{n\to\infty} \| x_{n+1} - x_n \| = \lim_{n\to\infty}(1-\beta_n) \| z_n - x_n \| = 0. \tag{3.14}$$

From (3.11), (3.13), (3.14) and condition (iii) we have

$$\lim_{n\to\infty} \| u_{n+1} - u_n \| = 0.$$

Since $x_{n+1} = a_n \gamma f(x_n) + \beta_n x_n + ((1 - \beta_n)I - a_n\mu F)W_nu_n$, we have

$$\begin{aligned}
\| x_n - W_nu_n \| &\le \| x_n - x_{n+1} \| + \| x_{n+1} - W_nu_n \| \\
&\le \| x_n - x_{n+1} \| + \alpha_n \| \gamma f(x_n) - \mu FW_nu_n \| + \beta_n \| x_n - W_nu_n \|,
\end{aligned}$$

that is

$$\| x_n - W_nu_n \| \le \frac{1}{1-\beta_n} \| x_n - x_{n+1} \| + \frac{\alpha_n}{1-\beta_n} \| \gamma f(x_n) - \mu FW_nu_n \| .$$

It follows that

$$\lim_{n\to\infty} \| x_n - W_nu_n \| = 0. \tag{3.15}$$

On the other hand, from (3.3) and (3.4) we get

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\le [(1-\beta_n-\alpha_n\tau) \| u_n - p \| + \beta_n \| x_n - p \| + \alpha_n \| \gamma f(x_n) - \mu Fp \|]^2 \\
&\le (1-\beta_n-\alpha_n\tau)\|u_n - p\|^2 + \beta_n\|x_n - p\|^2 + \tfrac{\alpha_n}{\tau}\|\gamma f(x_n) - \mu Fp\|^2 \\
&\le (1-\beta_n-\alpha_n\tau)[\|x_n - p\|^2 + r_n(r_n - 2\delta)\|Ax_n - Ap\|^2] + \beta_n\|x_n - p\|^2 \\
&\quad + \tfrac{\alpha_n}{\tau}\|\gamma f(x_n) - \mu Fp\|^2 \\
&= (1-\alpha_n\tau)\|x_n - p\|^2 + r_n(r_n - 2\delta)(1-\beta_n-\alpha_n\tau)\|Ax_n - Ap\|^2 \\
&\quad + \tfrac{\alpha_n}{\tau} \| \gamma f(x_n) - \mu Fp\|^2 \\
&\le \|x_n - p\|^2 + r_n(r_n - 2\delta)(1-\beta_n-\alpha_n\tau)\|Ax_n - Ap\|^2 + \tfrac{\alpha_n}{\tau}\|\gamma f(x_n) - \mu Fp\|^2,
\end{aligned}$$

and hence

$$\begin{aligned}
&r_n(2\delta - r_n)(1-\beta_n-\alpha_n\tau)\|Ax_n - Ap\|^2 \\
&\le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \tfrac{\alpha_n}{\tau}\|\gamma f(x_n) - \mu Fp\|^2 \\
&= (\| x_n - p \| + \| x_{n+1} - p \|) \| x_n - x_{n+1} \| + \tfrac{\alpha_n}{\tau}\|\gamma f(x_n) - \mu Fp\|^2.
\end{aligned}$$

Obviously, conditions (i), (ii), (iii) guarantee that $\alpha_n \to 0$, $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$ and $0 < \liminf_{n\to\infty} r_n \le \limsup_{n\to\infty} r_n < 2\delta$. Thus from $\| x_n - x_{n+1} \| \to 0$ we conclude that

$$\lim_{n\to\infty} \| Ax_n - Ap \| = 0. \tag{3.16}$$

Note that $T_r^{(\Theta,\varphi)}$ is firmly nonexpansive. Hence we have

$$\begin{aligned}
&\|u_n - p\|^2 \\
&= \left\|T_{r_n}^{(\Theta,\varphi)}(x_n - r_nAx_n) - T_{r_n}^{(\Theta,\varphi)}(p - r_nAp)\right\|^2 \\
&\le \langle (x_n - r_nAx_n) - (p - r_nAp), u_n - p\rangle \\
&= \tfrac{1}{2}[\|(x_n - r_nAx_n) - (p - r_nAp)\|^2 + \|u_n - p\|^2 - \|(x_n - r_nAx_n) - (p - r_nAp) - (u_n - p)\|^2] \\
&\le \tfrac{1}{2}[\|x_n - p\|^2 + \|u_n - p\|^2 - \|x_n - u_n - r_n(Ax_n - Ap)\|^2] \\
&= \tfrac{1}{2}[\|x_n - p\|^2 + \|u_n - p\|^2 - \|x_n - u_n\|^2 + 2r_n\langle Ax_n - Ap,\ x_n - u_n\rangle - r_n^2\|Ax_n - Ap\|^2],
\end{aligned}$$

which implies that

$$\|u_n - p\|^2 \le \|x_n - p\|^2 - \|x_n - u_n\|^2 + 2r_n \| Ax_n - Ap \| \| x_n - u_n \| . \tag{3.17}$$

Therefore, utilizing Lammas 2.5 and 2.7 we deduce from (3.17) that

$$\begin{aligned}
&\|x_{n+1} - p\|^2 \\
&= \|\alpha_n(\gamma f(x_n) - \mu Fp) + \beta_n(x_n - W_nu_n) + (I - \alpha_n\mu F)W_nu_n - (I - \alpha_n\mu F)W_np\|^2 \\
&\le \|(I - \alpha_n\mu F)W_nu_n - (I - \alpha_n\mu F)W_np + \beta_n(x_n - W_nu_n)\|^2 + 2\alpha_n\langle\gamma f(x_n) - \mu Fp, x_{n+1} - p\rangle \\
&\le [\|(I - \alpha_n\mu F)W_nu_n - (I - \alpha_n\mu F)W_np + \beta_n\|x_n - W_nu_n\|]^2 + 2\alpha_n\|\gamma f(x_n) - \mu Fp\|\,\|x_{n+1} - p\| \\
&\le [(1 - \alpha_n\tau)\|u_n - p\| + \beta_n\|x_n - W_nu_n\|]^2 + 2\alpha_n\|\gamma f(x_n) - \mu Fp\|\,\|x_{n+1} - p\| \\
&\le (\|u_n - p\| + \|x_n - W_nu_n\|)^2 + 2\alpha_n\|\gamma f(x_n) - \mu Fp\|\,\|x_{n+1} - p\| \\
&= \|u_n - p\|^2 + \|x_n - W_nu_n\|^2 + 2\|u_n - p\|\,\|x_n - W_nu_n\| + 2\alpha_n\|\gamma f(x_n) - \mu Fp\|\,\|x_{n+1} - p\| \\
&\le \|x_n - p\|^2 - \|x_n - u_n\|^2 + 2r_n \| Ax_n - Ap \| \| x_n - u_n \| + \|x_n - W_nu_n\|^2 \\
&\quad + 2 \| u_n - p \| \| x_n - W_nu_n \| + 2\alpha_n \| \gamma f(x_n) - \mu Fp \| \| x_{n+1} - p \| .
\end{aligned}$$

Then we have

$$\begin{aligned}
\|x_n - u_n\|^2 &\le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + 2r_n \| Ax_n - Ap \| \| x_n - u_n \| + \|x_n - W_nu_n\|^2 \\
&\quad + 2 \| u_n - p \| \| x_n - W_nu_n \| + 2\alpha_n \| \gamma f(x_n) - \mu Fp \| \| x_{n+1} - p \| \\
&\le (\| x_n - p \| + \| x_{n+1} - p \|) \| x_n - x_{n+1} \| + 2r_n \| Ax_n - Ap \| \| x_n - u_n \| \\
&\quad + \|x_n - W_nu_n\|^2 + 2 \| u_n - p \| \| x_n - W_nu_n \| + 2\alpha_n \| \gamma f(x_n) - \mu Fp \| \| x_{n+1} - p \| .
\end{aligned}$$

So, from (3.14)-(3.16) and $\alpha_n \to 0$, we have

$$\lim_{n\to\infty} \| x_n - u_n \| = 0.$$

Since

$$\| W_nu_n - u_n \| \le \| W_nu_n - x_n \| + \| x_n - u_n \|,$$

we also have

$$\lim_{n\to\infty} \| W_nu_n - u_n \| = 0.$$

Next, let us show that

$$\limsup_{n\to\infty}\langle(\mu F - \gamma f)x^*, x^* - x_n\rangle \le 0,$$

where $x^* = P_{\bigcap_{i=1}^N \mathrm{Fix}(S_i)\cap GMEP}(I-\mu F+\gamma f)x^*$ is a unique solution of the variational inequality (3.2). To show this, we can choose a subsequence $\{u_{n_i}\}$ of $\{u_n\}$ such that

$$\limsup_{n\to\infty}\langle(\mu F-\gamma f)x^*, x^*-u_n\rangle = \lim_{i\to\infty}\langle(\mu F-\gamma f)x^*, x^*-u_{n_i}\rangle.$$

Since $\{u_{n_i}\}$ is bounded, there exists a subsequence $\{u_{ij}\}$ of $\{u_{n_i}\}$ which converges weakly to w. Without loss of generality, we may assume that $u_{n_i}\rightharpoonup w$. From $\|W_nu_n - u_n\| \to 0$, we obtain $W_nu_{n_i}\rightharpoonup w$. Now we show that $w \in GMEP$. From $u_n = T_{r_n}^{(\Theta,\varphi)}(x_n - r_nAx_n)$, we know that

$$\Theta(u_n, y)+\varphi(y)-\varphi(u_n)+\langle Ax_n, y-u_n\rangle+\frac{1}{r_n}\langle y-u_n, u_n-x_n\rangle\geq 0,\ \forall y\in C.$$

From (H2) it follows that

$$\varphi(y)-\varphi(u_n)+\langle Ax_n, y-u_n\rangle+\frac{1}{r_n}\langle y-u_n, u_n-x_n\rangle\geq\Theta(y, u_n),\ \forall y\in C.$$

Replacing n by n_i, we have

$$\varphi(y)-\varphi(u_{n_i})+\langle Ax_{n_i}, y-u_{n_i}\rangle+\langle y-u_{n_i}, \frac{u_{n_i}-x_{n_i}}{r_{n_i}}\rangle\geq\Theta(y, u_{n_i}),\ \forall y\in C. \tag{3.18}$$

Put $u_t = ty + (1 - t)w$ for all $t \in (0, 1]$ and $y \in C$. Then, we have $u_t \in C$. So, from (3.18) we have

$$\begin{aligned}\langle u_t-u_{n_i}, Au_t\rangle &\geq \langle u_t-u_{n_i}, Au_t\rangle-\varphi(u_t)+\varphi(u_{n_i})-\langle u_t-u_{n_i}, Ax_{n_i}\rangle\\ &\quad-\langle u_t-u_{n_i}, \tfrac{u_{n_i}-x_{n_i}}{r_{n_i}}\rangle+\Theta(u_t, u_{n_i})\\ &=\langle u_t-u_{n_i}, Au_t-Au_{n_i}\rangle+\langle u_t-u_{n_i}, Au_{n_i}-Ax_{n_i}\rangle-\varphi(u_t)+\varphi(u_{n_i})\\ &\quad-\langle u_t-u_{n_i}, \tfrac{u_{n_i}-x_{n_i}}{r_{n_i}}\rangle+\Theta(u_t, u_{n_i}).\end{aligned}$$

Since $\|u_{n_i}-x_{n_i}\|\to 0$, we have $\|Au_{n_i}-Ax_{n_i}\|\to 0$. Further, from the monotonicity of A, we have $\langle u_t-u_{n_i}, Au_t-Au_{n_i}\rangle\geq 0$. So, from (H4), the weakly lower semicontinuity of φ, $\frac{u_{n_i}-x_{n_i}}{r_{n_i}}\to 0$ and $u_{n_i}\rightharpoonup w$, we have

$$\langle u_t-w, Au_t\rangle\geq-\varphi(u_t)+\varphi(w)+\Theta(u_t, w), \tag{3.19}$$

as $i \to \infty$. From (H1), (H4) and (3.19), we also have

$$\begin{aligned}0 &= \Theta(u_t, u_t)+\varphi(u_t)-\varphi(u_t)\\ &\leq t\Theta(u_t, y)+(1-t)\Theta(u_t, w)+t\varphi(y)+(1-t)\varphi(w)-\varphi(u_t)\\ &= t[\Theta(u_t, y)+\varphi(y)-\varphi(u_t)]+(1-t)[\Theta(u_t, w)+\varphi(w)-\varphi(u_t)]\\ &\leq t[\Theta(u_t, y)+\varphi(y)-\varphi(u_t)]+(1-t)\langle u_t-w, Au_t\rangle\\ &= t[\Theta(u_t, y)+\varphi(y)-\varphi(u_t)]+(1-t)t\langle y-w, Au_t\rangle,\end{aligned}$$

and hence

$$0\leq\Theta(u_t, y)+\varphi(y)-\varphi(u_t)+(1-t)\langle y-w, Au_t\rangle.$$

Letting $t \to 0$, we have, for each $y \in C$,

$$0\leq\Theta(w, y)+\varphi(y)-\varphi(w)+\langle y-w, Aw\rangle.$$

This implies that $w \in GMEP$.

We shall show $w \in \cap_{i=1}^{N} \mathrm{Fix}(S_i)$. To see this, we observe that we may assume (by passing to a further subsequence if necessary)

$$\lambda_{n_m,k} \to \lambda_k \in (0,1)\ (k = 1, 2, \ldots, N).$$

Let W be the W-mapping generated by $S_1,\ldots, S_N$ and $\lambda_1,\ldots, \lambda_N$. Then by Proposition 2.3, we have, for every $x \in H$,

$$W_{n_m}x \to Wx. \tag{3.20}$$

Moreover, from Proposition 2.2 it follows that $\mathrm{Fix}(W) = \cap_{i=1}^{N} \mathrm{Fix}(S_i)$. Assume that $w \notin \cap_{i=1}^{N} \mathrm{Fix}(S_i)$; then $w \neq Ww$. Since $w \in GMEP$, in terms of $\| x_n - W_n u_n \| \to 0$ and Opial's property of a Hilbert space, we conclude from (3.20) that

$$\begin{aligned}
\liminf_{m\to\infty} \| x_{n_m} - w \| &< \liminf_{m\to\infty} \| x_{n_m} - Ww \| \\
&\le \liminf_{m\to\infty} (\| x_{n_m} - W_{n_m}u_{n_m} \| + \| W_{n_m}u_{n_m} - W_{n_m}w \| + \| W_{n_m}w - Ww \|) \\
&= \liminf_{m\to\infty} \| W_{n_m}u_{n_m} - W_{n_m}w \| \\
&\le \liminf_{m\to\infty} \| u_{n_m} - w \| \\
&= \liminf_{m\to\infty} \| T_{r_{n_m}}^{(\Theta,\varphi)}(x_{n_m} - r_{n_m}Ax_{n_m}) - T_{r_{n_m}}^{(\Theta,\varphi)}(w - r_{n_m}Aw) \| \\
&\le \liminf_{m\to\infty} \| (x_{n_m} - r_{n_m}Ax_{n_m}) - (w - r_{n_m}Aw) \| \\
&= \liminf_{m\to\infty} \| x_{n_m} - w - r_{n_m}(Ax_{n_m} - Aw) \| \\
&\le \liminf_{m\to\infty} \| x_{n_m} - w \|,
\end{aligned}$$

due to the δ-inverse strong monotonicity of A. This is a contradiction. So, we get $w \in \bigcap_{i=1}^{N} \mathrm{Fix}(S_i)$. Therefore $w \in \bigcap_{i=1}^{N} \mathrm{Fix}(S_i) \cap GMEP$. Since $x^* = P_{\bigcap_{i=1}^{N} \mathrm{Fix}(S_i)\cap GMEP}(I - \mu F + \gamma f)x^*$, we have

$$\begin{aligned}
\limsup_{n\to\infty} \langle (\mu F - \gamma f)x^*, x^* - x_n \rangle &= \limsup_{n\to\infty} \langle (\mu F - \gamma f)x^*, x^* - u_n \rangle \\
&= \lim_{i\to\infty} \langle (\mu F - \gamma f)x^*, x^* - u_{n_i} \rangle \\
&= \langle (\mu F - \gamma f)x^*, x^* - w \rangle \le 0.
\end{aligned} \tag{3.21}$$

Finally, we prove that $\{x_n\}$ and $\{u_n\}$ converge strongly to x^*. From (3.1), utilizing Lemmas 2.5 and 2.7 we have

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&= \|\alpha_n(\gamma f(x_n) - \mu F x^*) + \beta_n(x_n - x^*) + ((1-\beta_n)I - \alpha_n\mu F)W_n u_n - ((1-\beta_n)I - \alpha_n \mu F)W_n x^*\|^2 \\
&\le \|\beta_n(x_n - x^*) + ((1-\beta_n)I - \alpha_n\mu F)W_n u_n - ((1-\beta_n)I - \alpha_n\mu F)W_n x^*\|^2 \\
&\quad + 2\alpha_n\langle \gamma f(x_n) - \mu F x^*, x_{n+1} - x^*\rangle \\
&\le [\beta_n \| x_n - x^* \| + \| ((1-\beta_n)\, I - \alpha_n\mu F)W_n u_n - ((1-\beta_n)I - \alpha_n\mu F)W_n x^* \|]^2 \\
&\quad + 2\alpha_n\langle \gamma f(x_n) - \mu F x^*, x_{n+1} - x^*\rangle \\
&\le [\beta_n \| x_n - x^* \| + \| ((1-\beta_n)\, I - \tfrac{\alpha_n}{1-\beta_n}\mu F)W_n u_n - (I - \tfrac{\alpha_n}{1-\beta_n}\mu F)W_n x^* \|]^2 \\
&\quad + 2\alpha_n\gamma\langle f(x_n) - f(x^*), x_{n+1} - x^*\rangle + 2\alpha_n\langle \gamma f(x^*) - \mu F x^*, x_{n+1} - x^*\rangle \\
&\le [\beta_n \| x_n - x^* \| + (1-\beta_n)(1 - \tfrac{\alpha_n}{1-\beta_n}\tau) \| x_n - x^* \|]^2 \\
&\quad + 2\alpha_n\gamma\rho \| x_n - x^* \| \| x_{n+1} - x^* \| + 2\alpha_n\langle \gamma f(x^*) - \mu F x^*, x_{n+1} - x^*\rangle \\
&\le (1-\alpha_n\tau)^2\|x_n - x^*\|^2 + \alpha_n\gamma\rho(\|x_n - x^*\|^2 + \|x_{n+1} - x^*\|^2) \\
&\quad + 2\alpha_n\langle \gamma f(x^*) - \mu F x^*, x_{n+1} - x^*\rangle \\
&= (1-\alpha_n\tau)^2\|x_n - x^*\|^2 + \alpha_n\gamma\rho\|x_n - x^*\|^2 + \alpha_n\gamma\rho\|x_{n+1} - x^*\|^2 \\
&\quad + 2\alpha_n\langle \gamma f(x^*) - \mu F x^*, x_{n+1} - x^*\rangle.
\end{aligned}$$

This implies that

$$
\begin{aligned}
\|x_{n+1} - x^*\|^2 &\le \frac{(1-\alpha_n\tau)^2 + \alpha_n\gamma\rho}{1-\alpha_n\gamma\rho}\|x_n - x^*\|^2 + \frac{2\alpha_n}{1-\alpha_n\gamma\rho}\langle \gamma f(x^*) - \mu F x^*,\ x_{n+1} - x^*\rangle \\
&= [1 - \frac{2(\tau-\gamma\rho)\alpha_n}{1-\alpha_n\gamma\rho}]\|x_n - x^*\|^2 + \frac{(\alpha_n\tau)^2}{1-\alpha_n\gamma\rho}\|x_n - x^*\|^2 \\
&\quad + \frac{2\alpha_n}{1-\alpha_n\gamma\rho}\langle \gamma f(x^*) - \mu F x^*,\ x_{n+1} - x^*\rangle \\
&\le [1 - \frac{2(\tau-\gamma\rho)\alpha_n}{1-\alpha_n\gamma\rho}]\|x_n - x^*\|^2 + \frac{2(\tau-\gamma\rho)\alpha_n}{1-\alpha_n\gamma\rho} \\
&\quad \times \{\tfrac{(\alpha_n\tau^2)M_1}{2(\tau-\gamma\rho)} + \tfrac{1}{\tau-\gamma\rho}\gamma f(x^*) - \mu F x^*,\ x_{n+1} - x^*\rangle\} \\
&= (1-\delta_n)\|x_n - x^*\|^2 + \delta_n\sigma_n,
\end{aligned}
$$

where $M_1 = \sup\{||x_n - p||^2: n \ge 1\}$, $\delta_n = \frac{2(\tau-\gamma\rho)\alpha_n}{1-\alpha_n\gamma\rho}$ and $\sigma_n = \frac{(\alpha_n\tau^2)M_1}{2(\tau-\gamma\rho)} + \frac{1}{\tau-\gamma\rho}\gamma f(x^*) - \mu F x^*, x_{n+1} - x^*\rangle$. It is easy to see that $\delta_n \to 0, \sum_{n=1}^{\infty}\delta_n = \infty$ and $\limsup_{n\to\infty}\sigma_n \le 0$. Hence, by Lemma 2.4, the sequence $\{x_n\}$ converges strongly to x^*. Consequently, we can obtain from $||x_n - u_n|| \to 0$ that $\{u_n\}$ also converges strongly to x^*. This completes the proof. □

Remark 3.1.

(i) The new technique of argument is applied to derive our Theorem 3.1. For instance, Lemma 2.7 for deriving the convergence of hybrid steepest-descent method plays an important role in proving the strong convergence of the sequences $\{x_n\}$, $\{u_n\}$ in our Theorem 3.1. In addition, utilizing Proposition 2.1 and $r_{n+1} - r_n \to 0$ we can obtain $\lim_{n\to\infty} \| T_{r_{n+1}}^{(\Theta,\varphi)}(x_n - r_nAx_n) - T_{r_n}^{(\Theta,\varphi)}(x_n - r_nAx_n) \| = 0$.

(ii) In order to show $w \in \cap_{i=1}^N \mathrm{Fix}(S_i)$, the proof of Theorem 3.2 [10] directly asserts that $||u_n - W_n u_n|| \to 0$ $(n \to \infty)$ implies $\| u_{n_j} - W_n u_{n_j} \| \to 0$ $(j \to \infty)$ for all n. Actually, this assertion seems impossible under their assumptions imposed on $\{\lambda_{n,i}\}_{i=1}^N$. However, following Colao, Marino and Xu's Step 7 of the proof in [[14], Theorem 3.1] and utilizing Proposition 2.3 (i.e., Lemma 2.8 in [14]), we successively derive $w \in \cap_{i=1}^N \mathrm{Fix}(S_i)$ by the condition $\{\lambda_{n,i}\}_{i=1}^N \subset [a, b]$ with $0 < a \le b < 1$.

Theorem 3.2. Let C be a nonempty closed convex subset of a real Hilbert space H. Let $A: H \to H$ be δ-inverse strongly monotone, $\Theta: C \times C \to \mathbf{R}$ be a bifunction satisfying assumptions (H1)-(H4) and $\phi: C \to \mathbf{R}$ be a lower semicontinuous and convex function with restriction (A1) or (A2) such that $GMEP \ne \emptyset$. Let $F: H \to H$ be a κ-Lipschitzian and η-strongly monotone operator with constants κ, $\eta > 0$ and $f: H \to H$ a ρ-Lipschitzian mapping with constant $\rho \ge 0$. Let $0 < \mu < 2\eta/\kappa^2$ and $0 \le \gamma\rho < \tau$, where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$. Suppose $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in (0, 1) and $\{r_n\}$ is a sequence in $(0, 2\delta]$. Given $x_1 \in H$ arbitrarily, suppose the sequences $\{x_n\}$ and $\{u_n\}$ are generated iteratively by

$$
\begin{cases}
\Theta(u_n, y) + \varphi(y) - \varphi(u_n) + \langle Ax_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge 0, & \forall y \in C, \\
x_{n+1} = \alpha_n\gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n\mu F)u_n, & \forall n \ge 1,
\end{cases}
\tag{3.22}
$$

where the sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{r_n\}$ satisfy the conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$;

(iii) $0 < \liminf_{n\to\infty} r_n \le \limsup_{n\to\infty} r_n < 2\delta$ and $\lim_{n\to\infty}(r_{n+1} - r_n) = 0$.

Then both $\{x_n\}$ and $\{u_n\}$ converge strongly to $x^* \in GMEP$, where $x^* = P_{GMEP}(I - \mu F + \gamma f)x^*$.

Proof. Put $S_i x = x$ for all $i = 1, 2,..., N$ and $x \in H$ and take the finite family of sequences $\{\lambda_{n,i}\}_{i=1}^N$ in $[a, b]$ with $0 < a \le b < 1$ such that $\lim_{n\to\infty}(\lambda_{n+1, i} - \lambda_{n, i}) = 0$ for all $i = 1, 2,..., N$. In this case, the W-mapping W_n generated by $S_1,..., S_N$ and $\lambda_{n,1}, \lambda_{n,2},..., \lambda_{n, N}$, is the identity mapping I of H. It is easy to see that all conditions of Theorem 3.1 are satisfied. Thus, the desired result follows from Theorem 3.1. □

Theorem 3.3. Let H be a real Hilbert space. Let $\{S_i\}_{i=1}^N$ be a finite family of nonexpansive mappings on H such that $\cap_{i=1}^N \mathrm{Fix}(S_i) \neq \emptyset$. Let $F: H \to H$ be a κ-Lipschitzian and η-strongly monotone operator with constants $\kappa, \eta > 0$ and $f: H \to H$ a ρ-Lipschitzian mapping with constant $\rho \ge 0$. Let $0 < \mu < 2\eta/\kappa^2$ and $0 \le \gamma\rho < \tau$, where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$. Suppose $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in (0, 1) and $\{\lambda_{n,i}\}_{i=1}^N$ is a sequence in $[a, b]$ with $0 < a \le b < 1$. For every $n \ge 1$, let W_n be the W-mapping generated by $S_1,..., S_N$ and $\lambda_{n,1}, \lambda_{n,2},..., \lambda_{n, N}$. Given $x_1 \in H$ arbitrarily, let $\{x_n\}$ be a sequence generated by

$$x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1 - \beta_n)I - \alpha_n \mu F)W_n x_n, \quad \forall n \ge 1,$$

where the sequences $\{\alpha_n\}$, $\{\beta_n\}$ and the finite family of sequences $\{\lambda_{n,i}\}_{i=1}^N$ satisfy the conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$;

(iii) $\lim_{n\to\infty}(\lambda_{n+1, i} - \lambda_{n, i}) = 0$ for all $i = 1, 2,..., N$.

Then $\{x_n\}$ converges strongly to $x^* \in \cap_{i=1}^N \mathrm{Fix}(S_i)$, where $x^* = P_{\cap_{i=1}^N \mathrm{Fix}(S_i)}(I - \mu F + \gamma f)x^*$.

Proof. Put $C = H$ and $r_n = 1$, and take $\Theta(x, y) = 0$, $Ax = 0$ and $\phi(x) = 0$ for all $x, y \in H$. Then $\Theta: H \times H \to \mathbf{R}$ is a bifunction satisfying assumptions (H1)-(H4) and $\phi: H \to \mathbf{R}$ is a lower semicontinuous and convex function with restriction (A1). Moreover the mapping $A: H \to H$ is δ-inverse strongly monotone for any $\delta > \frac{1}{2}$. In this case, from Theorem 3.1 we deduce that $u_n = x_n$, $0 < \liminf_{n\to\infty} r_n \le \limsup_{n\to\infty} r_n < 2\delta$ and $\lim_{n\to\infty}(r_{n+1} - r_n) = 0$. Beyond question, all conditions of Theorem 3.1 are satisfied. Therefore the conclusion follows. □

Acknowledgements

Lu-Chuan Ceng was partially supported by National Science Foundation of China (11071169), Innovation Program of Shanghai Municipal Education Commission (09ZZ133) and Leading Academic Discipline Project of Shanghai Normal University (DZL707), Sy-Ming Guu was partially supported by the grant NSC 97-2221-E-155-041-MY3, and Jen-Chih Yao was partially supported by the grant NSC 99-2221-E-037-007-MY3.

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Authors' contributions
LC conceived of the study and drafted the manuscript initially. SM participated in its design, coordination and finalized the manuscript. JC outlined the scope and design of the study. All authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

Received: 13 September 2011 Accepted: 6 June 2012 Published: 6 June 2012

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doi:10.1186/1687-1812-2012-92
Cite this article as: Ceng *et al.*: **Hybrid iterative method for finding common solutions of generalized mixed equilibrium and fixed point problems.** *Fixed Point Theory and Applications* 2012 **2012**:92.