# Delay-dependent stability analysis of neural networks with time-varying delay: a generalized free-weighting-matrix approach

Chuan-Ke Zhang[a,b], Yong He[a,*], Lin Jiang[b], Wen-Juan Lin[a], Min Wu[a]

[a]School of Automation, China University of Geosciences, Wuhan 430074, China

[b]Department of Electrical Engineering & Electronics, University of Liverpool, Liverpool L69 3GJ, United Kingdom

### Abstract

This paper investigates the delay-dependent stability problem of continuous neural networks with a bounded time-varying delay via Lyapunov-Krasovskii functional (LKF) method. This paper focuses on reducing the conservatism of stability criteria by estimating the derivative of the LKF more accurately. Firstly, based on several zero-value equalities, a generalized free-weighting-matrix (GFWM) approach is developed for estimating the single integral term. It is also theoretically proved that the GFWM approach is less conservative than the existing methods commonly used for the same task. Then, the GFWM approach is applied to investigate the stability of delayed neural networks, and several stability criteria are derived. Finally, three numerical examples are given to verify the advantages of the proposed criteria.

*Keywords:* Neural networks, time-varying delay, generalized free-weighting-matrix approach, stability

## 1. Introduction

Neural networks have been successfully applied in image processing, pattern recognition, associative memory, optimization problem, *etc.* [1]-[3]. For those applications, the artificial neural networks usually must be stable [6]. However, the finite switching speed of amplifiers and the inherent communication time between neurons inevitably cause time delays during the implementation of artificial neural networks [7]. These delays might lead to undesired dynamics like oscillation and instability. Therefore, it is an important job to determine the admissible maximal delay bound (AMDB) such that the delayed neural networks (DNNs) with a delay less than this bound remain stable. In the past few decades, such delay-dependent stability analysis problem has become a hot issue in the field of neural networks [4].

These delays are usually time-varying and Lyapunov-Krasovskii functional (LKF) method can easily handle such DNNs, thus, the LKF method has become the one of most popular methods for stability analysis and finding the AMDBs. The DNNs with bounded delays are asymptotically stable if there exists an LKF, which consists of state vector based quadratic terms, is positive definite and has a negative gradient over the time. Linear matrix inequalities (LMIs) based delay-dependent stability criteria can be easily used to check whether or not such LKF can be found for the DNNs. However, the criteria derived have some conservatism since they are only sufficient conditions. In order

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*Corresponding author: Yong He, Email: heyong08@cug.edu.cn

to find the AMDBs more accurately, one important issue in the related research is to develop new criteria with less conservatism. This paper also investigates the stability of DNNs following this direction.

By constructing a special form of LKF candidate, tractable LMI-based stability criteria are derived using necessary techniques to estimate the LKF and its derivative. Therefore, the construction and the treatment for the LKF are the basic issues related to how conservative the criteria are.

In the early research, the LKFs for the stability analysis of the DNNs were constructed by introducing delay-based single and double integral terms into the typical non-integral quadratic form of Lyapunov function for delay-free systems [5], [6], [8]-[14]. Later, researchers have developed many new LKFs by making the previous ones more general in three aspects.

1) Firstly, based on several subintervals divided from the whole delay region, some scholars have developed the delay-partition-based LKFs by replacing the original integral terms with multiple new integral terms with smaller domain of integration [16]-[30].

2) Secondly, by using various state vectors (current, delayed, and/or integrated state vectors *etc.*), some scholars have developed new LKFs by augmenting the quadratic terms of original LKFs [31]-[42].

3) Thirdly, since the triple integral term was found to be helpful for reducing the conservatism of stability criteria for linear time-delay systems [47], similar and/or extended forms have also been widely applied to stability analysis of various DNNs [49]-[58].

Although those LKFs have different forms, they all include a common term with the form of $\int_{-h}^{0}\int_{t+\theta}^{t} y^T(s)Ry(s)dsd\theta$ (Here $h > 0$ is the scalar, $y(s)$ is the system state-based vector, and matrix $R > 0$.). Then its derivative contains the term as follows

$$-\int_{t-h}^{t} y^T(s)Ry(s)ds \tag{1}$$

This term was directly dropped in the early literature [5], but such treatment is very conservative. Later, this term was retained to improve the results, in which case it must be estimated to represent the criterion in the form of tractable LMI. As mentioned in [43], the estimation of the above single integral term is strongly related to the conservatism of criteria. Therefore, the stability criteria of DNNs have been improved gradually by using more effective techniques for this estimation task.

The basic inequality was used to estimate the single integral term [6]. Since He *et al.* [44] proposed the free-weighting matrix (FWM) approach, which is more effective than the basic inequality, the FWM approach has been widely used in the stability analysis of DNNs [8]-[13], [28]-[32]. The slack matrices introduced by the FWM approach provide great freedom of the criteria.

An alternative method that estimates the original integral terms directly was also used in the stability analysis of DNNs. The criteria from this type of method are strongly linked to the inequalities used. At the beginning, the Jensen inequality has been wildly applied to analyze the stability of the DNNs [14]-[26], [33, 34, 51, 52]. In 2013, Seuret *et al.* [43] presented a Wirtinger-based inequality and proved that it is less conservative than the Jensen inequality. Since then, Wirtinger-based inequality has become the most popular method to estimate the single integral term during the investigation of DNNs [27], [37]-[41], [53]-[56].

Very recently, Zeng *et al.* proposed a free-matrix-based inequality (FMBI) in [45] and extended it to the research of DNNs [42, 50]. To the best of the authors' knowledge, this inequality is the least conservative among the existing inequalities for estimating single integral term. However, there is further room to be investigated using the FMBI. Some slack matrices in the FMBI do not seem to contribute to a reduction of the conservatism. In [42], the FMBI was only used to estimate the single integral term without any augmented vector, while the augmented integral term was still estimated via the Jensen inequality.

It can be expected that stability criteria with less conservatism will be obtained by developing and using a more effective approach to estimate the single integral term. This motivates the present research.

This paper further investigates delay-dependent stability of DNNs following the development of a more effective method to estimate the single integral term (1). The contributions of the paper are summarized as follows:

1) A general free-weighting-matrix (GFWM) approach is developed to estimate single integral term. Based on several zero-value equalities, a new estimation method, named as GFWM approach, is developed by following the basic estimation procedure of the FWM approach. And a new inequality is derived based on the GFWM approach (Lemma 5).

2) Necessary theoretical studies are carried out to compare the GFWM approach and several previous estimation methods. It is proved that the inequality obtained from the GFWM approach can encompass the Wirtinger-based inequality and the FMBI.

3) Several new stability criteria with less conservatism for the DNNs are derived. For generalized neural networks with a time-varying delay, based on two LKFs (one with delay-product-type terms and the other without similar terms), two stability criteria are derived by using the GFWM to estimate the single integral term appearing in the derivative of the LKFs.

The remainder of the paper is organized as follows. Section 2 gives the problem formulation and necessary preliminary. In Section 3, the development of the GFWM approach and its comparison to previous methods are discussed in detail. The GFWM approach is applied to a generalized DNN and several stability criteria are derived in Section 4. In Section 5, three numerical examples are used to demonstrate the benefits of the proposed criteria. Conclusions are given in Section 6.

*Notations*: Throughout this paper, the superscripts $T$ and $-1$ mean the transpose and the inverse of a matrix, respectively; $\mathcal{R}^n$ denotes the $n$-dimensional Euclidean space; $\mathcal{R}^{n\times m}$ is the set of all $n \times m$ real matrices; $\|\cdot\|$ refers to the Euclidean vector norm; $P > 0$ $(\geq 0)$ means that $P$ is a real symmetric and positive-definite (semi-positive-definite) matrix; diag$\{\cdots\}$ denotes a block-diagonal matrix; the symmetric term in a symmetric matrix is denoted by $*$; and $\mathrm{Sym}\{X\} = X + X^T$.

## 2. Problem formulation and preliminary

This section describes the problem to be investigated and gives related preliminaries.

### 2.1. Problem formulation

Consider the following generalized DNNs [35]:

$$\dot{y}(t) = -Ay(t) + W_0 g(Wy(t)) + W_1 g(Wy(t-d(t))) + J \tag{2}$$

where $y(t) = [y_1(t)\ y_2(t)\ \cdots\ y_n(t)]^T$ is the state vector associated with the $n$ neurons; $g(\cdot) = [g_1(\cdot)\ g_2(\cdot)\ \cdots\ g_n(\cdot)]^T$ represents the neuron activation function with $g(0)=0$; $A=\text{diag}\{a_1, a_2, \cdots, a_n\}>0$; $W$, $W_0$ and $W_1$ are the connection weight matrices; $J=[J_1\ J_2\ \cdots\ J_n]^T$ is a vector representing the bias; and $d(t)$ is a time-varying delay satisfying

$$0 \le d(t) \le h \tag{3}$$

and

$$Case I: \quad \dot{d}(t) \le \mu \tag{4}$$
$$Case II: \quad |\dot{d}(t)| \le \mu \tag{5}$$

where $h$ and $\mu$ are constants.

**Assumption 1.** *The neuron activation function $g_i(\cdot)$ is assumed to be bounded and satisfies the following condition:*

$$\sigma_i^- \le \frac{g_i(s_1) - g_i(s_2)}{s_1 - s_2} \le \sigma_i^+, \quad s_1 \ne s_2, i = 1, 2, \cdots, n \tag{6}$$

*where $\sigma_i^-$ and $\sigma_i^+$ are known real constants.*

**Remark 1.** *Constants $\sigma_i^+$ and $\sigma_i^-$ in Assumption 1 are allowed to be positive, negative, or zero, and many activation functions (monotonic or non-monotonic) satisfy the condition of (6) [15].*

Based on this assumption for the activation function, there exists an equilibrium point $y^*$ for the neural network, i.e., $0 = -Ay^* + W_0 g(Wy^*) + W_1 g(Wy^*) + J$. Using transformation $x(t) = y(t) - y^*$ [4], one can shift the equilibrium point $y^*$ of (2) to the origin and rewrite system (2) as:

$$\dot{x}(t) = -Ax(t) + W_0 f(Wx(t)) + W_1 f(Wx(t - d(t))) \tag{7}$$

where $f(.) = [f_1(\cdot)\ f_2(\cdot)\ \cdots\ f_n(\cdot)]^T$ and $f_i(W_{2i}x(t)) = g_i(W_{2i}x(t) + W_{2i}y^*) - g_i(W_{2i}y^*)$ with $f_i(0) = 0$ and $W_{2i}$ denoting the $i$-th row vector of the matrix $W$. Then,

it follows from (6) and $f_i(0) = 0$ that [51]

$$\sigma_i^- \le \frac{f_i(s_1) - f_i(s_2)}{s_1 - s_2} \le \sigma_i^+, \quad s_1 \ne s_2 \tag{8}$$
$$\sigma_i^- \le \frac{f_i(s)}{s} \le \sigma_i^+, \quad s \ne 0 \tag{9}$$

This paper is concerned with the delay-dependent stability of DNN (2). In order to determine the AMDBs more accurately, this paper aims to derive new delay-dependent stability criteria with conservatism as small as possible. As mentioned in Section I, the construction of LKFs and the treatment of their derivatives are two aspects related to conservatism. Since there are many different forms of LKFs, this paper mainly pays attention to the treatment of the single integral term that appears in the derivative of all LKFs.

*2.2. Preliminary*

The following lemmas are used in subsequent sections of this paper.

**Lemma 1.** *Wirtinger-based inequality [43]: For symmetric positive definite matrix $R \in \mathcal{R}^{n\times n}$, scalars $a < b$, and vector $\omega : [a, b] \mapsto \mathcal{R}^n$ such that the integrations concerned are well defined, then the following inequality holds*

$$\int_a^b \omega^T(s)R\omega(s)ds \geq \frac{1}{b-a}\begin{bmatrix}\chi_1\\ \chi_2\end{bmatrix}^T\begin{bmatrix}R & 0\\ 0 & 3R\end{bmatrix}\begin{bmatrix}\chi_1\\ \chi_2\end{bmatrix} \tag{10}$$

*where* $\chi_1 = \int_a^b \omega(s)ds$ *and* $\chi_2 = \chi_1 - \frac{2}{b-a}\int_a^b\int_a^s \omega(u)duds = -\chi_1 + \frac{2}{b-a}\int_a^b\int_s^b \omega(u)duds$.

**Lemma 2.** *Free-matrix-based inequality [42]: For symmetric matrices $R \in \mathcal{R}^{n\times n}$ and $Z_1, Z_3 \in \mathcal{R}^{3n\times 3n}$, any matrices $Z_2 \in \mathcal{R}^{3n\times 3n}$ and $N_1, N_2 \in \mathcal{R}^{3n\times n}$, such that*

$$\begin{bmatrix}Z_1 & Z_2 & N_1\\ * & Z_3 & N_2\\ * & * & R\end{bmatrix} > 0 \tag{11}$$

*and vector $\nu : [a, b] \mapsto \mathcal{R}^n$ such that the integration concerned are well defined, the following inequality holds*

$$\int_a^b \dot{\nu}^{\mathrm{T}}(s)R\dot{\nu}(s)\mathrm{d}s \geq -(b-a)\varsigma_0^T\left(\frac{3Z_1 + Z_3}{3}\right)\varsigma_0 - \mathrm{Sym}\{\varsigma_0^T N_1\varsigma_1 + \varsigma_0^T N_2\varsigma_2\} \tag{12}$$

*where* $\varsigma_0 = [\nu^T(b), \nu^T(a), \int_a^b \frac{\nu^T(s)}{b-a}ds]^T$, $\varsigma_1 = \nu(b) - \nu(a)$, *and* $\varsigma_2 = \nu(b) + \nu(a) - 2\int_a^b \frac{\nu(s)}{b-a}ds$.

**Lemma 3.** *Reciprocally convex inequality [59]: For any vectors $\beta_1$ and $\beta_2$, symmetric matrix $R$, any matrix $S$ satisfying $\begin{bmatrix}R & S\\ * & R\end{bmatrix} \geq 0$, and real scalar $0 \leq \alpha \leq 1$, then the following inequality holds*

$$\frac{1}{\alpha}\beta_1^T R\beta_1 + \frac{1}{1-\alpha}\beta_2^T R\beta_2 \geq \begin{bmatrix}\beta_1\\ \beta_2\end{bmatrix}^T\begin{bmatrix}R & S\\ * & R\end{bmatrix}\begin{bmatrix}\beta_1\\ \beta_2\end{bmatrix} \tag{13}$$

**Lemma 4.** *For any symmetric matrices $\Pi_0$, $\Pi_1$, and $\Pi_2$, and a scalar continuous function $\alpha(t) \in [0, \gamma]$ with $\gamma$ being constant, the following holds*

$$1): \left.\begin{array}{l}\gamma^2\Pi_0 + \gamma\Pi_1 + \Pi_2 \leq 0\\ \gamma\Pi_1 + \Pi_2 \leq 0\\ \Pi_2 \leq 0\end{array}\right\} \Longrightarrow \alpha^2(t)\Pi_0 + \alpha(t)\Pi_1 + \Pi_2 \leq 0 \tag{14}$$

$$2): \left.\begin{array}{l}\Pi_0 \geq 0\\ \gamma^2\Pi_0 + \gamma\Pi_1 + \Pi_2 \leq 0\\ \Pi_2 \leq 0\end{array}\right\} \Longrightarrow \alpha^2(t)\Pi_0 + \alpha(t)\Pi_1 + \Pi_2 \leq 0 \tag{15}$$

$$3): \left.\begin{array}{l}\Pi_0 \leq 0\\ \gamma\Pi_1 + \Pi_2 \leq 0\\ \Pi_2 \leq 0\end{array}\right\} \Longrightarrow \alpha^2(t)\Pi_0 + \alpha(t)\Pi_1 + \Pi_2 \leq 0 \tag{16}$$

*Proof.* Rewriting $\alpha^2(t)\Pi_0+\alpha(t)\Pi_1+\Pi_2$ yields

$$\alpha^2(t)\Pi_0+\alpha(t)\Pi_1+\Pi_2=\frac{\alpha(t)}{\gamma^2}\left[\alpha(t)(\gamma^2\Pi_0+\gamma\Pi_1+\Pi_2)\right]+\frac{\alpha(t)}{\gamma^2}\left[(\gamma-\alpha(t))(\gamma\Pi_1+\Pi_2)\right]+\frac{\gamma-\alpha(t)}{\gamma}\Pi_2$$

Obviously, 1) of Lemma can easily be obtained from the above representation. If $\Pi_0\geq 0$, $\gamma^2\Pi_0+\gamma\Pi_1+\Pi_2\leq 0\Longrightarrow\gamma\Pi_1+\Pi_2\leq 0$, then 2) of Lemma is obtained. If $\Pi_0\leq 0, \gamma\Pi_1+\Pi_2\leq 0\Longrightarrow\gamma^2\Pi_0+\gamma\Pi_1+\Pi_2\leq 0$, then 3) of Lemma is obtained. This completes the proof.

□

## 3. Estimation of single integral term via the GFWM approach

This section develops the GFWM approach to estimate the single integral term. The comparison of the GFWM approach and the existing commonly used methods (Wirtinger-based inequality and FMBI) is also given.

### 3.1. The GFWM approach

Based on several zero-value equalities, the GFWM approach is developed by following the basic estimation procedure of the FWM approach. More specifically, the following new inequality is established for single integral term.

**Lemma 5.** *(The GFWM-based inequality) For symmetric positive definite matrix $R\in\mathcal{R}^{n\times n}$, any matrices $L, M$, and vector $\omega:[a,b]\mapsto\mathcal{R}^n$ such that the integration concerned are well defined, then the following inequality holds*

$$\int_a^b\omega^T(s)R\omega(s)ds\geq-\mathrm{Sym}\{\chi_0^TL\chi_1+\chi_0^TM\chi_2\}-(b-a)\chi_0^T\left(\frac{3LR^{-1}L^T+MR^{-1}M^T}{3}\right)\chi_0 \tag{17}$$

*where $\chi_0$ is any vector and $\chi_1,\chi_2$ are defined in Lemma 1.*

*Proof.* For a function $\lambda(s)=k_1+k_2s$, the calculation based on integration by parts leads to

$$\int_a^b\lambda(s)\omega(s)ds=\lambda(a)\int_a^b\omega(s)ds+k_2\int_a^b\int_s^b\omega(u)duds$$

By setting $\lambda(a)=-1, k_2=\frac{2}{b-a}$, i.e. $\lambda(s)=\frac{-b-a}{b-a}+\frac{2}{b-a}s$, the above equality is rewritten as

$$\int_a^b\lambda(s)\omega(s)ds=\chi_2 \tag{18}$$

Then the following zero-value equality is obtained for any vector $\chi_0$ and any matrix $M$:

$$0=2\int_a^b\lambda(s)\chi_0^TM\omega(s)ds-2\chi_0^TM\chi_2 \tag{19}$$

Similarly, the following zero-value equalities are derived:

$$0=2\int_a^b\chi_0^TL\omega(s)ds-2\chi_0^TL\chi_1 \tag{20}$$

$$0=\int_a^b\chi_0^T(LR^{-1}L^T)\chi_0ds-(b-a)\chi_0^T(LR^{-1}L^T)\chi_0 \tag{21}$$

$$0=\int_a^b\lambda(s)\chi_0^T(MR^{-1}M^T)\lambda(s)\chi_0ds-\frac{b-a}{3}\chi_0^T(MR^{-1}M^T)\chi_0 \tag{22}$$

$$0 = 2\int_a^b \chi_0^T(LR^{-1}M^T)\lambda(s)\chi_0 ds \tag{23}$$

Based on the estimation idea of the FWM approach, by adding the right sides of (19)-(23) to the single integral term in inequality (17), the following equality is derived:

$$\int_a^b \begin{bmatrix} \chi_0 \\ \lambda(s)\chi_0 \\ \omega(s) \end{bmatrix}^T \begin{bmatrix} LR^{-1}L^T & LR^{-1}M^T & L \\ * & MR^{-1}M^T & M \\ * & * & R \end{bmatrix} \begin{bmatrix} \chi_0 \\ \lambda(s)\chi_0 \\ \omega(s) \end{bmatrix} ds = \int_a^b \omega^T(s)R\omega(s)ds \tag{24}$$

$$+\mathrm{Sym}\{\chi_0^T L\chi_1 + \chi_0^T M\chi_2\} + (b-a)\chi_0^T\left(\frac{3LR^{-1}L^T + MR^{-1}M^T}{3}\right)\chi_0$$

It follows from the Schur complement that

$$\begin{bmatrix} LR^{-1}L^T & LR^{-1}M^T & L \\ * & MR^{-1}M^T & M \\ * & * & R \end{bmatrix} \geq 0 \tag{25}$$

thus

$$\int_a^b \omega^T(s)R\omega(s)ds + \mathrm{Sym}\{\chi_0^T L\chi_1 + \chi_0^T M\chi_2\} + (b-a)\chi_0^T\left(\frac{3LR^{-1}L^T + MR^{-1}M^T}{3}\right)\chi_0 \geq 0$$

which implies (17). This completes the proof. □

**Remark 2.** *From the proof procedure, it can be found that the GFWM-based inequality is obtained by using several zero-value terms and following the similar idea of the FWM approach. The advantages of the GFWM approach compared with the FWM approach are reflected in two aspects: For the FWM approach [44], $\omega = \dot{v}$, and only two zero-value terms (18) and (21) are introduced, while the GFWM approach does not require $\omega = \dot{v}$ such that it can handle single integral term with more general form. And it includes more zero-value terms, namely, (19), (22), and (23), which would reduce the gap between original integral term and its estimated value.*

### 3.2. *The comparison investigation*

This part shows that the inequality obtained by the GFWM approach encompasses the ones given in Lemmas 1 and 2.

*(a) The GFWM-based inequality (17) encompasses the Wirtinger-based inequality (10)*

Letting $\chi_0^T L = -\frac{1}{b-a}\chi_1^T R$ and $\chi_0^T M = -\frac{3}{b-a}\chi_2^T R$ yields

$$-\mathrm{Sym}\{\chi_0^T L\chi_1 + \chi_0^T M\chi_2\} - (b-a)\chi_0^T\left(\frac{3LR^{-1}L^T + MR^{-1}M^T}{3}\right)\chi_0 = \frac{\chi_1^T R\chi_1 + 3\chi_2^T R\chi_2}{b-a} \tag{26}$$

Then inequality (17) reduces to inequality (10). Thus, the GFWM-based inequality (17) encompasses the Wirtinger-based inequality (10).

*(b) The GFWM-based inequality (17) encompasses the FMBI (12)*

First step: proving that (17) encompasses a new inequality (27). Let $\omega = \dot{v}$, then $\chi_1 = \varsigma_1, \chi_2 = \varsigma_2$. By setting $\chi_0 = \varsigma_0$, (17) is rewritten as

$$\int_a^b \dot{v}^T(s)R\dot{v}(s)ds \geq -\mathrm{Sym}\{\varsigma_0^T L\varsigma_1 + \varsigma_0^T M\varsigma_2\} - (b-a)\varsigma_0^T\left(\frac{3LR^{-1}L^T + MR^{-1}M^T}{3}\right)\varsigma_0 \tag{27}$$

Second step: proving that (27)$\Longleftrightarrow$(12).

- On one hand, if set $N_1 = L$, $N_2 = M$, $Z_1 = LR^{-1}L^T$, and $Z_3 = MR^{-1}M^T$, then (12) can be rewritten as

$$\begin{aligned}\int_a^b \dot{v}^{\mathrm{T}}(s)R\dot{v}(s)\mathrm{d}s &\geq -(b-a)\varsigma_0^T\left(\frac{3Z_1+Z_3}{3}\right)\varsigma_0 - \mathrm{Sym}\{\varsigma_0^T N_1\varsigma_1 + \varsigma_0^T N_2\varsigma_2\}\\ &= -(b-a)\varsigma_0^T\left(\frac{3LR^{-1}L^T + MR^{-1}M^T}{3}\right)\varsigma_0 - \mathrm{Sym}\{\varsigma_0^T L\varsigma_1 + \varsigma_0^T M\varsigma_2\}\end{aligned} \tag{28}$$

  Thus, (12)$\Longrightarrow$(27).

- On the other hand, if there exists matrices $X$, $Y$, and $Z$ satisfying $X \geq LR^{-1}L^T$, $Y \geq LR^{-1}M^T$, and $Z \geq MR^{-1}M^T$, then the following inequalities hold

$$\begin{bmatrix} X & Y & L\\ * & Z & M\\ * & * & R\end{bmatrix} \geq 0 \tag{29}$$

  and

$$\begin{aligned}\int_a^b \dot{v}^T(s)R\dot{v}(s)ds &\geq -\mathrm{Sym}\{\varsigma_0^T L\varsigma_1 + \varsigma_0^T M\varsigma_2\} - (b-a)\varsigma_0^T\left(\frac{3LR^{-1}L^T + MR^{-1}M^T}{3}\right)\varsigma_0\\ &\geq -\mathrm{Sym}\{\varsigma_0^T L\varsigma_1 + \varsigma_0^T M\varsigma_2\} - (b-a)\varsigma_0^T\left(\frac{3X+Z}{3}\right)\varsigma_0\end{aligned} \tag{30}$$

  Then, if set $L = N_1$, $M = N_2$, $X = Z_1$, $Y = Z_2$, and $Z = Z_3$, then inequalities (29) and (30) become inequalities (11) and (12). Thus, (27)$\Longrightarrow$(12), (11).

Therefore, the GFWM-based inequality (17) encompasses the FMBI (12).

**Remark 3.** *From the comparison studies, it can be found that the Wirtinger-based inequality is special case of the GFWM-based inequality and can be obtained by fixing some slack matrices. Thus, the GFWM approach is less conservative since the slack matrices provide additional freedom. Moreover, the equivalence of the FMBI and (27) shows that some slack matrices in the FMBI (*$Z_i, i = 1, 2, 3$*) do not contribution to reducing the conservatism. In addition, the FMBI can only handle the case of* $\omega = \dot{v}$.

## 4. Delay-dependent stability analysis of the DNNs

This section investigates the stability of DNN (2) by using the GFWM approach and derives three stability criteria. The following notations are introduced at first for simplifying the representation of subsequent parts:

$$\begin{aligned}
&h_d(t) := h - d(t), \quad x_d(t) := x(t-d(t)), \quad x_h(t) := x(t-h)\\
&f(t) := f(Wx(t)), \quad f_d(t) := f(Wx(t-d(t))), \quad f_h(t) := f(Wx(t-h))\\
&v_1(t) := \int_{t-d(t)}^{t} \frac{x(s)}{d(t)}ds, \quad v_2(t) := \int_{t-h}^{t-d(t)} \frac{x(s)}{h_d(t)}ds\\
&v_3(t) := \int_{t-d(t)}^{t}\int_s^t \frac{x(u)}{d(t)}duds \quad v_4(t) := \int_{t-h}^{t-d(t)}\int_s^{t-d(t)} \frac{x(u)}{h_d(t)}duds\\
&\xi_1(t) := [x^T(t)\ x_d^T(t)\ x_h^T(t)]^T \quad \xi_2(t) := [f^T(t)\ f_d^T(t)\ f_h^T(t)]^T\\
&\zeta(t) := [\xi_1^T(t)\ \ \xi_2^T(t)\ \ v_1^T(t)\ \ v_2^T(t)\ \ v_3^T(t)\ \ v_4^T(t)]^T
\end{aligned}$$

$$e_i := [0_{n\times(i-1)n}\ I_{n\times n}\ 0_{n\times(10-i)n}],\quad i=1,2,\cdots,10$$
$$e_s := [-A\ \ 0\ \ 0\ \ W_0\ \ W_1\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0]$$
$$e_g := [e_1^T\ \ e_2^T\ \ e_4^T\ \ e_5^T\ \ e_7^T\ \ e_8^T\ \ e_9^T\ \ e_{10}^T]^T,\qquad e_0 := [0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0]$$
$$\Sigma_1 := \mathrm{diag}\{\sigma_1^+,\sigma_2^+,\cdots,\sigma_n^+\},\qquad \Sigma_2 := \mathrm{diag}\{\sigma_1^-,\sigma_2^-,\cdots,\sigma_n^-\}$$

### 4.1. Case I: Stability of DNN (2) with the delay satisfying (4)

For DNN (2) with the delay satisfying (4), construct the following LKF candidate:

$$V_1(t)=\begin{bmatrix} x(t)\\ \int_{t-h}^{t}x(s)ds\end{bmatrix}^T P\begin{bmatrix} x(t)\\ \int_{t-h}^{t}x(s)ds\end{bmatrix}+\int_{t-d(t)}^{t}\varepsilon_1^T(s)Q_1\varepsilon_1(s)ds+\int_{t-h}^{t}\varepsilon_1^T(s)Q_2\varepsilon_1(s)ds+\int_{-h}^{0}\int_{t+\theta}^{t}\varepsilon_2^T(s)Z\varepsilon_2(s)dsd\theta$$
$$+\sum_{i=1}^{n}\int_0^{W_{2i}x}[\lambda_{1i}(\sigma_i^+ s-f_i(s))+\lambda_{2i}(f_i(s)-\sigma_i^- s)]ds \tag{31}$$

where

$$\varepsilon_1(t)=[x^T(t)\ f^T(t)]^T,\ \varepsilon_2(t)=[x^T(t)\ \dot{x}^T(t)]^T$$

and $P$, $Q_i, i=1,2$, and $Z$ are the symmetric positive definite matrices; and $\Lambda_i=\mathrm{diag}\{\lambda_{i1},\lambda_{i2},\cdots,\lambda_{in}\}, i=1,2$ are the symmetric diagonal matrices.

Based on LKF (31), the following stability criterion is derived by using the GFWM-based inequality (17) to derive single integral terms appearing in the derivative of the LKF.

**Theorem 1.** *For given scalars $h$ and $\mu$, DNN (2) with time delay satisfying (3) and (4) and activation function satisfying (6) is asymptotically stable, if there exist positive symmetric matrices $P,Q_1,Q_2,Z\in\mathcal{R}^{2n\times 2n}$, symmetric matrices $Z_a,Z_b\in\mathcal{R}^{n\times n}$; positive diagonal matrices $\Lambda_1,\Lambda_2,U_j,H_j\in\mathcal{R}^{n\times n}$, $j=1,2,3$; and any matrices $L_i,M_i\in\mathcal{R}^{8n\times 2n}$, $i=1,2$, such that the following LMIs hold*

$$\begin{bmatrix}\Psi_1(d(t))|_{d(t)=h}-h\Xi_5 & he_g^TL_1 & he_g^TM_1\\ * & -h\bar{Z}_a & 0\\ * & * & -3h\bar{Z}_a\end{bmatrix}\le 0 \tag{32}$$

$$\begin{bmatrix}\Psi_1(d(t))|_{d(t)=0}-h\Xi_7 & he_g^TL_2 & he_g^TM_2\\ * & -h\bar{Z}_b & 0\\ * & * & -3h\bar{Z}_b\end{bmatrix}\le 0 \tag{33}$$

*where*

$$\Psi_1(d(t))=\Upsilon_0+d(t)\Upsilon_1+h_d(t)\Upsilon_2 \tag{34}$$
$$\Upsilon_0=\Xi_0+\Xi_3$$
$$\Upsilon_1=\Xi_1+\Xi_1^T+\Xi_4+\Xi_4^T+\Xi_5$$
$$\Upsilon_2=\Xi_2+\Xi_2^T+\Xi_6+\Xi_6^T+\Xi_7$$
$$\Xi_0=\mathrm{Sym}\left\{\begin{bmatrix}e_1\\ e_0\end{bmatrix}^T P\begin{bmatrix}e_s\\ e_1-e_3\end{bmatrix}\right\}+\begin{bmatrix}e_1\\ e_4\end{bmatrix}^T(Q_1+Q_2)\begin{bmatrix}e_1\\ e_4\end{bmatrix}-\begin{bmatrix}e_3\\ e_6\end{bmatrix}^T Q_2\begin{bmatrix}e_3\\ e_6\end{bmatrix}-(1-\mu)\begin{bmatrix}e_2\\ e_5\end{bmatrix}^T Q_1\begin{bmatrix}e_2\\ e_5\end{bmatrix}+h\begin{bmatrix}e_1\\ e_s\end{bmatrix}^T Z\begin{bmatrix}e_1\\ e_s\end{bmatrix}$$

$$
\begin{aligned}
&+e_1^T Z_a e_1 + e_2^T (Z_b - Z_a) e_2 - e_3^T Z_b e_3 + \mathrm{Sym}\left\{\left[(\Sigma_1 W e_1 - e_4)^T \Lambda_1 + (e_4 - \Sigma_2 W e_1)^T \Lambda_2\right] W e_s\right\} \\
&+ \sum_{i=1}^{3} \mathrm{Sym}\left\{(\Sigma_1 W e_i - e_{i+3})^T H_i (e_{i+3} - \Sigma_2 W e_i)\right\} \\
&+ \sum_{i=1}^{2} \mathrm{Sym}\left\{[\Sigma_1 W (e_i - e_{i+1}) - (e_{i+3} - e_{i+4})]^T U_i [(e_{i+3} - e_{i+4}) - \Sigma_2 W (e_i - e_{i+1})]\right\} \\
&+\mathrm{Sym}\left\{[\Sigma_1 W (e_1 - e_3) - (e_4 - e_6)]^T U_3 \ [(e_4 - e_6) - \Sigma_2 W (e_1 - e_3)]\right\}
\end{aligned}
\tag{35}
$$

$$
\Xi_1 = \begin{bmatrix} e_0 \\ e_7 \end{bmatrix}^T P \begin{bmatrix} e_s \\ e_1 - e_3 \end{bmatrix} \tag{36}
$$

$$
\Xi_2 = \begin{bmatrix} e_0 \\ e_8 \end{bmatrix}^T P \begin{bmatrix} e_s \\ e_1 - e_3 \end{bmatrix} \tag{37}
$$

$$
\Xi_3 = \mathrm{Sym}\left\{ e_g^T L_1 \begin{bmatrix} e_0 \\ e_1 - e_2 \end{bmatrix} + e_g^T M_1 \begin{bmatrix} 2e_9 \\ e_1 + e_2 - 2e_7 \end{bmatrix} \right\} + \mathrm{Sym}\left\{ e_g^T L_2 \begin{bmatrix} e_0 \\ e_2 - e_3 \end{bmatrix} + e_g^T M_2 \begin{bmatrix} 2e_{10} \\ e_2 + e_3 - 2e_8 \end{bmatrix} \right\} \tag{38}
$$

$$
\Xi_4 = e_g^T L_1 \begin{bmatrix} e_7 \\ e_0 \end{bmatrix} + e_g^T M_1 \begin{bmatrix} -e_7 \\ e_0 \end{bmatrix} \tag{39}
$$

$$
\Xi_5 = e_g^T \left( L_1 \bar{Z}_a^{-1} L_1^T + \frac{1}{3} M_1 \bar{Z}_a^{-1} M_1^T \right) e_g \tag{40}
$$

$$
\Xi_6 = e_g^T L_2 \begin{bmatrix} e_8 \\ e_0 \end{bmatrix} + e_g^T M_2 \begin{bmatrix} -e_8 \\ e_0 \end{bmatrix} \tag{41}
$$

$$
\Xi_7 = e_g^T \left( L_2 \bar{Z}_b^{-1} L_2^T + \frac{1}{3} M_2 \bar{Z}_b^{-1} M_2^T \right) e_g \tag{42}
$$

$$
\bar{Z}_a = Z + \begin{bmatrix} 0 & Z_a \\ Z_a & 0 \end{bmatrix}, \quad \bar{Z}_b = Z + \begin{bmatrix} 0 & Z_b \\ Z_b & 0 \end{bmatrix} \tag{43}
$$

*Proof.* Firstly, differentiating $V_1(t)$ along the solutions of (7) yields

$$
\begin{aligned}
\dot{V}_1(t) \le\ & 2 \begin{bmatrix} x(t) \\ d(t) v_1(t) + h_d(t) v_2(t) \end{bmatrix}^T P \begin{bmatrix} \dot{x}(t) \\ x(t) - x_h(t) \end{bmatrix} \\
&+\varepsilon_1^T(t)(Q_1 + Q_2)\varepsilon_1(t) - \varepsilon_1^T(t-h) Q_2 \varepsilon_1(t-h) - (1-\mu)\varepsilon_1^T(t-d(t)) Q_1 \varepsilon_1(t-d(t)) \\
&+2\left\{[\Sigma_1 W x(t) - f(t)]^T \Lambda_1 + [f(t) - \Sigma_2 W x(t)]^T \Lambda_2\right\} W \dot{x}(t) + h \varepsilon_2^T(t) Z \varepsilon_2(t) - \int_{t-h}^{t} \varepsilon_2^T(s) Z \varepsilon_2(s) ds
\end{aligned}
\tag{44}
$$

Secondly, by taking into account the assumption of the activation function, (8) and (9), the following inequalities hold:

$$
\begin{aligned}
h_i(s) &:= 2\,[\Sigma_1 W x(s) - f(s)]^T H_i\, [f(s) - \Sigma_2 W x(s)] \ge 0 \\
u_i(s_1, s_2) &:= 2\,[\Sigma_1 W (x(s_1) - x(s_2)) - (f(s_1) - f(s_2))]^T U_i\, [(f(s_1) - f(s_2)) - \Sigma_2 W (x(s_1) - x(s_2))] \ge 0
\end{aligned}
$$

where

$$
H_i = \mathrm{diag}\{h_{1i}, h_{2i}, \cdots, h_{ni}\} \ge 0, i = 1, 2, 3
$$

$$U_j = \mathrm{diag}\{u_{1j}, u_{2j}, \cdots, u_{nj}\} \geq 0, j = 1, 2, 3$$

Thus, the following inequalities hold:

$$h_1(t)+h_2(t-d(t))+h_3(t-h) \geq 0 \tag{45}$$

$$u_1(t, t-d(t))+u_2(t-d(t), t-h)+u_3(t, t-h) \geq 0 \tag{46}$$

Thirdly, for symmetric matrices $Z_a, Z_b$, the following zero-value term is obtained [51]:

$$0 = x^T(t)Z_a x(t) - x_d^T(t)Z_a x_d(t) - 2\int_{t-d(t)}^{t} x^T(s)Z_a\dot{x}(s)ds + x_d^T(t)Z_b x_d(t) - x_h^T(t)Z_b x_h(t) - 2\int_{t-h}^{t-d(t)} x^T(s)Z_b\dot{x}(s)ds \tag{47}$$

Fourthly, introducing (45)-(47) into (44) and combining the $d(t)$- and $h_d(t)$-dependent terms yield

$$\dot{V}_1(t) \leq \zeta^T(t)\Gamma_1(d(t))\zeta(t) - \dot{V}_s(t) \tag{48}$$

where $\zeta(t)$ is defined in (31), and

$$\Gamma_1(d(t)) = \Xi_0 + d(t)(\Xi_1 + \Xi_1^T) + h_d(t)(\Xi_2 + \Xi_2^T)$$
$$\dot{V}_s(t) = \int_{t-d(t)}^{t} \varepsilon_2^T(s)\bar{Z}_a\varepsilon_2(s)ds + \int_{t-h}^{t-d(t)} \varepsilon_2^T(s)\bar{Z}_b\varepsilon_2(s)ds$$

and $\Xi_i, i = 0, 1, 2$ are defined in (35)-(37), $\bar{Z}_a, \bar{Z}_b$ are defined in (43).

Fifthly, for any matrices $L_i, M_i \in \mathcal{R}^{8n\times 2n}, i = 1, 2$, letting $\chi_0$ in (17) be

$$\chi_0 = \eta_0 = [x^T(t), x_d^T(t), f^T(t), f_d^T(t), v_1^T(t), v_2^T(t), v_3^T(t), v_4^T(t)]^T = e_g\zeta(t) \tag{49}$$

and using the GFWM-based inequality (17) to estimate the single integral term $\dot{V}_s(t)$ yield

$$\begin{aligned}\dot{V}_s(t) \geq & -\mathrm{Sym}\left\{\eta_0^T L_1\eta_1 + \eta_0^T M_1\eta_2\right\} - d(t)\eta_0^T\left(\frac{3L_1\bar{Z}_a^{-1}L_1^T + M_1\bar{Z}_a^{-1}M_1^T}{3}\right)\eta_0 \\ & -\mathrm{Sym}\left\{\eta_0^T L_2\eta_3 + \eta_0^T M_2\eta_4\right\} - (h - d(t))\eta_0^T\left(\frac{3L_2\bar{Z}_b^{-1}L_2^T + M_2\bar{Z}_b^{-1}M_2^T}{3}\right)\eta_0\end{aligned} \tag{50}$$

where

$$\eta_1 = \begin{bmatrix} d(t)v_1(t) \\ x(t)-x_d(t) \end{bmatrix}, \quad \eta_2 = \begin{bmatrix} -d(t)v_1(t) + 2v_3(t) \\ x(t)+x_d(t) - 2v_1(t) \end{bmatrix}, \quad \eta_3 = \begin{bmatrix} h_d(t)v_2(t) \\ x_d(t)-x_h(t) \end{bmatrix}, \quad \eta_4 = \begin{bmatrix} -h_d(t)v_2(t) + 2v_4(t) \\ x_d(t)+x_h(t) - 2v_2(t) \end{bmatrix}$$

which implies

$$-\dot{V}_s(t) \leq \zeta^T(t)\Gamma_2(d(t))\zeta(t) \tag{51}$$

where

$$\Gamma_2(d(t)) = \Xi_3 + d(t)(\Xi_4+\Xi_4^T+\Xi_5) + h_d(t)(\Xi_6+\Xi_6^T+\Xi_7)$$

and $\Xi_i, i = 3, 4, \ldots, 7$ are defined in (38)-(42).

Finally, combining (48) and (51) yields

$$\dot{V}_1(t) \leq \zeta^T(t)\Psi_1(d(t))\zeta(t) \tag{52}$$

where $\Psi_1(d(t))$ is defined in (34).

On the other hand, based on Schur complement, LMIs (32) and (33) leads to

$$\Psi_1(d(t))|_{d(t)=h} \leq 0, \ \ \Psi_1(d(t))|_{d(t)=0} \leq 0 \tag{53}$$

which is equivalent to $\Psi_1(d(t)) \leq 0$ based on the convex combination method [60].

Therefore, if LMIs (32) and (33) hold, then the following holds for a sufficiently small scalar $\epsilon > 0$:

$$\dot{V}_1(t) \leq -\epsilon \|x(t)\|^2 \tag{54}$$

which shows the asymptotical stability of DNN (2) with time delay satisfying (3) and (4) . This completes the proof. □

In order to clearly show the GFWM approach is more effective than the Wirtinger-based inequality, the following stability criterion is also obtained by using the LKF (31) and applying the Wirtinger-based inequality (10) to estimate the single integral term arises in the derivative of the LKF.

**Theorem 2.** *For given scalars $h$ and $\mu$, DNN (2) with time delay satisfying (3) and (4) and activation function satisfying (6) is asymptotically stable, if there exist positive symmetric matrices $P, Q_1, Q_2, Z \in \mathcal{R}^{2n\times 2n}$, symmetric matrices $Z_a, Z_b \in \mathcal{R}^{n\times n}$; positive diagonal matrices $\Lambda_1, \Lambda_2, U_j, H_j \in \mathcal{R}^{n\times n}$, $j = 1, 2, 3$; and any matrices $S_1, S_2 \in \mathcal{R}^{2n\times 2n}$, such that the following LMIs hold*

$$\Psi_3(d(t))|_{d(t)=0} \leq 0 \tag{55}$$

$$\Psi_3(d(t))|_{d(t)=h} \leq 0 \tag{56}$$

$$\Phi_1 = \begin{bmatrix} \bar{Z}_a & S_1 \\ * & \bar{Z}_b \end{bmatrix} > 0 \tag{57}$$

$$\Phi_2 = \begin{bmatrix} \bar{Z}_a & S_2 \\ * & \bar{Z}_b \end{bmatrix} > 0 \tag{58}$$

*where*

$$\Psi_3(d(t)) = \Upsilon_3 + d(t)\Upsilon_4 + h_d(t)\Upsilon_5 \tag{59}$$

$$\Upsilon_3 = \Xi_0 + \Xi_8$$

$$\Upsilon_4 = \Xi_1 + \Xi_1^T + \Xi_9$$

$$\Upsilon_5 = \Xi_2 + \Xi_2^T$$

$$\Xi_8 = -\frac{E_1^T \Phi_1 E_1 + 3E_3^T \Phi_2 E_3}{h} \tag{60}$$

$$\Xi_9 = -\frac{\mathrm{Sym}\{E_1^T \Phi_1 E_2 + 3E_3^T \Phi_2 E_4\}}{h} \tag{61}$$

$$E_1 = \begin{bmatrix} e_0 \\ e_1 - e_2 \\ he_8 \\ e_2 - e_3 \end{bmatrix}, E_2 = -E_4 = \begin{bmatrix} e_7 \\ e_0 \\ -e_8 \\ e_0 \end{bmatrix}, E_3 = \begin{bmatrix} 2e_9 \\ e_1 + e_2 - 2e_7 \\ 2e_{10} - he_8 \\ e_2 + e_3 - 2e_8 \end{bmatrix}$$

*Proof.* By constructing the LKF same to (31), its derivative is given as (48). Then by using the Wirtinger-based inequality (10) and Lemma 3 to estimate the single integral term $\dot{V}_s(t)$ yields

$$
\begin{aligned}
&-\int_{t-d(t)}^{t} \varepsilon_2^T(s)\bar{Z}_a\varepsilon_2(s)ds - \int_{t-h}^{t-d(t)} \varepsilon_2^T(s)\bar{Z}_b\varepsilon_2(s)ds \\
&\leq -\frac{1}{d(t)}\zeta^T(t)\left\{\begin{bmatrix} d(t)e_7 \\ e_1-e_2 \end{bmatrix}^T \bar{Z}_a \begin{bmatrix} d(t)e_7 \\ e_1-e_2 \end{bmatrix}\right\}\zeta(t) - \frac{1}{h_d(t)}\zeta^T(t)\left\{\begin{bmatrix} h_d(t)e_8 \\ e_2-e_3 \end{bmatrix}^T \bar{Z}_b \begin{bmatrix} h_d(t)e_8 \\ e_2-e_3 \end{bmatrix}\right\}\zeta(t) \\
&\quad -\frac{3}{d(t)}\zeta^T(t)\left\{\begin{bmatrix} 2e_9-d(t)e_7 \\ e_1+e_2-2e_7 \end{bmatrix}^T \bar{Z}_a \begin{bmatrix} 2e_9-d(t)e_7 \\ e_1+e_2-2e_7 \end{bmatrix}\right\}\zeta(t) - \frac{3}{h_d(t)}\zeta^T(t)\left\{\begin{bmatrix} 2e_{10}-h_d(t)e_8 \\ e_2+e_3-2e_8 \end{bmatrix}^T \bar{Z}_b \begin{bmatrix} 2e_{10}-h_d(t)e_8 \\ e_2+e_3-2e_8 \end{bmatrix}\right\}\zeta(t) \\
&\leq -\frac{1}{h}\zeta^T(t)\left\{\begin{bmatrix} d(t)e_7 \\ e_1-e_2 \\ h_d(t)e_8 \\ e_2-e_3 \end{bmatrix}^T \Phi_1 \begin{bmatrix} d(t)e_7 \\ e_1-e_2 \\ h_d(t)e_8 \\ e_2-e_3 \end{bmatrix}\right\}\zeta(t) - \frac{3}{h}\zeta^T(t)\left\{\begin{bmatrix} 2e_9-d(t)e_7 \\ e_1+e_2-2e_7 \\ 2e_{10}-h_d(t)e_8 \\ e_2+e_3-2e_8 \end{bmatrix}^T \Phi_2 \begin{bmatrix} 2e_9-d(t)e_7 \\ e_1+e_2-2e_7 \\ 2e_{10}-h_d(t)e_8 \\ e_2+e_3-2e_8 \end{bmatrix}\right\}\zeta(t) \\
&= -\frac{1}{h}\zeta^T(t)[E_1 + d(t)E_2]^T\Phi_1[E_1 + d(t)E_2]\zeta(t) - \frac{3}{h}\zeta^T(t)[E_3 + d(t)E_4]^T\Phi_2[E_3 + d(t)E_4]\zeta(t) \\
&= \zeta^T(t)(\Xi_8 + d(t)\Xi_9 + d^2(t)\Xi_{10})\zeta(t)
\end{aligned}
\tag{62}
$$

where $\Phi_1$ and $\Phi_2$ are defined in (57) and (58), respectively, $\Xi_8$ and $\Xi_9$ are defined in (60) and (61), respectively, and

$$\Xi_{10} = -\frac{E_2^T\Phi_1E_2 + 3E_4^T\Phi_2E_4}{h} \tag{63}$$

Then, combining (48) and (62) yields

$$\dot{V}_1(t) \leq \zeta^T(t)\Psi_2(d(t))\zeta(t) \tag{64}$$

where

$$
\begin{aligned}
\Psi_2(d(t)) &= \Psi_3(d(t)) + d^2(t)\Xi_{10} \\
&= \Upsilon_3 + h\Upsilon_5 + d(t)(\Upsilon_4 - \Upsilon_5) + d^2(t)\Xi_{10}
\end{aligned}
\tag{65}
$$

and $\Psi_3(d(t))$ is defined in (59). Obviously, $\Xi_{10} \leq 0$, thus, based on 3) of Lemma 4, $\Psi_2(d(t)) \leq 0$ requires the following holds

$$\Upsilon_3 + h\Upsilon_5 \leq 0, \quad \Upsilon_3 + h\Upsilon_5 + h(\Upsilon_4 - \Upsilon_5) \leq 0 \tag{66}$$

That is,

$$\Psi_3(d(t))|_{d(t)=0} \leq 0, \quad \Psi_3(d(t))|_{d(t)=h} \leq 0 \tag{67}$$

Therefore, if LMIs (55)-(58) hold, then the following holds for a sufficiently small scalar $\epsilon > 0$:

$$\dot{V}_1(t) \leq -\epsilon||x(t)||^2 \tag{68}$$

which shows the asymptotical stability of DNN (2) with time delay satisfying (3) and (4) . This completes the proof. □

**Remark 4.** *In [42], the FMBI was used to analyze the stability of DNN (2). Compared with the criterion derived in this paper, there are some drawbacks in [42]. Firstly, although the single integral term in* $\dot{V}_3(x_t)$, $-\int_{t-\tau}^{t}\dot{x}^T(s)Z_1\dot{x}(s)ds$, *was estimated by using FMBI, however, the single integral terms in* $\dot{V}_5(x_t)$, $-\int_{t-d(t)}^{t}\begin{bmatrix}x(s)\\ \dot{x}(s)\end{bmatrix}^T\begin{bmatrix}Z_2 & U_1\\ * & Z_3\end{bmatrix}\begin{bmatrix}x(s)\\ \dot{x}(s)\end{bmatrix}ds$ *and*
$-\int_{t-\tau}^{t-d(t)}\begin{bmatrix}x(s)\\ \dot{x}(s)\end{bmatrix}^T\begin{bmatrix}Z_2 & U_2\\ * & Z_3\end{bmatrix}\begin{bmatrix}x(s)\\ \dot{x}(s)\end{bmatrix}ds$,
*were still estimated via the Jensen inequality, which is more conservative than Wirtinger-based inequality and the GFWM-based inequality. Secondly, there exists an error during the discussion of* $d(t)$ *(i.e., Remark 3 therein). Specifically, in LMI (11) of Theorem 1,* $\Xi$ *can be rewritten as*

$$\Xi(d(t)) = \Omega_0 + d(t)\Omega_1 + d^2(t)\Omega_2 \tag{69}$$

*where* $\Omega_i, i = 0,1,2$ *are delay-independent matrices, and* $\Omega_2 = -\begin{bmatrix}e_7\\ e_0\\ -e_8\\ e_0\end{bmatrix}^T(\Phi_3 + \frac{\Phi_4}{\tau})\begin{bmatrix}e_7\\ e_0\\ -e_8\\ e_0\end{bmatrix}$ *which is obtained from* $\Pi_6^T\Phi_3\Pi_6$ *and* $\Pi_{21}^T\Phi_4\Pi_{21}$. *In Remark 3 therein the authors claimed that* $\Xi(d(t)) \leq 0$ *for all* $d(t) \in [0,\tau]$ *if* $\Xi(0) \leq 0$ *and* $\Xi(\tau) \leq 0$. *However, due to* $\Phi_3 + \frac{\Phi_4}{\tau} \geq 0$, *it is not correct based on 3) of Lemma 4 in this paper. Therefore, the results in [42] are incorrect and will not be listed in the following numerical studies. Note that the notations, Remark, and Theorem mentioned in this remark are all defined in [42] if not explicity stated.*

### 4.2. Case II: Stability of DNN (2) with the delay satisfying (5)

For DNN (2) with the delay satisfying (5), the following LKF candidate with delay-product-type terms is constructed:

$$V_2(t) = V_1(t)+V_3(t)+V_4(t)+V_5(t)-\begin{bmatrix}x(t)\\ \int_{t-h}^{t}x(s)ds\end{bmatrix}^T P\begin{bmatrix}x(t)\\ \int_{t-h}^{t}x(s)ds\end{bmatrix} \tag{70}$$

where

$$\begin{aligned}
V_3(t) &= \begin{bmatrix}x(t)\\ \int_{t-d(t)}^{t}x(s)ds\\ \int_{t-h}^{t-d(t)}x(s)ds\end{bmatrix}^T P_0\begin{bmatrix}x(t)\\ \int_{t-d(t)}^{t}x(s)ds\\ \int_{t-h}^{t-d(t)}x(s)ds\end{bmatrix}\\
V_4(t) &= d(t)\begin{bmatrix}x(t)\\ \int_{t-d(t)}^{t}\frac{x(s)}{d(t)}ds\end{bmatrix}^T P_1\begin{bmatrix}x(t)\\ \int_{t-d(t)}^{t}\frac{x(s)}{d(t)}ds\end{bmatrix}\\
V_5(t) &= (h-d(t))\begin{bmatrix}x(t)\\ \int_{t-h}^{t-d(t)}\frac{x(s)}{h_d(t)}ds\end{bmatrix}^T P_2\begin{bmatrix}x(t)\\ \int_{t-h}^{t-d(t)}\frac{x(s)}{h_d(t)}ds\end{bmatrix}
\end{aligned}$$

and $V_1(t)$ is defined in (31), and $P_i, i = 0,1,2$ are the symmetric positive definite matrices.

Based on LKF (70) and the GFWM-based inequality (17), the following stability criterion is derived.

**Theorem 3.** *For given scalars* $h$ *and* $\mu$, *DNN (2) with time delay satisfying (3) and (5) and activation function satisfying (6) is asymptotically stable, if there exist positive symmetric matrices* $P_0 \in \mathcal{R}^{3n\times 3n}$, $P_1, P_2, Q_1, Q_2, Z \in$

$\mathcal{R}^{2n\times 2n}$, *symmetric matrices* $Z_a, Z_b \in \mathcal{R}^{n\times n}$; *positive diagonal matrices* $\Lambda_1, \Lambda_2, U_j, H_j \in \mathcal{R}^{n\times n}$, $j = 1,2,3$; *and any matrices* $L_i, M_i \in \mathcal{R}^{8n\times 2n}$, $i = 1,2$, *such that the following LMIs hold*

$$\begin{bmatrix} \Psi_a - h\Xi_5 & he_g^T L_1 & he_g^T M_1 \\ * & -h\bar{Z}_a & 0 \\ * & * & -3h\bar{Z}_a \end{bmatrix} \leq 0 \tag{71}$$

$$\begin{bmatrix} \Psi_b - h\Xi_7 & he_g^T L_2 & he_g^T M_2 \\ * & -h\bar{Z}_b & 0 \\ * & * & -3h\bar{Z}_b \end{bmatrix} \leq 0 \tag{72}$$

$$\begin{bmatrix} \Psi_c - h\Xi_5 & he_g^T L_1 & he_g^T M_1 \\ * & -h\bar{Z}_a & 0 \\ * & * & -3h\bar{Z}_a \end{bmatrix} \leq 0 \tag{73}$$

$$\begin{bmatrix} \Psi_d - h\Xi_7 & he_g^T L_2 & he_g^T M_2 \\ * & -h\bar{Z}_b & 0 \\ * & * & -3h\bar{Z}_b \end{bmatrix} \leq 0 \tag{74}$$

*where*

$$\Psi_a = \Psi_4(d(t), \dot{d}(t))|_{d(t)=h, \dot{d}(t)=-\mu}$$

$$\Psi_b = \Psi_4(d(t), \dot{d}(t))|_{d(t)=0, \dot{d}(t)=-\mu}$$

$$\Psi_c = \Psi_4(d(t), \dot{d}(t))|_{d(t)=h, \dot{d}(t)=\mu}$$

$$\Psi_d = \Psi_4(d(t), \dot{d}(t))|_{d(t)=0, \dot{d}(t)=\mu}$$

$$\Psi_4(d(t), \dot{d}(t)) = \bar{\Upsilon}_0 + d(t)\bar{\Upsilon}_1 + h_d(t)\bar{\Upsilon}_2 + \Xi_{11}(d(t), \dot{d}(t)) + \Xi_{12}(d(t), \dot{d}(t)) + \Xi_{13}(d(t), \dot{d}(t)) \tag{75}$$

$$\bar{\Upsilon}_0 = \Xi_0 + \Xi_3 - \mathrm{Sym}\left\{ \begin{bmatrix} e_1 \\ e_0 \end{bmatrix}^T P \begin{bmatrix} e_s \\ e_1 - e_3 \end{bmatrix} \right\}$$

$$\bar{\Upsilon}_1 = \Xi_4 + \Xi_4^T + \Xi_5$$

$$\bar{\Upsilon}_2 = \Xi_6 + \Xi_6^T + \Xi_7$$

$$\Xi_{11}(d(t), \dot{d}(t)) = \mathrm{Sym}\left\{ \begin{bmatrix} e_1 \\ d(t)e_7 \\ h_d(t)e_8 \end{bmatrix}^T P_0 \begin{bmatrix} e_s \\ e_1 - (1-\dot{d}(t))e_2 \\ (1-\dot{d}(t))e_2 - e_3 \end{bmatrix} \right\} \tag{76}$$

$$\Xi_{12}(d(t), \dot{d}(t)) = \dot{d}(t) \begin{bmatrix} e_1 \\ e_7 \end{bmatrix}^T P_1 \begin{bmatrix} e_1 \\ e_7 \end{bmatrix} + \mathrm{Sym}\left\{ \begin{bmatrix} e_1 \\ e_7 \end{bmatrix}^T d(t)P_1 \begin{bmatrix} e_s \\ e_0 \end{bmatrix} \right\} + \mathrm{Sym}\left\{ \begin{bmatrix} e_1 \\ e_7 \end{bmatrix}^T P_1 \begin{bmatrix} e_0 \\ e_1 - (1-\dot{d}(t))e_2 - \dot{d}(t)e_7 \end{bmatrix} \right\} \tag{77}$$

$$\Xi_{13}(d(t), \dot{d}(t)) = -\dot{d}(t) \begin{bmatrix} e_1 \\ e_8 \end{bmatrix}^T P_2 \begin{bmatrix} e_1 \\ e_8 \end{bmatrix} + \mathrm{Sym}\left\{ \begin{bmatrix} e_1 \\ e_8 \end{bmatrix}^T h_d(t)P_2 \begin{bmatrix} e_s \\ e_0 \end{bmatrix} \right\} + \mathrm{Sym}\left\{ \begin{bmatrix} e_1 \\ e_8 \end{bmatrix}^T P_2 \begin{bmatrix} e_0 \\ (1-\dot{d}(t))e_2 - e_3 + \dot{d}(t)e_8 \end{bmatrix} \right\} \tag{78}$$

*and* $\Xi_i, i = 0, 3, 4, \ldots, 7$, *are defined in (35), (38)-(42).*

*Proof.* The derivatives of the $V_i(t), i = 3, 4, 5$ can be obtained as

$$\begin{aligned}
\dot{V}_3(t) &= 2\begin{bmatrix} x(t) \\ \int_{t-d(t)}^{t} x(s)ds \\ \int_{t-h}^{t-d(t)} x(s)ds \end{bmatrix}^T P_0 \begin{bmatrix} \dot{x}(t) \\ x(t) - (1-\dot{d}(t))x_d(t) \\ (1-\dot{d}(t))x_d(t) - x_h(t) \end{bmatrix} \\
&= \zeta^T(t)\left(\Xi_{11}(d(t), \dot{d}(t))\right)\zeta(t) \qquad (79) \\
\dot{V}_4(t) &= \dot{d}(t)\begin{bmatrix} x(t) \\ \int_{t-d(t)}^{t} \frac{x(s)}{d(t)}ds \end{bmatrix}^T P_1 \begin{bmatrix} x(t) \\ \int_{t-d(t)}^{t} \frac{x(s)}{d(t)}ds \end{bmatrix} + 2\begin{bmatrix} x(t) \\ \int_{t-d(t)}^{t} \frac{x(s)}{d(t)}ds \end{bmatrix}^T d(t)P_1 \begin{bmatrix} \dot{x}(t) \\ \frac{x(t)-(1-\dot{d}(t))x_d(t)-\dot{d}(t)\int_{t-d(t)}^{t}\frac{x(s)}{d(t)}ds}{d(t)} \end{bmatrix} \\
&= \zeta^T(t)\left(\Xi_{12}(d(t), \dot{d}(t))\right)\zeta(t) \qquad (80) \\
\dot{V}_5(t) &= -\dot{d}(t)\begin{bmatrix} x(t) \\ \int_{t-h}^{t-d(t)} \frac{x(s)}{h_d(t)}ds \end{bmatrix}^T P_2 \begin{bmatrix} x(t) \\ \int_{t-h}^{t-d(t)} \frac{x(s)}{h_d(t)}ds \end{bmatrix} + 2\begin{bmatrix} x(t) \\ \int_{t-h}^{t-d(t)} \frac{x(s)}{h_d(t)}ds \end{bmatrix}^T h_d(t)P_2 \begin{bmatrix} \dot{x}(t) \\ \frac{(1-\dot{d}(t))x_d(t)-x_h(t)+\dot{d}(t)\int_{t-h}^{t-d(t)}\frac{x(s)}{h_d(t)}ds}{h_d(t)} \end{bmatrix} \\
&= \zeta^T(t)\left(\Xi_{13}(d(t), \dot{d}(t))\right)\zeta(t) \qquad (81)
\end{aligned}$$

where $\Xi_i, i = 11, 12, 13$, are defined in (76)-(78).

Combining the proof of Theorem 1, the derivative of the $V_2(t)$ can be estimated as

$$\dot{V}_2(t) \leq \zeta^T(t)\Psi_4(d(t), \dot{d}(t))\zeta(t) \qquad (82)$$

where $\Psi_4(d(t), \dot{d}(t))$ is defined in (75).

It can be found that the $\Psi_4(d(t), \dot{d}(t))$ can be presented as

$$\Psi_4(d(t), \dot{d}(t)) = \Omega_3 + \dot{d}(t)\Omega_4 + d(t)[\Omega_5 + \dot{d}(t)\Omega_6] \qquad (83)$$

where $\Omega_i, i = 3, 4, 5, 6$ are delay-independent matrices. Thus, $\Psi_4(d(t), \dot{d}(t)) \leq 0$ requires the following holds

$$\Psi_4(d(t), \dot{d}(t))|_{d(t)\in\{0,h\},\dot{d}(t)\in\{-\mu,\mu\}} \leq 0 \qquad (84)$$

which can be guaranteed by LMIs (71)-(74) based on Schur complement.

Therefore, if LMIs (71)-(74) hold, then the following holds for a sufficiently small scalar $\epsilon > 0$:

$$\dot{V}_2(t) \leq -\epsilon\|x(t)\|^2 \qquad (85)$$

which shows the asymptotical stability of DNN (2) with time delay satisfying (3) and (5) . This completes the proof. □

To verify the contribution of delay-product-type terms $V_4(t)$ and $V_5(t)$, the stability criterion obtained from Theorem 3 by setting $P_1 = 0$ and $P_2 = 0$ (i.e., $V_4(t) = 0$ and $V_5(t) = 0$) is given as follows.

**Corollary 1.** *For given scalars $h$ and $\mu$, DNN (2) with time delay satisfying (3) and (5) and activation function satisfying (6) is asymptotically stable, if there exist positive symmetric matrices $P_0 \in \mathcal{R}^{3n\times 3n}$, $Q_1, Q_2, Z \in \mathcal{R}^{2n\times 2n}$, symmetric matrices $Z_a, Z_b \in \mathcal{R}^{n\times n}$; positive diagonal matrices $\Lambda_1, \Lambda_2, U_j, H_j \in \mathcal{R}^{n\times n}$, $j = 1, 2, 3$; and any matrices $L_i, M_i \in \mathcal{R}^{8n\times 2n}$, $i = 1, 2$, such that LMIs (71)-(74) with $P_i = 0, i = 1, 2$ hold.*

**Remark 5.** *In Section 4.1, based on a simple augmented LKF, the comparison of Theorem 1 and Theorem 2 is given to show the advantage of the GFWM approach. In Section 4.2, the GFWM approach is extended to a new LKF with some delay-product-type terms, whose contribution to conservatism reduction can be verified through the comparison of Theorem 3 and Corollary 1. In fact, as mentioned in Section 1, the estimation of the single term is unavoidable for different LKFs. Therefore, it can be expected that better stability criteria can be obtained by extending the GFWM approach to other LKFs with more general forms. Moreover, the GFWM approach can be applied to investigate other problems, such as the analysis of dissipative [61, 62] and/or exponential stability [63, 64], and the DNNs with infinite delay [65] and/or leakage delays [66, 67].*

**Remark 6.** *It is worthy pointing out that the GFWM-based criteria developed in this paper achieve the conservatism reduction at the cost of many slack matrices, which may increase the calculation complexity. Although such problem may be solved with the development of high performance computer, it is also important to develop some new criteria that can reduce the conservatism without introducing additional decision variables. For example, the novel integral inequalities developed in [46] can achieve this objective and will be extended into the study of the DNNs in our future work.*

## 5. Numerical examples

In this section, three numerical examples are given to show the advantages of the obtained criteria. As mentioned in Section 1, the important aim of the stability analysis of DNNs is to determine the AMDBs. And the stability criterion that provides larger AMDBs is less conservative than the one that gives smaller ones. Therefore, the advantages of the proposed criteria are demonstrated via the comparison of the AMDBs calculated by various criteria.

### *5.1. Results comparison*

**Example 1.** *Consider DNN (2) with the following parameters:*

$$A = \mathrm{diag}\{1.5, 0.7\}, \quad W = \mathrm{diag}\{1, 1\}, \quad W_0 = \begin{bmatrix} 0.0503 & 0.0454 \\ 0.0987 & 0.2075 \end{bmatrix}, \; W_1 = \begin{bmatrix} 0.2381 & 0.9320 \\ 0.0388 & 0.5062 \end{bmatrix}$$

$$\Sigma_1 = \begin{bmatrix} 0.3 & 0 \\ 0 & 0.8 \end{bmatrix}, \; \Sigma_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \; J = \begin{bmatrix} 0.4 \\ 0.2 \end{bmatrix}$$

In order to verify the advantages of the proposed method, the AMDBs of the delay satisfying $\dot{d}(t) \le \mu$ with respect to various $\mu$ obtained by the proposed criteria are given in Table 1, where the AMDBs calculated by stability criteria in the literature are also listed for comparison. The following observations are summarized from the results.

- It can be easily found that the proposed stability criteria can produce the larger AMDBs for all cases than those given in the existing literature. It shows that the proposed criteria are indeed less conservative than the ones in the literature.
- Theorem 1 and Theorem 2 are respectively derived via the proposed GFWM approach and the Wirtinger inequality based on the same LKF, and the AMDBs of Theorem 1 are larger than the ones of Theorem 2. That is, the GFWM approach is more effective than the widely used Wirtinger inequality approach, which matches the theoretical analysis in Section 3.

- The comparison between the AMDBs calculated by of Theorem 3 (with delay-product-type terms) and the ones of Corollary 1 (without delay-product-type terms) clearly verifies the delay-product-type terms, $V_4(t)$ and $V_5(t)$ in (70), are effective in the reduction of conservatism.

- The comparison between the results of Theorem 1 (for the case of $\dot{d}(t) \leq \mu$) and the ones of Theorem 3 (for the case of $|\dot{d}(t)| \leq \mu$) shows that the additional information of delay changing rate, i.e., the lower bound of the delay changing rate, is helpful to further reduce the conservatism of the results.

- The comparison of NDVs included by different stability criteria listed in Table 1 shows that the GFWM-based inequality achieves the reduction of conservatism at the cost of the increase of decision variables.

**Example 2.** *Consider DNN (2) with the following parameters:*

$$A = \mathrm{diag}\{7.3458, 6.9987, 5.5949\}, \quad W = \begin{bmatrix} 13.6014 & -2.9616 & -0.6936 \\ 7.4736 & 21.6810 & 3.2100 \\ 0.7290 & -2.6334 & -20.1300 \end{bmatrix}, \quad W_0 = \mathrm{diag}\{0, 0, 0\}, \quad W_1 = \mathrm{diag}\{1, 1, 1\}$$

$$\Sigma_1 = \mathrm{diag}\{0.3680, 0.1795, 0.2876\}, \quad \Sigma_2 = \mathrm{diag}\{0, 0, 0\}, \quad J = \begin{bmatrix} 0.4 & 0.2 & 0.3 \end{bmatrix}^T$$

The DNN described above is a static neural network, and its stability has been widely studied in [14], [25], [32], [37], [38], [41], [49], [53], [55]. For different $\mu$, the AMDBs calculated via the criteria presented in this paper and the ones in the literature are summarized in Table 2, where '-' represents that the allowable upper bounds for the corresponding cases are not provided in those literatures. The results given reveal that the GFWM-based Theorem 1 is more effective than both the stability criteria in existing literature and the one developed via the Wirtinger inequality.

**Example 3.** *Consider DNN (2) with the following parameters:*

$$A = \mathrm{diag}\{1.2769, 0.6231, 0.9230, 0.4480\}, \quad W = \mathrm{diag}\{1, 1, 1, 1\}$$

$$W_0 = \begin{bmatrix} -0.0373 & 0.4852 & -0.3351 & 0.2336 \\ -1.6033 & 0.5988 & -0.3224 & 1.2352 \\ 0.3394 & -0.0860 & -0.3824 & -0.5785 \\ -0.1311 & 0.3253 & -0.9534 & -0.5015 \end{bmatrix}, \quad W_1 = \begin{bmatrix} 0.8674 & -1.2405 & -0.5325 & 0.0220 \\ 0.0474 & -0.9164 & 0.0360 & 0.9816 \\ 1.8495 & 2.6117 & -0.3788 & 0.8428 \\ -2.0413 & 0.5179 & 1.1734 & -0.2775 \end{bmatrix}$$

$$\Sigma_1 = \mathrm{diag}\{0.1137, 0.1279, 0.7994, 0.2368\}, \quad \Sigma_2 = \mathrm{diag}\{0, 0, 0, 0\}, \quad J = [0.4, 0.2, 0.3, 0.1]^T$$

This example has often been used to compare the conservativeness of the stability criteria in the literature [20, 35, 25, 52, 36, 29, 21, 23, 42, 40]. Table 3 gives the corresponding AMDBs with respect to different $\mu$ obtained by the proposed stability criteria as well as the ones given in literature. The stability criteria in [9, 12], derived based on the FWM approach, lead to smaller AMDBs than Theorem 1 does. It shows that the GFWM approach is more effective than the FWM approach, which verifies the statement in Remark 1. The LKFs with more general form, such delay-partition-based LKFs [20, 16, 25, 29, 21, 40, 23], augmented LKF [33], or the LKF with triple integral term [52], have improved the results, while they are still more conservative than the ones obtained by the proposed criteria since the Jensen inequality used therein is more conservative than the proposed GFWM approach.

### 5.2. *Simulation verification*

From the parameters of the DNNs, the equilibrium points of them can be obtained as $y^* = [0.6760\ 0.9077]^T$, $y^* = [0.0948\ 0.0532\ \ 0.0261]^T$, and $y^* = [0.1501\ 0.3471\ 0.3037\ 0.2401]^T$, respectively. From tables, the DNNs are stable for the cases: Example 1, $\mu = 0.4$, and $h = 9.7430$; Example 2, $\mu = 0.1$, and $h = 1.1118$; and Example 3, $\mu = 0.1$, and $h = 4.3583$. Thus, simulation studies for the following three cases are given:

- Example 1: $g(y) = \begin{bmatrix} 0.3\tanh(y_1) \\ 0.8\tanh(y_2) \end{bmatrix}$, $y(t) = [0.8, 0.5]^T$, $t \in [-9.7430, 0]$; $d(t) = \frac{9.7430}{2} + \frac{9.7430}{2}\sin\left(\frac{0.8}{9.7430}t\right)$;
- Example 2: $g(y) = \begin{bmatrix} 0.3680\tanh(y_1) \\ 0.1795\tanh(y_2) \\ 0.2876\tanh(y_3) \end{bmatrix}$ $y(t) = [0.2, 0.1, 0.3]^T$, $t \in [-1.1118, 0]$; $d(t) = \frac{1.1118}{2} + \frac{1.1118}{2}\sin\left(\frac{0.2}{1.1118}t\right)$;
- Example 3: $g(y) = [0.1137\tanh(y_1),\ 0.1279\tanh(y_2), 0.7994\tanh(y_3),\ 0.2368\tanh(y_4)]^T$, $y(t) = [0.3, 0.1, 0.2, 0.4]^T$, $t \in [-4.3583, 0]$; $d(t) = \frac{4.3583}{2} + \frac{4.3583}{2}\sin\left(\frac{0.2}{4.3583}t\right)$;

The responses of the DNNs are shown in Figs. 1-3, and the results show that the DNNs are stable at their equilibrium points, which verifies the effectiveness of the proposed methods.

## 6. Conclusions

This paper has developed a novel GFWM approach to analyze the delay-dependent stability of continuous DNN with a bounded time-varying delay, and several new stability criteria with less conservatism have been established. The improvement of the proposed stability criteria is benefit from the development of the GFWM approach, which can estimate the single integral term arising in the derivative of the LKF more accurately. It has been theoretically proved that the GFWM approach encompasses the widely used Wirtinger-based inequality and the recently presently FMBI approach. Finally, the comparison of the AMDBs for three numerical examples calculated based on the proposed criteria and the existing ones has clearly verified the advantages of the proposed criteria.

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Table 1: The AMDBs $h$ for various $\mu$ (Example 1)

| Cases | Criteria | $\mu$ | | | | NDVs |
|---|---|---|---|---|---|---|
| | | 0.40 | 0.45 | 0.50 | 0.55 | |
| $\dot{d}(t) \leq \mu$ | [11] | 3.9972 | 3.276 | 3.059 | 2.9814 | $12.5n^2 + 5.5n$ |
| | [28] (Th.1) | 4.39 | 3.67 | 3.46 | 3.41 | $35.5n^2 + 7.5n$ |
| | [25] (Th.4, $\rho = 0.5$) | 4.5023 | 3.7588 | 3.5472 | 3.4885 | $12.5n^2 + 23.5n$ |
| | [52] (Th.1) | 4.5543 | 3.8364 | 3.5583 | 3.4110 | $63.5n^2 + 11.5n$ |
| | [51] (Co.1) | 4.8748 | 4.2702 | 4.0551 | 3.9369 | $30.5n^2 + 23.5n$ |
| | [36] (Th.1) | 5.1029 | 4.1100 | 3.6855 | 3.4434 | $30.5n^2 + 8.5n$ |
| | [48] (Th.1) | 5.2420 | 4.4301 | 4.1055 | 3.9231 | $87.5n^2 + 11.5n$ |
| | [55] (Co.3) | 7.4203 | 6.6190 | 6.3428 | 6.2095 | $11.5n^2 + 13.5n$ |
| | Theorem 2 | 6.7883 | 6.1800 | 5.9623 | 5.8481 | $17n^2 + 13n$ |
| | Theorem 1 | 8.3498 | 7.3817 | 7.0219 | 6.8156 | $73n^2 + 13n$ |
| $\lvert\dot{d}(t)\rvert \leq \mu$ | [58] (Th.2.1) | 8.9704 | 7.6635 | 7.1554 | 6.8550 | $59.5n^2 + 20.5n$ |
| | [51] (Th.3) | 9.7094 | 7.7523 | 6.8570 | 6.2977 | $42n^2 + 27n$ |
| | Corollary 1 | 12.3721 | 10.0868 | 9.2340 | 8.7839 | $75.5n^2 + 13.5n$ |
| | Theorem 3 | 13.8671 | 11.1174 | 10.0050 | 9.4157 | $79.5n^2 + 15.5n$ |

Table 2: The AMDBs $h$ for various $\mu$ (Example 2)

| Cases | Criteria | $\mu$ | | | | NDVs |
|---|---|---|---|---|---|---|
| | | 0.0 | 0.1 | 0.5 | 0.9 | |
| $\dot{d}(t) \leq \mu$ | [25] (Th.4, $\rho = 0.5$) | 1.7683 | 1.0426 | 0.4313 | – | $12.5n^2 + 23.5n$ |
| | [38] (Co.1) | 1.8764 | 1.1127 | 0.4464 | – | $26n^2 + 17n$ |
| | [53] (Co.1) | 1.5575 | 0.9430 | 0.4417 | 0.3632 | $38n^2 + 19n$ |
| | [14] (Th.2) | 1.3323 | 0.8245 | 0.3733 | 0.2343 | $2.5n^2 + 5.5n$ |
| | [32] (Co.2) | 1.3323 | 0.8402 | 0.4264 | 0.3214 | $8.5n^2 + 6.5n$ |
| | [49] (Co.1) | – | 0.8411 | 0.4296 | 0.3227 | $25.5n^2 + 8.5n$ |
| | [55] (Co.3) | 1.5857 | 0.9567 | 0.4432 | – | $11.5n^2 + 13.5n$ |
| | [41] (Co.1) | 1.6386 | 0.9956 | 0.4464 | 0.3800 | $104.5n^2 + 17.5n$ |
| | Theorem 2 | 1.6124 | 0.9727 | 0.4442 | 0.3662 | $17n^2 + 13n$ |
| | Theorem 1 | 1.7302 | 1.0453 | 0.4486 | 0.3938 | $73n^2 + 13n$ |
| $\lvert\dot{d}(t)\rvert \leq \mu$ | [37] (Pro.2) | 1.8899 | 1.1114 | 0.4514 | – | $17n^2 + 14n$ |
| | Corollary 1 | 1.8899 | 1.1132 | 0.4920 | 0.4700 | $75.5n^2 + 13.5n$ |
| | Theorem 3 | 1.8899 | 1.1135 | 0.4922 | 0.4701 | $79.5n^2 + 15.5n$ |

Table 3: The AMDBs $h$ for various $\mu$ (Example 3)

| Cases | Criteria | $\mu$ | | | NDVs |
|---|---|---|---|---|---|
| | | 0.1 | 0.5 | 0.9 | |
| $\dot{d}(t) \leq \mu$ | [20] (Th.2 $\rho = 0.6$) | 3.3574 | 2.5915 | 2.1306 | $15n^2 + 7n$ |
| | [35] (Pro.2) | 3.5204 | 2.7167 | 2.2141 | $10.5n^2 + 8.5n$ |
| | [25] (Th.4, $\rho = 0.5$) | 3.8739 | 2.7415 | 2.3011 | $12.5n^2 + 23.5n$ |
| | [52] (Th.1) | 3.623 | 2.965 | 2.352 | $63.5n^2 + 11.5n$ |
| | [36] (Th.1) | 3.4984 | 2.7243 | 2.2029 | $30.5n^2 + 8.5n$ |
| | [29] (Th.3, $m = 2$) | 3.7665 | 2.6814 | 2.2274 | $17n^2 + 8n$ |
| | [21] (Th.1, $m = 2$) | 3.8428 | 2.7081 | 2.2485 | $7n^2 + 9n$ |
| | [55] (Co.3) | 4.1838 | 3.1510 | 2.8347 | $11.5n^2 + 13.5n$ |
| | [23] (Th.1, $m = 2$) | 4.1840 | 2.8387 | 2.3423 | $20n^2 + 11n$ |
| | [42] (Th.1) | 4.1903 | 3.0779 | 2.8268 | $66.5n^2 + 18.5n$ |
| | [40] (Co.3.1) | 4.2143 | 3.1059 | 2.7494 | $13.5n^2 + 6.5n$ |
| | Theorem 2 | 4.1268 | 3.0778 | 2.7619 | $17n^2 + 13n$ |
| | Theorem 1 | 4.2778 | 3.2152 | 2.9361 | $73n^2 + 13n$ |
| $\lvert\dot{d}(t)\rvert \leq \mu$ | [26] (Th.1) | 3.7515 | – | 2.4628 | $12.5n^2 + 9.5n$ |
| | [51] (Th.3) | 3.9337 | 3.5307 | 3.2627 | $42n^2 + 27n$ |
| | Corollary 1 | 4.3939 | 3.5657 | 3.3591 | $75.5n^2 + 13.5n$ |
| | Theorem 3 | 4.4167 | 3.5986 | 3.3755 | $79.5n^2 + 15.5n$ |

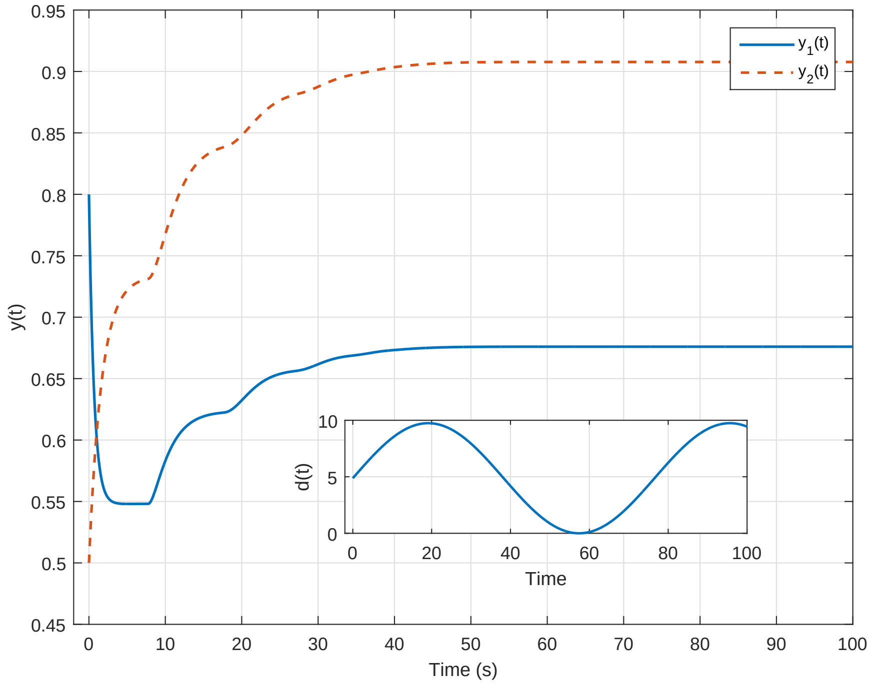


Figure 1: State trajectories of the DNN of Example 1.

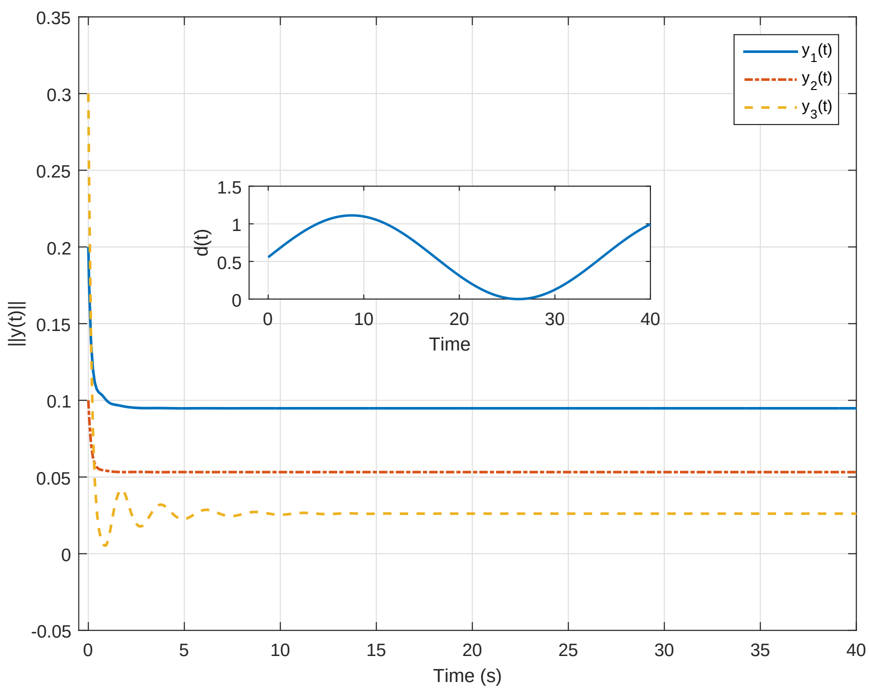


Figure 2: State trajectories of the DNN of Example 2.

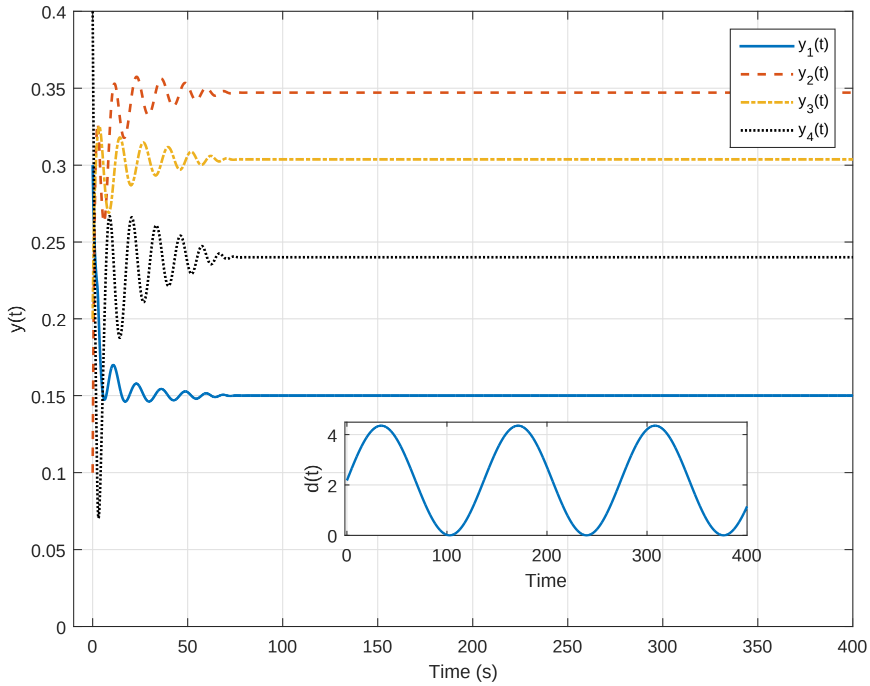


Figure 3: State trajectories of the DNN of Example 3.