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# Remarks on stability of some inhomogeneous functional equations

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*Dedicated to Professor János Aczél on the occasion of his ninetieth birthday*

**Abstract.** This is an expository paper in which we present some simple observations on the stability of some inhomogeneous functional equations. In particular, we state several stability results for the inhomogeneous Cauchy equation

$$f(x + y) = f(x) + f(y) + d(x, y)$$

and for the inhomogeneous forms of the Jensen and linear functional equations.

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## 1. Introduction

In this paper $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ denote the sets of positive integers, integers, rationals, reals and complex numbers, respectively; $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $\mathbb{R}_+ := [0, \infty)$.

Let us recall that the problem of stability of functional equations was motivated by a question of Ulam asked in 1940 and an answer to it published by Hyers in [21]. Since then numerous papers on this subject have been published and we refer to [4,11,22,25–27] for more details, some discussions, further references and examples of very recent results.

One of the most classical results, concerning the stability of the Cauchy equation

$$f(x + y) = f(x) + f(y), \tag{1.1}$$

can be stated as follows.

Birkhäuser

**Theorem 1.1.** *Let $E_1$ and $E_2$ be normed spaces and $c \geq 0$ and $p \neq 1$ be fixed real numbers. Assume also that $f : E_1 \to E_2$ is a mapping satisfying*

$$\|f(x+y) - f(x) - f(y)\| \leq c(\|x\|^p + \|y\|^p), \qquad x, y \in E_1 \setminus \{0\}. \tag{1.2}$$

*If $p \geq 0$ and $E_2$ is complete, then there exists a unique solution $T : E_1 \to E_2$ of (1.1) such that*

$$\|f(x) - T(x)\| \leq \frac{c\|x\|^p}{|2^{p-1} - 1|}, \qquad x \in E_1 \setminus \{0\}. \tag{1.3}$$

*If $p < 0$, then $f$ is additive [i.e., it is a solution to (1.1)].*

It is composed of the results in [1,8,20,21,29]. Also, it is known (see [20,22]) that for $p = 1$ an analogous result is not valid. Moreover, it has been proved in [5] that estimation (1.3) is optimal (for $E_1 = E_2 = \mathbb{R}$).

There arises a natural question whether analogous results can be proved for the inhomogenous Cauchy equation

$$g(x+y) = g(x) + g(y) + d(x, y). \tag{1.4}$$

The equation has drawn the attention of several authors and been studied already for various spaces and forms of $d$ in, e.g., [3,12–17,19,23,24].

In this expository paper we present some simple remarks motivated by that issue. We believe that they are new and can be of some interest for researchers investigating that field and related areas.

In particular, we show that the following result is valid.

**Theorem 1.2.** *Let $E_1$ and $E_2$ be normed spaces, $d : E_1^2 \to E_2$ and $c, p \in \mathbb{R}$. Assume that (1.4) admits a solution $f_0 : E_1 \to E_2$. Then the following three statements are valid.*

(*a*) *If $p \geq 0, p \neq 1$, and $E_2$ is complete, then for every $f : E_1 \to E_2$, satisfying*

$$\|f(x+y) - f(x) - f(y) - d(x, y)\| \leq c(\|x\|^p + \|y\|^p), \qquad x, y \in E_1 \setminus \{0\}, \tag{1.5}$$

*there exists a unique solution $g : E_1 \to E_2$ of (1.4) such that*

$$\|f(x) - g(x)\| \leq \frac{c\|x\|^p}{|2^{p-1} - 1|}, \qquad x \in E_1 \setminus \{0\}. \tag{1.6}$$

*Moreover, that estimation is optimal when $E_1 = \mathbb{R}$; namely there exists a function $f : \mathbb{R} \to E_2$ such that*

$$\|f(x+y) - f(x) - f(y) - d(x, y)\| \leq c(|x|^p + |y|^p), \qquad x, y \in \mathbb{R},$$

$$\|f(x) - f_0(x)\| = \frac{c|x|^p}{|2^{p-1} - 1|}, \qquad x \in \mathbb{R}. \tag{1.7}$$

(*b*) *If $p < 0$, then every $f : E_1 \to E_2$ satisfying (1.5) is a solution of (1.4).*

(c) *If* $E_1 = E_2 = \mathbb{R}$*, then for each real* $c_0 > 0$ *there is* $f : \mathbb{R} \to \mathbb{R}$ *such that*

$$|f(x + y) - f(x) - f(y) - d(x, y)| \le c_0(|x| + |y|), \qquad x, y \in \mathbb{R}, \tag{1.8}$$

*and*

$$\sup_{x\in\mathbb{R}\setminus\{0\}} \frac{|f(x) - h(x)|}{|x|} = \infty \tag{1.9}$$

*for each solution* $h : \mathbb{R} \to \mathbb{R}$ *of* (1.4)*.*

The statement (c) shows that, in the case $p = 1$, Eq. (1.4) demonstrates an analogous lack of stability as (1.1) (cf. [20,22]).

## 2. General observations

We start with some general observations. In what follows $S$ is a nonempty set, $(X, +)$ is a commutative group, and we define a binary operation $+$ in $X^S$ (the family of all functions mapping $S$ into $X$) in the usual way by: $(f + g)(x) := f(x) + g(x)$ for $f, g \in X^S$, $x \in S$. Clearly $(X^S, +)$ is a group.

Let us introduce the following technical definition ($2^X$ stands for the family of all subsets of $X$).

**Definition 2.1.** Let $n \in \mathbb{N}$, $P \subset S^n$ be nonempty, $\Phi : P \to 2^X \setminus \{\emptyset\}$, $B \subset S$, $\Psi : S \setminus B \to 2^X \setminus \{\emptyset\}$, $\mathcal{F}_1, \mathcal{F}_2$ be functions mapping $\mathcal{D} \subset X^S$ into $X^P$, and $\mathcal{U} \subset \mathcal{D}$ be nonempty. We say that the conditional equation

$$\mathcal{F}_1\varphi(x_1, \ldots, x_n) = \mathcal{F}_2\varphi(x_1, \ldots, x_n), \qquad (x_1, \ldots, x_n) \in P, \tag{2.1}$$

is $(\Phi, \Psi)$−stable in $\mathcal{U}$ provided for any $\varphi_0 \in \mathcal{U}$ with

$$\mathcal{F}_1\varphi_0(x_1, \ldots, x_n) - \mathcal{F}_2\varphi_0(x_1, \ldots, x_n) \in \Phi(x_1, \ldots, x_n), \quad (x_1, \ldots, x_n) \in P, \tag{2.2}$$

there exists a solution $\varphi \in \mathcal{D}$ of Eq. (2.1) such that

$$\varphi_0(x) - \varphi(x) \in \Psi(x), \qquad x \in S \setminus B. \tag{2.3}$$

Moreover, if for every $\varphi_0 \in \mathcal{U}$, satisfying (2.2), there is exactly one solution $\varphi \in \mathcal{D}$ of (2.1), fulfilling (2.3), then we say that Eq. (2.1) is $(\Phi, \Psi)$—stable in $\mathcal{U}$ with uniqueness.

If $\mathcal{U} = \mathcal{D}$, then we omit the part 'in $\mathcal{U}$' and simply say '$(\Phi, \Psi)$−stable'.

Let $n \in \mathbb{N}$, $P \subset S^n$ be nonempty, $\mathcal{U} \subset \mathcal{D}$ be two subgroups of the group $(X^S, +)$ and $\mathcal{H} : \mathcal{D} \to X^P$ be additive, i.e.,

$$\mathcal{H}(f + g)(x_1, \ldots, x_n) = \mathcal{H}f(x_1, \ldots, x_n) + \mathcal{H}g(x_1, \ldots, x_n), \quad f, g \in \mathcal{D}, (x_1, \ldots, x_n) \in P.$$

We have the following (very simple, but very useful) observation.

**Theorem 2.2.** *Let $B \subset S$, $\Phi : P \to 2^X \setminus \{\emptyset\}$, $\Psi : S \setminus B \to 2^X \setminus \{\emptyset\}$ and $\mu : P \to X$. Suppose that the equation*

$$\mathcal{H}f(x_1, \dots, x_n) = \mu(x_1, \dots, x_n), \qquad (x_1, \dots, x_n) \in P, \tag{2.4}$$

*admits a solution $f_0 \in \mathcal{U}$. Then the equation*

$$\mathcal{H}f(x_1, \dots, x_n) = 0, \qquad (x_1, \dots, x_n) \in P, \tag{2.5}$$

*is $(\Phi, \Psi)$–stable in $\mathcal{U}$ (with uniqueness) if and only if Eq. (2.4) is $(\Phi, \Psi)$–stable in $\mathcal{U}$ (with uniqueness).*

*Proof.* Assume first that Eq. (2.4) is $(\Phi, \Psi)$-stable in $\mathcal{U}$. Let $g \in \mathcal{U}$ satisfy the condition

$$\mathcal{H}g(x_1, \dots, x_n) \in \Phi(x_1, \dots, x_n), \qquad (x_1, \dots, x_n) \in P. \tag{2.6}$$

Write $g_0 := g + f_0$. Then $g_0 \in \mathcal{U}$ and

$$\mathcal{H}g_0(x_1, \dots, x_n) - \mu(x_1, \dots, x_n) = \mathcal{H}g(x_1, \dots, x_n) \in \Phi(x_1, \dots, x_n), \\ (x_1, \dots, x_n) \in P.$$

Hence, there exists a solution $h_0 \in \mathcal{D}$ of Eq. (2.4) such that

$$g_0(x) - h_0(x) \in \Psi(x), \qquad x \in S \setminus B.$$

Clearly, $h := h_0 - f_0 \in \mathcal{D}$ is a solution to (2.5) and

$$g(x) - h(x) = g_0(x) - h_0(x) \in \Psi(x), \qquad x \in S \setminus B.$$

The proof of the necessary condition is analogous. But for the convenience of readers, we present it below. So, assume that Eq. (2.5) is $(\Phi, \Psi)$–stable in $\mathcal{U}$. Let $g_0 \in \mathcal{U}$ satisfy

$$\mathcal{H}g_0(x_1, \dots, x_n) - \mu(x_1, \dots, x_n) \in \Phi(x_1, \dots, x_n) \\ (x_1, \dots, x_n) \in P. \tag{2.7}$$

Write $g := g_0 - f_0$. Then

$$\mathcal{H}g(x_1, \dots, x_n) = \mathcal{H}g_0(x_1, \dots, x_n) - \mu(x_1, \dots, x_n) \in \Phi(x_1, \dots, x_n), \\ (x_1, \dots, x_n) \in P.$$

Hence, there exists a solution $h \in \mathcal{D}$ of Eq. (2.5) such that

$$g(x) - h(x) \in \Psi(x), \qquad x \in S \setminus B.$$

Clearly, $h_0 := h + f_0$ is a solution to (2.4) and

$$g_0(x) - h_0(x) = g(x) - h(x) \in \Psi(x), \qquad x \in S \setminus B.$$

Assume now that Eq. (2.4) is $(\Phi, \Psi)$-stable in $\mathcal{U}$ with uniqueness. Let $g \in \mathcal{U}$ satisfy condition (2.6) and $h, h' \in \mathcal{D}$ be solutions to (2.5) such that

$$g(x) - h(x), g(x) - h'(x) \in \Psi(x), \qquad x \in S \setminus B.$$

Write $g_0 := g + f_0$, $h_0 := h + f_0$ and $h'_0 := h' + f_0$. Then (2.7) holds, $h_0$ and $h'_0$ are solutions to (2.4) and

$$g_0(x) - h_0(x), g_0(x) - h'_0(x) \in \Psi(x), \qquad x \in S \setminus B.$$

Consequently, $h_0 = h'_0$, whence $h = h'$. This completes the proof of the sufficient condition concerning uniqueness.

The proof of the converse implication is analogous. □

*Remark* 2.3. Apparently, the assumption of Theorem 2.2 that Eq. (2.4) admits a solution $f_0 \in \mathcal{U}$ is quite natural at least in the case when $\mathcal{U} = \mathcal{D}$. Otherwise, if (2.4) does not have any solution $f_0 \in \mathcal{D}$, then it seems that we can consider it to be not stable, provided there exist some functions $g_0 \in \mathcal{U}$ satisfying (2.7); in such a case we could speak of trivial $(\Phi, \Psi)$–nonstability. Clearly, without this assumption, we could deduce from Theorem 2.2 that the existence of a function $g_0 \in \mathcal{U}$ satisfying (2.7) implies the existence of a solution $f \in \mathcal{D}$ of (2.4). The subsequent example shows that sometimes this is not the case, which makes the necessity of the assumption more convincing.

*Example.* Let $E_1$ and $E_2$ be normed spaces, $d : E_1^2 \to E_2$, $c, p, r, s \in \mathbb{R}$, $p < 0$, $c > 0$, $s + r < 0$, $d(x, x) \neq 0$ for some $x \neq 0$, and let one of the following two inequalities be fulfilled:

$$d(x, y) \le c\big(\|x\|^p + \|y\|^p\big) =: \phi_1(x, y), \qquad x, y \in E_1 \setminus \{0\},$$
$$d(x, y) \le c\|x\|^s\|y\|^r =: \phi_2(x, y), \qquad x, y \in E_1 \setminus \{0\}.$$

Then, according to [8, Corollary 4.3] and [10, Corollary 4.2], functional Eq. (1.4) has no solutions $g : E_1 \to E_2$. But, on the other hand, each additive $h : E_1 \to E_2$ fulfils the inequality

$$\begin{aligned}\|h(x + y) - h(x) - h(y) - d(x, y)\| &= \|d(x, y)\| \\ &\le \phi(x, y), \qquad x, y \in E_1 \setminus \{0\},\end{aligned}$$

for $\phi := \phi_i$ with suitable $i \in \{1, 2\}$, i.e.,

$$h(x + y) - h(x) - h(y) - d(x, y) \in \Phi(x, y), \qquad x, y \in E_1 \setminus \{0\},$$

with

$$\Phi(x, y) := B(0, \phi_i(x, y)), \qquad x, y \in E_1 \setminus \{0\},$$

where $B(0, r) := \{x \in E_2 : \|x\| \le r\}$ for $r \in \mathbb{R}_+$.

## 3. Proof of Theorem 1.2

The proof is actually a routine in view of Theorems 1.1 and 2.2. But for the convenience of readers we present the main steps. In particular, we need the following simple observation.

**Lemma 3.1.** *If a function $h : E_1 \to E_2$ fulfils the condition*

$$h(x+y) = h(x) + h(y), \qquad x, y \in E_1 \setminus \{0\}, \tag{3.1}$$

*then it is additive.*

*Proof.* Let $h : E_1 \to E_2$ satisfy (3.1). Take $x \in E_1 \setminus \{0\}$. Then

$$h(x) = h(2x - x) = h(2x) + h(-x) = 2h(x) + h(-x),$$

whence $h(-x) = -h(x)$ and next

$$h(0) = h(x - x) = h(x) - h(x) = 0.$$

Thus we have proved that $h$ is additive. □

Now, we are ready to prove the theorem. First we show statement $(a)$. So, fix $p \geq 0, p \neq 1$. In view of Theorem 1.1 the conditional functional equation

$$h(x+y) = h(x) + h(y), \qquad (x, y) \in P := (E_1 \setminus \{0\})^2, \tag{3.2}$$

is $(\Phi, \Psi)$-stable with uniqueness for

$$\Phi(x,y) := B\big(0, c(\|x\|^p + \|y\|^p)\big), \quad \Psi(x) := B\Big(0, \frac{c\|x\|^p}{\big|2^{p-1} - 1\big|}\Big),$$
$$x, y \in E_1 \setminus \{0\}.$$

Hence, according to Theorem 2.2 $(\beta)$ (with $n = 2$, $S = E_1$, $X = E_2$, $B = \{0\}$, $\mu = d$, and $\mathcal{H}h(x, y) := h(x+y) - h(x) - h(y)$ for $x, y \in E_1$ and $h : E_1 \to E_2$), the conditional functional equation

$$g(x+y) = g(x) + g(y) + d(x,y), \qquad x, y \in P, \tag{3.3}$$

is $(\Phi, \Psi)$-stable with uniqueness.

Moreover, if $g : E_1 \to E_2$ is a solution of Eq. (3.3), then $g_0 := g - f_0$ fulfils

$$g_0(x+y) = g_0(x) + g_0(y), \qquad x, y \in E_1 \setminus \{0\},$$

whence, by Lemma 3.1, it is additive, which means that $g$ is a solution to Eq. (1.4).

To complete the proof of $(a)$, assume that $E_1 = \mathbb{R}$. Take $u_0 \in E_2$ with $\|u_0\| = 1$ and define functions $f_1 : \mathbb{R} \to \mathbb{R}$ and $f : \mathbb{R} \to E_2$ by

$$f_1(x) = \frac{sgn(x)\, c|x|^p}{|2^{p-1} - 1|}, \qquad f(x) = f_1(x)u_0 + f_0(x), \qquad x \in \mathbb{R}.$$

Then (1.7) holds and

$$\|f(x+y) - f(x) - f(y) - d(x,y)\| = \|(f_1(x+y) - f_1(x) - f_1(y))u_0\|$$
$$= |f_1(x+y) - f_1(x) - f_1(y)|, \qquad x, y \in \mathbb{R}.$$

This ends the proof of $(a)$, because by [5, Theorem 2]

$$|f_1(x+y) - f_1(x) - f_1(y)| \leq c(|x|^p + |y|^p), \qquad x, y \in \mathbb{R}.$$

Statement $(b)$ follows at once from Lemma 3.1 and Theorems 1.1 and 2.2 $(\beta)$ (analogously to statement $(a)$), with

$$\Phi(x,y) := B\big(0, c(\|x\|^p + \|y\|^p)\big), \quad \Psi(x) := \{0\}, \qquad x, y \in E_1 \setminus \{0\}.$$

Finally, we prove $(c)$. Let $E_1 = E_2 = \mathbb{R}$. Fix $c_0 > 0$. According to the results in [20], there is $f_1 : \mathbb{R} \to \mathbb{R}$ such that

$$|f_1(x+y) - f_1(x) - f_1(y)| \le |x| + |y|, \qquad x, y \in \mathbb{R},$$

and

$$\sup_{x \in \mathbb{R}\setminus\{0\}} \frac{|f_1(x) - h_1(x)|}{|x|} = \infty$$

for each additive $h_1 : \mathbb{R} \to \mathbb{R}$. Write $f := c_0 f_1 + f_0$. Then

$$\begin{aligned} |f(x+y) - f(x) - f(y) - d(x,y)| &= c_0 |f_1(x+y) - f_1(x) - f_1(y)| \\ &\le c_0(|x| + |y|), \qquad x, y \in \mathbb{R}. \end{aligned}$$

Take a solution $h : \mathbb{R} \to \mathbb{R}$ of (1.4). Then $h_1 := h - f_0$ is additive and

$$\sup_{x \in \mathbb{R}\setminus\{0\}} \frac{|f(x) - h(x)|}{|x|} = \sup_{x \in \mathbb{R}\setminus\{0\}} c_0 \frac{|f_1(x) - c_0^{-1} h_1(x)|}{|x|} = \infty.$$

This ends the proof.

## 4. Further consequences

In this part we show some further examples of applications of Theorem 2.2. Let us start with a result which follows from [7, Corollary 12].

**Theorem 4.1.** *Let $E$ be a real linear topological space, $F$ be a Banach space, $K$ be a subgroup of the group $(F, +)$, $C \subset F$ be nonempty, closed and convex, $C = -C$, and*

$$\inf_{x \in C, y \in K\setminus\{0\}} \|4x - y\| > 0. \tag{4.1}$$

*Suppose that $g : E \to F$ is continuous at a point $x_0 \in E$ and satisfies the condition*

$$g(x+y) - g(x) - g(y) \in K + C, \qquad x, y \in E. \tag{4.2}$$

*Then there exists an additive $h : E \to F$ such that*

$$g(x) - h(x) \in K + C, \qquad x \in E.$$

*Moreover, if $C$ is bounded, then $h$ is unique.*

Theorems 4.1 and 2.2 yield the following.

**Corollary 4.2.** *Let $E$, $F$, $K$, $C$ be as in Theorem 4.1, $d : E^2 \to F$ and $x_0 \in E$. Suppose that Eq. (1.4) admits a solution $f_0 : E \to F$ that is continuous at $x_0$. Then for every function $g : E \to F$ that is continuous at $x_0$ and satisfies the condition*

$$g(x+y) - g(x) - g(y) - d(x,y) \in K + C, \qquad x, y \in E,$$

*there exists a solution $h : E \to F$ of Eq. (1.4) such that*

$$g(x) - h(x) \in K + C, \qquad x \in E.$$

*Moreover, if $C$ is bounded, then $h$ is unique.*

*Proof.* Theorem 4.1 implies that Eq. (1.4) is $(\Phi, \Psi)$-stable in $\mathcal{U}$ with $\mathcal{D} = F^E$, $\Psi(x) = \Phi(x,y) = K + C$ for every $x, y \in E$ and $\mathcal{U}$ being the family of all functions $f \in F^E$ that are continuous at $x_0$; moreover that stability is with uniqueness when $C$ is bounded. So, it is enough to use Theorem 2.2 with $n = 2$, $S = E$, $X = F$, and $B = \emptyset$. □

Now we present several examples of hyperstability results (see [11] for further information on this issue). Let us start with the following remark.

*Remark* 4.3. It is well known (see, e.g., [24] or [16,31]) that, under the assumptions of Theorem 1.2, Eq. (1.4) admits a solution $f_0 : E_1 \to E_2$ if and only if $d$ is symmetric (i.e., $d(x,y) = d(y,x)$ for $x, y \in E_1$) and fulfils the cocycle equation

$$d(x+y,z) + d(x,y) = d(x,y+z) + d(y,z). \tag{4.3}$$

Examples of very useful related results can be found in [3,12–15,17,19,23,24].

The next theorem can be easily deduced from [9, Proposition 2.2].

**Theorem 4.4.** *Let $E$ and $Y$ be normed spaces, $\dim E > 2$, $g : E \to Y$, and $p \neq 1$ and $L_0$ be positive real numbers with*

$$\|g(x+y) - g(x) - g(y)\| \le L_0 \big| \|x+y\|^2 - \|x-y\|^2 \big|^p, \qquad x, y \in E. \tag{4.4}$$

*Then $g$ is additive.*

It yields the following.

**Corollary 4.5.** *Let $E$ and $Y$ be normed spaces, $\dim E > 2$, $g : E \to Y$, $p \neq 1$ and $L_0$ be positive real numbers, $d : E^2 \to Y$ be symmetric and fulfil the cocycle equation (4.3), and*

$$\|g(x+y) - g(x) - g(y) - d(x,y)\| \\ \le L_0 \big| \|x+y\|^2 - \|x-y\|^2 \big|^p, \qquad x, y \in E. \tag{4.5}$$

*Then $g$ is a solution to (1.4).*

*Proof.* Equation (1.4) admits a solution $f_0 : E \to Y$ in view of Remark 4.3. So, it is enough to use Theorem 2.2 analogously as in the proof of Corollary 4.2 with $\mathcal{U} = \mathcal{D} = F^E$, $\Phi(x, y) = B(0, L_0\big|\|x + y\|^2 - \|x - y\|^2\big|^p)$ and $\Psi(x) = \{0\}$ for every $x, y \in E$. □

The next result was proved in [28, Theorem 2] and concerns the linear functional equation (in two variables).

**Theorem 4.6.** *Assume that $E$ is a normed space over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, $Y$ is a Banach space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, $a, b \in \mathbb{F} \setminus \{0\}$, $A, B \in \mathbb{K}$, $\theta, p \in \mathbb{R}$, $p < 0$, $\theta \geq 0$, and $f : E \to Y$ satisfies the inequality*

$$\|f(ax + by) - Af(x) - Bf(y)\| \leq \theta(\|x\|^p + \|y\|^p), \qquad x, y \in E \setminus \{0\}. \tag{4.6}$$

*Then*

$$f(ax + by) = Af(x) + Bf(y), \qquad x, y \in E \setminus \{0\}. \tag{4.7}$$

We will derive from it a hyperstability result for the inhomogeneous version of the linear equation. To this end we need yet the following.

**Lemma 4.7.** *Assume that $E$ is a linear space over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, $Y$ is a linear space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, $a, b \in \mathbb{F} \setminus \{0\}$, $A, B \in \mathbb{K}$, and $f : E \to Y$ satisfies (4.7). Then $f$ satisfies the equation*

$$f(ax + by) = Af(x) + Bf(y) \tag{4.8}$$

*for every $x, y \in E$.*

*Proof.* Let $e := f(0)$. Then, in view of (4.7), we get

$$e = f(0) = f(abx - bax) = Af(bx) + Bf(-ax), \qquad x \in E \setminus \{0\}.$$

This implies that

$$Bf(-b^{-1}z) = e - Af(a^{-1}z), \qquad z \in E \setminus \{0\}. \tag{4.9}$$

Consequently, by (4.7) and (4.9), for every $x, y \in E \setminus \{0\}$

$$\begin{aligned} f(x) &= f(y + x - y) = f(aa^{-1}y + bb^{-1}(x - y)) \\ &= Af(a^{-1}y) + Bf(b^{-1}(x - y)) \\ &= Af(a^{-1}y) + Bf(ab^{-1}a^{-1}x - bb^{-2}y) \\ &= Af(a^{-1}y) + BAf(b^{-1}a^{-1}x) + B^2f(-b^{-2}y) \\ &= Af(a^{-1}y) + BAf(b^{-1}a^{-1}x) + B(e - Af(a^{-1}b^{-1}y)), \end{aligned}$$

whence

$$f(x) - BAf(b^{-1}a^{-1}x) - Be = A(f(a^{-1}y) - Bf(a^{-1}b^{-1}y)), \tag{4.10}$$

which means that there is $d \in Y$ such that

$$f(z) - Bf(b^{-1}z) = d, \qquad z \in E \setminus \{0\}. \tag{4.11}$$

Analogously, for every $x, y \in E \setminus \{0\}$, we get

$$\begin{aligned} f(x) &= f(bb^{-1}y + aa^{-1}(x-y)) \\ &= Bf(b^{-1}y) + BAf(b^{-1}a^{-1}x) + A(e - Bf(a^{-1}b^{-1}y)) \end{aligned}$$

and next

$$f(x) - BAf(b^{-1}a^{-1}x) - Ae = B(f(b^{-1}y) - Af(a^{-1}b^{-1}y)), \tag{4.12}$$

which means that there is $c \in Y$ such that

$$f(z) - Af(a^{-1}z) = c, \qquad z \in E \setminus \{0\}. \tag{4.13}$$

Clearly, (4.11) and (4.13) can be rewritten in the form

$$f(az) = Af(z) + c, \qquad f(bz) = Bf(z) + d, \qquad z \in E \setminus \{0\}. \tag{4.14}$$

Consequently, from (4.7) we obtain

$$f(ax + by) = Af(x) + Bf(y) = f(ax) - c + f(by) - d, \qquad x, y \in E \setminus \{0\},$$

whence the function $f_0 := f - c - d$ satisfies

$$f_0(x + y) = f_0(x) + f_0(y), \qquad x, y \in E \setminus \{0\}. \tag{4.15}$$

Analogously as in the proof of Lemma 3.1 we show that $f_0$ must be additive, which means that $f_0(0) = 0$ and consequently

$$e = f(0) = f_0(0) + c + d = c + d. \tag{4.16}$$

Note that (4.10), (4.11) and (4.13) yield

$$\begin{aligned} c + Ad &= f(x) - Af(a^{-1}x) + A(f(a^{-1}x) - Bf(b^{-1}a^{-1}x)) \\ &= f(x) - BAf(b^{-1}a^{-1}x) = Ad + Be \end{aligned} \tag{4.17}$$

with any fixed $x \neq 0$. Analogously, from (4.12) we deduce that $d + Bc = Bc + Ae$, whence and from (4.17) we get

$$d = Ae, \qquad c = Be. \tag{4.18}$$

This and (4.14) imply that

$$f(ax) = Af(x) + Bf(0), \qquad f(bx) = Af(0) + Bf(x), \qquad x \in E \setminus \{0\}.$$

Finally, by (4.16) and (4.18),

$$f(0) = e = c + d = Ae + Be = Af(0) + Bf(0). \tag{4.19}$$

Thus we have proved that (4.8) holds also when $x = 0$ or $y = 0$. $\square$

Now, we are in a position to prove the following.

**Corollary 4.8.** *Assume that $E$ is a normed space over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, $Y$ is a normed space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, $a, b \in \mathbb{F} \setminus \{0\}$, $A, B \in \mathbb{K}$, $\theta, p \in \mathbb{R}$, $p < 0$, $\theta \geq 0$, $d : E^2 \to Y$, and $g : E \to Y$ satisfies the inequality*

$$\begin{aligned} &\|g(ax + by) - Ag(x) - Bg(y) - d(x, y)\| \\ &\qquad \leq \theta(\|x\|^p + \|y\|^p), \qquad x, y \in E \setminus \{0\}. \end{aligned} \tag{4.20}$$

*Assume that the functional equation*

$$f(ax + by) = Af(x) + Bf(y) + d(x, y), \qquad x, y \in E, \tag{4.21}$$

*has a solution $f_0 : E \to Y$. Then $g$ is a solution to (4.21).*

*Proof.* Clearly, if $Y$ is not complete, then without loss of generality we can replace it by its completion. Thus, by Theorem 2.2 and Theorem 4.6,

$$g(ax + by) = Ag(x) + Bg(y) + d(x, y), \qquad x, y \in E \setminus \{0\}. \tag{4.22}$$

Hence condition (4.7) holds for $f := g - f_0$. Consequently, Lemma 4.7 implies that $f$ is a solution to (4.21), which means that $g$ is a solution to (4.21). □

*Remark* 4.9. It is easily seen that (under the assumptions of Corollary 4.8) in the case where $A + B \neq 1$ and $d$ is a constant function, $d(x, y) \equiv c$, (4.21) admits a constant solution of the form

$$f_0(x) = \frac{c}{1 - A - B}, \qquad x \in E.$$

Therefore Corollary 4.8 also generalizes [28, Corollary 3].

Other examples of solutions to (4.21), with some particular form of $d$, can be found in [6].

From [2, Theorem 5] we obtain the following hyperstability result for the Jensen equation.

**Theorem 4.10.** *Assume that $E$ and $Y$ are normed spaces, $U \subset E$, $0 \in U$, $2U = U$, $\gamma, p, q \in (0, \infty)$, $p + q \neq 1$, and $f : U \to Y$ satisfies the inequality*

$$\begin{aligned} &\left\| f\Big(\frac{1}{2}(x + y)\Big) - \frac{1}{2}(f(x) + f(y)) \right\| \leq \gamma \|x\|^p \|y\|^q, \\ &\qquad\qquad x, y \in U, \frac{1}{2}(x + y) \in U. \end{aligned} \tag{4.23}$$

*Then*

$$f\Big(\frac{1}{2}(x + y)\Big) = \frac{1}{2}(f(x) + f(y)), \qquad x, y \in U, \frac{1}{2}(x + y) \in U. \tag{4.24}$$

The subsequent corollary can be easily derived from this.

**Corollary 4.11.** *Assume that $E$ and $Y$ are normed spaces, $U \subset E$, $0 \in U$, $2U = U$, $\gamma, p, q \in (0, \infty)$, $p + q \neq 1$, and $f : U \to Y$ and $d : U^2 \to Y$ satisfy the inequality*

$$\left\| f\Big(\frac{1}{2}(x+y)\Big) - \frac{1}{2}(f(x) + f(y)) - d(x,y) \right\| \leq \gamma \|x\|^p \|y\|^q, \qquad x, y \in U, \frac{1}{2}(x+y) \in U. \tag{4.25}$$

*Suppose that the functional equation*

$$f\Big(\frac{1}{2}(x+y)\Big) = \frac{1}{2}(f(x) + f(y)) + d(x,y), \qquad x, y \in U, \frac{1}{2}(x+y) \in U, \tag{4.26}$$

*admits a solution $f_0 : U \to Y$. Then $f$ is a solution to (4.26).*

*Proof.* We deduce the conclusion from Theorem 2.2 (with $S = U$ and $P = \{(x,y) \in U^2 : x + y \in 2U\}$) and Theorem 4.10, analogously as in the proofs of previous results. □

We end the paper with an example concerning orthogonally additive mappings. In the remaining part of the paper, $(E, \langle \cdot, \cdot \rangle)$ is an inner product space. As usual, given $x, y \in E$, we write $x \perp y$ provided $\langle x, y \rangle = 0$. The next theorem can be easily deduced from the main result in [18].

**Theorem 4.12.** *Let $Y$ be a Banach space, $\epsilon \in \mathbb{R}_+$, and $f : E \to Y$ satisfy the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon, \qquad x \perp y. \tag{4.27}$$

*Then there exists a unique solution $g : E \to Y$ of the conditional equation*

$$g(x+y) = g(x) + g(y), \qquad x \perp y, \tag{4.28}$$

*such that*

$$\|f(x) - g(x)\| \leq 5\epsilon, \qquad x \in E. \tag{4.29}$$

Theorem 4.12 enables us to prove the following.

**Corollary 4.13.** *Let $Y$ be a Banach space, $\epsilon \in \mathbb{R}_+$, $d : E^2 \to Y$ be symmetric and fulfil the cocycle equation (4.3), and $f : E \to Y$ satisfy the inequality*

$$\Big\| f(x+y) - f(x) - f(y) - d(x,y) \| \leq \epsilon, \qquad x \perp y. \tag{4.30}$$

*Then there exists a unique solution $g : E \to Y$ of the conditional equation*

$$g(x+y) = g(x) + g(y) + d(x,y), \qquad x \perp y, \tag{4.31}$$

*such that*

$$\|f(x) - g(x)\| \leq 5\epsilon, \qquad x \in E. \tag{4.32}$$

*Proof.* Since $d : E^2 \to Y$ is a solution of the cocycle equation (4.3), there exists a solution $g_0 : E \to Y$ of (1.4) (see Remark 4.3), which clearly is a solution to (4.31), as well. So, it is enough to use Theorem 2.2 (with $P = \{(x, y) \in E^2 : x \perp y\}$) and Theorem 4.12. □

We can find in [4,11,22,25–27] numerous further examples of stability results for various equations that can be extended to their inhomogeneous versions, in a similar way as in this paper (by applying Theorem 2.2). In general, it is very easy to find suitable reasonings.



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