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Fixed points of conditionally F-contractions in complete metric-like spaces

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Abstract

In this paper, we introduce the notion of a conditionally F-contraction in the setting of complete metric-like spaces and we investigate the existence of fixed points of such mappings. Our results unify, extend, and improve several results in the literature.

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1 Introduction and preliminaries

Recently, Wardowski [1] introduced the notion of a F-contraction mapping and investigated the existence of fixed points for such mappings. The results of Wardowski [1] extend and unify several fixed point results in the literature including the celebrated Banach contraction mapping principle.

In this paper, we present the notion of conditionally F-contractions of various types and we investigate the existence of a fixed point for such mappings in metric-like spaces. We also present some criteria for the uniqueness of a fixed point.

Throughout the paper, $\mathbb{N}$ and $\mathbb{N}_0$ denote the set of positive integers and the set of nonnegative integers. Similarly, let $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{R}_0^+$ represent the set of reals, positive reals, and the set of nonnegative reals, respectively.

Definition 1.1 [1] Let $\mathcal{F}$ be the family of all functions $F: (0, \infty) \to \mathbb{R}$ such that

(F1) F is strictly increasing, *i.e.* for all $x, y \in \mathbb{R}^+$ such that $x < y$, $F(x) < F(y)$;

(F2) for each sequence $\{\alpha_n\}_{n=1}^{\infty}$ of positive numbers, $\lim_{n \to \infty} \alpha_n = 0$ if and only if $\lim_{n \to \infty} F(\alpha_n) = -\infty$;

(F3) there exists $k \in (0, 1)$ such that $\lim_{\alpha \to 0^+} \alpha^k F(\alpha) = 0$.

Definition 1.2 [1] Let (X, d) be a metric space. A mapping $T: X \to X$ is said to be a F-contraction on (X, d) if there exist $F \in \mathcal{F}$ and $\tau > 0$ such that

$$\forall x, y \in X, \quad \left[d(Tx, Ty) > 0 \Rightarrow \tau + F\big(d(Tx, Ty)\big) \leq F\big(d(x, y)\big) \right]. \tag{1}$$

Remark 1.3 From (F1) and (1) it is easy to conclude that every F-contraction is necessarily continuous.

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Recently, Piri and Kumam [2] extended the result of Wardowski [1] by replacing the condition (F3) in Definition 1.1 with the following one:

(F3′) F is continuous on $(0, \infty)$.

Let $\mathfrak{F}$ denote the family of all functions $F : \mathbb{R}_+ \to \mathbb{R}$ which satisfy conditions (F1), (F2) and (F3′).

Under this new set-up, they proved a fixed point result that generalized the result of Wardowski [1].

Definition 1.4 [2] Let (X, d) be a metric space and let $F \in \mathfrak{F}$. A mapping $T : X \to X$ is said to be a F-Suzuki-contraction if there exist $F \in \mathfrak{F}$ and $\tau > 0$ such that for all $x, y \in X$ with $Tx \neq Ty$

$$\frac{1}{2} d(x, Tx) < d(x, y) \quad \Rightarrow \quad \tau + F\big(d(Tx, Ty)\big) \leq F\big(d(x, y)\big).$$

Wardowski and Van Dung [3] introduced the notion of a F-weak contraction and proved a fixed point theorem for F-weak contractions.

Definition 1.5 [3] Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be a F-weak contraction on (X, d) if there exist $F \in \mathcal{F}$ and $\tau > 0$ such that, for all $x, y \in X$ satisfying $d(Tx, Ty) > 0$, the following holds:

$$\tau + F\big(d(Tx, Ty)\big) \leq F\left(\max\left\{ d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2} \right\} \right). \tag{2}$$

Wardowski and Van Dung [3] gave an example to show that their result was a proper extension of results in the literature.

There are many papers in the literature that generalize the notion of metric spaces as well as the Banach contraction mapping principle (see [4–11] and the references therein). The notion of a metric-like space was introduced by Hitzler [12] and re-introduced by Amini-Harandi in [13].

Definition 1.6 (See [12]) Let X be a non-empty set. A mapping $d : X \times X \to \mathbb{R}_0^+$ is said to be a metric-like (dislocated) on X, if for all $x, y, z \in X$ the following conditions are satisfied:

(D1) if $d(x, y) = 0$ then $x = y$.
(D2) $d(x, y) = d(y, x)$.
(D3) $d(x, y) \leq d(x, z) + d(z, y)$.

The pair (X, d) is called a dislocated (metric-like) space.

Notice that if we replace the condition (D3) with

(D3*) $d(x, y) \leq d(x, z) + d(z, y) - d(z, z)$

in Definition 1.6 then, (X, d) turns to be a partial metric space (PMS). Fore more details see *e.g.* [14–18].

Remark 1.7 (See [13]) Every partial metric is metric-like (dislocate).

A sequence $\{x_n\}_{n=1}^{\infty}$, in a metric-like space (X, d),

(a) converges to $x \in X$ if $\lim_{n \to \infty} d(x_n, x) = d(x, x)$,

(b) is called Cauchy in (X, d), if $\lim_{n,m \to \infty} d(x_n, x_m)$ exists and is finite.

A metric-like space (X, d) is said to be complete if and only if every Cauchy sequence $\{x_n\}_{n=1}^{\infty}$ in X converges to $x \in X$ so that

$$\lim_{n,m \to \infty} d(x_n, x_m) = \lim_{n \to \infty} d(x_n, x) = d(x, x).$$

We recall some basic definitions and crucial results on the topic. In this paper, we follow the notation of Amini-Harandi [13].

Definition 1.8 (See [13]) Let (X, d) be a metric-like space and let U be a subset of X. We say U is a d-open subset of X, if for all $x \in X$ there exists $r > 0$ such that $B_d(x, r) \subseteq U$. Also, $V \subseteq X$ is a d-closed subset of X if $(X \backslash V)$ is a d-open subset of X.

Lemma 1.9 (See [19]) *Let (X, d) be a metric-like space. Then*:

(A) *if $d(x, y) = 0$ then $d(x, x) = d(y, y) = 0$;*

(B) *if $\{x_n\}$ be a sequence such that $\lim_{n \to \infty} d(x_n, x_{n+1}) = 0$, then we have,*

$$\lim_{n \to \infty} d(x_n, x_n) = \lim_{n \to \infty} d(x_{n+1}, x_{n+1}) = 0;$$

(C) *if $x \neq y$ then $d(x, y) > 0$;*

(D) *$d(x, x) \leq \frac{2}{n} \sum_{i=1}^{i=n} d(x, x_i)$ holds for all $x_i, x \in X$ where $1 \leq i \leq n$;*

(E) *if $\{x_n\}$ is a sequence in a d-closed subset V of X with $x_n \to x$ as $n \to \infty$, then $x \in V$;*

(F) *if $\{x_n\}$ is a sequence in X such that $x_n \to x$ as $n \to \infty$ and $d(x, x) = 0$, then $\lim_{n \to \infty} d(x_n, y) = d(x, y)$ for all $y \in X$.*

Definition 1.10 Let (X, d) and (Y, ρ) be metric-like spaces and $\{x_n\}_{n=1}^{\infty}$ be a sequence in X such that $x_n \to x$. A mapping $f : X \to Y$ is said to be continuous at a point $x \in X$ if $f(x_n) \to f(x)$

2 Main results

We begin this section with the following definition.

Definition 2.1 Let (X, d) be a metric-like space. A mapping $T : X \to X$ is said to be a conditionally F-contraction of type (A) if there exist $F \in \mathfrak{F}$ and $\tau > 0$ such that, for all $x, y \in X$ with $d(Tx, Ty) > 0$,

$$\frac{1}{2} d(x, Tx) < d(x, y) \quad \Rightarrow \quad \tau + F\big(d(Tx, Ty)\big) \leq F\big(M_T(x, y)\big), \tag{3}$$

where

$$M_T(x, y) \max \left\{ d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{4} \right\}.$$

Theorem 2.2 *Let (X, d) is a complete metric-like space. If T is a conditionally F-contraction of type* (A), *then T has a fixed point $x^* \in X$.*

Proof Take $x \in X$ and construct a sequence $\{x_n\}$ as follows:

$$x_n = Tx_{n-1} = T^n x \quad \text{for all } n \in \mathbb{N} \text{ where } x_0 = x. \tag{4}$$

If there exists $n_* \in \mathbb{N}$ such that $d(x_{n_*}, Tx_{n_*}) = 0$ then $x_* = x_{n_*}$ becomes a fixed point which completes the proof. Consequently, in the rest of the proof, we assume that, for every $n \in \mathbb{N}$,

$$0 < d(x_n, Tx_n). \tag{5}$$

Hence, from (5), we have

$$\frac{1}{2} d(x_n, Tx_n) < d(x_n, Tx_n) \quad \text{for all } n \in \mathbb{N}. \tag{6}$$

Since T is conditionally F-contraction (note $d(Tx_n, T(Tx_n)) = d(x_{n+1}, Tx_{n+1}) > 0$), from the inequality (6), we have

$$
\begin{aligned}
\tau + F\big(d\big(Tx_n, T^2 x_n\big)\big) &\leq F\bigg(\max\bigg\{ d(x_n, Tx_n), d(x_n, Tx_n), d\big(Tx_n, T^2 x_n\big), \\
&\qquad\qquad \frac{d(x_n, T^2 x_n) + d(Tx_n, Tx_n)}{4} \bigg\}\bigg) \\
&\leq F\bigg(\max\bigg\{ d(x_n, Tx_n), d\big(Tx_n, T^2 x_n\big), \\
&\qquad\qquad \frac{d(x_n, Tx_n) + d(Tx_n, T^2 x_n) + d(Tx_n, x_n) + d(x_n, Tx_n)}{4} \bigg\}\bigg) \\
&= F\bigg(\max\bigg\{ d(x_n, Tx_n), d\big(Tx_n, T^2 x_n\big), \\
&\qquad\qquad \frac{3d(x_n, Tx_n) + d(Tx_n, T^2 x_n)}{4} \bigg\}\bigg) \\
&= F\big(\max\big\{ d(x_n, Tx_n), d\big(Tx_n, T^2 x_n\big)\big\}\big). \tag{7}
\end{aligned}
$$

If there exists $n \in \mathbb{N}$ such that $\max\{d(x_n, Tx_n), d(Tx_n, T^2 x_n)\} = d(Tx_n, T^2 x_n)$, then (7) becomes

$$\tau + F\big(d\big(Tx_n, T^2 x_n\big)\big) \leq F\big(d\big(Tx_n, T^2 x_n\big)\big),$$

which is a contradiction. Thus, we conclude that

$$\max\big\{ d(x_n, Tx_n), d\big(Tx_n, T^2 x_n\big)\big\} = d(x_n, Tx_n),$$

for all $n \in \mathbb{N}$. Hence, the inequality (7) turns into

$$F\big(d\big(Tx_n, T^2 x_n\big)\big) \leq F\big(d(x_n, Tx_n)\big) - \tau \quad \text{for all } n \in \mathbb{N}, \tag{8}$$

which is equivalent to

$$F\big(d(x_{n+1}, Tx_{n+1})\big) \leq F\big(d(x_n, Tx_n)\big) - \tau \quad \text{for all } n \in \mathbb{N}.$$

Iteratively, we find that

$$F\big(d(x_n, Tx_n)\big) \leq F\big(d(x_{n-1}, Tx_{n-1})\big) - \tau$$
$$\leq F\big(d(x_{n-2}, Tx_{n-2})\big) - 2\tau$$
$$\leq F\big(d(x_{n-3}, Tx_{n-3})\big) - 3\tau$$
$$\vdots$$
$$\leq F\big(d(x_0, Tx_0)\big) - n\tau. \tag{9}$$

From (9), we obtain $\lim_{m \to \infty} F(d(x_n, Tx_n)) = -\infty$, which together with (F2) gives

$$\lim_{n \to \infty} d(x_n, Tx_n) = \lim_{m \to \infty} d(x_n, x_{n+1}) = 0. \tag{10}$$

Now, we claim that

$$\lim_{n,m \to \infty} d(x_n, x_m) = 0. \tag{11}$$

Arguing by contradiction, we assume that there exist $\epsilon > 0$ and sequences $\{p(n)\}_{n=1}^{\infty}$ and $\{q(n)\}_{n=1}^{\infty}$ of natural numbers such that

$$p(n) > q(n) > n, \qquad d(x_{p(n)}, x_{q(n)}) \geq \epsilon, \qquad d(x_{p(n)-1}, x_{q(n)}) < \epsilon \quad \text{for all } n \in \mathbb{N}. \tag{12}$$

From the triangle inequality, we get

$$\epsilon \leq d(x_{p(n)}, x_{q(n)}) \leq d(x_{p(n)}, x_{p(n)-1}) + d(x_{p(n)-1}, x_{q(n)})$$
$$\leq d(x_{p(n)}, x_{p(n)-1}) + \epsilon$$
$$= d(x_{p(n)-1}, Tx_{p(n)-1}) + \epsilon \quad \text{for all } n \in \mathbb{N}. \tag{13}$$

Thus from (10), (13), and the sandwich theorem, we get

$$\lim_{n \to \infty} d(x_{p(n)}, x_{q(n)}) = \epsilon. \tag{14}$$

Again by the triangle inequality, for all $n \in \mathbb{N}$, we have the following two inequalities:

$$d(x_{p(n)}, x_{q(n)}) \leq d(x_{p(n)}, x_{p(n)+1}) + d(x_{p(n)+1}, x_{q(n)+1}) + d(x_{q(n)+1}, x_{q(n)}) \tag{15}$$

and

$$d(x_{p(n)+1}, x_{q(n)+1}) \leq d(x_{p(n)+1}, x_{p(n)}) + d(x_{p(n)}, x_{q(n)}) + d(x_{q(n)}, x_{q(n)+1}). \tag{16}$$

Letting $n \to \infty$ in the inequalities (15) and (16), using (10) and (14), we obtain

$$\lim_{n \to \infty} d(x_{p(n)+1}, x_{q(n)+1}) = \epsilon. \tag{17}$$

From (10) and (12), there exists $N_2 \in \mathbb{N}$ such that

$$\frac{1}{2} d(x_{p(n)}, Tx_{p(n)}) < \frac{\epsilon}{2} < d(x_{p(n)}, x_{q(n)}), \quad \forall n > N_2.$$

Note from (17) for n large enough (*i.e.* $n \geq N_3 \geq N_2$ say) we have $d(Tx_{p(n)}, Tx_{q(n)}) = d(x_{p(n)+1}, x_{q(n)+1}) > 0$. Since T is a conditionally F-contractive mapping of type (A), we have (with $n \geq N_3$)

$$\tau + F\big(d(Tx_{p(n)}, Tx_{q(n)})\big)$$

$$\leq F\bigg(\max\bigg\{ d(x_{p(n)}, x_{q(n)}), d(x_{p(n)}, Tx_{p(n)}), d(x_{q(n)}, Tx_{q(n)}),$$

$$\frac{d(x_{p(n)}, Tx_{q(n)}) + d(x_{q(n)}, Tx_{p(n)})}{4} \bigg\} \bigg)$$

$$\leq F\bigg(\max\bigg\{ d(x_{p(n)}, x_{q(n)}), d(x_{p(n)}, Tx_{p(n)}), d(x_{q(n)}, Tx_{q(n)}),$$

$$\frac{d(x_{p(n)}, Tx_{p(n)}) + d(x_{q(n)}, Tx_{q(n)}) + 2d(x_{q(n)}, x_{p(n)})}{4} \bigg\} \bigg)$$

$$\leq F\big(\max\{ d(x_{p(n)}, x_{q(n)}), d(x_{p(n)}, Tx_{p(n)}), d(x_{q(n)}, Tx_{q(n)}),$$

$$\max\big\{ d(x_{p(n)}, x_{q(n)}), d(x_{p(n)}, Tx_{p(n)}), d(x_{q(n)}, Tx_{q(n)}) \big\} \big)$$

$$= F(\max\big\{ d(x_{p(n)}, x_{q(n)}), d(x_{p(n)}, Tx_{p(n)}), d(x_{q(n)}, Tx_{q(n)}) \big\}. \tag{18}$$

Letting $n \to \infty$ in the inequality above and using (10), (14), and (F3′) we obtain

$$\tau + F(\varepsilon) \leq F(\varepsilon), \tag{19}$$

a contradiction since $\tau > 0$. Hence

$$\lim_{m,n \to \infty} d(x_n, x_m) = 0.$$

Therefore, we conclude that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in X. Now (X, d) is a complete metric-like space, so there exists $x^* \in X$ such that

$$d\big(x^*, x^*\big) = \lim_{n \to \infty} d\big(x_n, x^*\big) = \lim_{n,m \to \infty} d(x_n, x_m) = 0. \tag{20}$$

Now note

$$d\big(x^*, Tx^*\big) \leq d\big(x^*, x_{n+1}\big) + d\big(x_{n+1}, Tx^*\big)$$

$$\leq d\big(x^*, x_{n+1}\big) + d\big(x_{n+1}, x^*\big) + d\big(x^*, Tx^*\big)$$

$$= 2d\big(x^*, x_{n+1}\big) + d\big(x^*, Tx^*\big).$$

Thus from (20) and the sandwich theorem, we get

$$\lim_{n\to\infty}\left[d\left(x^*, x_{n+1}\right) + d\left(x_{n+1}, Tx^*\right)\right] = d\left(x^*, Tx^*\right).$$

Now this and (20) yield

$$\lim_{n\to\infty} d\left(x_{n+1}, Tx^*\right) = d\left(x^*, Tx^*\right). \tag{21}$$

We now prove that, for every $n \in \mathbb{N}$,

$$\frac{1}{2}d(x_n, Tx_n) < d\left(x_n, x^*\right) \quad \text{or} \quad \frac{1}{2}d\left(Tx_n, T^2x_n\right) < d\left(Tx_n, x^*\right), \quad \forall n \in \mathbb{N}. \tag{22}$$

Arguing by contradiction, we assume that there exists $m \in \mathbb{N}$ such that

$$\frac{1}{2}d(x_m, Tx_m) \geq d\left(x_m, x^*\right) \quad \text{and} \quad \frac{1}{2}d\left(Tx_m, T^2x_m\right) \geq d\left(Tx_m, x^*\right). \tag{23}$$

Now from (8) and (F$_1$), we have

$$d\left(Tx_m, T^2x_m\right) < d(x_m, Tx_m). \tag{24}$$

It follows from (23) and (24) that

$$\begin{aligned}
d(x_m, Tx_m) &\leq d\left(x_m, x^*\right) + d\left(x^*, Tx_m\right) \\
&\leq \frac{1}{2}d(x_m, Tx_m) + \frac{1}{2}d\left(Tx_m, T^2x_m\right) \\
&< \frac{1}{2}d(x_m, Tx_m) + \frac{1}{2}d(x_m, Tx_m) \\
&= d(x_m, Tx_m),
\end{aligned}$$

which is a contradiction. Hence (22) holds.

Suppose, now, part (I) of (22) is satisfied and $d(x^*, Tx^*) > 0$. Note from (21) there exists $N_1 \in \mathbb{N}$ such that $d(Tx_n, Tx^*) = d(x_{n+1}, Tx^*) > 0$ for $n \geq N_1$. Then from our assumption (with $n \geq N_1$) we have

$$\begin{aligned}
\tau + F\left(d\left(x_{n+1}, Tx^*\right)\right) &= \tau + F\left(d\left(Tx_n, Tx^*\right)\right) \\
&\leq F\left(\max\left\{d\left(x_n, x^*\right), d(x_n, Tx_n), d\left(x^*, Tx^*\right),\right.\right. \\
&\qquad \left.\left.\frac{d(x_n, Tx^*) + d(x^*, Tx_n)}{4}\right\}\right) \\
&\leq F\left(\max\left\{d\left(x_n, x^*\right), d(x_n, Tx_n), d\left(x^*, Tx^*\right),\right.\right. \\
&\qquad \left.\left.\frac{2d(x_n, x^*) + d(x^*, Tx^*) + d(x_n, Tx_n)}{4}\right\}\right).
\end{aligned} \tag{25}$$

From (10) and (21), there exists $N_3 \in \mathbb{N}$ (with $N_3 \geq N_1$) such that for all $n \geq N_3$

$$\max\left\{ d(x_n, x^*), d(x_n, Tx_n), d(x^*, Tx^*), \frac{2d(x_n, x^*) + d(x^*, Tx^*) + d(x_n, Tx_n)}{4} \right\}$$
$$= d(x^*, Tx^*).$$

Now from (25), we get

$$\tau + F\big(d\big(x_{n+1}, Tx^*\big)\big) \leq F\big(d\big(x^*, Tx^*\big)\big), \quad \forall n \geq N_3. \tag{26}$$

From (F3′) and (21), by taking the limit as $n \to \infty$ in (26), we obtain

$$\tau + F\big(d\big(x^*, Tx^*\big)\big) \leq F\big(d\big(x^*, Tx^*\big)\big),$$

which is a contradiction.

Now suppose part (II) of (22) is true, and $d(x^*, Tx^*) > 0$. Note from (21) there exists $N_2 \in \mathbb{N}$ such that $d(T(Tx_n), Tx^*) = d(x_{n+2}, Tx^*) > 0$ for $n \geq N_2$. Then from our assumption (with $n \geq N_2$) we have

$$\tau + F\big(d\big(x_{n+2}, Tx^*\big)\big) = \tau + F\big(d\big(T^2 x_n, Tx^*\big)\big)$$
$$\leq F\Bigg(\max\Bigg\{ d\big(Tx_n, x^*\big), d\big(Tx_n, T^2 x_n\big), d\big(x^*, Tx^*\big),$$
$$\frac{d(Tx_n, Tx^*) + d(x^*, T^2 x_n)}{4} \Bigg\} \Bigg)$$
$$\leq F\Bigg(\max\Bigg\{ d\big(Tx_n, x^*\big), d\big(Tx_n, T^2 x_n\big), d\big(x^*, Tx^*\big),$$
$$\frac{2d(Tx_n, x^*) + d(x^*, Tx^*) + d(Tx_n, T^2 x_n)}{4} \Bigg\} \Bigg)$$
$$= F\Bigg(\max\Bigg\{ d\big(x_{n+1}, x^*\big), d(x_{n+1}, Tx_{n+1}), d\big(x^*, Tx^*\big),$$
$$\frac{2d(x_{n+1}, x^*) + d(x^*, Tx^*) + d(x_{n+1}, Tx_{n+1})}{4} \Bigg\} \Bigg). \tag{27}$$

From (10) and (21), there exists $N_4 \in \mathbb{N}$ (with $N_4 \geq N_2$) such that for all $n \geq N_4$

$$\max\left\{ d\big(x_{n+1}, x^*\big), d(x_{n+1}, Tx_{n+1}), d\big(x^*, Tx^*\big), \frac{2d(x_{n+1}, x^*) + d(x^*, Tx^*) + d(x_{n+1}, Tx_{n+1})}{4} \right\}$$
$$= d(x^*, Tx^*).$$

From (27), we get

$$\tau + F\big(d\big(x_{n+2}, Tx^*\big)\big) \leq F\big(d\big(x^*, Tx^*\big)\big), \quad \forall n \geq N_4. \tag{28}$$

From (F3) and (21), taking the limit as $n \to \infty$ in (28), we obtain

$$\tau + F\big(d\big(x^*, Tx^*\big)\big) \leq F\big(d\big(x^*, Tx^*\big)\big),$$

which is a contradiction. Hence, we conclude that x^* is a fixed point of T.   □

Definition 2.3 Let (X, d) be a metric-like space. A mapping $T : X \to X$ is said to be a conditionally F-contraction of type (B) if there exist $F \in \mathfrak{F}$ and $\tau > 0$ such that, for all $x, y \in X$ with $d(Tx, Ty) > 0$,

$$\frac{1}{2} d(x, Tx) < d(x, y) \quad \Rightarrow \quad \tau + F\big(d(Tx, Ty)\big) \leq F\big(\max\{d(x, y), d(x, Tx), d(y, Ty)\}\big). \quad (29)$$

Definition 2.4 Let (X, d) be a metric-like space. A mapping $T : X \to X$ is said to be a conditionally F-contraction of type (C) if there exist $F \in \mathfrak{F}$ and $\tau > 0$ such that, for all $x, y \in X$ with $d(Tx, Ty) > 0$,

$$\frac{1}{2} d(x, Tx) < d(x, y) \quad \Rightarrow \quad \tau + F\big(d(Tx, Ty)\big) \leq F\big(d(x, y)\big). \quad (30)$$

Theorem 2.5 *Let (X, d) be a complete metric-like space. If T is a conditionally F-contraction of type* (B), *then T has a fixed point $x^* \in X$.*

Proof Following the proof in Theorem 2.2, we easily conclude the result.   □

Theorem 2.6 *Let (X, d) be a complete metric-like space. If T is a conditionally F-contraction of type* (C), *then T has a fixed point $x^* \in X$.*

Proof It can easily be derived by following the proof in Theorem 2.2.   □

Next we consider an example to illustrate our main result. We consider a mapping T which is not continuous, so not an F-contraction but it is a conditionally F-contraction of type (C).

Example 2.7 Consider $X = \{0, 1, 2\}$. Let $d : X \times X \to [0, \infty)$ be a mapping defined by

$$d(0, 0) = d(1, 1) = 0, \qquad d(2, 2) = 5/2,$$
$$d(0, 2) = d(2, 0) = 2, \qquad d(1, 2) = d(2, 1) = 3,$$
$$d(0, 1) = d(1, 0) = 3/2.$$

It is clear that d is a metric-like. Note that $d(2, 2) \neq 0$, so d is not a metric. Clearly, (X, d) is a complete metric-like space. Let $T : X \to X$ be given by

$$T0 = 0 = T1 \quad \text{and} \quad T2 = 1.$$

Suppose that $F(\alpha) = \frac{-1}{\alpha} + \alpha \in \mathfrak{F}$ and $\tau \in (0, 1/2)$. Since T is not continuous, T is not a F-contraction by Remark 1.3.

We will consider the inequality

$$\frac{1}{2} d(x, Tx) < d(x, y), \quad (31)$$

where $x, y \in X$ with $d(Tx, Ty) > 0$ and the inequality

$$\tau + F\big(d(Tx, Ty)\big) \leq F\big(d(x, y)\big) \quad (32)$$

for those $x, y \in X$ with $d(Tx, Ty) > 0$ which satisfy (31).

Case 1: Let $x = 0$. Now $d(T0, T0) = d(T0, T1) = d(0, 0) = 0$ so we need only consider $y = 2$ in (31) and (32). Now (31) is true since

$$\frac{1}{2}d(0, T0) = 0 < d(0, 2) = 2.$$

Also note

$$d(T0, T2) = d(0, 1) = \frac{3}{2} < 2 = d(0, 2).$$

Now inequality (32) is satisfied since

$$\tau + F\big(d(T0, T2)\big) = \tau - \frac{1}{d(T0, T2)} + d(T0, T2)$$

$$\leq \tau - \frac{1}{d(0, 2)} + \frac{3}{2} \leq \frac{1}{2} - \frac{1}{d(0, 2)} + \frac{3}{2}$$

$$= -\frac{1}{d(0, 2)} + 2 = -\frac{1}{d(0, 2)} + d(0, 2) = F\big(d(0, 2)\big).$$

Case 2: Let $x = 1$. Now $d(T0, T1) = d(T1, T1) = d(0, 0) = 0$ so we need only consider $y = 2$ in (31) and (32). Now (31) is true since

$$\frac{1}{2}d(1, T1) = 0 < d(1, 2) = 3.$$

Also note

$$d(T1, T2) = d(0, 1) = \frac{3}{2} < 3 = d(1, 2).$$

Now inequality (32) is satisfied since

$$\tau + F\big(d(T1, T2)\big) = \tau - \frac{1}{d(T1, T2)} + d(T1, T2)$$

$$\leq \tau - \frac{1}{d(1, 2)} + \frac{3}{2} \leq \frac{1}{2} - \frac{1}{d(1, 2)} + \frac{3}{2} = -\frac{1}{d(1, 2)} + 2$$

$$\leq -\frac{1}{d(1, 2)} + 3 = -\frac{1}{d(1, 2)} + d(1, 2) = F\big(d(1, 2)\big).$$

Case 3: Let $x = 2$. Now $d(T2, T2) = 0$ so we need only consider the case $y \in \{0, 1\}$. Note $d(T2, T1) = d(1, 0) > 0$, $d(T2, T0) = d(1, 0) > 0$, and also note

$$\frac{1}{2}d(2, T2) = \frac{1}{2}d(1, 2) = \frac{3}{2}$$

and

$$d(2, 0) = 2, \qquad d(2, 1) = 3,$$

so (31) holds.

Note

$$d(T2, T0) = d(1, 0) = \frac{3}{2} < 2 = d(2, 0)$$

and

$$d(T2, T1) = d(1, 0) = \frac{3}{2} < 3 = d(2, 1),$$

so

$$\tau + F\big(d(T2, T0)\big) \leq F\big(d(2, 0)\big)$$

follows as in Case 1 and

$$\tau + F\big(d(T2, T1)\big) \leq F\big(d(2, 1)\big)$$

follows as in Case 2.

Hence T is a conditionally F-contraction of type (C). It is clear that 0 is the fixed point of T.

3 Consequences

In [14, 20], Matthews introduced the notion of partial metric, a generalization of a metric, as a part of the study of denotational semantics of dataflow networks.

Definition 3.1 (See [14]) Let X be a non-empty set. A mapping $p: X \times X \to \mathbb{R}_0^+$ is said to be a partial metric on X if for all $x, y, z \in X$ the following conditions are satisfied:

(p$_1$) $x = y$ if and only if $p(x, x) = p(x, y) = p(y, y)$;
(p$_2$) $p(x, x) \leq p(x, y)$;
(p$_3$) $p(x, y) = p(y, x)$;
(p$_4$) $p(x, z) \leq p(x, y) + p(y, z) - p(y, y)$.

In this case, the pair (X, p) is called a partial metric space (PMS).

Notice that the function $d_p : X \times X \to \mathbb{R}^+$ defined by $d_p(x, y) = 2p(x, y) - p(x, x) - p(y, y)$ satisfies the conditions of a metric on X. Each partial metric p on X generates a T_0 topology τ_p on X, whose base is a family of open p-balls $\{B_p(x, \varepsilon) : x \in X, \varepsilon > 0\}$ where $B_p(x, \varepsilon) = \{y \in X : p(x, y) \leq p(x, x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$. Consequently, it is easy to consider several topological concepts. A sequence $\{x_n\}$ in the PMS (X, p) converges to the limit x if $p(x, x) = \lim_{n \to \infty} p(x, x_n)$ and is said to be a Cauchy sequence if $\lim_{n,m \to \infty} p(x_n, x_m)$ exists and is finite. A PMS (X, p) is called complete if every Cauchy sequence $\{x_n\}$ in X converges with respect to τ_p, to a point $x \in X$ such that $p(x, x) = \lim_{n,m \to \infty} p(x_n, x_m)$. For more details, see *e.g.* [14, 20–40] and the related references therein.

Lemma 3.2 (See *e.g.* [21, 22]) *Let (X, p) be a complete PMS. Then*
  (A) *If $p(x, y) = 0$ then $x = y$.*
  (B) *If $x \neq y$, then $p(x, y) > 0$.*
  (C) *A sequence $\{x_n\}$ is a Cauchy sequence in the PMS (X, p) if and only if it is a Cauchy sequence in the metric space (X, d_p).*

(D) *A PMS (X, p) is complete if and only if the metric space (X, d_p) is complete. Moreover,*

$$\lim_{n \to \infty} d_p(x, x_n) = 0 \quad \Leftrightarrow \quad p(x, x) = \lim_{n \to \infty} p(x, x_n) = \lim_{n,m \to \infty} p(x_n, x_m). \tag{33}$$

(E) *Assume $x_n \to z$ as $n \to \infty$ in a PMS (X, p) such that $p(z, z) = 0$. Then*
$\lim_{n \to \infty} p(x_n, y) = p(z, y)$ *for every $y \in X$.*

Now we derive the analog of Theorem 2.2 in the context of partial metric spaces. In fact, in the following theorem we conclude not only the existence of a fixed point of the given mapping but also the uniqueness.

Theorem 3.3 *Let (X, p) be a complete partial metric space and let $T : X \to X$ be a self-mapping. Suppose that there exist $F \in \mathfrak{F}$ and $\tau > 0$ such that, for all $x, y \in X$ with $p(Tx, Ty) > 0$,*

$$\frac{1}{2} p(x, Tx) < p(x, y)$$

$$\Rightarrow \quad \tau + F\big(p(Tx, Ty)\big) \le F\left(\max\left\{ p(x, y), p(x, Tx), p(y, Ty), \frac{p(x, Ty) + p(y, Tx)}{2} \right\} \right). \tag{34}$$

Then T has a unique fixed point $x^* \in X$.

Proof Since every partial metric space is a metric-like space, we obtain the proof by following the proof in Theorem 2.2. Note that the expression $\frac{p(x, Ty) + p(y, Tx)}{4}$ in the inequality (3) is replaced by $\frac{p(x, Ty) + p(y, Tx)}{2}$ in the inequality (34). This difference arises due to assumptions (p_2) and (p_4) of partial metric spaces. Hence, taking (p_2) and (p_4) into account, following the proof in Theorem 2.2 yields the existence of a fixed point $(x^* \in X)$ of T.

We now show the uniqueness of the fixed point of T. Suppose there is another fixed point $y^* \in X$ of T, such that $x^* \ne y^*$. Thus from Lemma 3.2, we have $p(x^*, y^*) > 0$. From (p_2), we have

$$\frac{1}{2} p\big(x^*, Tx^*\big) = \frac{1}{2} p\big(x^*, x^*\big) < p\big(x^*, x^*\big) \le p\big(x^*, y^*\big).$$

Thus, from (p_2), we obtain (note $p(Tx^*, Ty^*) = p(x^*, y^*) > 0$)

$$\tau + F\big(p\big(x^*, y^*\big)\big) = \tau + F\big(p\big(Tx^*, Ty^*\big)\big)$$

$$\le F\left(\max\left\{ p\big(x^*, y^*\big), p\big(x^*, Tx^*\big), p\big(y^*, Ty^*\big), \frac{p(x^*, Ty^*) + p(y^*, Tx^*)}{2} \right\} \right)$$

$$= F\left(\max\left\{ p\big(x^*, y^*\big), p\big(x^*, x^*\big), p\big(y^*, y^*\big), \frac{p(x^*, y^*) + p(y^*, x^*)}{2} \right\} \right)$$

$$\le F\left(\max\left\{ p\big(x^*, y^*\big), p\big(x^*, y^*\big), p\big(x^*, y^*\big) \frac{p(x^*, y^*) + p(y^*, x^*)}{2} \right\} \right)$$

$$= F\big(p\big(x^*, y^*\big)\big).$$

This is a contradiction, and hence $x^* = y^*$. □

The following two theorems can be obtained easily by repeating the steps in the proof of Theorem 3.3.

Theorem 3.4 *Let (X,p) be a complete partial metric space and let $T : X \to X$ be a self-mapping. Suppose that there exist $F \in \mathfrak{F}$ and $\tau > 0$ such that, for all $x,y \in X$ with $p(Tx,Ty) > 0$,*

$$\frac{1}{2}p(x,Tx) < p(x,y) \quad \Rightarrow \quad \tau + F\big(p(Tx,Ty)\big) \le F\big(\max\{p(x,y),p(x,Tx),p(y,Ty)\}\big). \quad (35)$$

Then T has a unique fixed point $x^* \in X$.

Theorem 3.5 *Let (X,p) be a complete partial metric space and let $T : X \to X$ be a self-mapping. Suppose that there exist $F \in \mathfrak{F}$ and $\tau > 0$ such that, for all $x,y \in X$ with $p(Tx,Ty) > 0$,*

$$\frac{1}{2}p(x,Tx) < p(x,y) \quad \Rightarrow \quad \tau + F\big(p(Tx,Ty)\big) \le F\big(p(x,y)\big). \quad (36)$$

Then T has a unique fixed point $x^* \in X$.

Remark 3.6 On can also easily conclude that the analog of Theorem 3.3-Theorem 3.5 in the context of metric spaces since each metric space is a partial metric space.

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