A NOTE ON ASYMPTOTIC STABILITY OF DELAY DIFFERENCE SYSTEMS

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For the linear delay difference system $x_{n+1} - x_n = Ax_{n-k}$, where A is a 2×2 real constant matrix and k is a nonnegative integer, we present an explicit necessary and sufficient condition for the asymptotic stability of the zero solution of this system in terms of $\det A$, $\operatorname{tr} A$, and the delay k.

1. Introduction

In this paper, we are concerned with the asymptotic stability of the zero solution of the linear delay difference system

$$x_{n+1} - x_n = Ax_{n-k}, \quad n = 0, 1, 2, \ldots, \tag{1.1}$$

where A is a 2×2 real constant matrix and k is a nonnegative integer.

In the scalar case, Levin and May [7] showed that *the zero solution of the delay difference equation $x_{n+1} - x_n = -ax_{n-k}$ is asymptotically stable if and only if*

$$0 < a < 2\sin\frac{\pi/2}{2k+1} \left(= 2\cos\frac{k\pi}{2k+1} \right). \tag{1.2}$$

This nice result is proved by using the fact that the zero solution of the linear difference equation is asymptotically stable if and only if all the roots of its associated characteristic equation are inside the unit disk. Here, the Schur-Cohn criterion (see [2, 5]) and the Jury criterion (see [3]) are known to be effective tools for determining the asymptotic stability of linear difference systems. However, several kinds of the necessary and sufficient conditions established by the above criteria are too much complicated even to verify the condition (1.2). In fact, we need some careful root analysis of the characteristic equation in and on the unit circle to get the condition (1.2); see [6, 7, 8].

The purpose of this paper is to give an explicit necessary and sufficient condition for the asymptotic stability of the zero solution of the system (1.1) in terms of $\det A$, $\operatorname{tr} A$, and the delay k. As an application, we investigate the local asymptotic stability of delay difference systems of Lotka-Volterra type. For the general background of delay difference systems, one can refer to recent books [1, 2, 4].

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2. Main result

Our main result is stated as follows.

Theorem 2.1. *The zero solution of (1.1) is asymptotically stable if and only if*

$$2\sqrt{\det A}\sin\left((2k+1)\sin^{-1}\left(\frac{\sqrt{\det A}}{2}\right)\right) < -\operatorname{tr}A < 2\sin\frac{\pi/2}{2k+1} + \frac{\det A}{2\sin((\pi/2)/(2k+1))},$$
$$0 < \det A < 4\sin^2\frac{\pi/2}{2k+1}. \tag{2.1}$$

Remark 2.2. In case $A = \operatorname{diag}[-a,-a]$, one can easily verify that the condition (2.1) is equivalent to the condition (1.2) because of $2\sqrt{\det A} = -\operatorname{tr}A$.

Remark 2.3. In case $k = 1$, it follows from Theorem 2.1 that the zero solution of (1.1) is asymptotically stable if and only if

$$\begin{gathered} -(\det A)^2 + 3\det A < -\operatorname{tr}A < 1 + \det A, \\ 0 < \det A < 1. \end{gathered} \tag{2.2}$$

Remark 2.4. Let $k = 0$ and let $A = B - I$, where B is a 2×2 real constant matrix and I is the 2×2 indentity matrix. Then one can easily see that the system (1.1) becomes

$$x_{n+1} = Bx_n \tag{2.3}$$

and the condition (2.1) is reduced to

$$\begin{gathered} 1 - \det B > 0, \\ 1 + \operatorname{tr}B + \det B > 0, \\ 1 - \operatorname{tr}B + \det B > 0, \end{gathered} \tag{2.4}$$

namely,

$$|\operatorname{tr}B| < 1 + \det B < 2 \tag{2.5}$$

because $\det A = \det B - \operatorname{tr}B + 1$ and $\operatorname{tr}A = \operatorname{tr}B - 2$. This coincides with the Schur-Cohn necessary and sufficient condition for the asymptotic stability of the zero solution of (2.3); see, for example, [2, Theorem 4.15].

In order to prove Theorem 2.1, we need the following lemmas which deal with the two special cases

$$A = -\rho\begin{pmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{pmatrix}, \quad \rho\in\mathbb{R},\ 0 < |\theta| \le \frac{\pi}{2}, \tag{2.6}$$

$$A = -\begin{pmatrix}a_1 & b \\ 0 & a_2\end{pmatrix}, \quad a_1, a_2, b \in \mathbb{R}. \tag{2.7}$$

Lemma 2.5 (see [8]). *Suppose that the matrix A is given by (2.6). Then the zero solution of (1.1) is asymptotically stable if and only if*

$$0 < \rho < 2\sin\frac{\pi/2 - |\theta|}{2k+1}. \tag{2.8}$$

Lemma 2.6 (see [8]). *Suppose that the matrix A is given by (2.7). Then the zero solution of (1.1) is asymptotically stable if and only if*

$$0 < a_1 < 2\sin\frac{\pi/2}{2k+1}, \qquad 0 < a_2 < 2\sin\frac{\pi/2}{2k+1}. \tag{2.9}$$

Proof of Theorem 2.1. Let λ be an eigenvalue of A. Then we have $\det(\lambda I - A) = 0$ or

$$\lambda^2 - (\operatorname{tr} A)\lambda + \det A = 0. \tag{2.10}$$

There are two possible cases to consider.

Case 1. The matrix A has complex eigenvalues $-\rho(\cos\theta \pm i\sin\theta)$. We may assume that $\rho \in \mathbb{R} \setminus \{0\}$ and $0 < |\theta| \le \pi/2$ by choosing the values of ρ and θ again, if necessary.

In this case, it follows that

$$(\operatorname{tr} A)^2 - 4\det A < 0. \tag{2.11}$$

Then there exists a nonsingular matrix P such that

$$P^{-1}AP = -\rho\begin{pmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{pmatrix}, \tag{2.12}$$

and hence, by the transformation $x_n = Py_n$, the system (1.1) can be written as

$$y_{n+1} - y_n = P^{-1}APy_{n-k}. \tag{2.13}$$

Note that the asymptotic stability of the zero solution of (1.1) is equivalent to that of (2.13). Therefore, by virtue of Lemma 2.5, we will show that the zero solution of (1.1) is asymptotically stable if and only if

$$\begin{gathered} 2\sqrt{\det A}\sin\left((2k+1)\sin^{-1}\left(\frac{\sqrt{\det A}}{2}\right)\right) < -\operatorname{tr} A < 2\sqrt{\det A}, \\ 0 < \det A < 4\sin^2\frac{\pi/2}{2k+1}. \end{gathered} \tag{2.14}$$

To this end, it is sufficient to verify that the condition (2.8) is equivalent to the condition (2.14). Under the relation (2.12), we have

$$\det A = \rho^2, \qquad \operatorname{tr} A = -2\rho\cos\theta. \tag{2.15}$$

Since $\det A > 0$ and $|\operatorname{tr} A/(2\sqrt{\det A})| < 1$ by (2.11), we get

$$\rho = \pm\sqrt{\det A}, \qquad |\theta| = \cos^{-1}\left(-\frac{\operatorname{tr} A}{2\rho}\right). \tag{2.16}$$

Thus, the condition (2.8) can be written as

$$0 < \frac{\sqrt{\det A}}{2} < \sin \frac{\pi/2 - \cos^{-1}\left(-\operatorname{tr} A/(2\sqrt{\det A})\right)}{2k+1}, \tag{2.17}$$

or equivalently,

$$\begin{aligned}
&(2k+1)\sin^{-1}\left(\frac{\sqrt{\det A}}{2}\right) < \frac{\pi}{2} - \cos^{-1}\left(-\frac{\operatorname{tr} A}{2\sqrt{\det A}}\right)\\
&\Longleftrightarrow \cos\left(\frac{\pi}{2} - (2k+1)\sin^{-1}\left(\frac{\sqrt{\det A}}{2}\right)\right) < -\frac{\operatorname{tr} A}{2\sqrt{\det A}}\\
&\Longleftrightarrow 2\sqrt{\det A}\sin\left((2k+1)\sin^{-1}\left(\frac{\sqrt{\det A}}{2}\right)\right) < -\operatorname{tr} A.
\end{aligned} \tag{2.18}$$

Also, it follows from (2.17) that

$$0 < \det A < 4\sin^2 \frac{\pi/2 - \cos^{-1}\left(-\operatorname{tr} A/(2\sqrt{\det A})\right)}{2k+1} < 4\sin^2 \frac{\pi/2}{2k+1}, \tag{2.19}$$

which, together with the conditions (2.11) and (2.17), is equivalent to the condition (2.14). Consequently, we obtain that the condition (2.8) is equivalent to the condition (2.14).

Case 2. The matrix A has real eigenvalues $-a_1, -a_2$.

In this case, it follows that

$$(\operatorname{tr} A)^2 - 4\det A \geq 0. \tag{2.20}$$

Then there exists a nonsingular matrix Q such that

$$Q^{-1}AQ = -\begin{pmatrix} a_1 & b \\ 0 & a_2 \end{pmatrix}, \quad \text{for some } b \in \mathbb{R}, \tag{2.21}$$

and hence, by the transformation $x_n = Qy_n$, the system (1.1) can be written as

$$y_{n+1} - y_n = Q^{-1}AQy_{n-k}. \tag{2.22}$$

Note that the asymptotic stability of the zero solution of (1.1) is equivalent to that of (2.22). Therefore, by virtue of Lemma 2.6, we will show that the zero solution of (1.1) is asymptotically stable if and only if

$$\begin{gathered}
2\sqrt{\det A} \leq -\operatorname{tr} A < 2\sin\frac{\pi/2}{2k+1} + \frac{\det A}{2\sin\left((\pi/2)/(2k+1)\right)},\\
0 < \det A < 4\sin^2\frac{\pi/2}{2k+1}.
\end{gathered} \tag{2.23}$$

To this end, it is sufficient to verify that the condition (2.9) is equivalent to the condition (2.23). Under the relation (2.21), we have

$$\det A = a_1 a_2, \qquad \operatorname{tr} A = -(a_1 + a_2). \tag{2.24}$$

Without loss of generality, we may assume that $a_1 \leq a_2$. Then we get

$$a_1 = \frac{-\operatorname{tr} A - \sqrt{(\operatorname{tr} A)^2 - 4 \det A}}{2}, \qquad a_2 = \frac{-\operatorname{tr} A + \sqrt{(\operatorname{tr} A)^2 - 4 \det A}}{2}. \tag{2.25}$$

Thus, the condition (2.9) can be written as

$$\begin{gathered} 0 < \frac{-\operatorname{tr} A - \sqrt{(\operatorname{tr} A)^2 - 4 \det A}}{2}, \\ \frac{-\operatorname{tr} A + \sqrt{(\operatorname{tr} A)^2 - 4 \det A}}{2} < 2 \sin \frac{\pi/2}{2k+1}. \end{gathered} \tag{2.26}$$

It follows from the first inequality of (2.26) that

$$\begin{gathered} \sqrt{(\operatorname{tr} A)^2 - 4 \det A} < -\operatorname{tr} A \\ \Longleftrightarrow -\operatorname{tr} A > 0, \quad \det A > 0. \end{gathered} \tag{2.27}$$

Also, it follows from the second inequality of (2.26) that

$$\begin{aligned} &\sqrt{(\operatorname{tr} A)^2 - 4 \det A} < 4 \sin \frac{\pi/2}{2k+1} + \operatorname{tr} A \\ &\Longleftrightarrow 4 \sin \frac{\pi/2}{2k+1} + \operatorname{tr} A > 0, \qquad -4 \det A < 16 \sin^2 \frac{\pi/2}{2k+1} + 8 \sin \frac{\pi/2}{2k+1} \operatorname{tr} A \\ &\Longleftrightarrow -\operatorname{tr} A < 4 \sin \frac{\pi/2}{2k+1}, \qquad -\operatorname{tr} A < 2 \sin \frac{\pi/2}{2k+1} + \frac{\det A}{2 \sin ((\pi/2)/(2k+1))}. \end{aligned} \tag{2.28}$$

Therefore, under (2.20), the condition (2.26) is equivalent to

$$\begin{gathered} \det A > 0, \\ 0 < -\operatorname{tr} A < 4 \sin \frac{\pi/2}{2k+1}, \\ 0 < -\operatorname{tr} A < 2 \sin \frac{\pi/2}{2k+1} + \frac{\det A}{2 \sin ((\pi/2)/(2k+1))}. \end{gathered} \tag{2.29}$$

We here claim that

$$4 \sin \frac{\pi/2}{2k+1} > 2 \sin \frac{\pi/2}{2k+1} + \frac{\det A}{2 \sin ((\pi/2)/(2k+1))}, \tag{2.30}$$

namely,

$$\det A < 4 \sin^2 \frac{\pi/2}{2k+1}. \tag{2.31}$$

Indeed, if not, then we have

$$\det A \geq 4 \sin^2 \frac{\pi/2}{2k+1} > \frac{1}{4} (\operatorname{tr} A)^2 \tag{2.32}$$

by using the second inequality of (2.29). This contradicts the condition (2.20), and hence, the conditions (2.20) and (2.29) are equivalent to the condition (2.23). Consequently, we obtain that the condition (2.9) is equivalent to the condition (2.23).

From the argument above, we therefore conclude that the zero solution of (1.1) is asymptotically stable if and only if the condition (2.14) or (2.23) holds, that is, the condition (2.1) holds. This completes the proof. □

Now, we investigate the local asymptotic stability of the positive equilibrium of the Lotka-Volterra difference system

$$\begin{aligned} x_{n+1} &= x_n \exp\left[r_1\left(1 - x_{n-k} - \mu_1 y_{n-k}\right)\right], \\ y_{n+1} &= y_n \exp\left[r_2\left(1 - \mu_2 x_{n-k} - y_{n-k}\right)\right], \end{aligned} \tag{2.33}$$

with initial conditions

$$\begin{aligned} &x_{-s} \geq 0, \quad s = 0,1,\dots,k, \quad x_0 > 0, \\ &y_{-s} \geq 0, \quad s = 0,1,\dots,k, \quad y_0 > 0, \end{aligned} \tag{2.34}$$

where r_1, r_2, μ_1, and μ_2 are positive constants and k is a nonnegative integer. We assume that the system (2.33) has the (unique) positive equilibrium

$$(x^*, y^*) = \left(\frac{1-\mu_1}{1-\mu_1\mu_2}, \frac{1-\mu_2}{1-\mu_1\mu_2}\right). \tag{2.35}$$

Then, linearizing the system (2.33) around (x^*, y^*), one can easily get

$$\begin{aligned} x_{n+1} - x_n &= -r_1 x^*\left(x_{n-k} + \mu_1 y_{n-k}\right), \\ y_{n+1} - y_n &= -r_2 y^*\left(\mu_2 x_{n-k} + y_{n-k}\right). \end{aligned} \tag{2.36}$$

It is known that if the zero solution of the linearized system (2.36) is asymptotically stable, then the positive equilibrium (x^*, y^*) of the nonlinear system (2.33) is locally asymptotically stable, and thus, we have the following result by Theorem 2.1.

Corollary 2.7. *Assume that the system (2.33) has the positive equilibrium (x^*, y^*). If the matrix A given by*

$$A = \begin{pmatrix} -r_1 x^* & -r_1 x^* \mu_1 \\ -r_2 y^* \mu_2 & -r_2 y^* \end{pmatrix} = -\frac{1}{1-\mu_1\mu_2}\begin{pmatrix} r_1(1-\mu_1) & r_1\mu_1(1-\mu_1) \\ r_2\mu_2(1-\mu_2) & r_2(1-\mu_2) \end{pmatrix} \tag{2.37}$$

satisfies the condition (2.1), then the positive equilibrium (x^*, y^*) of (2.33) is locally asymptotically stable.

Remark 2.8. We are unable to directly apply Lemma 2.5 or 2.6 to the linearized system (2.36) because the eigenvalues of A given by (2.37) are much complicated.

Recently, Tang et al. [9] have shown that under $\mu_1 < 1$ and $\mu_2 < 1$, the positive equilibrium (x^*, y^*) of (2.33) with (2.34) is globally attractive, provided that

$$r_j(k+1) \leq \frac{3(1-\mu)}{2(1+\mu)}, \quad j = 1,2, \tag{2.38}$$

where $\mu = \max\{\mu_1, \mu_2\}$. In the case where $k = 1$, $r_1 = r_2 = r$, and $\mu_1 = \mu_2 = \mu < 1$, we claim that the condition (2.38) is also a sufficient condition for the local asymptotic stability of the positive equilibrium (x^*, y^*) of (2.33) with (2.34). In fact, the condition (2.38) is reduced to $r < 3(1-\mu)/4(1+\mu)$, while, by Remark 2.3, one can easily verify that the zero solution of the linearized system (2.36) is asymptotically stable if and only if $r < 1$; and so our claim is valid. Consequently, in this case, these above facts show that the condition (2.38) is a sufficient condition for the global asymptotic stability of the positive equilibrium (x^*, y^*) of (2.33) with (2.34).

Remark 2.9. By virtue of Corollary 2.7, we believe that under $\mu_1 < 1$ and $\mu_2 < 1$, the condition (2.38) is a sufficient condition for the global asymptotic stability of the positive equilibrium (x^*, y^*) of (2.33) with (2.34).

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