Marginal Indemnification Function Formulation for Optimal Reinsurance

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Abstract

In this paper, we propose to combine the Marginal Indemnification Function (MIF) formulation and the Lagrangian dual method to solve optimal reinsurance model with distortion risk measure and distortion reinsurance premium principle. The MIF method exploits the absolute continuity of admissible indemnification functions and formulates optimal reinsurance model into a functional linear programming of determining an optimal measurable function valued over a bounded interval. The MIF method was recently introduced to analyze the reinsurance model but without premium budget constraint. In this paper, a Lagrangian dual method is applied to combine with MIF to solve for optimal reinsurance solutions under premium budget constraint. Compared with the existing literature, the proposed integrated MIF-based Lagrangian dual method provides a more technically convenient and transparent solution to the optimal reinsurance design. To demonstrate the practicality of the proposed method, analytical solution is derived on a particular reinsurance model that involves minimizing Conditional Value at Risk (a special case of distortion function) and with the reinsurance premium being determined by the inverse-S shaped distortion principle.

JEL Classification: C61; G22; G32.

Key Words: optimal reinsurance; marginal indemnification function; Lagrangian dual method; distortion risk measure; inverse-S shaped distortion premium principle.

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1 Introduction

Reinsurance, if exploited appropriately, can be an effective risk management tool for an insurer. In a typical reinsurance contract, the insurer pays a certain amount of premium to a reinsurer in return for some indemnification when losses occur from a designated risk. This indemnification is always a function of the risk, and the premium is determined by the resulting indemnification function, together with a given premium principle. While a higher stipulated indemnification implies a lower risk exposure to the insurer, this is achieved at the expense of a higher upfront reinsurance premium. This demonstrates the classical tradeoff between risk retention and risk transfer and the problem of optimal reinsurance is to address the optimal risk sharing between insurer and reinsurance for a given prescribed objective and constraints.

Two pioneering works on optimal reinsurance are attributed to Borch (1960) and Arrow (1963). Borch (1960) demonstrated that the stop-loss reinsurance is the best contract if the insurer measures risks by variance and the reinsurer prices risks by the expected value premium principle. Arrow (1963) also showed that the stop-loss reinsurance is an optimal one if the insurer is an expected utility maximizer under the assumption of the expected value premium principle. These fundamental results have been generalized in a number of interesting and important directions. Just to name a few, Kaluszka (2001) extended the Borch's result by considering the mean-variance premium principle, while Young (1999) elaborated Arrow's result by taking Wang's premium principle into account.

More recently, there is a surge of interest in formulating the optimal reinsurance problem involving more sophistical risk measures such as Value at Risk (VaR), Conditional Value at Risk (CVaR) and more generally distortion risk measures. Fo example, Cai and Tan (2007), Cai et al. (2008), Cheung (2010) and Tan et al. (2011) discussed the minimization of VaR and CVaR of the insurer's total risk exposure with expected value premium principle. Cheung et al. (2014) further explored Tan et al. (2011)'s results under the general law-invariant convex risk measure. Balbás et al. (2009) also studied the optimal reinsurance problem when risk is measured by a general risk measure. Chi and Tan (2013), and Chi and Weng (2013) considered VaR and CVaR with premium principles which preserve the convex ordering. Zheng et al. (2014) designed the optimal reinsurance contract under distortion risk measure, but assuming that the distortion function is piecewise concave or convex. Cui et al. (2013) studied a general model involving the distortion risk measure and the distortion premium principle. Cheung and Lo (2015) extended the model of Cui et al. (2013) to a cost-benefit framework. Assa (2015) demonstrated that the optimal reinsurance model of Cui et al. (2013), without the premium constraint, can be tackled more elegantly via a marginal indemnification function (MIF) formulation.

The primary objective of the present paper is to extend the MIF-based method so as it can be used to derive analytically the solution to the distortion risk measure based reinsurance model in the presence of a premium budget constraint. It is well-known that in many optimization problems, the complexity of the optimization problem can be significantly increased by merely imposing a constraint. In particular, one often discovers that while an optimization procedure can be used to solved an unconstrained optimization problem analytically, the same procedure may no long be applicable when a constraint is imposed on the model. This is precisely the issue with the method of MIF proposed by Assa (2015). As demonstrated in Assa (2015) that without the premium constraint the MIF formulation elegantly solves the reinsurance model of Cui et al. (2013). The same method, however, cannot be readily used in the presence of the premium budget.

The MIF method makes full use of the absolute continuity of admissible ceded loss functions. It is well known that an absolute continuous function over real line is, out of a Lebesgue null set, differentiable. The derivative of the ceded loss function is called marginal indemnification function, because it measures the increase in ceded loss per unit of increase in the group-up loss. It should be pointed out that Balbás et al. (2015) have independently proposed the MIF formulation for optimal reinsurance models, though the term "MIF" was not used. The authors considered a general mean-risk reinsurance model under uncertainty of the group-up risk and formulated the reinsurance model with the derivative of the retained loss function being the decision variable, which they referred to as "sensitivity". Moreover, they proposed to impose a lower bound on the decision variable to effectively eliminate the moral hazard from the insurer. This translates to an upper bound on the MIF in our formulation. For $\widetilde{\mathbb{E}}_\rho$-translation invariant risk measures (which satisfy subadditivity), Balbás et al. (2015) developed two duality methods of transforming optimal reinsurance models, which may be non-linear, into functional linear programming problems.

The objective of this paper is to demonstrate that by integrating MIF with a Lagrangian method, one can derive explicit optimal reinsurance policies for problems with a budget constraint on reinsurance premium and bounds on the derivative of admissible ceded loss functions. Compared to the approach of Cui et al. (2013) for solving the same reinsurance model with premium budget constraint, our proposed integrated MIF and Lagranging method possesses at least the following three advantages. Firstly, it is simpler and more transparent. More specifically, the approach of Cui et al. (2013) critically depends on a pre-conjectured candidate solution. This implies we need to first guess an optimal solution and then apply certain comparison analysis to prove its optimality. Their method, therefore, requires us to have a preconception on the shape of the optimal solution in order to justify its optimality. Our integrated method, on the other hand, does not require any preanalysis on the shape of optimal solutions. Secondly, due to the nature of the procedure in searching for the solution developed by Cui et al. (2013), it is difficult to discuss the uniqueness of optimal solution. In contrast, the uniqueness of solution can be easily studied, and the non-uniqueness of solutions can also be uncovered from our optimization procedure. Thirdly, even if bounds are imposed on the derivative of the admissible ceded loss functions, our proposed integrated method can similarly be used to derive the explicit solutions of the models.

To highlight the practicality of our proposed solution, we consider a particular reinsurance model that minimizes CVaR (a special distortion risk measure) and with the premium being dictated by the inverse-S shaped distortion (ISSD) premium principle. The ISSD premium principle is a distortion premium principle with a distortion function such that it has derivative which changes from being strictly decreasing to being strictly increasing derivative at a certain point. Thus, it encompasses both the concave and convex distortion premium principles as special cases. Indeed, as it will become clear in Section 5, the optimal solutions for either a concave or a convex distortion premium principle can be recovered from those we obtained for the ISSD premium principle as special cases.

Another important feature of the ISSD premium principle is its economic interpretation in that the insurance provider may overweight not only large losses but also small ones in underwriting the insurance risks. This is consistent with the empirically observed phenomena in psychological experiments (Quiggin 1982, 1992; Tversky and Kahneman 1992; Tversky and Fox 1995; Gonzalez and Wu, 1999). Furthermore, Kaluszka and Krzeszowiec (2012) introduced a premium principle from the perspective of the Cumulative Prospect Theory (CPT). The ISSD premium principle can also be viewed as a special CPT premium principle corresponding to a linear utility function and a zero reference point. Unlike the concave distortion premium principle, the ISSD premium principle has not received much attention in the actuarial literature. This, in part, can be attributed to its relatively new concept and its short history. Other reasons could be due to the possibility that the ISSD introduces additional technical hurdles, such as non-convex order property, for solving optimization problems.

The rest of the paper proceeds as follows. In Section 2, we formally specify our optimal reinsurance models and develop their corresponding MIF's. Section 3 gives the optimal solutions for the model without premium budget constraint. In Section 4, we integrate the Lagrangian dual method with the MIF formulation and derive explicit solutions for the model with reinsurance premium budget constraint. In Section 5, we demonstrate the practicality of our proposed approach by resorting to a specialized example involving risk measure CVaR and reinsurance premium principle ISSD. Section 6 concludes the paper.

2 Model Setup

Throughout the paper, all the random variables are defined on a common probability space $(\Omega, F, \mathbb{P})$. The indicator function is denoted by $\mathbf{1}_A(s)$, i.e., $\mathbf{1}_A(s) = 1$ for $s \in A$ and $\mathbf{1}_A(s) = 0$ for $s \notin A$. The capital letter X is exclusively used to denote the non-negative random variable for which the insurer seeks reinsurance coverage and $M \triangleq \text{esssup} X$. For convenience, the domain of the random variable X is consistently denoted by $[0, M]$. While this suggests that the domain of X is a bounded interval, it should be emphasized that all the results obtained in the paper hold even if $\text{esssup} X = \infty$; i.e. even if $[0, M]$ is replaced by $[0, \infty)$.

The problem of optimal reinsurance is concerned with the optimal partitioning of X into $f(X)$ and $r(X)$ such that $X = f(X) + r(X)$, where f and r are two measurable functions defined over $[0, M]$. Here $f(X)$ represents the portion of loss that is ceded to a reinsurer and $r(X)$ is the residual loss that is retained by the insurer. The functions f and r are respectively referred to as "indemnification function" (or "ceded loss function") and "retained loss function". While the presence of reinsurance reduces the retained loss of the insurer, it incurs an additional cost to the insurer in the form of reinsurance premium. Generally, this premium is positive and is a functional of $f(X)$, say $\Pi(f(X))$ for some functional Π. Therefore, the total risk exposure of the insurer in the presence of reinsurance is given by

$$T_f = r(X) + \Pi(f(X)). \tag{1}$$

2.1 Admissible Set

It is common in the literature (e.g., Chi and Tan, 2013; Cui et al., 2013; Assa, 2015) to consider the following admissible set of ceded loss functions:

$$\mathcal{F}_0 \triangleq \{f : [0, M] \mapsto [0, M] \,|\, f(0) = 0, 0 \leq f(x) - f(y) \leq x - y, \;\; y < x \text{ with } y, x \in [0, M]\}, \tag{2}$$

where we note that every function $f \in \mathcal{F}_0$ is absolutely continuous, and thus, it is almost everywhere differentiable on $[0, M]$, i.e., there exists a Lebesgue integrable function h such that

$$f(x) = \int_0^x h(z)dz, \;\; x \in [0, M]. \tag{3}$$

Here $h(z)$ is the slope of the ceded loss function f at z, and thus, we must have $h(z) \in [0, 1]$, $z \in [0, M]$. The function $h(z)$ can be interpreted as the "marginal indemnification" from an increase of the loss X. Thus, the function h is referred to as a "marginal indemnification function (MIF)". Obviously, two MIF's only differing each other over a Lebesgue null set result in the same ceded loss function f everywhere.

The admissible set $\mathcal{F}_0$ is typically justified by the argument of avoiding moral hazard. Note that each ceded loss function in $\mathcal{F}$ is non-decreasing and any increment in compensation is always less than or equal to the increment in loss, hence potentially reducing moral hazard for both the insurer and reinsurer.

With the above admissible set $\mathcal{F}_0$, a stop-loss or closely related contracts frequently solve a variety of optimal reinsurance models. As pointed by Balbás et al. (2015), in practice reinsurers rarely accept these solutions due to the lack of incentives of the insurer to verify claims beyond some thresholds. To rectify this, Balbás et al. (2015) proposed to impose a strictly positive lower bound on the derivative of admissible retained loss functions, which is equivalent to imposing an upper bound on the derivative h of admissible ceded loss function.

In the present paper, we follow the argument of Balbás et al. (2015) and consider the following admissible set of MIF's:

$$\mathcal{H} \triangleq \{h : [0, M] \mapsto \mathbb{R} \,|\, h_0 \leq h \leq h_1 \;\; \text{a.e., and } h \text{ is Lebesgue measurable}\}, \tag{4}$$

where h_0 and h_1 are two constants with $0 \leq h_0 < h_1 \leq 1$. Accordingly, the admissible set of ceded loss functions is given by

$$\mathcal{F} = \left\{ f : [0, M] \mapsto [0, M] \,\middle|\, f(x) = \int_0^x h(z)dz, \;\; x \in [0, M], \;\; h \in \mathcal{H} \right\}. \tag{5}$$

Using the representation (3), it is easy to check that the admissible set $\mathcal{F}$ can be equivalently written as

$$\begin{aligned}\mathcal{F} \triangleq \{ & f:[0,M]\mapsto[0,M]\,|\,f(0)=0,\\ & h_0\cdot(x-y)\le f(x)-f(y)\le h_1\cdot(x-y),\ \forall\ y<x \text{ with } y,x\in[0,M]\}.\end{aligned} \tag{6}$$

2.2 Distortion Risk Measure and Distortion Premium Principle

In this paper, the optimal reinsurance is defined as those policies which minimize a certain distortion risk measure of the insurer's total risk exposure T_f over the feasible set $\mathcal{F}$ defined in (5), or equivalently, (6). Before introducing the distortion risk measure, it is useful to define the following set of functions:

$$\mathcal{G} \triangleq \left\{ g:[0,1]\to[0,1] \middle| g(t) \text{ is non-decreasing and left continuous}, \ g(0)=0 \text{ and } g(1)=1 \right\}.$$

The left continuity imposed in the definition of $\mathcal{G}$ is to facilitate the subsequence analysis. As we will see shortly, such an assumption retains Value-at-Risk (VaR) and Conditional Value-at-Risk (CVaR) in the class of distortion risk measures. Formally, the distortion risk measures is defined as follows.

Definition 2.1 *A a distortion risk measure ρ^g of a random variable Z with a distortion function $g\in\mathcal{G}$ is defined as a Choquet integral:*

$$\rho^g(Z)=\int_0^\infty g(1-F_Z(t))dt+\int_{-\infty}^0 [g(1-F_Z(t))-1]\,dt, \tag{7}$$

where F_Z denotes the cumulative distribution function of random variable Z, provided of the existence of the integrals.

We note that (7) reduces to

$$\rho^g(Z)=\int_0^\infty g(1-F_Z(t))dt$$

for nonnegative random variables Z. We also note the following properties of distortion risk measures:

- **Commonotonic Additivity:** $\rho^g(X+Y)=\rho^g(X)+\rho^g(Y)$ for two commonotonic random variables X and Y;
- **Translation Invariance:** $\rho^g(Z+C)=\rho^g(Z)+C$ for any constant C and random variable Z such that the resulting integral is well defined;
- **Monotonicity:** $\rho^g(X)\le\rho^g(Y)$ whenever $X\le Y$ a.s.

The distortion risk measure encompasses many interesting risk measures as its special cases. Two prominent examples are VaR and CVaR, which are respectively defined as follows.

Definition 2.2 *The* VaR *and* CVaR *of a random variable* Z *at a confidence level* α *(with* $0 < \alpha < 1$*) are respectively defined as*

$$\mathrm{VaR}_\alpha(Z) \triangleq \inf\{z \in \mathbb{R} : \mathbb{P}(Z \leq z) \geq \alpha\},$$

and

$$\mathrm{CVaR}_\alpha(Z) \triangleq \frac{1}{1-\alpha}\int_\alpha^1 \mathrm{VaR}_s(Z)ds,$$

provided that the integral exists.

For any distortion function $g \in \mathcal{G}$, define $\widetilde{g}(s) = 1 - g(1-s)$, which is right continuous. Then, the distortion risk measure ρ^g can be equivalently expressed as

$$\rho^g(Z) \triangleq \int_0^1 \mathrm{VaR}_s(Z)d\,\widetilde{g}(s); \tag{8}$$

see, for example, Theorem 6 in Jhaene et al. (2012). Therefore, VaR_α and CVaR_α, for some $\alpha \in (0,1)$, are two distortion risk measures, respectively corresponding to $\widetilde{g}(s) = \mathbf{1}_{[\alpha,1]}(s)$ and $\widetilde{g}(s) = \frac{s-\alpha}{1-\alpha}\mathbf{1}_{[\alpha,1]}(s)$, or equivalently $g(t) = \mathbf{1}_{(1-\alpha,1]}(t)$ and $g(t) = \min\{1, \frac{t}{1-\alpha}\}$. The family of spectral risk measures, which is defined by Acerbi (2002) and in a form of $\int_0^1 \mathrm{VaR}_s(Z)\phi(s)ds$ for some probability density ϕ on $(0,1)$, also obviously falls into the class of distortion risk measures.

We assume that the reinsurance premium in (1) is also a distortion principle as defined below.

Definition 2.3 *The distortion premium principle is defined as*

$$\pi^g(Y) \triangleq (1+\theta)\rho^g(Y),$$

where the constant $\theta \geq 0$ *and* $g \in \mathcal{G}$.

In the above definition, θ is the safety loading of the reinsurer. When $g(x) = x$, the distortion premium principle recovers the expected value premium principle. Furthermore, when the distortion function is concave with $\theta = 0$, the distortion principle recovers Wang's premium principle.

2.3 Optimal Reinsurance Model

We now formulate the two optimal reinsurance models that we will analyze in this paper. These models are described as problems 2.1 and 2.2 below.

Problem 2.1

$$\begin{cases} \inf\limits_{f \in \mathcal{F}} & \rho^{g_1}(T_f) \\ \text{s.t.} & \pi^{g_2}(f(X)) \leq \pi_0, \end{cases} \tag{9}$$

where g_1 and g_2 are two distortion functions from $\mathcal{G}$, $\pi_0 > 0$ is a constant denoting the reinsurance premium budget. The presence of the premium constraint signifies that the insurer is willing to seek reinsurance protection as long as the cost does not exceed π_0. We assume $\pi_0 \in \left[\pi^{g_2}(h_0 X), \pi^{g_2}(h_1 X)\right]$, because, from the definition of $\mathcal{F}$ in (6), it is obvious that $\pi^{g_2}(f(X)) \in \left[\pi^{g_2}(h_0 X), \pi^{g_2}(h_1 X)\right]$ for any $f \in \mathcal{F}$.

As it will become clear shortly, when the reinsurance premium budget π_0 is larger than a certain threshold, the solution to problem 2.1 coincides with the following problem 2.2 where no premium budget constraint is imposed.

Problem 2.2

$$\inf_{f \in \mathcal{F}} \quad \rho^{g_1}(T_f). \tag{10}$$

In the above two problems, we assume that $\rho^{g_1}(X)$ and $\rho^{g_2}(X)$ are finite so that, for every $f \in \mathcal{F}$,

$$\begin{aligned}
\rho^{g_1}(T_f) &= \rho^{g_1}(r(X) + \Pi(f(X))) \\
&= \rho^{g_1}(r(X)) + (1+\theta)\rho^{g_2}(f(X)) \\
&\leq \rho^{g_1}(X) + (1+\theta)\rho^{g_2}(X) \\
&< \infty,
\end{aligned}$$

where the first inequality is due to the monotonicity of distortion risk measure.

Remark 2.1 *In principle, if we could solve problem 2.1 for a general* $\pi_0 \in \left[\pi^{g_2}(h_0 X), \pi^{g_2}(h_1 X)\right]$*, the solution to problem 2.2 follows trivially by taking* $\pi_0 = \pi^{g_2}(h_1 X)$*. As pointed earlier, analyzing problem 2.1 is much more technically involved compared to problem 2.1 due to the existence of the premium constraint.*

Remark 2.2 *While problem 2.2 closely resembles to that studied by Assa (2015) and Cui et al. (2013), it has some distinctive differences. Notably, lower bound* h_0 *and upper bound* h_1 *are imposed on the derivative of ceded loss functions (see (4)) in problem 2.2. By setting* $h_0 = 0$ *and* $h_1 = 1$*, problem 2.2 reduces to the model studied by Assa (2015) and Cui et al. (2013). By imposing an upper bound* $h_1 \in (0, 1)$ *on the derivative of ceded loss functions, this has the effect of ensuring the optimal retained loss is strictly increasing. As advocated by Balbás et al. (2015), this in turn has the advantage of reducing moral hazard of the insurer.*

Prompted by the difficulty of directly applying the MIF-based method of Assa (2015) of solving a constrained optimal reinsurance model; i.e. problem 2.1 with the premium budget constraint, this paper alleviates this issue by integrating the MIF formulation with a Lagrangian method.

Comparing our proposed integrated method to the construction method of Cui et al. (2013) for solving problems 2.1 and 2.2, our proposed method has the advantage of not requiring any preconception on the shape of optimal solutions. Moreover, our method allows us to study the uniqueness of solution (see Corollary 3.1 and Proposition 4.2). Furthermore, our integrated method can similarly be used for deriving solutions to our reinsurance model, even if there are bounds on the derivative of admissible ceded loss functions.

2.4 MIF Formulation

By exploiting the equivalent expression of $\mathcal{F}$ given in (5), this subsection transforms problems 2.1 and 2.2 into their equivalence problems in terms of MIF. The following lemma plays a critical role in accomplishing this objective.

Lemma 2.1 *Given any $g \in \mathcal{G}$ and $f \in \mathcal{F}$, there exists $h \in \mathcal{H}$, independent of g, such that*

$$f(x) = \int_0^x h(z)dz, \quad x \in [0, M], \quad \text{and } \rho^g(f(X)) = \int_0^M g\left(1 - F_X(z)\right) h(z)dz.$$

PROOF: We first assume $M < \infty$. For any $f \in \mathcal{F}$, let us define

$$f^{-1}(t) := \inf\{z \geq 0 : f(z) \geq t\}, \ \ t \in [0, f(M)].$$

Then, the two events $\{f(X) \geq t\}$ and $\{X \geq f^{-1}(t)\}$ are identical for every $t \in [0, f(M)]$, since $f \in \mathcal{F}$ is non-decreasing and continuous. $\mathbb{P}(f(X) > t)$ and $\mathbb{P}(f(X) \geq t)$, as two functions of t on $[0, f(M)]$, differ from each other at most over a countable set. These facts imply

$$\rho^g(f(X)) = \int_0^{f(M)} g\big(\mathbb{P}(f(X) > t\big)dt = \int_0^{f(M)} g\big(\mathbb{P}(f(X) \geq t\big)dt = \int_0^{f(M)} g\big(\mathbb{P}(X \geq f^{-1}(t))\big)dt. \quad (11)$$

Further note that $f(f^{-1}(t)) = t$ for any $t \in [0, f(M)]$, whereby applying a change-of-variable and invoking the representation (3) yield

$$\rho^g(f(X)) = \int_0^{f^{-1}(f(M))} g\big(\mathbb{P}(X \geq z)\big)df(z) = \int_0^M g\big(\mathbb{P}(X \geq z)\big)df(z) = \int_0^M g\big(\mathbb{P}(X \geq z)\big)h(z)dz, \quad (12)$$

where the second equality holds due to the fact that $f^{-1}(f(M)) \leq M$ and that, if $f^{-1}(f(M)) < M$, f must equal to the constant $f(M)$ on $[f^{-1}(f(M)), M]$.

Next, we assume $M = \infty$ and denote $f(M) = \lim_{x\uparrow\infty} f(x)$. We recall that, in this case, "$[0, M]$" is interpreted as "$[0, \infty)$" throughout the paper. If $f(M) < \infty$, the proof in the above still holds. If $f(M) = \infty$, we define $f^{-1}(t)$ on $[0, \infty)$. Then, all the equalities in (11) and (12) still hold if we replace M, $f(M)$ and $f^{-1}(f(M))$ by ∞, and replace "$[0, f(M)]$" by "$[0, \infty)$". Thus, this completes the proof. □

It follows from Lemma 2.1 that, for any $f \in \mathcal{F}$, there exists a function $h \in \mathcal{H}$, independent of g_1 and g_2, with $f(x) = \int_0^x h(z)dz$, $\forall \ x \in [0, M]$, such that the reinsurance premium $\pi^{g_2}(f(X))$ can be represented as

$$\pi^{g_2}(f(X)) = (1+\theta)\rho^{g_2}(f(X)) = (1+\theta)\int_0^M g_2\left(1 - F_X(z)\right) h(z)dz, \quad (13)$$

and the objective in problem 2.1 can be rewritten as

$$\rho^g(T_f) = \rho^{g_1}(X - f(X) + \pi^{g_2}(f(X)))$$

$$
\begin{aligned}
&= \rho^{g_1}(X) - \rho^{g_1}(f(X)) + (1+\theta)\rho^{g_2}(f(X)) \\
&= \rho^{g_1}(X) - \int_0^M g_1(1-F_X(z))h(z)dz + (1+\theta)\int_0^M g_2(1-F_X(z))h(z)dz \\
&= \rho^{g_1}(X) - \int_0^M \psi(F_X(z))h(z)dz, \qquad (14)
\end{aligned}
$$

where

$$
\psi(t) = g_1(1-t) - (1+\theta)g_2(1-t), \;\; t \in [0,1]. \qquad (15)
$$

Since $\rho^{g_1}(X)$ is a constant, it suffices to analyze the term $\int_0^M \psi(F_X(z))h(z)dz$ for optimal solutions of problems 2.1 and 2.2. As a consequence, problems 2.1 and 2.2 can be respectively transformed into a MIF formulation as follows:

Problem 2.3

$$
\begin{cases}
\sup\limits_{h\in\mathcal{H}} & U(h) \triangleq \int_0^M \psi(F_X(z))h(z)dz \\
\text{s.t.} & \int_0^M g_2\left(1-F_X(z)\right)h(z)dz \leq \pi_1,
\end{cases} \qquad (16)
$$

where $\pi_1 = \frac{\pi_0}{1+\theta} \in \left[h_0\rho^{g_2}(X), h_1\rho^{g_2}(X)\right]$;

Problem 2.4

$$
\sup_{h\in\mathcal{H}} \int_0^M \psi(F_X(z))h(z)dz. \qquad (17)
$$

Remark 2.3 *The transformed problems 2.3 and 2.4 are well-posed in the sense that the supremum is finite, because the objectives in both problems are bounded from above for every* $h \in \mathcal{H}$. *More specifically, every* $h \in \mathcal{H}$ *is bounded by* h_1 *from the above, and thus,*

$$
U(h) \leq \int_0^M \left[g_1(1-F_X(z))\cdot h(z)\right]dz \leq h_1\rho^{g_1}(X) < \infty.
$$

In Sections 3 and 4, we will respectively obtain feasible MIF which attains the supremum of the objective for problems 2.3 and 2.4.

A formal result on the equivalence between the MIF formulation (i.e., problems 2.3 and 2.4) and the formulation in terms of ceded loss functions (i.e., problems 2.1 and 2.2) is summarized in the following proposition.

Proposition 2.1 *An element* h^* *solves problem 2.3 (or problem 2.4) if and only if* $f^*(x) = \int_0^x h^*(z)dz$ *for* $x \in [0, M]$, *solves problem 2.1 (problem 2.2).*

Proof: We only show the relationship between problems 2.1 and 2.3 as the result can be similarly proved for the other pair of problems. We furnish the proof by a contradiction argument. We first consider the "if" part. Assume that $h^* \in \mathcal{H}$, which is the derivative of f^*, is not an optimal solution to problem 2.3, i.e., there exists another element, say $\widehat{h} \in \mathcal{H}$, satisfying $U(\widehat{h}) > U(h^*)$. Then, we define $\widehat{f}(x) = \int_0^x \widehat{h}(z)dz$, $\forall\, x \in [0, M]$, to get

$$\rho^{g_1}(X) - \rho^{g_1}(T_{\widehat{f}}) = U(\widehat{h}) > U(h^*) = \rho^{g_1}(X) - \rho^{g_1}(T_{f^*}),$$

which means that f^* cannot be a solution to problem 2.1. Therefore, if f^* solves problem 2.1, its derivative h^* must be a solution to problem 2.3).

To show the "only if" part, we assume that f^* as given is not a solution to problem 2.1. Then, there must exist another element $\widehat{f} \in \mathcal{F}$ with $\rho^{g_1}(T_{\widehat{f}}) < \rho^{g_1}(T_{f^*})$, which combines with the equation (14), further implies $U(\widehat{h}) > U(f^*)$, where $\widehat{h}$ is the derivative of $\widehat{f}$. This in turn implies that h^* is not a solution to problem 2.3, leading to a contradiction. Therefore, if h^* solves problem 2.3, $f^*(x) = \int_0^x h^*(z)dz \;\forall\, x \in [0, M]$ must be one solution to problem 2.1, and thus, the proof is complete. □

3 Optimal Reinsurance without Premium Constraint

In this section, we first focus on solving problem 2.4, which by Proposition 2.1, is equivalent to solving problem 2.2. Here and thereafter, we use μ to denote the Lebesgue measure.

Theorem 3.1 *$h^* \in \mathcal{H}$ solves problem 2.4 if and only if it admits the following representation:*

$$h^*(z) \overset{a.e.}{=} \begin{cases} h_1, & \text{if } \psi(F_X(z)) > 0, \\ \kappa(z), & \text{if } \psi(F_X(z)) = 0, \\ h_0, & \text{if } \psi(F_X(z)) < 0, \end{cases} \tag{18}$$

where the domain of h^* is $[0, M]$, and $\kappa(z)$ can be any Lebesgue measurable and $[h_0, h_1]$-valued function on $\{z \in [0, M] : \psi(F_X(z)) = 0\}$.

Proof: Let $l \in \mathcal{H}$ be a function which differs from h^* on a Lebesgue set $A \subset \{z \in [0, M] : \psi(F_X(z)) \neq 0\}$ with $\mu(A) > 0$. We need to show $U(l) < U(h^*)$. Indeed,

$$\begin{aligned}
&U(l) - U(h^*) \\
&= \int_0^M \psi(F_X(z))l(z)dz - \int_0^M \psi(F_X(z))h^*(z)dz \\
&= \int_{\{z\in[0,M]|\psi(F_X(z))>0\}} \psi(F_X(z))\big[l(z) - h^*(z)\big]dz + \int_{\{z\in[0,M]|\psi(F_X(z))<0\}} \psi(F_X(z))\big[l(z) - h^*(z)\big]dz \\
&= \int_{\{z\in[0,M]|\psi(F_X(z))>0\}} \psi(F_X(z))\big[l(z) - h_1\big]dz + \int_{\{z\in[0,M]|\psi(F_X(z))<0\}} \psi(F_X(z))\big[l(z) - h_0\big]dz < 0,
\end{aligned}$$

and hence the proof is complete. ∎

Remark 3.1 *In the optimal control theory, if an optimal control switches from one extreme point of the feasible set to the other (i.e., is strictly never in between the bounds), then that control is referred to as a bang-bang solution. A bang-bang solution is often optimal for a (functional) linear programming problem, and it is indeed the case for our problem 2.4. To see this, we take either* $\kappa(z) \equiv h_0$ *or* $\kappa(z) \equiv h_1$ *in (18), the optimal solution* h^* *is obviously an extreme point of the feasible set of problem 2.4. The concept of bang-bang solutions have been previously pointed out by Balbás et al. (2015) in the context of optimal reinsurance design. As pointed our earlier, Balbás et al. (2015) developed two dual methods of transforming a class of mean-risk reinsurance models which may be non-linear into functional linear programming problems. The existence of bang-bang solutions have been studied by the authors in Balbás et al. (2015).*

Remark 3.2 *By setting* $h_0 = 0$ *and* $h_1 = 1$*, Theorem 3.1 recovers the optimal solutions obtained by Assa (2015).*

By Proposition 2.1, the optimal ceded loss functions for problem 2.2 can be obtained as

$$f^*(x) = \int_0^x h^*(z)dz, \ \ \forall \ x \in [0, M],$$

where h^* can be any MIF as given in (18). In particular, if we take $\kappa(z) = h_0$ on the set $\{z \in [0, M] : \psi(F_X(z)) = 0\}$ in (18), then the MIF $\widetilde{h}$ defined by

$$\widetilde{h}(z) \triangleq h_1 \mathbf{1}_{\{\psi(F_X(z))>0\}} + h_0 \mathbf{1}_{\{\psi(F_X(z))\leq 0\}}, \ \ z \in [0, M], \tag{19}$$

solves problem 2.4, and $\widetilde{f}$ defined by

$$\widetilde{f}(x) = h_1 \mu(\{z \in [0, x] \,|\psi(F_X(z)) > 0\}) + h_0 \mu(\{z \in [0, x] \,|\psi(F_X(z)) \leq 0\}), \ \ x \in [0, M], \tag{20}$$

is one optimal ceded loss function of problem 2.2.

Corollary 3.1 *The optimal solution to problem 2.2 is unique on* $[0, M]$*, if and only if*

$$\mu(\{z \in [0, M] \,|\psi(F_X(z)) = 0\}) = 0.$$

Proof: The result trivially follows from Proposition 2.1 and Theorem 3.1. ∎

4 Optimal Solutions with Premium Constraint

In this section, we analyze the solutions of problem 2.1. In view of Proposition 2.1, we first focus on problem 2.3. Once we derive a solution h^* for problem 2.3, function f^* defined by $f^*(x) = \int_0^x h(z)dz$, $\forall\, x \in [0, M]$, is an optimal ceded loss function for problem 2.1. To proceed, let us denote

$$\widetilde{\pi} \triangleq \int_0^M g_2\left(1 - F_X(z)\right)\widetilde{h}(z)dz, \tag{21}$$

where $\widetilde{h}$ is given in (19), which is a solution of problem 2.4. If $\pi_1 \geq \widetilde{\pi}$, then $\widetilde{h}$ is obviously an optimal solution to problem 2.3, and thus, the ceded loss function $\widetilde{f}$ given in (20) solves problem 2.1 in this case.

In the subsequent analysis we assume $\pi_1 < \widetilde{\pi}$ to exclude the known case. We resort to a Lagrangian dual method to solve problem 2.3. This entails introducing a multiplier λ and the following auxiliary problem:

$$\inf_{h\in\mathcal{H}} \quad V(\lambda, h) \triangleq \int_0^M \psi(F_X(z))h(z)dz - \lambda\Big(\int_0^M g_2\left(1 - F_X(z)\right)h(z)dz - \pi_1\Big). \tag{22}$$

The connection between problem 2.3 and auxiliary problem (22) will be developed in Lemma 4.1 and Theorem 4.1.

Noticing the definition of ψ in (15), we write $V(\lambda, h) = \int_0^M \psi_\lambda(F_X(z))h(z)dz + \lambda\pi_1$, where

$$\psi_\lambda(t) = g_1(1-t) - (1+\theta+\lambda)g_2(1-t), \;\; t \in [0, 1]. \tag{23}$$

Thus, following the same reasoning as in the proof of Theorem 3.1, we obtain the solutions of auxiliary problem (22) as given by:

$$h_\lambda(z) \overset{a.e.}{=} \begin{cases} h_1, & \text{if } z \in A_\lambda \\ \kappa_\lambda(z), & \text{if } z \in B_\lambda, \\ h_0, & \text{if } z \in C_\lambda, \end{cases} \tag{24}$$

where the domain of h_λ is $[0, M]$, and κ_λ is any Lebesgue measurable and $[h_0, h_1]$-valued function, and

$$\begin{cases} A_\lambda = \{z \in [0, M]\,|\psi_\lambda(F_X(z)) > 0\}, \\ B_\lambda = \{z \in [0, M]\,|\psi_\lambda(F_X(z)) = 0\}, \\ C_\lambda = \{z \in [0, M]\,|\psi_\lambda(F_X(z)) < 0\}. \end{cases} \tag{25}$$

Lemma 4.1 below provides sufficient conditions for a solution of the form (24) with certain special value of λ to solve problem 2.3. The existence of such λ will be proved in Theorem 4.1 in the sequel.

Lemma 4.1 *Assume that there exists a constant $\lambda^* \geq 0$ such that h_{λ^*} solves problem (22) for $\lambda = \lambda^*$ and*

$$\int_0^M g_2\left(1 - F_X(z)\right)h_{\lambda^*}(z)dz = \pi_1. \tag{26}$$

Then, $h^* \triangleq h_{\lambda^*}$ solves problem 2.3.

Proof: We denote the optimal value of problem 2.3 by $u(\pi_1)$. Then, it follows

$$\begin{aligned}
u(\pi_1) &= \sup_{\substack{h\in\mathcal{H} \\ \int_0^M g_2(1-F_X(z))h(z)dz\le\pi_1}} U(h) \\
&\le \sup_{\substack{h\in\mathcal{H} \\ \int_0^M g_2(1-F_X(z))h(z)dz\le\pi_1}} \left[U(h) - \lambda^*\left(\int_0^M g_2\left(1-F_X(z)\right)h(z)dz - \pi_1\right)\right] \\
&\le \sup_{h\in\mathcal{H}} \left[U(h) - \lambda^*\left(\int_0^M g_2\left(1-F_X(z)\right)h(z)dz - \pi_1\right)\right] \\
&= U(h_{\lambda^*}) \le u(\pi_1),
\end{aligned}$$

where we apply the fact that h_{λ^*} is feasible to problem 2.3. Hence, $h^* := h_{\lambda^*}$ solves problem 2.3. □

Remark 4.1 *To establish an analytical solution h^*, we need to determine the specific value of λ^* and a function κ_{λ^*} to satisfy (26). By (24), we take $\kappa_\lambda(z) = a$ for some $a \in [h_0, h_1]$ and write*

$$\begin{aligned}
\Delta(\lambda, a) &\triangleq \int_0^M g_2(1-F_X(z))h_\lambda(z)dz \\
&= h_1\int_{A_\lambda} g_2(1-F_X(z))dz + a\int_{B_\lambda} g_2(1-F_X(z))dz + h_0\int_{C_\lambda} g_2(1-F_X(z))dz,
\end{aligned}$$

which can be viewed as a function of λ and a. This motivates us to follow a two-step procedure to determine λ^* and κ_{λ^*}. Firstly, we identify the value of λ^* as the λ such that $\Delta(\lambda, h_1)$ is closest to π_1, and secondly, we stick to the determined λ^* and take a $[h_0, h_1]$-valued constant function for κ_{λ^*} such that equation (26) is satisfied. The proof of part (a) in Theorem 4.1 below indeed follows such a procedure.

The following Proposition 4.1 shows that, when the set $\{z \in [0, M]\,|\psi(F_X(z)) = 0\}$ is a μ-null set, the condition (26) is also necessary for h_{λ^*} to solve problem 2.3 in the case of $\pi_1 < \widetilde{\pi}$.

Proposition 4.1 *Assume $\pi_1 < \widetilde{\pi}$, and $\mu\{z \in [0, M]\,|\psi(F_X(z)) = 0\} = 0$. Then, any solution h^* of problem 2.3 must satisfy $\int_0^M g_2\left(1-F_X(z)\right)h^*(z)dz = \pi_1$.*

Proof: By Corollary 3.1, if $\mu\{z \in [0, M]\,|\psi(F_X(z)) = 0\} = 0$, then the optimal solution of problem 2.4 is unique almost everywhere, and given by $\widetilde{h}$ as defined in (19), which satisfies

$$\int_0^M g_2\left(1-F_X(z)\right)\widetilde{h}(z)dz = \widetilde{\pi} > \pi_1. \tag{27}$$

We prove the desired result by a contradiction argument. Assume that an optimal solution h^* of problem 2.3 satisfies

$$\int_0^M g_2\left(1-F_X(z)\right)h^*(z)dz < \pi_1.$$

This, combined with (27), implies that there exists a constant $\beta \in (0,1)$ to satisfy

$$\int_0^M g_2\left(1-F_X(z)\right)\left(\beta h^*(z)+(1-\beta)\widetilde{h}(z)\right)dz=\pi_1.$$

We observe that $\beta h^*+(1-\beta)\widetilde{h}$ is a feasible solution of problem 2.3. Moreover, since $\widetilde{h}$ is the unique optimal solution of problem 2.4, $U\big(\beta h^*+(1-\beta)\widetilde{h}\big)=\beta U(h^*)+(1-\beta)U(\widetilde{h})>U(h^*)$, which contradicts to the optimality of h^*, and thus the proof is complete. □

To invoke Lemma 4.1 for optimal solutions of problem 2.3, we need to show that there indeed exists a constant $\lambda^*\geq 0$ for a function h_{λ^*} defined in (24) with some function κ_{λ^*} that satisfies equation (26). To this end, for $\lambda \geq 0$ and $a\in[h_0,h_1]$, we define function

$$h_{\lambda,a}(z):=h_1\mathbf{1}_{A_\lambda}(z)+a\mathbf{1}_{B_\lambda}(z)+h_0\mathbf{1}_{C_\lambda}(z),\ \ z\in[0,M]. \tag{28}$$

Obviously, $h_{\lambda,a}$ solves (22) for any constant $a\in[h_0,h_1]$. The existence of λ^* is summarized in part (a) of Theorem 4.1. Part (b) of the theorem is simply a restatement of the optimality of $\widetilde{h}$ as we have justified in the beginning of the section, and such a restatement is present for convenient development in section 5.

Theorem 4.1 *(a) Given any $\pi_1<\widetilde{\pi}$, there exists $\lambda^*\geq 0$ and $a^*\in[h_0,h_1]$ for h_{λ^*,a^*} to satisfies* (26) *and thus, it solves problem 2.3.*
(b) For $\pi_1\geq\widetilde{\pi}$, $h_{0,h_0}=h_1\mathbf{1}_{A_0}+h_0\mathbf{1}_{B_0\cup C_0}$ solves problem 2.3.

PROOF: See Appendix. □

Remark 4.2 *We note that the optimal MIF's h_{λ^*,a^*} in part (a) of Theorem 4.1 for the case of $\pi_1\leq\widetilde{\pi}$ satisfy* (26) *and thus, in view of* (13)*, the corresponding optimal reinsurance contracts must saturate the premium budget constraint in this case. In contrast, the optimal MIF obtained in part (b) of Theorem 4.1 for the case of $\pi_1>\widetilde{\pi}$ is given by h_{0,h_0}, which is identical to $\widetilde{h}$ given in* (19) *and thus, in view of* (21)*, the corresponding reinsurance premium is equal to $\widetilde{\pi}$. In this case, the obtained optimal reinsurance contracts do not saturate the premium budget constraint.*

In summary, Theorem (4.1) *implies that the optimal reinsurance contracts do not exhaust a large premium budget but they do saturate a small premium budget constraint. Such a result has been frequently observed in the literature (e.g., Tan, et al., 2011).*

Proposition 4.2 below can be used to verify the uniqueness of solution of problem 2.3. Some interesting sufficient conditions to satisfy (29) are given in Remarks 4.3 and 4.4.

Proposition 4.2 *Assume $\pi_1<\widetilde{\pi}$, and*

$$\mu(\{z\in[0,M]\,|\,\psi_\lambda(F_X(z))=0\})=0\ \ \textit{for any}\ \ \lambda\geq 0. \tag{29}$$

Then, the optimal solution to problem 2.3 is unique almost everywhere.

PROOF: Since $\mu(\{z \in [0, M] \,|\, \psi_\lambda(F_X(z)) = 0\}) = 0$ for any $\lambda \geq 0$, by Theorem 4.1, there exists $\lambda^* \geq 0$ such that the resulting $h_{\lambda^*}(z) = h_1 \mathbf{1}_{A_{\lambda^*}}(z) + h_0 \mathbf{1}_{C_{\lambda^*}}(z)$ satisfies $\int_0^M g_2(1 - F_X(z))\, h_{\lambda^*}(z)dz = \pi_1$ and h_{λ^*} solves problem 2.3. Let $\widehat{h}$ be another solution to problem 2.3, which differs from h_{λ^*} on a Lebesgue set B with $\mu(B) > 0$.

We recall from (25) that

$$A_{\lambda^*} = \{z \in [0, M] \,|\, \psi_{\lambda^*}(F_X(z)) > 0\}, \ \text{ and } \ C_{\lambda^*} = \{z \in [0, M] \,|\, \psi_{\lambda^*}(F_X(z)) < 0\}.$$

Then, obviously $\psi(F_X(z)) > \lambda^* g_1(1 - F_X(z))$ on A_{λ^*} and $\psi(F_X(z)) < \lambda^* g_1(1 - F_X(z))$ on C_{λ^*}. Moreover, since $h_{\lambda^*} = h_1 \mathbf{1}_{A_{\lambda^*}} + h_0 \mathbf{1}_{C_{\lambda^*}}$, we have $h_{\lambda^*} - \widehat{h} = h_1 - \widehat{h} \geq 0$ on A_{λ^*}, and $h_{\lambda^*} - \widehat{h} = h_0 - \widehat{h} \leq 0$ on C_{λ^*}. Consequently, if further noticing the condition of $\mu\{z \in [0, M] \,|\, \psi_{\lambda^*}(F_X(z)) = 0\} = 0$, we obtain

$$\begin{aligned}
&\int_0^M \psi(F_X(z)) h_{\lambda^*}(z)dz - \int_0^M \psi(F_X(z)) \widehat{h}(z)dz \\
=& \int_{A_{\lambda^*}} \psi(F_X(z)) \Big(h_{\lambda^*}(z) - \widehat{h}(s)\Big) dz + \int_{C_{\lambda^*}} \psi(F_X(z)) \Big(h_{\lambda^*}(z) - \widehat{h}(z)\Big) dz \\
=& \int_{A_{\lambda^*} \cap B} \psi(F_X(z)) \Big(h_{\lambda^*}(z) - \widehat{h}(z)\Big) dz + \int_{C_{\lambda^*} \cap B} \psi(F_X(z)) \Big(h_{\lambda^*}(z) - \widehat{h}(z)\Big) dz \\
>& \int_{A_{\lambda^*} \cap B} \lambda^* g_2(1 - F_X(z)) \Big(h_{\lambda^*}(z) - \widehat{h}(z)\Big) dz + \int_{C_{\lambda^*} \cap B} \lambda^* g_2(1 - F_X(z)) \Big(h_{\lambda^*}(z) - \widehat{h}(z)\Big) dz \\
=& \int_{A_{\lambda^*}} \lambda^* g_2(1 - F_X(z)) \Big(h_{\lambda^*}(z) - \widehat{h}(z)\Big) dz + \int_{C_{\lambda^*}} \lambda^* g_2(1 - F_X(z)) \Big(h_{\lambda^*}(z) - \widehat{h}(z)\Big) dz \\
=& \int_0^M \lambda^* g_2(1 - F_X(z)) \Big(h_{\lambda^*}(z) - \widehat{h}(z)\Big) dz \\
=& 0,
\end{aligned}$$

which contradicts to the optimality of h_{λ^*} and thus, the solution to problem 2.3 is unique almost everywhere. □

Remark 4.3 *In this remark, we make some comments on the condition* (29)*. Note that, from* (23)*,*

$$\psi_\lambda(F_X(z)) = g_1(1 - F_X(z)) - (1 + \theta + \lambda) g_2(1 - F_X(z)), \ \ z \in [0, M].$$

We assume that $F_X(x)$ is strictly increasing. Then, a sufficient condition for (29) *is given by*

$$\mu\{t \in [0, 1] \,|\, g_1(t) = c \cdot g_2(t)\} = 0, \forall \ c \geq 1 + \theta. \tag{30}$$

Define

$$w(t) \triangleq \frac{g_1(t)}{g_2(t)}, \ \ t \in (c, 1], \ \ \text{where } \ c \triangleq \sup\{z \in [0, 1] \,|\, g_2(z) = 0\}. \tag{31}$$

Then, a further sufficient condition for (30) *in terms of $w(t)$ is given as below:*

(a) If $c = 0$ and $(0,1]$ has a finite partition with $(0,1] = \cup_{i=1}^{\infty}(\alpha_i, \alpha_{i+1}]$ such that $w(t)$ is either strictly increasing or strictly decreasing on each interval $(\alpha_i, \alpha_{i+1}]$.

(b) If $c > 0$, $g_1(t) > 0$ for $t > 0$ and $(c,1]$ has a finite partition with $(c,1] = \cup_{i=1}^{\infty}(\beta_i, \beta_{i+1}]$ such that $w(t)$ is either strictly increasing or strictly decreasing on each interval $(\beta_i, \beta_{i+1}]$.

The analysis of optimal solutions in section 5 critically depends on $w(1-t)$ as a function of t; see the function k defined in (33).

Remark 4.4 *In this remark, we make further comments on the condition (29) for $\rho^{g_1} = \mathrm{VaR}_\alpha$ and $\rho^{g_1} = \mathrm{CVaR}_\alpha$, respectively, in the objective of problem 2.1, where $\alpha \in (0,1)$. We still assume that $F_X(x)$ is strictly increasing so that (30) is a sufficient for (29). Then, we have the following observations:*

(a) If $\rho^{g_1} = \mathrm{VaR}_\alpha$, then $g_1(t) = \mathbf{1}_{(1-\alpha,\,1]}(t)$ and thus, (30) holds for any $g_2 \in \mathcal{G}$ with

$$g_2(t) > 0 \ \text{for} \ \ t \in (0, 1-\alpha] \ \text{ and } \ \mu\{t \in (1-\alpha, 1]\,|g_2(t) = a\} = 0 \ \text{ for any } \ a \in \left(0, \ \frac{1}{1+\theta}\right].$$

(b) If $\rho^{g_1} = \mathrm{CVaR}_\alpha$, then $g_1(t) = \min\{1, \frac{t}{1-\alpha}\}$. Therefore, (30) is satisfied by any $g_2 \in \mathcal{G}$ with

$$\mu\{t \in [0, 1-\alpha]\,|g_2(t) = at\} = 0 \ \text{ for any } \ a \in \left(0, \ \frac{1}{(1-\alpha)(1+\theta)}\right],$$

and $\mu\{t \in (1-\alpha, 1]\,|g_2(t) = a\} = 0$ for any $a \in \left(0, \ \frac{1}{1+\theta}\right]$.

To complete this section, we conclude the optimal ceded loss functions for problem 2.1 by invoking Proposition 2.1. Let $\lambda^* \geq 0$ and $a^* \in [h_0, h_1]$ be the two constants in Theorem 4.1 such that h_{λ^*,a^*} solves problem 2.3. For $x \in [0, M]$, we denote $A_{\lambda^*,x} = \{z \in [0,x] : \psi_{\lambda^*}(F_X(z)) > 0\}$, $B_{\lambda^*,x} = \{z \in [0,x] : \psi_{\lambda^*}(F_X(z)) = 0\}$ and $C_{\lambda^*,x} = \{z \in [0,x] : \psi_{\lambda^*}(F_X(z)) < 0\}$. Then, combining Proposition 2.1 and Theorem 4.1, we obtain optimal ceded loss function for problem 2.1 with a reinsurance budget of $\pi_1 < \widetilde{\pi}$ as follows:

$$f_{\lambda^*,a^*}(x) = \int_0^x h_{\lambda^*,a^*}(z)dz = h_1\mu\left(A_{\lambda^*,x}\right) + a^*\mu\left(B_{\lambda^*,x}\right) + h_0\mu\left(C_{\lambda^*,x}\right), \ \ x \in [0, M].$$

5 CVaR Minimization with ISSD Premium Principle

In this section, we apply the results from the preceding section to study optimal reinsurance treaties where the distortion risk measure is CVaR and the reinsurance premium is computed by the Inverse-S Shaped Distortion (ISSD) principle. The ISSD premium principle is a special distortion premium principle with an inverse-S shaped distortion function. It has interesting economic meaning in pricing insurance contracts. It can be seen shortly that the optimal solutions under general concave distortion premium principle can be easily retrieved from those obtained under the ISSD premium principle.

We will focus on the derivation of the optimal MIF to problem 2.3, because the optimal indemnification functions to problem 2.1 can be easily obtained by an integration once we derive the optimal MIF. Moreover, the MIF can equally tell us the exact shape of the optimal indemnification function if not better.

5.1 ISSD Premium Principle

The ISSD premium principle is defined as $\pi^g = (1+\theta)\rho^g$ with a loading factor θ and an ISSD distortion function satisfying those conditions in Definition 5.1 below.

Definition 5.1 *(ISSD Function) A distortion function g is called an ISSD function, if and only if it satisfies the following conditions:*

(1) It is a continuous and strictly increasing mapping from [0,1] onto [0,1] and twice differentiable in the interior;

(2) There exists $b \in (0,1)$ such that $g'(\cdot)$ is strictly decreasing on $(0,b)$ and strictly increasing on $(b,1)$;

(3) $g'(0) \triangleq \lim_{x\downarrow 0} g'(x) > 1$ and $g'(1) \triangleq \lim_{x\uparrow 1} g'(x) > 1$.

As proposed by Tversky and Kahneman (1992), a plausible (and popular) ISSD function is of the form

$$g_\gamma(x) = \frac{x^\gamma}{(x^\gamma + (1-x)^\gamma)^{\frac{1}{\gamma}}},$$

where γ is a parameter. Figure 1 displays this inverse-S shaped distortion function for $\gamma = 0.5$. As noted by Rieger and Wang (2006) and Ingersoll (2008), this probability distortion function is increasing and inverse-S shaped for any $\gamma \in (0.279, 1)$.

To understand the economic meaning of an ISSD premium principle, we apply a simple transform to derive

$$\pi^g(Y) = \int_0^\infty s g'(1 - F_Y(z)) dF_Y(z). \tag{32}$$

When g is concave, it indicates that the insurance provider puts more weights on great losses (bad outcomes) than small ones (good ones) in pricing risks. In contrast, when g is SSID, $g'(0) > 1$ and $g'(1) > 1$ and thus, the insurance provider overweights not only large losses but also small ones, which is consistent with the empirically observed phenomena in psychological experiments (Quiggin 1982,1992; Tversky and Kahneman 1992; Tversky and Fox 1995; Gonzalez and Wu, 1999).

Moreover, Kaluszka and Krzeszowiec (2012) introduce a premium principle which relies on Cumulative Prospect Theory (CPT). The ISSD premium principle can be viewed as a special CPT

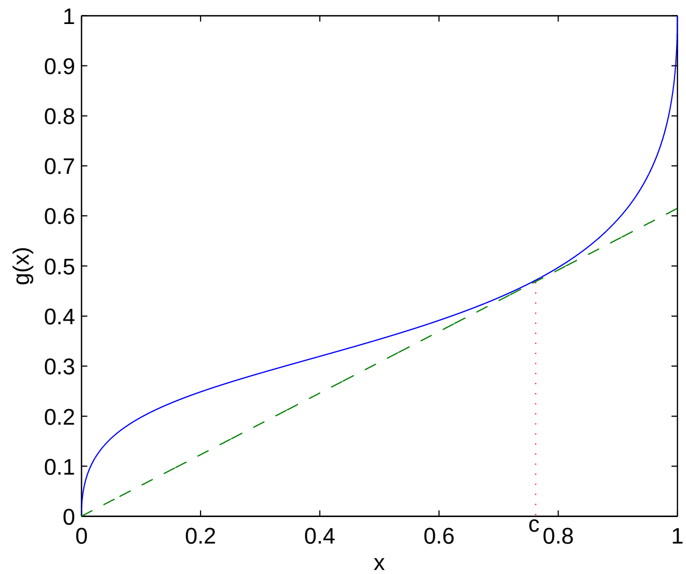

Figure 1: This is an inverse-S shaped probability distortion function which satisfies Definition 5.1. The point c is explained in Lemma 5.1.

premium principle corresponding to a linear utility function and a zero reference point. Numerical estimates of the probability distortion function are studied in Abdellaoui (2000) and Wu and Gonzalez (1996).

By introducing $\eta(x;g) \triangleq g(x)/x$ on $x \in (0,1]$ for a given ISSD distortion function, its property, which is stated in Lemma 5.1 below, will be useful in our subsequent discussion.

Lemma 5.1 *If g is an ISSD function, then there exists a unique point $c \in [b,1]$ such that $\eta(\cdot;g)$ is strictly decreasing from 0 to c and strictly increasing from c to 1.*

PROOF: Firstly, $\eta(\cdot;g)$ is a continuous function on $(0,1]$. Moreover, $\eta(\cdot;g)$ is strictly decreasing on $(0,b]$ since $g'(\cdot)$ is strictly decreasing (or equivalently $g(\cdot)$ is strictly convex) on $(0,b]$. Then, we take the derivative of $\eta(x;g)$ with respect to x on $(b,1)$ to get $\eta'(x;g) = \frac{g'(x)x-g(x)}{x^2}$. So, we only need to know the sign of $m(x) \triangleq g'(x)x - g(x)$. Now, $m'(x) = g''(x)x > 0$ for $x \in (b,1)$ which means that $m(x)$ is a strictly increasing function on $x \in (b,1)$. Thus, the desired result follows. □

From Definition 5.1, an ISSD distortion function becomes a concave distortion function as $b \to 1$, and in this case, the point c at which $\eta'(x;g)$ changes its sign as given in Lemma 5.1 is equal to 1. As we can see shortly, the optimal solutions under the CVaR_α and an ISSD premium principle depend on such point c, and accordingly, the optimal solutions for a concave distortion premium principle can be obtained by replacing c with 1 in those solutions obtained for an ISSD principle below.

5.2 Optimal Solutions for CVaR Minimization

For presentation convenience, we assume that F_X is strictly increasing in the present section, unless otherwise stated. It is well known that CVaR_α is a special distortion risk measure with a distortion function $g_1(s) = \min\{\frac{s}{1-\alpha}, 1\}$ so that

$$g_1(1-s) = \begin{cases} 1, & \text{if } 0 \leq s \leq \alpha, \\ \frac{1-s}{1-\alpha}, & \text{if } \alpha < s \leq 1. \end{cases}$$

Assume that g_2 in the reinsurance premium principle π^{g_2} is an ISSD function as defined in Definition 5.1. By denoting

$$k(t) \triangleq \frac{g_1(1-t)}{g_2(1-t)} = \begin{cases} \frac{1}{g_2(1-t)}, & \text{if } 0 \leq t \leq \alpha, \\ \frac{1-t}{(1-\alpha)g_2(1-t)}, & \text{if } \alpha < t < 1, \end{cases} \tag{33}$$

and $k(1) = \frac{1}{(1-\alpha)g_2'(0^+)}$, the function ψ defined in (15) is then given by

$$\psi(t) = g_2(1-t)\Big(k(t) - (1+\theta)\Big), \;\; t \in [0,1].$$

Since $F_X(z)$ is strictly increasing, the set $\{z \in [0, M] \,|\, \psi(F_X(z)) = 0\}$ has a zero Lebesgue measure, and thus, according to (18), the optimal MIF to the unconstrained problem 2.4 is given by $\widetilde{h}(z) = h_1 \mathbf{1}_{\{\psi(F_X(z))>0\}} + h_0 \mathbf{1}_{\{\psi(F_X(z))\leq 0\}}$. Moreover, the function ψ_λ defined in (23) is given by

$$\psi_\lambda(t) = g_2(1-t)\Big(k(t) - (1+\theta+\lambda)\Big), \;\; t \in [0,1]. \tag{34}$$

We further denote

$$\widehat{\alpha} = \min\{1-\alpha, c\} \;\text{ and }\; s^* = \mathrm{VaR}_{1-\widehat{\alpha}}(X).$$

Lemmas 5.2 and 5.3 state, respectively, how functions $k(t)$ and $k(F_X(z))$ change their increasing/decreasing patten at the point of $1-\widehat{\alpha}$ and s^*. These results are useful in analyzing the optimal reinsurance contracts.

Lemma 5.2 *The function k is strictly increasing on $[0, 1-\widehat{\alpha}]$ and strictly decreasing on $[1-\widehat{\alpha}, 1]$.*

PROOF: First, we consider the case of $1-\alpha \leq c$ so that $\widehat{\alpha} = 1-\alpha$. Since $g_2(1-t)$ is strictly decreasing on $[0,1]$, k is strictly increasing on $[0,\alpha]$. Moreover, from Lemma 5.1, $\frac{g_2(1-t)}{1-t}$ is strictly increasing on $[\alpha, 1]$ as $1-\alpha \leq c$, and hence k is strictly decreasing on $[\alpha, 1]$. Second, we assume $1-\alpha > c$ so that $\widehat{\alpha} = c$ and $\alpha < 1-c$. Since g_2 is strictly increasing, $k(t) = 1/g_2(1-t)$ is strictly increasing on $[0,\alpha]$. Moreover, from Lemma 5.1, $\frac{1}{k(t)} = (1-\alpha)\frac{g_2(1-s)}{1-s}$ is strictly decreasing on $[\alpha, 1-c]$. Thus, k is strictly increasing on $[0, 1-c]$. Further, from Lemma 5.1, $\frac{g_2(1-s)}{1-s}$ is strictly increasing on $[1-c, 1]$ and thus k is strictly decreasing on $[1-c, 1]$. $\square$

To proceed, recall that $M \triangleq \mathrm{esssup} X$. Since $F_X(s)$ is strictly increasing and $g_2(s)$ is continuous, Lemma 5.2 and equation (34) imply that the set $\mu(\{z \in [0, M] \,|\, \psi_\lambda(F_X(z)) = 0\}) = 0$ for any $\lambda \geq 0$. Therefore, according Proposition 4.2, the solution to problem 2.3 is unique almost everywhere.

Lemma 5.3 *The function $k(F_X(s))$ is strictly increasing on $s \in [0, s^*)$ and $k(F_X(s))$ is strictly decreasing on $s \in [s^*, M]$.*

PROOF: For $s < s^* = \mathrm{VaR}_{1-\widehat{\alpha}}(X)$, we have $F_X(s) < 1-\hat{\alpha}$. Thus, by Lemma 5.2 and the assumption that $F_X(\cdot)$ is strictly increasing, we conclude that $k(F_X(s))$ is strictly increasing on $s \in [0, s^*)$. For $s > s^*$, we get $F_X(s) > F_X(s^*) \geq 1 - \hat{\alpha}$, and hence, $k(F_X(s))$ is strictly decreasing on $s \in (s^*, M]$. □

Generally, the maximum of $k(F_X(s))$ may not be attainable, which occurs when $\lim_{s\uparrow s^*} k(F_X(s)) > k(F_X(s^*))$. In view of Lemma 5.3, let

$$\widetilde{k} = \max\left\{ \lim_{s\uparrow s^*} k(F_X(s)), k(F_X(s^*))\right\}.$$

To derive the optimal solutions to our reinsurance model, the analysis in section 4 indicates that we need to consider the set $\{z \in [0, M] \,|\, \psi_\lambda(F_X(z)) > 0\}$ for an optimal MIF while (34) additionally suggests that we need to take into account the value of $k(F_X(z))$. Together with Lemma 5.3, this implies that it is necessary to determine the optimal reinsurance on a case-by-case basis depending on the relative magnitude of the following four critical values:

$$\widetilde{k}, \;\; k_0 \triangleq k(F_X(0)), \;\; k_M \triangleq k(F_X(M)), \;\; 1 + \theta.$$

Figure 2 enumerates all the cases that we need to consider. For each of these cases, the corresponding proposition that gives the optimal solution is also provided.

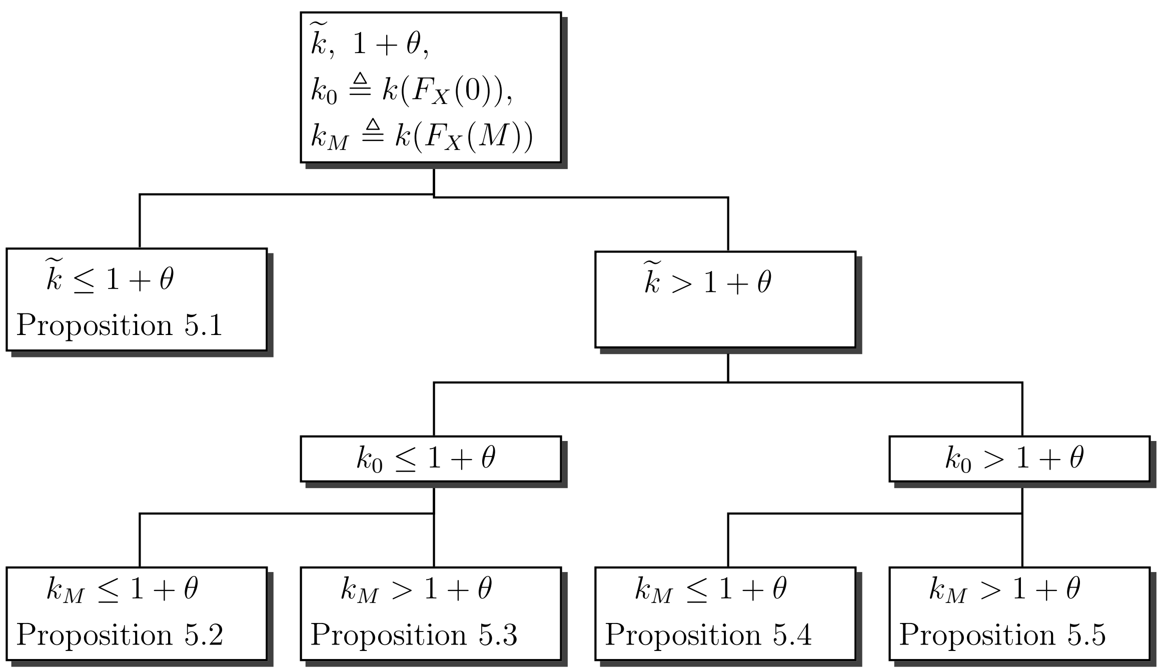

Figure 2: Possible cases for the optimal reinsurance

We now analyze the optimal solution on the first case.

Proposition 5.1 *If $\widetilde{k} \leq 1+\theta$, an optimal solution to problem 2.3 is given by $h^*(z) = h_0$ for $z \in [0, M]$.*

PROOF: When $\widetilde{k} \leq 1+\theta$, $\psi_\lambda(t) \leq 0$, $\forall\ t \in [0,1]$, and $A_\lambda = \emptyset$, for any $\lambda \geq 0$. The result therefore follows from Theorem 4.1. □

For the case of $\widetilde{k} > 1+\theta$, we need to discuss a few further subcases based on the relative magnitude of k_0 and k_M compared to $1+\theta$. If $k_0 \leq 1+\theta$ and $k_M \leq 1+\theta$, then there exists $p \in [0, \mathrm{VaR}_{1-\hat{\alpha}}(X)]$ and $q \in [\mathrm{VaR}_{1-\hat{\alpha}}(X), M]$ such that $k(F_X(s)) \leq 1+\theta$ for $s \in [0,p) \bigcup [q, M]$ and $k(F_X(s)) \geq 1+\theta$ for $s \in [p, q)$.

Proposition 5.2 *Assume that $\widetilde{k} > 1+\theta$, $k_0 \leq 1+\theta$ and $k_M \leq 1+\theta$ hold.*
(i) If $\pi_1 \geq \widetilde{\pi}$, one optimal solution to problem 2.3 is given by

$$h^*(z) = \begin{cases} h_0, & \text{if } 0 < z < p, \\ h_1, & \text{if } p < z < q, \\ h_0, & \text{if } q < z < M. \end{cases}$$

(ii) If $\pi_1 < \widetilde{\pi}$, the optimal solution to problem 2.3 is given by

$$h^*(z) = \begin{cases} h_0, & \text{if } 0 < z < l, \\ h_1, & \text{if } l < z < n, \\ h_0, & \text{if } n < z < M, \end{cases}$$

where $l \in [p, \mathrm{VaR}_{1-\hat{\alpha}}(X)]$, $n \in [\mathrm{VaR}_{1-\hat{\alpha}}(X), M]$ and β are such that $k(F_X(s)) \leq \beta$ for $s \in [0,l) \bigcup [n, M]$, $k(F_X(s)) \geq \beta$ for $s \in [l, n)$ and $h_0 \int_0^l g_2(1-F_X(z))dz + h_1 \int_l^n g_2(1-F_X(z))dz + h_0 \int_n^M g_2(1-F_X(z))dz = \pi_1$.

PROOF: The proof of part (i) follows from Theorem 4.1 (b), where $A_0 = (p, q)$. For the part (ii), the existence of l, n and β follows from Lemma 5.3. Denote by $\lambda^* = \beta - 1 - \theta$, it is easy to show that h^* in part (ii) satisfies (24) with $\lambda = \lambda^*$. The residual result follows easily from Theorem 4.1 (a). □

In the case of $\widetilde{k} > 1+\theta$ and $k_M > 1+\theta$, there exists $p \in [0, \mathrm{VaR}_{1-\hat{\alpha}}(X)]$ such that $k(F_X(s)) \leq 1+\theta$ for $s \in [0,p)$, and $k(F_X(s)) \geq 1+\theta$ for $s \in [p, M]$ and there exists $q \in [0, \mathrm{VaR}_{\hat{\alpha}}(X)]$ such that $k(F_X(s)) \leq k_M$ for $s \in [0,q)$ and $k(F_X(s)) \geq k_M$ for $s \in [q, M]$. Let us denote

$$\hat{\pi} = h_0 \int_0^q g_2(1-F_X(z))dz + h_1 \int_q^M g_2(1-F_X(z))dz.$$

Proposition 5.3 *Assume that $\widetilde{k} > 1+\theta$, $k_0 \le 1+\theta$ and $k_M > 1+\theta$ hold.*
(i) If $\widetilde{\pi} \le \pi_1$, then the optimal solution to problem 2.3 is $h^* = \widetilde{h}$, which is given as

$$h^*(z) = \begin{cases} h_0, & \text{if } 0 < z < p, \\ h_1, & \text{if } p < z < M. \end{cases}$$

(ii) If $\hat{\pi} \le \pi_1 < \widetilde{\pi}$, then the optimal solution to problem 2.3 is

$$h^*(z) = \begin{cases} h_0, & \text{if } 0 < z < d, \\ h_1, & \text{if } d < z < M, \end{cases}$$

where d is such that $h_0 \int_0^d g_2(1-F_X(z))dz + h_1 \int_d^M g_2(1-F_X(z))dz = \pi_1$.
(iii) If $\pi_1 < \hat{\pi}$, then the optimal solution to problem 2.3 is

$$h^*(z) = \begin{cases} h_0, & \text{if } 0 < z < l, \\ h_1, & \text{if } l < z < n, \\ h_0, & \text{if } n < z < M, \end{cases}$$

where $l \in [p, \mathrm{VaR}_{1-\hat{\alpha}}(X)]$, $n \in [\mathrm{VaR}_{1-\hat{\alpha}}(X), M]$ and β are such that $k(F_X(s)) \le \beta$ for $s \in [0,l) \bigcup [n, M]$, $k(F_X(s)) \ge \beta$ for $s \in [l, n)$ and $h_0 \int_0^l g_2(1-F_X(z))dz + h_1 \int_l^n g_2(1-F_X(z))dz + h_0 \int_n^M g_2(1-F_X(z))dz = \pi_1$.

Proof: The proof is similar to that of Proposition 5.2 and hence is omitted. Noted that λ^* in parts (ii) and (iii) are $k(F_X(d)) - 1 - \theta$ and $\beta - 1 - \theta$ respectively. □

Remark 5.1 *Proposition 5.3 confirms that when the budget is not sufficiently high enough, the optimal reinsurance policy will change from the stop-loss contract to a one layer contract, i.e., a contract of stop-loss with an upper limit.*

If $k_0 > 1+\theta$ and $k_M \le 1+\theta$, then there exists $p \in [VaR_{1-\hat{\alpha}}(X), M]$ such that $k(F_X(s)) \ge 1+\theta$ for $s \in [0, q)$ and $k(F_X(s)) \le 1+\theta$ for $s \in [q, M]$ and there exists $q \in [VaR_{1-\hat{\alpha}}(X), M]$ such that $k(F_X(s)) \ge k_0$ for $s \in [0, q)$ and $k(F_X(s)) \le k_0$ for $s \in [q, M]$. By denoting $\hat{\pi} = h_1 \int_0^q g_2(1-F_X(z))dz + h_0 \int_q^M g_2(1-F_X(z))dz$, we have the following proposition.

Proposition 5.4 *Assume that $\widetilde{k} > 1+\theta$, $k_0 > 1+\theta$ and $k_M \le 1+\theta$ hold.*
(i) If $\widetilde{\pi} \le \pi_1$, then the optimal solution to problem 2.3 is $h^* = \widetilde{h}$, which is given as

$$h^*(z) = \begin{cases} h_1, & \text{if } 0 < z < p, \\ h_0, & \text{if } p < z < M. \end{cases}$$

(ii) If $\hat{\pi} \le \pi_1 < \widetilde{\pi}$, then the optimal solution to problem 2.3 is

$$h^*(z) = \begin{cases} h_1, & \text{if } 0 < z < d, \\ h_0, & \text{if } d < z < M, \end{cases}$$

where d is such that $h_1 \int_0^d g_2(1 - F_X(z))dz + h_0 \int_d^M g_2(1 - F_X(z))dz = \pi_1$.
(ii) If $\pi_1 < \hat{\pi}$, then the optimal solution to problem 2.3 is

$$h^*(z) = \begin{cases} h_0, & \text{if } 0 < z < l, \\ h_1, & \text{if } l < z < n, \\ h_0, & \text{if } n < z < M, \end{cases}$$

where $l \in [p, \mathrm{VaR}_{1-\hat{\alpha}}(X)]$, $n \in [\mathrm{VaR}_{1-\hat{\alpha}}(X), M]$ and β are such that $k(F_X(s)) \le \beta$ for $s \in [0, l) \bigcup [n, M]$, $k(F_X(s)) \ge \beta$ for $s \in [l, n)$ and $h_0 \int_0^l g_2(1 - F_X(z))dz + h_1 \int_l^n g_2(1 - F_X(z))dz + h_0 \int_n^M g_2(1 - F_X(z))dz = \pi_1$.

Proof: The proof is the same as that of Proposition 5.3 and hence is omitted. □

If $k_0 > 1 + \theta$ and $k_M > 1 + \theta$, without loss of generality, we assume that $k_0 \le k_M$. Then, then there exists $p \in [0, VaR_{1-\hat{\alpha}}(X)]$ such that $k(F_X(s)) \le k_M$ for $s \in [0, p)$ and $k(F_X(s)) \ge k_M$ for $s \in [p, M]$. By setting $\hat{\pi} = h_0 \int_0^p g_2(1 - F_X(z))dz + h_1 \int_p^M g_2(1 - F_X(z))dz$, we obtain the following proposition, where the proof is the same as above.

Proposition 5.5 *Assume that $\widetilde{k} > 1 + \theta$, $k_0 > 1 + \theta$ and $k_M > 1 + \theta$ hold.*
(i) If $\pi_1 = \mathbb{E}[X]$, then the optimal solution to problem 2.3 is $h^*(z) = 1$ for $z \in [0, M]$.
(ii) If $\hat{\pi} \le \pi_1 < \mathbb{E}[X]$, then the optimal solution to problem 2.3 is

$$h^*(z) = \begin{cases} h_0, & \text{if } 0 < z < d, \\ h_1, & \text{if } d < z < M, \end{cases}$$

where d is such that $h_0 \int_0^d g_2(1 - F_X(z))dz + h_1 \int_d^M g_2(1 - F_X(z))dz = \pi_1$.
(ii) If $\pi_1 < \hat{\pi}$, then the optimal solution to problem 2.3 is

$$h^*(z) = \begin{cases} h_0, & \text{if } 0 < z < l, \\ h_1, & \text{if } l < z < n, \\ h_0, & \text{if } n < z < M, \end{cases}$$

where $l \in [p, \mathrm{VaR}_{1-\hat{\alpha}}(X)]$, $n \in [\mathrm{VaR}_{1-\hat{\alpha}}(X), M]$ and β are such that $k(F_X(s)) \le \beta$ for $s \in [0, l) \bigcup [n, M]$, $k(F_X(s)) \ge \beta$ for $s \in [l, n)$ and $h_0 \int_0^l g_2(1 - F_X(z))dz + h_1 \int_l^n g_2(1 - F_X(z))dz + h_0 \int_n^M g_2(1 - F_X(z))dz = \pi_1$.

6 Conclusion

The quest for optimal reinsurance has remained a fascinating subject to academicians and practitioners. Over the last few decades many reinsurance models were proposed to provide better guidance for insurers to manage their risk. This paper contributed to the literature by providing better approach in solving sophisticated optimal reinsurance model whereby the distorted preference of an insurer's total risk exposure is minimized under the general distortion risk measure while subject to a budget constraint on premium. To eliminate the moral hazard which potentially occur on the resulting reinsurance contracts, general lower and upper bounds are imposed on the derivative of admissible ceded loss functions. Such optimal reinsurance is quite general. For example, the distortion risk measure includes VaR, CVaR and spectral risk measure as special cases while the distortion premium principle includes the expected value principle and Wang's premium principle as special cases. We solved this problem explicitly by using marginal indemnification function formulation in conjunction with the method of Lagrange. Compared to the existing method, our proposed method has the advantages of simplicity and transparent. As another added advantage of our method, the uniqueness of the optimal reinsurance policy can also be analyzed. By resorting to a well-specified optimal reinsurance model with CVaR as the risk measure and the inverse-S shaped distortion function as the premium principle, the proposed method was used to derive explicitly the optimal solutions.

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Appendix

Proof of Proposition 4.1. (a) Let $\phi(\lambda, a) \triangleq \int_0^M g_2\left(1 - F_X(z)\right) h_{\lambda,a}(z)dz$, $\lambda \geq 0$ and $a \in [h_0, h_1]$, where $h_{\lambda,a}$ is given in (28). We denote $C_p = \{z \in [0, M] \,|g_2(1 - F_X(z)) > 0\}$ and $C_0 = \{z \in [0, M] \,|g_2(1 - F_X(z)) = 0\}$. It is easy to check

$$\lim_{\lambda \to \infty} \mu(A_\lambda \cap C_p) = 0 \text{ and } \lim_{\lambda \to \infty} \mu(B_\lambda \cap C_p) = 0.$$

Further note that $g_2(1 - F_X(z)) = 0$ on C_0. Thus,

$\phi(\lambda, a)$

$$= h_1 \int_{A_\lambda} g_2(1-F_X(z))dz + a \int_{B_\lambda} g_2(1-F_X(z))dz + h_0 \int_{C_\lambda} g_2(1-F_X(z))dz$$
$$= (h_1 - h_0) \int_{A_\lambda} g_2(1-F_X(z))dz + (a-h_0) \int_{B_\lambda} g_2(1-F_X(z))dz + h_0 \int_0^M g_2(1-F_X(z))dz$$
$$= (h_1 - h_0) \int_{A_\lambda \cap C_p} g_2(1-F_X(z))dz + (a-h_0) \int_{B_\lambda \cap C_p} g_2(1-F_X(z))dz + h_0 \int_0^M g_2(1-F_X(z))dz$$

Combining the above two displays, we get $\lim_{\lambda\to\infty} \phi(\lambda, a) = h_0 \int_0^M g_2(1-F_X(z))dz$ for any $a \in [h_0, h_1]$. Moreover, for $\lambda = 0$ and $a = h_0$, it follows that $h_{\lambda,a} = \widetilde{h}$, which means $\phi(0, h_0) = \widetilde{\pi}$.

It is easy to check

$$\lim_{\gamma\to\lambda^+} \{z \in [0, M] : \psi_\gamma(F_X(z)) > 0\} = \{z \in [0, M] : \psi_\lambda(F_X(z)) > 0\},$$

and

$$\lim_{\gamma\to\lambda^-} \{z \in [0, M] : \psi_\gamma(F_X(z)) \geq 0\} = \{z \in [0, M] : \psi_\lambda(F_X(z)) \geq 0\}.$$

Thus, we apply the Dominated Convergence Theorem to obtain

$$\lim_{\gamma\to\lambda^+} \phi(\gamma, h_0) = \phi(\lambda, h_0), \text{ and } \lim_{\gamma\to\lambda^-} \phi(\gamma, h_1) = \phi(\lambda, h_1).$$

Define $\lambda^* = \sup S_{\pi_1}$, where

$$S_{\pi_1} = \{\lambda \geq 0 : \phi(\lambda, h_1) \geq \pi_1\}.$$

On one hand, given $\epsilon > 0$, by the supremum property of λ^*, there exists some $\lambda \in (\lambda^* - \epsilon, \lambda^*]$ and $\lambda \in S_{\pi_1}$ such that

$$\phi(\lambda^* - \epsilon, h_1) \geq \phi(\lambda, h_1) \geq \pi_1,$$

where the first inequality is due to the non-increasing property of $\phi(\lambda, a)$ as a function of λ. On the other hand, the superium property of λ^* implies $\lambda^* + \epsilon \notin S_\pi$ and thus,

$$\phi(\lambda^* + \epsilon, h_0) \leq \phi(\lambda^* + \epsilon, h_1) < \pi_1.$$

Letting $\epsilon \to 0$ in the last two displays yields

$$\phi(\lambda^*, h_0) \leq \pi_1 \leq \phi(\lambda^*, h_1). \tag{A.1}$$

Moreover, it is clear from the definition of $\phi(\lambda, a)$ and $h_{\lambda,a}$ that $\phi(\lambda^*, a)$ is a continuous function of a. Thus, it follows form (A.1) and the Intermediate Value Theorem, there exists some $a^* \in [h_0, h_1]$ such that $\phi(\lambda^*, a^*) = \pi_1$, as desired.

(b) The result has been clearly shown at the beginning of section 4. □

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