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A note on (h, q)-Boole polynomials

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Abstract

Kim *et al.* (Appl. Math. Inf. Sci. 9(6):1-6, 2015) consider the q-extensions of Boole polynomials. In this paper, we consider Witt-type formula for the q-Boole polynomials with weights and derive some new interesting identities and properties of those polynomials and numbers from the Witt-type formula which are related to special polynomials and numbers.

Keywords: (h, q)-Euler polynomials; (h, q)-Boole numbers and polynomials; p-adic invariant integral on $\mathbb{Z}_p$

1 Introduction

Let p be a fixed odd prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ will, respectively, denote the ring of p-adic rational integers, the field of p-adic rational numbers, and the completions of algebraic closure of $\mathbb{Q}_p$. The p-adic norm is defined by $|p|_p = \frac{1}{p}$.

When one talks of q-extension, q is variously considered as an indeterminate, a complex $q \in \mathbb{C}$, or p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, one normally assumes that $|q| < 1$. If $q \in \mathbb{C}_p$, then we assume that $|q-1|_p < p^{-\frac{1}{p-1}}$ so that $q^x = \exp(x \log q)$ for each $x \in \mathbb{Z}_p$. Throughout this paper, we use the notation

$$[x]_{-q} = \frac{1-(-q)^x}{1-(-q)}.$$

Note that $\lim_{q\to -1}[x]_{-q} = x$ for each $x \in \mathbb{Z}_p$.

Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable functions on $\mathbb{Z}_p$. For $f \in UD(\mathbb{Z}_p)$, the *p-adic invariant integral on $\mathbb{Z}_p$* is defined by Kim as follows:

$$I_{-q}(f) = \int_{\mathbb{Z}_p} f(x)\,d\mu_{-q}(x) = \lim_{N\to\infty} \frac{1}{[p^N]_{-q}} \sum_{x=0}^{p^N-1} f(x)(-q)^x \quad \text{(see [1–5])}. \tag{1.1}$$

Let f_1 be the translation of f with $f_1(x) = f(x+1)$. Then, by (1.1), we get

$$I_{-q}(f_1) + I_{-q}(f) = [2]_q f(0). \tag{1.2}$$

As is well known, the *Stirling number of the first kind* is defined by

$$(x)_n = x(x-1)\cdots(x-n+1) = \sum_{l=0}^{n} S_1(n,l)x^l, \tag{1.3}$$

and the *Stirling number of the second kind* is given by the generating function:

$$\left(e^t-1\right)^m = m!\sum_{l=m}^{\infty} S_2(l,m)\frac{t^l}{l!} \quad \text{(see [6, 7])}. \tag{1.4}$$

It is well known that the (h,q)-*Euler polynomials* are defined by the generating function:

$$\left(\frac{q+1}{q^h e^t+1}\right)e^{xt} = \sum_{n=0}^{\infty} E_{n,q}(x|h)\frac{t^n}{n!} \quad \text{(see [8])}, \tag{1.5}$$

where h is an integer. When $x=0$ and $h=0$, $E_{n,q}(0|h)=E_{n,q}(h)$ are called the *ordinary* q*-Euler numbers*.

Recently, DS Kim and T Kim introduced the *Changhee polynomials of the first kind* are defined by the generating function:

$$\frac{2}{2+t}(1+t)^x = \sum_{n=0}^{\infty} Ch_n(x)\frac{t^n}{n!} \quad \text{(see [1, 9–11])}, \tag{1.6}$$

and T Kim *et al.* defined the q*-Changhee polynomials* as follows:

$$\frac{[2]_q}{q(1+t)+1}(1+t)^x = \sum_{n=0}^{\infty} Ch_{n,q}(x)\frac{t^n}{n!} \quad \text{(see [9, 11, 12])}. \tag{1.7}$$

As is well known, the *Boole polynomials* are defined by the generating function:

$$\sum_{n=0}^{\infty} Bl_n(x|\lambda)\frac{t^n}{n!} = \frac{1}{1+(1+t)^\lambda} \quad \text{(see [7, 13])}.$$

When $\lambda=1$, $2Bl_n(x|1)=Ch_n(x)$ are Changhee polynomials. In [11], Kim *et al.* consider the q-analog of Boole polynomials, and found some new and interesting identities related to special polynomials, and Y Do and D Lim investigated the properties of (h,q)-Daehee numbers and polynomials, which are defined by

$$\int_{\mathbb{Z}_p} q^{-hy}(x+y)_n\,d\mu_q(y) \quad \text{(see [14])}.$$

In this paper, we consider Witt-type formula for the q-Boole polynomials with weights and derive some new interesting identities and properties of those polynomials and numbers from the Witt-type formula which are related to special polynomials and numbers.

2 q-Analog of Boole polynomials with weight

In this section, we assume that $t\in\mathbb{C}_p$ with $|t|_p<p^{-\frac{1}{p-1}}$, $\lambda\in\mathbb{Z}_p$ with $\lambda\neq 0$ and $h\in\mathbb{Z}$. From (1.2), we have

$$\int_{\mathbb{Z}_p} q^{(h-1)y}(1+t)^{x+\lambda y}\,d\mu_{-q}(y) = \frac{1+q}{q^h(1+t)^\lambda+1}(1+t)^x = \sum_{n=0}^{\infty}[2]_q Bl_{n,q}(x|h,\lambda)\frac{t^n}{n!}, \tag{2.1}$$

where $Bl_{n,q}(x|h,\lambda)$ are the (h,q)-Boole polynomials which are defined by

$$\frac{1}{q^h(1+t)^\lambda+1}(1+t)^x=\sum_{n=0}^{\infty}Bl_{n,q}(x|h,\lambda)\frac{t^n}{n!}. \tag{2.2}$$

By (2.1), we can derive the following equation:

$$\int_{\mathbb{Z}_p}q^{(h-1)y}\binom{x+\lambda y}{n}\,d\mu_{-q}=\frac{1+q}{n!}Bl_{n,q}(x|h,\lambda). \tag{2.3}$$

In the special case $x=0$, $Bl_{n,q}(0|h,\lambda)=Bl_{n,q}(h,\lambda)$ are called the (h,q)-*Boole numbers*.

Note that

$$\begin{aligned}(1+t)^{x+\lambda y}&=e^{(x+\lambda y)\log(1+t)}\\&=\sum_{n=0}^{\infty}\frac{(x+\lambda y)^n}{n!}\bigl(\log(1+t)\bigr)^n\\&=\sum_{n=0}^{\infty}\frac{(x+\lambda y)^n}{n!}m!\sum_{m=n}^{\infty}S_1(m,n)\frac{t^m}{m!}\\&=\sum_{n=0}^{\infty}\left\{\sum_{m=0}^{n}(x+\lambda y)^m S_1(n,m)\right\}\frac{t^n}{n!}.\end{aligned} \tag{2.4}$$

The (h,q)-*Euler polynomials* are defined by the generating function:

$$\frac{1+q}{q^he^t+1}e^{xt}=\sum_{n=0}^{\infty}E_{n,q}(x|h)\frac{t^n}{n!}. \tag{2.5}$$

Note that $\lim_{q\to1}E_{n,q}(x|1)=E_n(x)$. When $x=0$, $E_n(0|h)=E_{n,q}(h)$ are called the (h,q)-*Euler numbers*.

By (1.2), we can derive easily the following equation:

$$\int_{\mathbb{Z}_p}q^{(h-1)y}e^{(x+y)t}\,d\mu_{-q}(y)=\frac{1+q}{q^he^t+1}e^{xt}=\sum_{n=0}^{\infty}E_{n,q}(x|h)\frac{t^n}{n!}. \tag{2.6}$$

Since

$$\int_{\mathbb{Z}_p}q^{(h-1)y}e^{(x+y)t}\,d\mu_{-q}(y)=\sum_{n=0}^{\infty}\int_{\mathbb{Z}_p}q^{(h-1)y}(x+y)^n\,d\mu_{-q}(y)\frac{t^n}{n!},$$

by (2.5), we have

$$\int_{\mathbb{Z}_p}q^{(h-1)y}(x+y)^n\,d\mu_{-q}(y)=E_{n,q}(x|h)\quad(n\ge0). \tag{2.7}$$

From (2.4) and (2.7), we get

$$\begin{aligned}&\int_{\mathbb{Z}_p} q^{(h-1)y}(1+t)^{x+\lambda y}\,d\mu_{-q}(y)\\&\quad=\sum_{n=0}^{\infty}\left\{\sum_{m=0}^{n}\int_{\mathbb{Z}_p} q^{(h-1)y}(x+\lambda y)^m\,d\mu_{-q}(y)S_1(n,m)\right\}\frac{t^n}{n!}\\&\quad=\sum_{n=0}^{\infty}\left\{\sum_{m=0}^{n}\lambda^m E_{m,q}\left(\frac{x}{\lambda}|h\right)S_1(n,m)\right\}\frac{t^n}{n!}.\end{aligned}\tag{2.8}$$

Thus, by (2.2), (2.3), and (2.8), we obtain the following theorem.

Theorem 2.1 *For $n \geq 0$, we have*

$$Bl_{n,q}(x|h,\lambda)=\frac{1}{[2]_q}\sum_{m=0}^{n}\lambda^m E_{m,q}\left(\frac{x}{\lambda}\middle|h\right)S_1(n,m)$$

and

$$\int_{\mathbb{Z}_p} q^{(h-1)y}\binom{x+\lambda y}{n}\,d\mu_{-q}=\frac{[2]_q}{n!}Bl_{n,q}(x|h,\lambda).$$

By Theorem 2.1, we note that

$$Bl_{n,q}(x|h,\lambda)=\frac{1}{[2]_q}\int_{\mathbb{Z}_p} q^{(h-1)y}(x+\lambda y)_n\,d\mu_{-q}(y),$$

where $(x)_n = x(x-1)\cdots(x-n+1)$. When $\lambda = 1$ and $h = 0$, we have

$$Bl_{n,q}(x|0,1)=\frac{1}{[2]_q}\int_{\mathbb{Z}_p} q^{-1}(x+y)^n\,d\mu_{-q}(y).\tag{2.9}$$

In [13], Arici *et al.* defined the *q-analog of Changhee polynomials* by the generating function:

$$\sum_{n=0}^{\infty}Ch_n(x|q)\frac{t^n}{n!}=\frac{[2]_q}{[2]_t+1}(1+t)^x.\tag{2.10}$$

By (2.10), we have

$$\int_{\mathbb{Z}_p} q^{-y}(1+t)^{x+y}\,d\mu_{-q}(y)=\frac{[2]_q}{[2]_t+1}(1+t)^x=\sum_{n=0}^{\infty}Ch_n(x|q)\frac{t^n}{n!}.\tag{2.11}$$

By (1.6) and (2.10), we note that

$$\frac{[2]_q}{2}Ch_n(x)=Ch_n(x|q).\tag{2.12}$$

From (2.11), we get

$$\int_{\mathbb{Z}_p} q^{-1}(x+y)_n\,d\mu_{-q}(y)=Ch_n(x|q).\tag{2.13}$$

By (2.9), (2.12), and (2.13), we have

$$Bl_{n,q}(x|0,1) = \frac{1}{[2]_q} Ch_n(x|q) = \frac{1}{2} Ch_n(x).$$

By replacing t as $e^t - 1$ in (2.1), we derive the following equations:

$$\begin{aligned} \frac{1+q}{q^h e^{\lambda t}+1} e^{xt} &= \sum_{n=0}^{\infty} [2]_q Bl_{n,q}(x|h,\lambda) \frac{1}{n!} \left(e^t - 1\right)^n \\ &= \sum_{n=0}^{\infty} [2]_q Bl_{n,q}(x|h,\lambda) \frac{1}{n!} n! \sum_{m=n}^{\infty} S_2(m,n) \frac{t^m}{m!} \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{n} [2]_q Bl_{m,q}(x|h,\lambda) S_2(n,m) \frac{t^n}{n!} \end{aligned} \tag{2.14}$$

and

$$\frac{1+q}{q^h e^{\lambda t}+1} e^{xt} = \frac{1+q}{q^h e^{\lambda t}+1} e^{(\frac{x}{\lambda})\lambda t} = \sum_{n=0}^{\infty} E_{n,q}\left(\frac{x}{\lambda}\middle|h\right) \lambda^m \frac{t^m}{m!}. \tag{2.15}$$

Hence, by (2.14) and (2.15), we obtain the following theorem.

Theorem 2.2 *For $n \geq 0$, we have*

$$\sum_{m=0}^{n} Bl_{m,q}(x|h,\lambda) S_2(n,m) = \frac{\lambda^m}{q+1} E_{n,q}\left(\frac{x}{\lambda}\middle|h\right).$$

From now on, we define the $(h_1,\ldots,h_r,q)$-*Boole numbers of the first kind* as follows:

$$\begin{aligned} &[2]_q^r Bl_{n,q}^{(h_1,\ldots,h_r)}(\lambda) \\ &\quad = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{h_1+\cdots+h_r-r} \left(\lambda(x_1+\cdots+x_r)\right)_n d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_r) \quad (n \geq 0). \end{aligned} \tag{2.16}$$

By (2.16), we have

$$\begin{aligned} &[2]_q^r \sum_{n=0}^{\infty} Bl_{n,q}^{(h_1,\ldots,h_r)}(\lambda) \frac{t^n}{n!} \\ &\quad = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \sum_{n=0}^{\infty} q^{h_1+\cdots+h_r-r} \binom{\lambda(x_1+\cdots+x_r)}{n} t^n \, d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_r) \\ &\quad = \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{h_1+\cdots+h_r-r} (1+t)^{\lambda(x_1+\cdots+x_k)} \, d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_r) \\ &\quad = \prod_{i=1}^{r} \left(\frac{1+q}{q^{h_i}(1+t)^{\lambda}+1}\right) \\ &\quad = (1+q)^r \sum_{n=0}^{\infty} \left(\sum_{l_1+\cdots+l_r=n} \binom{n}{l_1,\ldots,l_r} B_{i_1,q}(h,\lambda)\cdots B_{i_r,q}(h,\lambda)\right) \frac{t^n}{n!}. \end{aligned} \tag{2.17}$$

Thus, by (2.17), we obtain the following corollary.

Corollary 2.3 *For $n \geq 0$, we have*

$$Bl_{n,q}^{(h_1,\dots,h_r)}(\lambda) = \sum_{l_1+\cdots+l_r=n} \binom{n}{l_1,\dots,l_r} B_{i_1,q}(h,\lambda)\cdots B_{i_r,q}(h,\lambda).$$

The $(h_1,\dots,h_r,q)$-*Euler polynomials* are defined by the generating function to be

$$\begin{aligned}
&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} q^{h_1+\cdots+h_r-r} e^{(x_1+\cdots+x_r+x)t}\, d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_r) \\
&\quad = \prod_{i=1}^{r}\left(\frac{1+q}{q^{h_i}e^t+1}\right)e^{xt} \\
&\quad = \sum_{n=0}^{\infty} E_{n,q}(x|h_1,\dots,h_r)\frac{t^n}{n!}.
\end{aligned} \tag{2.18}$$

By (2.18), we have

$$\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} q^{h_1+\cdots+h_r-r}(x_1+\cdots+x_r+x)^n\, d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_r) = E_{n,q}(x|h_1,\dots,h_r).$$

In the special case $x = 0$, $E_{n,q}(0|h_1,\dots,h_r) = E_{n,q}(h_1,\dots,h_r)$ are called the $(h_1,\dots,h_r,q)$-*Euler numbers*.

From (1.5) and (2.16), we note that

$$\begin{aligned}
&(1+q)^r Bl_{n,q}^{(h_1,\dots,h_r)}(\lambda) \\
&\quad = \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} q^{h_1+\cdots+h_r-r}\big(\lambda(x_1+\cdots+x_r)\big)_n\, d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_r) \\
&\quad = \sum_{l=0}^{n} S_1(n,l)\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} q^{h_1+\cdots+h_r-r}\lambda^l(x_1+\cdots+x_r)^l\, d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_r) \\
&\quad = \sum_{l=0}^{n} S_1(n,l)\lambda^l E_{l,q}(h_1,\dots,h_r).
\end{aligned} \tag{2.19}$$

Therefore, by (2.19), we obtain the following theorem.

Theorem 2.4 *For $n \geq 0$, we get*

$$Bl_{n,q}^{(h_1,\dots,h_r)}(\lambda) = \frac{1}{(1+q)^r}\sum_{l=0}^{n} S_1(n,l)\lambda^l E_{l,q}(h_1,\dots,h_r).$$

By replacing t by $e^t - 1$ in (2.17), we have

$$\begin{aligned}
[2]_q^r \sum_{n=0}^{\infty} Bl_{n,q}^{(h_1,\dots,h_r)}(\lambda)\frac{(e^t-1)^n}{n!} &= \prod_{i=1}^{r}\left(\frac{1+q}{q^{h_i}e^{\lambda t}+1}\right) \\
&= \sum_{n=0}^{\infty} E_{n,q}(h_1,\dots,h_r)\lambda^n\frac{t^n}{n!}
\end{aligned} \tag{2.20}$$

and

$$[2]_q^r \sum_{n=0}^{\infty} Bl_{n,q}^{(h_1,\ldots,h_r)}(\lambda)\frac{1}{n!}\left(e^t-1\right)^n = [2]_q^r \sum_{n=0}^{\infty} Bl_{n,q}^{(h_1,\ldots,h_r)}(\lambda)\sum_{m=n}^{\infty} S_2(m,n)\frac{t^m}{m!}$$
$$= [2]_q^r \sum_{n=0}^{\infty}\left\{\sum_{m=0}^{n} Bl_{m,q}^{(h_1,\ldots,h_r)}(\lambda)S_2(n,m)\right\}\frac{t^n}{n!}. \tag{2.21}$$

Hence, by (2.20) and (2.21), we obtain the following theorem.

Theorem 2.5 *For $n \geq 0$, we have*

$$\frac{\lambda^n}{[2]_q^r}E_{n,q}(h_1,\ldots,h_r) = \sum_{m=0}^{n} Bl_{m,q}^{(h_1,\ldots,h_r)}(\lambda)S_2(n,m).$$

Let us define the $(h_1,\ldots,h_r,q)$-*Boole polynomials of the first kind* as follows:

$$[2]_q^r Bl_{n,q}^{(h_1,\ldots,h_r)}(x|\lambda)$$
$$= \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} q^{h_1+\cdots+h_r-r}\big(\lambda(x_1+\cdots+x_r)+x\big)_n\,d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_r), \tag{2.22}$$

where $n \geq 0$ and $r \in \mathbb{N}$. By (2.22), we can derive the generating function of the $(h_1,\ldots,h_r,q)$-Boole polynomials of the first kind as follows:

$$[2]_q^r \sum_{n=0}^{\infty} Bl_{n,q}^{(h_1,\ldots,h_r)}(x|\lambda)\frac{t^n}{n!}$$
$$= \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} q^{h_1+\cdots+h_r-r}(1+t)^{\lambda(x_1+\cdots+x_r)+x}\,d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_r)$$
$$= \prod_{i=1}^{r}\left(\frac{1+q}{q^{h_i}(1+t)^{\lambda}+1}\right)(1+t)^x. \tag{2.23}$$

By (2.23), we can see easily

$$\prod_{i=1}^{r}\left(\frac{1+q}{q^{h_i}(1+t)^{\lambda}+1}\right)(1+t)^x$$
$$= [2]_q^r\left(\sum_{n=0}^{\infty} Bl_{n,q}^{(h_1,\ldots,h_r)}(\lambda)\frac{t^n}{n!}\right)\left(\sum_{m=0}^{\infty}\binom{x}{m}t^m\right)$$
$$= [2]_q^r\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n} m!\binom{x}{m}\frac{n!}{(n-m)!m!}Bl_{n-m,q}^{(h_1,\ldots,h_r)}(\lambda)\right)\frac{t^n}{n!}$$
$$= [2]_q^r\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n} m!\binom{x}{m}\binom{n}{m}Bl_{n-m,q}^{(h_1,\ldots,h_r)}(\lambda)\right)\frac{t^n}{n!}$$
$$= [2]_q^r\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}Bl_{n-m,q}^{(h_1,\ldots,h_r)}(\lambda)(x)_m\right)\frac{t^n}{n!}. \tag{2.24}$$

By (2.23) and (2.24), we obtain the following theorem.

Theorem 2.6 *For $n \geq 0$, we have*

$$Bl_{n,q}^{(h_1,\ldots,h_r)}(x|\lambda) = \sum_{m=0}^{n} \binom{n}{m} Bl_{n-m,q}^{(h_1,\ldots,h_r)}(\lambda)(x)_m.$$

Replacing t as $e^t - 1$ in (2.23), we get

$$\begin{aligned}
[2]_q^r \sum_{n=0}^{\infty} Bl_{n,q}^{(h_1,\ldots,h_r)}(x|\lambda) \frac{1}{n!} \left(e^t - 1\right)^n &= \prod_{i=1}^{n} \left(\frac{1+q}{q^{h_i} e^{\lambda t} + 1} \right) e^{xt} \\
&= \sum_{n=0}^{\infty} E_{n,q}^{(h_1,\ldots,h_r)} \left(\frac{x}{\lambda} \right) \lambda^n \frac{t^n}{n!}
\end{aligned} \tag{2.25}$$

and

$$\begin{aligned}
&[2]_q^r \sum_{n=0}^{\infty} Bl_{n,q}^{(h_1,\ldots,h_r)}(x|\lambda) \frac{(e^t - 1)^n}{n!} \\
&\quad = [2]_q^r \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} Bl_{m,q}^{(h_1,\ldots,h_r)}(x|\lambda) S_2(n,m) \right) \frac{t^n}{n!}.
\end{aligned} \tag{2.26}$$

Hence, by (2.25) and (2.26), we obtain the following theorem.

Theorem 2.7 *For $n \geq 0$, we have*

$$\sum_{m=0}^{n} Bl_{m,q}^{(h_1,\ldots,h_r)}(x|\lambda) S_2(n,m) = \frac{\lambda^n}{[2]_q^r} E_{n,q}^{(h_1,\ldots,h_r)} \left(\frac{x}{\lambda} \right).$$

From (2.23), we get

$$\begin{aligned}
&[2]_q^r Bl_{n,q}^{(h_1,\ldots,h_r)}(x|\lambda) \\
&\quad = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{h_1+\cdots+h_r-r} \big(\lambda(x_1 + \cdots + x_r) + x\big)_n \, d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_r) \\
&\quad = \sum_{l=0}^{n} S_1(n,l) \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{h_1+\cdots+h_r-r} \big(\lambda(x_1 + \cdots + x_r) + x\big)^l \, d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_r) \\
&\quad = \sum_{l=0}^{n} S_1(n,l) \lambda^l E_{n,q}^{(h_1,\ldots,h_r)} \left(\frac{x}{\lambda} \right).
\end{aligned} \tag{2.27}$$

Thus, by (2.27), we obtain the following theorem.

Theorem 2.8 *For $n \geq 0$, we have*

$$Bl_{n,q}^{(h_1,\ldots,h_r)}(x|\lambda) = \frac{1}{[2]_q^r} \sum_{l=0}^{n} S_1(n,l) \lambda^l E_{n,q}^{(h_1,\ldots,h_r)} \left(\frac{x}{\lambda} \right).$$

Now, we define the (h,q)-*Boole polynomials of the second kind* as follows:

$$\widehat{Bl}_{n,q}(x|h,\lambda) = \frac{1}{[2]_q} \int_{\mathbb{Z}_p} q^{(h-1)y} (-\lambda y + x)_n \, d\mu_{-q}(y) \quad (n \geq 0). \tag{2.28}$$

By (2.28), we have

$$\begin{aligned}\widehat{Bl}_{n,q}(x|h,\lambda) &= \frac{1}{[2]_q}\sum_{l=0}^{n}(-\lambda)^l S_1(n,l)\int_{\mathbb{Z}_p}\left(y-\frac{x}{\lambda}\right)^l d\mu_{-q}(y)\\ &= \frac{1}{[2]_q}\sum_{l=0}^{n}(-\lambda)^l S_1(n,l)E_{l,q}\left(-\frac{x}{\lambda}\right).\end{aligned} \tag{2.29}$$

In the special case $x=0$, $\widehat{Bl}_{n,q}(0|h,\lambda)=\widehat{Bl}_{n,q}(h,\lambda)$ are called the *(h,q)-Boole numbers of the second kind*. From (2.29), we can derive the generating function of $\widehat{Bl}_{n,q}(x|\lambda)$ as follows:

$$\begin{aligned}\sum_{n=0}^{\infty}\widehat{Bl}_{n,q}(x|h,\lambda)\frac{t^n}{n!} &= \frac{1}{[2]_q}\int_{\mathbb{Z}_p} q^{(h-1)y}(1+t)^{-\lambda y+x}\,d\mu_{-q}(y)\\ &= \frac{(1+t)^{\lambda}}{q^h+(1+t)^{\lambda}}(1+t)^x.\end{aligned} \tag{2.30}$$

By replacing t by e^t-1 in (2.30), we have

$$\begin{aligned}\sum_{n=0}^{\infty}\widehat{Bl}_{n,q}(x|h,\lambda)\frac{(e^t-1)^n}{n!} &= \frac{e^{\lambda t}}{q^h+e^{\lambda t}}e^{xt}\\ &= \frac{1}{1+q}\sum_{n=0}^{\infty}(-\lambda)^n E_{n,q}\left(-\frac{\lambda}{x}\middle|h\right)\frac{t^n}{n!}\end{aligned} \tag{2.31}$$

and

$$\sum_{n=0}^{\infty}\widehat{Bl}_{n,q}(x|h,\lambda)\frac{(e^t-1)^n}{n!} = \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\widehat{Bl}_{m,q}(x|h,\lambda)S_2(n,m)\right)\frac{t^n}{n!}. \tag{2.32}$$

By (2.31) and (2.32), we obtain the following theorem.

Theorem 2.9 *For $n\geq 0$, we have*

$$\widehat{Bl}_{n,q}(x|h,\lambda) = \frac{1}{[2]_q}\sum_{l=0}^{n}(-\lambda)^l S_1(n,l)E_{l,q}\left(-\frac{x}{\lambda}\right)$$

and

$$\frac{1}{[2]_q}(-\lambda)^n E_{n,q}\left(-\frac{\lambda}{x}\middle|h\right) = \sum_{m=0}^{n}\widehat{Bl}_{m,q}(x|h,\lambda)S_2(n,m).$$

For $h_1,\ldots,h_r\in\mathbb{Z}$, we define the *$(h_1,\ldots,h_r,q)$-Boole polynomials of the second kind* as follows:

$$\begin{aligned}&\widehat{Bl}_{n,q}^{(h_1,\ldots,h_r)}(x|\lambda)\\ &\quad= \frac{1}{q+1}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} q^{(h_1+\cdots+h_r-r)y}\bigl(-\lambda(x_1+\cdots+x_r)+x\bigr)_n\,d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_r).\end{aligned} \tag{2.33}$$

By (2.33), we can derive the generating function of the $(h_1,\ldots,h_r,q)$-Boole polynomials of the second kind as follows:

$$\begin{aligned}\sum_{n=0}^{\infty}\widehat{Bl}_{n,q}^{(h_1,\ldots,h_r)}(x|\lambda)\frac{t^n}{n!} &= \frac{1}{(1+q)^r}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+t)^{-\lambda x_1-\cdots-\lambda x_r+x}\,d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_r)\\ &= \prod_{i=1}^{r}\left(\frac{(1+t)^{\lambda}}{q^{h_i}+(1+t)^{\lambda}}\right)(1+t)^x\\ &= \prod_{i=1}^{r}\left(\frac{1}{q^{h_i}(1+t)^{-\lambda}+1}\right)(1+t)^x\\ &= \sum_{n=0}^{\infty}Bl_{n,q}^{(h_1,\ldots,h_r)}(x|-\lambda)\frac{t^n}{n!}.\end{aligned} \tag{2.34}$$

Hence, by (2.34), we obtain the following proposition.

Proposition 2.10 *For $n\geq 0$, we have*

$$\widehat{Bl}_{n,q}^{(h_1,\ldots,h_r)}(x|\lambda) = Bl_{n,q}^{(h_1,\ldots,h_r)}(x|-\lambda).$$

Note that

$$\begin{aligned}\frac{(-1)^n[2]_q}{n!}Bl_{n,q}(x|h,\lambda) &= (-1)^n\int_{\mathbb{Z}_p}q^{(h-1)y}\binom{x+\lambda y}{n}\,d\mu_{-q}(y)\\ &= \int_{\mathbb{Z}_p}q^{(h-1)y}\binom{-x-\lambda y+n-1}{n}\,d\mu_{-q}(y)\\ &= \int_{\mathbb{Z}_p}q^{(h-1)y}\sum_{m=0}^{n}\binom{-x-\lambda y}{m}\binom{n-1}{n-m}\,d\mu_{-q}(y)\\ &= \sum_{m=0}^{n}\binom{n-1}{n-m}\int_{\mathbb{Z}_p}q^{(h-1)y}\binom{-x-\lambda y}{m}\,d\mu_{-q}(y)\\ &= [2]_q\sum_{m=0}^{n}\binom{n-1}{n-m}\frac{\widehat{Bl}_{m,q}(-x|h,\lambda)}{m!},\end{aligned} \tag{2.35}$$

and, by a similar method, we get

$$\begin{aligned}\frac{(-1)^n[2]_q}{n!}\widehat{Bl}_{n,q}(x|h,\lambda) &= (-1)^n\int_{\mathbb{Z}_p}q^{(h-1)y}\binom{x-\lambda y}{n}\,d\mu_{-q}(y)\\ &= [2]_q\sum_{m=0}^{n}\binom{n-1}{n-m}\frac{Bl_{m,q}(-x|h,\lambda)}{m!}.\end{aligned} \tag{2.36}$$

By (2.35) and (2.36), we obtain the following theorem.

Theorem 2.11 *For $n \geq 0$, we have*

$$\frac{(-1)^n}{n!} Bl_{n,q}(x|h,\lambda) = \sum_{m=0}^{n} \binom{n-1}{n-m} \frac{\widehat{Bl}_{m,q}(-x|h,\lambda)}{m!}$$

and

$$\frac{(-1)^n}{n!} \widehat{Bl}_{n,q}(x|h,\lambda) = \sum_{m=0}^{n} \binom{n-1}{n-m} \frac{Bl_{m,q}(-x|h,\lambda)}{m!}.$$

By Theorem 2.11, we obtain the following corollary.

Corollary 2.12 *For $n \geq 0$, we have*

$$Bl_{n,q}(x|h,\lambda) = \sum_{m=0}^{n} \sum_{k=0}^{m} (-1)^{n+m} \binom{n}{n-m, m-k, k} (n-1)_{l-1} Bl_{k,q}(x|h,\lambda)$$

where $\binom{n}{p,q,r} = \frac{n!}{p!q!r!}$, $p + q + r = n$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors contributed equally to the manuscript and typed, read, and approved the final manuscript.

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