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Uniform mean convergence theorems for hybrid mappings in Hilbert spaces

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Abstract

Using the notion of sequences of means on the Banach space of all bounded real sequences, we prove mean and uniform mean convergence theorems for pointwise convergent sequences of hybrid mappings in Hilbert spaces.

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1 Introduction

Using the notion of asymptotically invariant sequences of means on l^∞, we obtain a mean convergence theorem for pointwise convergent sequences of hybrid mappings in Hilbert spaces. By assuming the strong regularity on the sequences of means, we also obtain a uniform mean convergence theorem.

In 1975, Baillon [1] established a nonlinear ergodic theorem for nonexpansive mappings in Hilbert spaces. Several results related to Baillon's ergodic theorem have been obtained since then; see, for instance, [2–8] and the references therein. Especially, using the notion of asymptotically invariant nets of means on semitopological semigroups, Hirano, Kido, and Takahashi [4] and Lau, Shioji, and Takahashi [5] generalized Baillon's ergodic theorem to commutative and noncommutative semigroups of nonexpansive mappings in Banach spaces, respectively.

On the other hand, Akatsuka, Aoyama, and Takahashi [9] obtained another generalization of Baillon's ergodic theorem for pointwise convergent sequences of nonexpansive mappings in Hilbert spaces. Their result was applied to the problem of approximating common fixed points of countable families of nonexpansive mappings. Recently, the authors [10] generalized some results in [9] for pointwise convergent sequences of hybrid mappings in the sense of [11].

The aim of the present paper is to obtain further generalizations of the results in [9, 10] by using a sequence $\{\mu_n\}$ of means on l^∞. In particular, by assuming the strong regularity on $\{\mu_n\}$, we prove a uniform mean convergence theorem (Theorem 3.5) for pointwise convergent sequences of hybrid mappings in Hilbert spaces.

Our paper is organized as follows. In Section 2, we recall some definitions and some preliminary results. In Section 3, we prove mean convergence theorems by using sequences of means on l^∞; see Theorems 3.4 and 3.5. In Section 4, we obtain some consequences of Theorem 3.5; see Theorems 4.1, 4.2, and 4.3. In Section 5, we give two applications of Theorem 4.3.

2 Preliminaries

Throughout the present paper, every linear space is real. We denote the sets of all nonnegative integers and all real numbers by $\mathbb{N}$ and $\mathbb{R}$, respectively. For a Banach space X, the conjugate space of X is denoted by X^*. We denote the norms of X and X^* by $\|\cdot\|$. For a sequence $\{x_n\}$ of a Banach space X and $x \in X$, strong and weak convergence of $\{x_n\}$ to x are denoted by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. For a sequence $\{x_n^*\}$ of X^* and $x^* \in X^*$, weak* convergence of $\{x_n^*\}$ to x^* is also denoted by $x_n^* \overset{*}{\rightharpoonup} x^*$. The inner product of a Hilbert space H is denoted by $\langle \cdot, \cdot \rangle$. For a subset A of a Hilbert space H, the closure of the convex hull of A is denoted by $\overline{\mathrm{co}}\, A$.

Let C be a nonempty subset of a Hilbert space H, and let $\lambda \in \mathbb{R}$. A mapping $T\colon C \to H$ is said to be λ-hybrid [11] if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + 2(1-\lambda)\langle x - Tx, y - Ty\rangle \tag{2.1}$$

for all $x, y \in C$. It is obvious that the following hold: T is 1-hybrid if and only if it is nonexpansive; T is 0-hybrid if and only if it is nonspreading in the sense of [12]; T is 1/2-hybrid if and only if it is a hybrid mapping in the sense of [13]. It is also known that if T is firmly nonexpansive, then it is λ-hybrid for all $\lambda \in [0,1]$; see [11, Lemma 3.1]. It should be noted that if $T\colon C \to H$ is λ-hybrid for some $\lambda > 1$, then T is the identity mapping on C. Indeed, by setting $x = y \in C$ in (2.1), we have

$$0 \leq 2(1-\lambda)\|x - Tx\|^2. \tag{2.2}$$

Since $1 - \lambda < 0$, we obtain $Tx = x$.

We denote the set of all λ-hybrid mappings of C into H by $H_\lambda(C,H)$. We also denote by $H_\lambda(C)$ the set of all λ-hybrid mappings of C into itself. The set of all fixed points of a mapping $T\colon C \to H$ is denoted by $F(T)$. A mapping $T\colon C \to H$ is said to be quasi-nonexpansive if $F(T)$ is nonempty and $\|u - Tx\| \leq \|u - x\|$ for all $u \in F(T)$ and $x \in C$. It is well known that $F(T)$ is closed and convex if $T\colon C \to H$ is quasi-nonexpansive and C is closed and convex. It is obvious that if $T \in H_\lambda(C,H)$ for some $\lambda \in \mathbb{R}$ and $F(T)$ is nonempty, then T is quasi-nonexpansive. We denote the identity mapping on C by I or T^0, where $T\colon C \to H$ is a mapping.

Let C be a nonempty closed convex subset of a Hilbert space H. Then for each $x \in H$, there exists a unique $z_x \in C$ such that $\|z_x - x\| = \min_{y\in C}\|y - x\|$. The metric projection P_C of H onto C is defined by $P_C x = z_x$ for all $x \in H$. For $x \in H$ and $z \in C$, the following holds:

$$z = P_C x \quad \Longleftrightarrow \quad \sup_{y\in C}\langle y - z, x - z\rangle \leq 0. \tag{2.3}$$

We know the following lemma.

Lemma 2.1 ([14, Lemma 3.2]) *Let S be a nonempty closed convex subset of a Hilbert space H and $\{x_n\}$ a sequence of H such that $\|u - x_{n+1}\| \leq \|u - x_n\|$ for all $u \in S$ and $n \in \mathbb{N}$. Then $\{P_S x_n\}$ converges strongly.*

Let l^∞ be the Banach space of all bounded real sequences with supremum norm. For $\mu \in (l^\infty)^*$ and $f = (f(0), f(1), \dots) \in l^\infty$, the value $\mu(f)$ is also denoted by

$$[\mu]_k f(k). \tag{2.4}$$

A bounded linear functional μ on l^∞ is said to be a mean on l^∞ if $\|\mu\| = \mu(e) = 1$, where $e = (1, 1, \dots)$. It is known that if μ is a mean on l^∞, then $\mu(f) \le \mu(g)$ whenever $f, g \in l^\infty$ satisfy $f(k) \le g(k)$ for all $k \in \mathbb{N}$. It is also known that the Hahn-Banach theorem ensures that there exists a mean μ on l^∞ such that

$$[\mu]_k f(k+1) = [\mu]_k f(k) \tag{2.5}$$

for all $f \in l^\infty$, where $(f(k+1)) = (f(1), f(2), \dots)$; see [8, Theorem 1.4.3]. Such a mean μ is called a Banach limit. If μ is a Banach limit and $f \in l^\infty$ is convergent, then $\mu(f) = \lim_k f(k)$.

For $p \in \mathbb{N}$, the bounded linear operator r_p of l^∞ into itself is defined by $(r_p f)(k) = f(k+p)$ for all $f \in l^\infty$ and $k \in \mathbb{N}$. The conjugate operator of r_p is denoted by r_p^*; that is, it is the bounded linear operator of $(l^\infty)^*$ into itself defined by $(r_p^* \mu)(f) = \mu(r_p f)$ for all $\mu \in (l^\infty)^*$ and $f \in l^\infty$. A sequence $\{\mu_n\}$ of means on l^∞ is said to be asymptotically invariant if $r_1^* \mu_n - \mu_n \overset{*}{\rightharpoonup} 0$, that is,

$$\lim_{n\to\infty} [\mu_n]_k \bigl(f(k+1) - f(k)\bigr) = 0 \tag{2.6}$$

for all $f \in l^\infty$. It is also said to be strongly regular if $\|r_1^* \mu_n - \mu_n\| \to 0$, that is,

$$\lim_{n\to\infty} \sup_{\|f\| \le 1} \bigl|[\mu_n]_k \bigl(f(k+1) - f(k)\bigr)\bigr| = 0. \tag{2.7}$$

Some examples of strongly regular sequences of means on l^∞ are shown in Sections 4 and 5. See [15] on asymptotically invariant nets of means and [2, 4–8] on the nonlinear ergodic theory for nonexpansive mappings with asymptotically invariant nets of means. The following lemma is well known.

Lemma 2.2 *Let $\{\mu_n\}$ be an asymptotically invariant sequence of means on l^∞ and $\{\mu_{n_\alpha}\}$ a subnet of $\{\mu_n\}$ such that $\mu_{n_\alpha} \overset{*}{\rightharpoonup} \mu \in (l^\infty)^*$. Then μ is a Banach limit.*

For the sake of completeness, we give the proof.

Proof Since the norm of $(l^\infty)^*$ is weakly* lower semicontinuous and $\|\mu_n\| = 1$ for each $n \in \mathbb{N}$, we have $\|\mu\| \le \liminf_\alpha \|\mu_{n_\alpha}\| = 1$. On the other hand, since $\mu_{n_\alpha} \overset{*}{\rightharpoonup} \mu$ and $\mu_n(e) = 1$ for each $n \in \mathbb{N}$, we obtain $\mu(e) = \lim_\alpha \mu_{n_\alpha}(e) = 1$. This implies that $1 = \mu(e) \le \|\mu\|$. Hence, μ is a mean on l^∞.

Fix $f \in l^\infty$. Since $\mu_{n_\alpha} \overset{*}{\rightharpoonup} \mu$ and $\{\mu_n\}$ is asymptotically invariant, we have

$$[\mu]_k \bigl(f(k+1) - f(k)\bigr) = \lim_\alpha [\mu_{n_\alpha}]_k \bigl(f(k+1) - f(k)\bigr) = 0. \tag{2.8}$$

Thus, μ is a Banach limit. □

Let H be a Hilbert space, μ a mean on l^∞, and $\{x_n\}$ a bounded sequence of H. Since the functional $y \mapsto [\mu]_k \langle x_k, y\rangle$ belongs to H^*, Riesz's theorem ensures that there corresponds a unique $z \in H$ such that

$$[\mu]_k \langle x_k, y\rangle = \langle z, y\rangle \tag{2.9}$$

for all $y \in H$; see [7, Theorem 1] and [8, Section 3.3]. We denote such a point z by

$$G(\{x_k\}, \mu) \quad \text{or} \quad G_\mu(\{x_k\}). \tag{2.10}$$

In other words, it is a unique element of H such that

$$[\mu]_k \langle x_k, y\rangle = \langle G(\{x_k\}, \mu), y\rangle \tag{2.11}$$

for all $y \in H$. In this case, it is known that $G(\{x_k\}, \mu) \in \overline{\mathrm{co}}\{x_n : n \in \mathbb{N}\}$; see [7, 8] for more details. It is easy to see that if μ is a Banach limit and $\{x_n\}$ is a sequence of H which converges weakly to $p \in H$, then $G(\{x_k\}, \mu) = p$. We need the following lemma in the proof of Theorem 3.1.

Lemma 2.3 *Let H be a Hilbert space, $\{x_n\}$ a bounded sequence of H, $\{y_n\}$ a strongly convergent sequence of H, and (β_n) a convergent sequence of real numbers. Then $[\mu]_n(\beta_n \langle x_n - x_{n+1}, y_n\rangle) = 0$ for each Banach limit μ.*

Proof Let μ be a Banach limit. Set $y = \lim_n y_n$ and $\beta = \lim_n \beta_n$. Since μ is a Banach limit and the second and third terms of the right-hand side of the equality

$$\begin{aligned} &\beta_n \langle x_n - x_{n+1}, y_n\rangle \\ &\quad = \beta \langle x_n - x_{n+1}, y\rangle + \beta \langle x_n - x_{n+1}, y_n - y\rangle + (\beta_n - \beta)\langle x_n - x_{n+1}, y_n\rangle \end{aligned} \tag{2.12}$$

tend to 0, we have $[\mu]_n(\beta_n \langle x_n - x_{n+1}, y_n\rangle) = \beta[\mu]_n \langle x_n - x_{n+1}, y\rangle = 0$. □

3 Mean convergence theorems

In this section, we show mean convergence theorems for a pointwise convergent sequence of mappings in $\bigcup_{\lambda \in \mathbb{R}} H_\lambda(C)$.

Throughout this section, we suppose the following conditions:

- C is a nonempty closed convex subset of a Hilbert space H;
- (λ_n) is a sequence of real numbers which tends to $\lambda \in \mathbb{R}$;
- $\{T_n\}$ is a sequence of mappings such that $T_n \in H_{\lambda_n}(C)$ for all $n \in \mathbb{N}$ and $\{T_n x\}$ converges strongly for all $x \in C$;
- T is a mapping of C into itself defined by $Tx = \lim_n T_n x$ for all $x \in C$;
- $\{x_n\}$ is a sequence of C defined by $x_0 \in C$ and $x_{n+1} = T_n x_n$ for all $n \in \mathbb{N}$.

Motivated by [7–10, 12], we first show the following fundamental theorem.

Theorem 3.1 *If $\{x_n\}$ is bounded, then $G(\{x_k\}, \mu)$ is a fixed point of T for each Banach limit μ.*

Proof Let μ be a Banach limit. Set $z = G(\{x_k\}, \mu)$. Since $z \in \overline{\mathrm{co}}\{x_n : n \in \mathbb{N}\}$ and C is closed and convex, we have $z \in C$. By assumption,

$$M = \sup_{n\in\mathbb{N}}\bigl(\|T_n z - Tz\| + 2\|x_{n+1} - T_n z\|\bigr) \tag{3.1}$$

is finite. Since each T_n is λ_n-hybrid, we have

$$\begin{aligned} &\|x_{n+1} - Tz\|^2 \\ &\quad = \|x_{n+1} - T_n z\|^2 + \|T_n z - Tz\|^2 + 2\langle x_{n+1} - T_n z, T_n z - Tz\rangle \\ &\quad \le \|T_n x_n - T_n z\|^2 + M\|T_n z - Tz\| \\ &\quad \le \|x_n - z\|^2 + 2(1-\lambda_n)\langle x_n - x_{n+1}, z - T_n z\rangle + M\|T_n z - Tz\| \end{aligned} \tag{3.2}$$

for all $n \in \mathbb{N}$. By Lemma 2.3, we have

$$[\mu]_n\bigl((1-\lambda_n)\langle x_n - x_{n+1}, z - T_n z\rangle\bigr) = 0. \tag{3.3}$$

By (3.2), (3.3), and $T_n z \to Tz$, we obtain

$$[\mu]_n\|x_n - Tz\|^2 = [\mu]_n\|x_{n+1} - Tz\|^2 \le [\mu]_n\|x_n - z\|^2. \tag{3.4}$$

On the other hand, by the definition of z, we also know that

$$\begin{aligned} [\mu]_n\|x_n - z\|^2 &= [\mu]_n\bigl(\|x_n - Tz\|^2 + \|Tz - z\|^2 + 2\langle x_n - Tz, Tz - z\rangle\bigr) \\ &= [\mu]_n\|x_n - Tz\|^2 + \|Tz - z\|^2 + 2\langle z - Tz, Tz - z\rangle \\ &= [\mu]_n\|x_n - Tz\|^2 - \|Tz - z\|^2. \end{aligned} \tag{3.5}$$

It follows from (3.4) and (3.5) that $0 \le -\|Tz - z\|^2$. Therefore, z is a fixed point of T. □

Using Lemma 2.1 and Theorem 3.1, we next show the following theorem.

Theorem 3.2 *Suppose that $F(T)$ is nonempty and $F(T) = \bigcap_{n=0}^{\infty} F(T_n)$. Then $\{x_n\}$ is bounded, $\{P_{F(T)}x_n\}$ is strongly convergent, and*

$$G\bigl(\{x_k\}, \mu\bigr) = \lim_{n\to\infty} P_{F(T)}x_n \tag{3.6}$$

for each Banach limit μ.

Proof Let μ be a Banach limit. It is obvious that $T \in H_\lambda(C)$. Hence, $F(T)$ is a nonempty closed convex subset of H, and hence $P_{F(T)}$ is well defined. We denote $P_{F(T)}$ by P. Since each T_n is quasi-nonexpansive and $F(T) \subset F(T_n)$, we have

$$\|u - x_{n+1}\| = \|u - T_n x_n\| \le \|u - x_n\| \tag{3.7}$$

for all $u \in F(T)$ and $n \in \mathbb{N}$. It also follows from (3.7) that $\{x_n\}$ is bounded. According to Theorem 3.1, we know that $G(\{x_k\}, \mu)$ is a fixed point of T. Using Lemma 2.1 and (3.7), we also know that $\{Px_n\}$ converges strongly to some $w \in F(T)$.

Set $z = G(\{x_k\}, \mu)$. By the definition of P and (3.7), we have

$$\|Px_{n+1} - x_{n+1}\| \le \|Px_n - x_{n+1}\| \le \|Px_n - x_n\| \tag{3.8}$$

for all $n \in \mathbb{N}$. On the other hand, it follows from $z \in F(T)$ and (2.3) that

$$\langle z - Px_n, x_n - Px_n\rangle \le 0 \tag{3.9}$$

for all $n \in \mathbb{N}$. This gives us that

$$\begin{aligned}
&\langle z - w, x_n - Px_n\rangle \\
&\quad = \langle Px_n - w, x_n - Px_n\rangle + \langle z - Px_n, x_n - Px_n\rangle \\
&\quad \le \langle Px_n - w, x_n - Px_n\rangle \\
&\quad \le \|Px_n - w\| \|x_n - Px_n\|
\end{aligned} \tag{3.10}$$

for all $n \in \mathbb{N}$. By (3.8) and (3.10), we have

$$\langle z - w, x_n - Px_n\rangle \le \|Px_n - w\| \|x_0 - Px_0\| \tag{3.11}$$

for all $n \in \mathbb{N}$. Consequently, we obtain

$$\begin{aligned}
\|z - w\|^2 &= [\mu]_n \langle z - w, x_n\rangle - \lim_{n\to\infty} \langle z - w, Px_n\rangle \\
&= [\mu]_n \langle z - w, x_n\rangle - [\mu]_n \langle z - w, Px_n\rangle \\
&= [\mu]_n \langle z - w, x_n - Px_n\rangle \\
&\le [\mu]_n \big(\|Px_n - w\| \|x_0 - Px_0\|\big) = 0.
\end{aligned} \tag{3.12}$$

Therefore, $z = w$. □

As a direct consequence of Theorems 3.1 and 3.2, we can obtain the following corollary for a single hybrid mapping.

Corollary 3.3 *Suppose that $x \in C$ and $S \in H_\gamma(C)$ for some $\gamma \in \mathbb{R}$. Then the following hold*:

(i) *if $\{S^n x\}$ is bounded, then $F(S)$ is nonempty and $G(\{S^k x\}, \mu)$ is a fixed point of S for each Banach limit μ*;

(ii) *if $F(S)$ is nonempty, then $\{S^n x\}$ is bounded, $\{P_{F(S)} S^n x\}$ is strongly convergent, and*

$$G\big(\big\{S^k x\big\}, \mu\big) = \lim_{n\to\infty} P_{F(S)} S^n x \tag{3.13}$$

for each Banach limit μ.

Using the notion of an asymptotically invariant sequence of means on l^∞, we next show the following mean convergence theorem.

Theorem 3.4 *Suppose that $F(T)$ is nonempty and $F(T) = \bigcap_{n=0}^{\infty} F(T_n)$. Let $\{\mu_n\}$ be an asymptotically invariant sequence of means on l^∞. Then the sequence*

$$\{G_{\mu_n}(\{x_k\})\}_{n\in\mathbb{N}} \tag{3.14}$$

converges weakly to the strong limit of $\{P_{F(T)}x_n\}$.

Proof By Theorem 3.2, we know that $\{x_n\}$ is bounded and $\{P_{F(T)}x_n\}$ converges strongly to some $w \in F(T)$.

Let $\{z_n\}$ be the sequence defined by $z_n = G_{\mu_n}(\{x_k\})$ for all $n \in \mathbb{N}$. Since $z_n \in \overline{\text{co}}\{x_k : k \in \mathbb{N}\}$ for all $n \in \mathbb{N}$, the sequence $\{z_n\}$ is bounded. Let u be any weak subsequential limit of $\{z_n\}$. Then we have a subsequence $\{z_{n_i}\}$ of $\{z_n\}$ such that $z_{n_i} \rightharpoonup u$. It follows from $\|\mu_{n_i}\| = 1$ that there exists a subnet $\{\mu_{n_{i_\alpha}}\}$ of $\{\mu_{n_i}\}$ such that $\mu_{n_{i_\alpha}} \overset{*}{\rightharpoonup} \mu \in (l^\infty)^*$. Since $\{\mu_{n_i}\}$ is asymptotically invariant, Lemma 2.2 implies that μ is a Banach limit.

By Theorem 3.2, we know that

$$G(\{x_k\}, \mu) = \lim_{n\to\infty} P_{F(T)}x_n = w. \tag{3.15}$$

This gives us that

$$\langle z_{n_{i_\alpha}}, y\rangle = [\mu_{n_{i_\alpha}}]_k \langle x_k, y\rangle \to [\mu]_k \langle x_k, y\rangle = \langle G(\{x_k\}, \mu), y\rangle = \langle w, y\rangle \tag{3.16}$$

for all $y \in H$. Thus, $\{z_{n_{i_\alpha}}\}$ converges weakly to w. On the other hand, since $z_{n_i} \rightharpoonup u$ and $\{z_{n_{i_\alpha}}\}$ is a subnet of $\{z_{n_i}\}$, we know that $z_{n_{i_\alpha}} \rightharpoonup u$. Accordingly, we have $u = w$. Thus, $\{z_n\}$ converges weakly to $w = \lim_n P_{F(T)}x_n$. □

As in the proof of [5, the corollary of Theorem 2], we can also show the following uniform mean convergence theorem in the case when the strong regularity of $\{\mu_n\}$ is assumed.

Theorem 3.5 *Suppose that $F(T)$ is nonempty and $F(T) = \bigcap_{n=0}^{\infty} F(T_n)$. Let $\{\mu_n\}$ be a strongly regular sequence of means on l^∞. Then the sequence*

$$\{G_{r_p^*\mu_n}(\{x_k\})\}_{n,p\in\mathbb{N}} \tag{3.17}$$

converges weakly to the strong limit of $\{P_{F(T)}x_n\}$ as $n \to \infty$ uniformly in $p \in \mathbb{N}$.

Proof Set $z_{n,p} = G_{r_p^*\mu_n}(\{x_k\})$ for all $n, p \in \mathbb{N}$. It is easy to see that $r_p^*\mu_n$ is also a mean on l^∞ for all $n, p \in \mathbb{N}$, and hence $\{z_{n,p}\}$ is well defined. By Theorem 3.2, $\{P_{F(T)}x_n\}$ converges strongly to some $w \in F(T)$.

We show that for each $y \in H$ and $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $n, p \in \mathbb{N}$ and $n \geq N$ imply that $|\langle z_{n,p} - w, y\rangle| < \varepsilon$. Suppose that this assertion does not hold. Then there exist $y_0 \in H$, $\varepsilon_0 > 0$, a strictly increasing sequence $\{n_i\}$ of $\mathbb{N}$, and a sequence $\{p_i\}$ of $\mathbb{N}$ such that

$$|\langle z_{n_i,p_i} - w, y_0\rangle| \geq \varepsilon_0 \tag{3.18}$$

for all $i \in \mathbb{N}$.

Set $\eta_i = r^*_{p_i}\mu_{n_i}$ for all $i \in \mathbb{N}$. Then $\{\eta_i\}$ is asymptotically invariant. Indeed, if $f \in l^\infty$, then we have

$$\begin{aligned}
\big|[\eta_i]_k\big(f(k+1)-f(k)\big)\big| &= \big|\big[r^*_{p_i}\mu_{n_i}\big]_k\big(f(k+1)-f(k)\big)\big| \\
&= \big|[\mu_{n_i}]_k\big(f(k+1+p_i)-f(k+p_i)\big)\big| \\
&= \big|[\mu_{n_i}]_k f(k+p_i+1)-[\mu_{n_i}]_k f(k+p_i)\big| \\
&= \big|\big[r^*_1\mu_{n_i}-\mu_{n_i}\big]_k f(k+p_i)\big| \\
&\le \big\|r^*_1\mu_{n_i}-\mu_{n_i}\big\| \cdot \sup_{k\in\mathbb{N}}\big|f(k+p_i)\big| \\
&\le \big\|r^*_1\mu_{n_i}-\mu_{n_i}\big\| \|f\|
\end{aligned} \tag{3.19}$$

for all $i \in \mathbb{N}$. Thus, it follows from the strong regularity of $\{\mu_n\}$ and (3.19) that $\lim_i[\eta_i]_k(f(k+1)-f(k)) = 0$. Hence, $\{\eta_i\}$ is asymptotically invariant.

By the definitions of $\{z_{n,p}\}$ and $\{\eta_i\}$, we have $z_{n_i,p_i} = G_{\eta_i}(\{x_k\})$ for all $i \in \mathbb{N}$. By Theorem 3.4, $\{z_{n_i,p_i}\}$ converges weakly to w as $i \to \infty$. This contradicts (3.18). □

As a direct consequence of Theorems 3.4 and 3.5, we obtain the following corollary for a single hybrid mapping.

Corollary 3.6 *Suppose that* $x \in C$, $S \in H_\gamma(C)$ *for some* $\gamma \in \mathbb{R}$, *and* $F(S)$ *is nonempty. Let* $\{\mu_n\}$ *be a sequence of means on* l^∞. *Then the following hold*:

(i) *if* $\{\mu_n\}$ *is asymptotically invariant, then the sequence* $\{G_{\mu_n}(\{S^k x\})\}_{n\in\mathbb{N}}$ *converges weakly to the strong limit of* $\{P_{F(S)}S^n x\}$;

(ii) *if* $\{\mu_n\}$ *is strongly regular, then the sequence* $\{G_{\mu_n}(\{S^{k+p}x\}_k)\}_{n,p\in\mathbb{N}}$ *converges weakly to the strong limit of* $\{P_{F(S)}S^n x\}$ *as* $n \to \infty$ *uniformly in* $p \in \mathbb{N}$.

4 Consequences of Theorem 3.5

In this section, using the techniques in [2, 4, 6–8], we obtain some consequences of Theorem 3.5. Throughout this section, we suppose that C, H, (λ_n), λ, $\{T_n\}$, T, and $\{x_n\}$ are the same as in Section 3 and $\bigcap_{n=0}^\infty F(T_n) = F(T) \neq \emptyset$.

We first obtain the following theorem for Cesàro means of sequences.

Theorem 4.1 *The sequence* $\{(n+1)^{-1}\sum_{k=0}^n x_{k+p}\}_{n,p\in\mathbb{N}}$ *converges weakly to the strong limit of* $\{P_{F(T)}x_n\}$ *as* $n \to \infty$ *uniformly in* $p \in \mathbb{N}$.

Proof Let $\{\mu_n\}$ be the sequence of means on l^∞ defined by

$$\mu_n(f) = \frac{1}{n+1}\sum_{k=0}^n f(k) \tag{4.1}$$

for all $n \in \mathbb{N}$ and $f \in l^\infty$. It is well known that $\{\mu_n\}$ is strongly regular and

$$G_{r^*_p\mu_n}\big(\{x_k\}\big) = \frac{1}{n+1}\sum_{k=0}^n x_{k+p} \tag{4.2}$$

for each $n, p \in \mathbb{N}$; see, for instance, [2, Theorem 5.1, 4, Theorem 5] and [8, Section 3.5]. Therefore, Theorem 3.5 implies the conclusion. □

Remark 4.1 In [10, Theorem 4.1], it was shown that $\{z_{n,0}\}$ in Theorem 4.1 converges weakly to the strong limit of $\{P_{F(T)}x_n\}$.

We next obtain the following theorem.

Theorem 4.2 *Let (ρ_n) be a sequence of $(0,1)$ such that $\rho_n \to 1$. Then the sequence $\{(1-\rho_n)\sum_{k=0}^{\infty}\rho_n^k x_{k+p}\}_{n,p\in\mathbb{N}}$ converges weakly to the strong limit of $\{P_{F(T)}x_n\}$ as $n\to\infty$ uniformly in $p\in\mathbb{N}$.*

Proof Let $\{\mu_n\}$ be the sequence of means on l^∞ defined by

$$\mu_n(f) = (1-\rho_n)\sum_{k=0}^{\infty}\rho_n^k f(k) \tag{4.3}$$

for all $n\in\mathbb{N}$ and $f\in l^\infty$. It is well known that $\{\mu_n\}$ is strongly regular and

$$G_{r_p^*\mu_n}\big(\{x_k\}\big) = (1-\rho_n)\sum_{k=0}^{\infty}\rho_n^k x_{k+p} \tag{4.4}$$

for each $n,p\in\mathbb{N}$; see, for instance, [2, Theorem 5.2] and [8, Section 3.5]. Therefore, Theorem 3.5 implies the conclusion. □

By using a strongly regular matrix introduced in [16], we can obtain the following theorem which actually generalizes Theorems 4.1 and 4.2.

Theorem 4.3 *Let $(q_{n,k})_{n,k\in\mathbb{N}}$ be a sequence of real numbers such that*

(A1) $q_{n,k}\ge 0$ *for all* $n,k\in\mathbb{N}$;
(A2) $\sum_{k=0}^{\infty}q_{n,k}=1$ *for all* $n\in\mathbb{N}$;
(A3) $\lim_n\sum_{k=0}^{\infty}|q_{n,k}-q_{n,k+1}|=0$.

Then the sequence $\{\sum_{k=0}^{\infty}q_{n,k}x_{k+p}\}_{n,p\in\mathbb{N}}$ converges weakly to the strong limit of $\{P_{F(T)}x_n\}$ as $n\to\infty$ uniformly in $p\in\mathbb{N}$.

Proof Let $\{\mu_n\}$ be the sequence of means on l^∞ defined by

$$\mu_n(f) = \sum_{k=0}^{\infty}q_{n,k}f(k) \tag{4.5}$$

for all $n\in\mathbb{N}$ and $f\in l^\infty$. It is well known that $\{\mu_n\}$ is strongly regular and

$$G_{r_p^*\mu_n}\big(\{x_k\}\big) = \sum_{k=0}^{\infty}q_{n,k}x_{k+p} \tag{4.6}$$

for each $n,p\in\mathbb{N}$; see, for instance, [2, Theorem 5.3] and [4, Theorem 7].

For the sake of completeness, we give the proof of this fact. It follows from (A1) that

$$q_{n,0} = |q_{n,0}| \le \sum_{k=0}^{m}|q_{n,k}-q_{n,k+1}| + q_{n,m+1} \tag{4.7}$$

for all $n, m \in \mathbb{N}$. It follows from (A2) that $\lim_k q_{n,k} = 0$ for all $n \in \mathbb{N}$. Thus, letting $m \to \infty$ in (4.7), we have

$$q_{n,0} \le \sum_{k=0}^{\infty} |q_{n,k} - q_{n,k+1}| \tag{4.8}$$

for all $n \in \mathbb{N}$. It also holds that

$$\begin{aligned} \|r_1^* \mu_n - \mu_n\| &= \sup_{\|f\| \le 1} |\mu_n(r_1 f - f)| \\ &= \sup_{\|f\| \le 1} \left| \sum_{k=0}^{\infty} q_{n,k} \bigl(f(k+1) - f(k)\bigr) \right| \\ &= \sup_{\|f\| \le 1} \left| \sum_{k=0}^{\infty} (q_{n,k} - q_{n,k+1}) f(k+1) - q_{n,0} f(0) \right| \\ &\le \sum_{k=0}^{\infty} |q_{n,k} - q_{n,k+1}| + q_{n,0} \end{aligned} \tag{4.9}$$

for all $n \in \mathbb{N}$.

By (4.8), (4.9), and (A3), we have

$$\|r_1^* \mu_n - \mu_n\| \le 2 \sum_{k=0}^{\infty} |q_{n,k} - q_{n,k+1}| \to 0 \tag{4.10}$$

and hence $\{\mu_n\}$ is strongly regular. On the other hand, if $n, p \in \mathbb{N}$, then we have

$$[r_p^* \mu_n]_k \langle x_k, y \rangle = [\mu_n]_k \langle x_{k+p}, y \rangle = \sum_{k=0}^{\infty} q_{n,k} \langle x_{k+p}, y \rangle = \left\langle \sum_{k=0}^{\infty} q_{n,k} x_{k+p}, y \right\rangle \tag{4.11}$$

for all $y \in H$. Thus, (4.6) holds. Therefore, Theorem 3.5 implies the conclusion. □

5 Applications

In this final section, we give two applications of Theorem 4.3. We first obtain a corollary for a single λ-hybrid mapping; see Corollary 5.1. We next study the problem of finding common fixed points of sequences of nonexpansive mappings; see Corollary 5.3.

Throughout this section, we suppose that (β_n) is a sequence of $(0,1)$ satisfying $\beta_n \to 0$ and $(q_{n,k})_{n,k \in \mathbb{N}}$ is the sequence of real numbers defined by $q_{0,0} = 1$, $q_{0,k} = 0$ $(k \ge 1)$, and

$$q_{n,k} = \begin{cases} n^{-1}(1 - \beta_n) & (0 \le k \le n-1); \\ \beta_n & (k = n); \\ 0 & (k \ge n+1) \end{cases}$$

for $n \ge 1$. The sequence $(q_{n,k})_{n,k \in \mathbb{N}}$ obviously satisfies (A1)-(A3) in Theorem 4.3.

Corollary 5.1 *Let C be a nonempty closed convex subset of a Hilbert space H, $T \in H_\lambda(C)$ for some $\lambda \in \mathbb{R}$ such that $F(T)$ is nonempty, and (α_n) a sequence of $[0,1)$ such that $\alpha_n \to 0$.*

Let $\{x_n\}$ be the sequence of C defined by $x_0 \in C$ and

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tx_n$$

for $n \in \mathbb{N}$. Then $\{\sum_{k=0}^{n} q_{n,k}x_{k+p}\}_{n,p\in\mathbb{N}}$ converges weakly to the strong limit of $\{P_{F(T)}x_n\}$ as $n \to \infty$ uniformly in $p \in \mathbb{N}$.

Proof Let $\{T_n\}$ be the sequence of mapping of C into itself defined by

$$T_n = \alpha_n I + (1-\alpha_n)T \tag{5.1}$$

for all $n \in \mathbb{N}$. Then it is clear that $x_{n+1} = T_n x_n$ for all $n \in \mathbb{N}$ and $T_n x \to Tx$ for all $x \in C$. It is also clear that $F(T_n) = F(T)$ for all $n \in \mathbb{N}$ and hence $\emptyset \neq F(T) = \bigcap_{n=0}^{\infty} F(T_n)$.

Since $T \in H_\lambda(C)$, we know that

$$\begin{aligned}\|T_n x - T_n y\|^2 &\leq \alpha_n\|x-y\|^2 + (1-\alpha_n)\|Tx-Ty\|^2\\ &\leq \|x-y\|^2 + 2(1-\lambda)(1-\alpha_n)\langle x-Tx, y-Ty\rangle\\ &= \|x-y\|^2 + 2\bigl(1-\bigl(\lambda+(1-\lambda)\alpha_n\bigr)\bigr)\langle x-Tx, y-Ty\rangle\end{aligned}$$

for all $n \in \mathbb{N}$ and $x, y \in C$. Thus, by setting $\lambda_n = \lambda + (1-\lambda)\alpha_n$ for all $n \in \mathbb{N}$, we know that $T_n \in H_{\lambda_n}(C)$ for all $n \in \mathbb{N}$. It is clear that $\lambda_n \to \lambda$.

Since $q_{n,k} = 0$ for $k \geq n+1$, it also holds that $\sum_{k=0}^{n} q_{n,k}x_{k+p} = \sum_{k=0}^{\infty} q_{n,k}x_{k+p}$ for all $n, p \in \mathbb{N}$. Consequently, Theorem 4.3 implies the conclusion. □

In order to obtain our final result, we need the following theorem, which was originally shown in strictly convex Banach spaces.

Lemma 5.2 ([17, Lemma 3]) *Let C be a nonempty closed convex subset of a Hilbert space H, $\{T_n\}$ a sequence of nonexpansive mappings of C into H such that $\bigcap_{n=0}^{\infty} F(T_n)$ is nonempty, and (γ_n) a sequence of $(0,1)$ such that $\sum_{k=0}^{\infty} \gamma_k = 1$. Then the mapping $T = \sum_{k=0}^{\infty} \gamma_k T_k$ is a nonexpansive mapping of C into H such that $F(T) = \bigcap_{k=0}^{\infty} F(T_k)$.*

Remark 5.1 If $T_n(C) \subset C$ for all $n \in \mathbb{N}$ in Lemma 5.2, then $T(C) \subset C$. Indeed, for each $x \in C$, we have

$$Tx = \lim_{N\to\infty} \frac{1}{\sum_{j=0}^{N} \gamma_j} \sum_{k=0}^{N} \gamma_k T_k x \in C \tag{5.2}$$

and hence T is a self-mapping on C.

As in the proof of [9, Theorem 3.7], we can show the following corollary.

Corollary 5.3 *Let C be a nonempty closed convex subset of a Hilbert space H, $\{S_n\}$ a sequence of nonexpansive mappings of C into itself such that $F = \bigcap_{k=0}^{\infty} F(S_k)$ is nonempty, and (γ_n) a sequence of $(0,1)$ such that $\sum_{n=0}^{\infty} \gamma_n = 1$. Let $\{x_n\}$ be the sequence of C defined by*

$x_0 \in C$ and

$$x_{n+1} = \sum_{k=0}^{n} \gamma_k S_k x_n + \left(1 - \sum_{k=0}^{n} \gamma_k\right) S_{n+1} x_n \tag{5.3}$$

for $n \in \mathbb{N}$. Then $\{\sum_{k=0}^{n} q_{n,k} x_{k+p}\}_{n,p\in\mathbb{N}}$ converges weakly to the strong limit of $\{P_F x_n\}$ as $n \to \infty$ uniformly in $p \in \mathbb{N}$.

Proof Let $\{T_n\}$ be the sequence of mappings of C into itself defined by

$$T_n = \sum_{k=0}^{n} \gamma_k S_k + \left(1 - \sum_{k=0}^{n} \gamma_k\right) S_{n+1} \tag{5.4}$$

for all $n \in \mathbb{N}$. It is clear that $x_{n+1} = T_n x_n$ for all $n \in \mathbb{N}$. Since each T_n is nonexpansive, we know that $T_n \in H_1(C)$ for all $n \in \mathbb{N}$.

By Lemma 5.2 and Remark 5.1, the mapping $T = \sum_{k=0}^{\infty} \gamma_k S_k$ is a nonexpansive mapping of C into itself such that $F(T) = F$. Since F is nonempty by assumption, so is $F(T)$. By Lemma 5.2, we also know that $F(T_n) = \bigcap_{k=0}^{n+1} F(S_k)$ and hence we have

$$\bigcap_{n=0}^{\infty} F(T_n) = \bigcap_{n=0}^{\infty} \bigcap_{k=0}^{n+1} F(S_k) = F = F(T). \tag{5.5}$$

It remains to be seen that $T_n x \to Tx$ for all $x \in C$. Fix $x \in C$. Since F is nonempty, we can fix $p \in F$. Since $\|p - S_k x\| \le \|p - x\|$ for all $k \in \mathbb{N}$, we know that $L = \sup_{k\in\mathbb{N}} \|S_k x\|$ is finite. By $\sum_{k=0}^{\infty} \gamma_k = 1$ and the definitions of T and T_n, we also know that

$$\begin{aligned} \|Tx - T_n x\| &= \left\| \sum_{k=0}^{\infty} \gamma_k S_k x - \sum_{k=0}^{n} \gamma_k S_k x - \left(1 - \sum_{k=0}^{n} \gamma_k\right) S_{n+1} x \right\| \\ &= \left\| \sum_{k=n+1}^{\infty} \gamma_k S_k x - \left(1 - \sum_{k=0}^{n} \gamma_k\right) S_{n+1} x \right\| \\ &\le \sum_{k=n+1}^{\infty} \gamma_k \|S_k x\| + \left(1 - \sum_{k=0}^{n} \gamma_k\right) \|S_{n+1} x\| \\ &\le 2L\left(1 - \sum_{k=0}^{n} \gamma_k\right) \to 0 \end{aligned} \tag{5.6}$$

as $n \to \infty$. Thus, $T_n x \to Tx$. Consequently, Theorem 4.3 implies the conclusion. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

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