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MEASUREMENT OF THE STATE VECTOR

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SUMMARY

Methods are described which may be used to transform the equations of motion of a vehicle to a standard form, measure the state of the system when described in this standard form, and remove first flexure mode effects from pitch rate signals. The methods are described by the use of illustrative examples.

INTRODUCTION

In state vector control of linear systems it is assumed that the differential equations of motion for the plant can be written as n linear first-order equations where n is the order of the system. If time-optimal or average power optimal control is to be used, each of the n state variables must actually be measured.

When the plant is an aerial or space vehicle, the differential equations of motion that serve as the starting point in the control synthesis problem are neither first order nor written in terms of state variables that can be measured with existing sensors. Because of this, it is necessary to apply certain transformations to the original equations. A procedure will be demonstrated on the pitch equations of motion for a flexible aerial vehicle. It will be assumed that gyros are to be used as the primary measuring devices. Accelerometers or strain gages could be used as well.

There are three transformations to be considered:

- A transformation to put the equations of motion in a standard form involving no derivatives higher than first
- A transformation to permit measurement of the body bending displacements
- A transformation to permit the measurement of angle of attack without an external probe

Then a scheme is described which may be used to obtain a measurement of a vehicle pitch rate that is free from the effects of the first flexure mode. This scheme involves blending the signals from two rate gyros located fore and aft of the first bending antinode of the vehicle.

The method for measuring the state vector is from an internal report of the Minneapolis-Honeywell Regulator Company (ref. 1). The procedure for obtaining a pitch rate signal free from first mode flexure effects is taken from ref. 2.

SYMBOLS

CG	Center of gravity
M	Pitching moment (positive up)
U	Forward speed
X	A force along the x axis (positive forward)
Z	A force along the z axis (positive down)
g	Gravitational constant
$q = \dot{\theta}$	Pitch rate
w	Velocity along the z axis (positive down)
x	Distance along the longitudinal axis (positive forward)
$z_{(\ )}$	Deflection of the ( ) body bending mode
$\triangleq$	By definition
α	Angle of attack
δe	Elevator deflection
δT	Throttle deflection
θ	Pitch angle (positive nose up)

Subscripts

δe, δT, U, α, $\dot{\alpha}$, q, w, z_i, $\dot{z}_i$, $\ddot{z}_i$, $\dot{\theta}$ indicate partial derivative.

STANDARD FORM FOR THE EQUATIONS OF MOTION

As stated before, the differential equations describing a linear plant which is being forced by r forcing functions can always be written as an equivalent set of n linear first-order differential equations of the form

$$\frac{dx_1}{dt} = \dot{x}_1 = a_{11}(t)x_1 + a_{12}(t)x_2 + \ldots + a_{1n}(t)x_n + b_{11}(t)u_1 + \ldots + b_{1r}(t)u_r$$

$$\frac{dx_2}{dt} = \dot{x}_2 = a_{21}(t)x_1 + a_{22}(t)x_2 + \ldots + a_{2n}(t)x_n + b_{21}(t)u_1 + \ldots + b_{2r}(t)u_r$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\frac{dx_n}{dt} = \dot{x}_n = a_{n1}(t)x_1 + a_{n2}(t)x_2 + \ldots + a_{nn}(t)x_n + b_{n1}(t)u_1 + \ldots + b_{nr}(t)u_r \tag{1}$$

The x_i's are referred to as the state variables, and the transformation given by equation (1) is called the point transformation. When the system equations are written in this form they will be said to be in standard form. This point transformation can be written as a vector differential equation of the form

$$\frac{dx}{dt} = Ax + Bu \tag{2}$$

To demonstrate how a set of equations is put in the form of equation (1) two simple examples are presented. Besides being useful in themselves, the two examples should be sufficient to demonstrate the general procedure and provide motivation for treatment of the aeroelastic vehicle which is presented later.

A Single-Degree-of-Freedom System

The differential equation of motion for a fourth-order single-degree-of-freedom system can be written as

$$\ddot{\ddot{x}} + a_3\dddot{x} + a_2\ddot{x} + a_1\dot{x} + a_o x = bu \tag{3}$$

If

$$\begin{aligned} x_1 &\triangleq x \\ x_2 &\triangleq \dot{x}_1 \\ x_3 &\triangleq \dot{x}_2 \\ x_4 &\triangleq \dot{x}_3 \end{aligned} \tag{4}$$

equation (3) can be written as a system of first-order equations:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a_o & -a_1 & -a_2 & -a_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ b \end{bmatrix} u \tag{5}$$

When written as a vector differential equation, equation (5) becomes

$$\dot{x} = Ax + Bu \tag{6}$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a_o & -a_1 & -a_2 & -a_3 \end{bmatrix}; \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ b \end{bmatrix}; \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

Three-Degree-of-Freedom Equations for Longitudinal Motion of a Rigid Aircraft

More effort is generally required to reduce multi-degree-of-freedom systems to the standard form than is required for single-degree-of-freedom systems. The pitch equations of motion for a rigid aircraft illustrate the difficulty without requiring matrix manipulations.

The equations of motion are

$$\dot{U} - X_U U - X_\alpha \alpha + g\theta - X_{\delta T}\delta T = 0$$

$$-\frac{1}{U_o} Z_U U + \dot{\alpha} - Z_w \alpha - \dot{\theta} - \frac{1}{U_o} Z_{\delta e}\delta e = 0 \qquad (7)$$

$$- M_U U - M_{\dot{\alpha}} \dot{\alpha} - M_\alpha \alpha + \ddot{\theta} - M_q \dot{\theta} - M_{\delta e}\delta e - M_{\delta T}\delta T = 0$$

In the single-degree-of-freedom case, the equation could be reduced to standard form by simply defining the derivatives as new variables. Trying this again, let

$$x_1 \triangleq \theta;\ x_2 \triangleq \dot{\theta};\ x_3 \triangleq \alpha;\ x_4 \triangleq U;\ u_1 \triangleq \delta e;\ u_2 \triangleq \delta_T \qquad (8)$$

Then equations (7) become

$$\dot{x}_4 - X_U x_4 - X_\alpha x_3 + g x_1 - X_{\delta T} u_2 = 0$$

$$-\frac{1}{U_o} Z_U x_4 + \dot{x}_3 - Z_w x_3 - x_2 - \frac{1}{U_o} Z_{\delta e}\ u_1 = 0 \qquad (9)$$

$$- M_U x_4 - M_{\dot{\alpha}} \dot{x}_3 - M_\alpha x_3 + \dot{x}_2 - M_q x_2 - M_{\delta e}\ u_1 - M_{\delta T}\ u_2 = 0$$

The first and second equations are well set; they can be solved for $\dot{x}_4$ and $\dot{x}_3$ in terms of zero-order derivatives. The third has two first-derivative terms; therefore neither $\dot{x}_3$ nor $\dot{x}_2$ can be solved for in terms of zero-order derivatives. Since there is already an equation for $\dot{x}_3$, the obvious solution is to eliminate $\dot{x}_3$ from the third equation by use of the second equation. The third equation then becomes

$$\left(-M_U - M_{\dot\alpha}\frac{Z_U}{U_o}\right)x_4 + \left(-M_\alpha - M_{\dot\alpha}\, Z_w\right)x_3 + \dot x_2 + \left(-M_q - M_{\dot\alpha}\right)x_2 +$$

$$\left(-M_{\delta e} - M_{\dot\alpha}\frac{Z_{\delta e}}{U_o}\right)u_1 - M_{\delta T}\, u_2 = 0 \qquad (10)$$

Now the equations can be written out by using the first of equations (8), the first two equations of (9), and equation (10).

$$\begin{bmatrix} \dot x_1 \\ \dot x_2 \\ \dot x_3 \\ \dot x_4 \end{bmatrix} = \begin{bmatrix} 0 & a_{12} & 0 & 0 \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & a_{32} & a_{33} & a_{34} \\ a_{41} & 0 & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ b_{21} & b_{22} \\ b_{31} & 0 \\ 0 & b_{42} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \qquad (11)$$

where

$$\begin{array}{lll} a_{12} = 1 & a_{33} = Z_w & b_{21} = M_{\dot\alpha}\dfrac{Z_{\delta e}}{U_o} + M_{\delta e} \\ a_{22} = M_{\dot\alpha} + M_q & a_{34} = \dfrac{Z_U}{U_o} & b_{22} = M_{\delta T} \\ a_{23} = M_{\dot\alpha}\, Z_w + M_\alpha & a_{41} = -g & b_{31} = \dfrac{Z_{\delta e}}{U_o} \\ a_{24} = M_U + M_{\dot\alpha}\dfrac{Z_U}{U_o} & a_{43} = X_\alpha & b_{42} = X_{\delta T} \\ a_{32} = 1 & a_{44} = X_U & \end{array} \qquad (12)$$

It should be clear from this example that had all derivatives appeared in every equation, it would have been more difficult to put the equations in standard form.

Elimination of Derivatives of the Forcing Function

In some instances the right sides of the plant equations naturally contain derivatives of the forcing functions. If a multi-degree-of-freedom system is written as a single equation by using Laplace transform techniques, derivatives of the forcing function will appear on the right. The right side of the differential equation is equivalent to the numerator in Laplace transform notation.

Numerator or right-side plant dynamics in conjunction with relay-type jumps in the forcing functions would require any analysis or synthesis technique to accommodate derivatives of impulse functions. This problem can be avoided formally by transferring to a set of coordinates in which only zero-order forcing terms appear.

Equation (13) contains derivatives of the forcing function f(t); that is, the differential equation is of the form

$$\frac{d^n x}{dt^n} + a_1(t)\frac{d^{n-1}x}{dt^{n-1}} + a_2(t)\frac{d^{n-2}x}{dt^{n-2}} + \ldots + a_{n-1}(t)\frac{dx}{dt} + a_n(t)\,x$$

$$= b_o(t)\frac{d^n f}{dt^n} + b_1(t)\frac{d^{n-1}f}{dt^{n-1}} + \ldots + b_n(t)\,f(t) \tag{13}$$

In the case of constant coefficients this equation corresponds to a transfer function of the form

$$\frac{X(s)}{F(s)} = \frac{N(s)}{D(s)}$$

where N(s) and D(s) are finite polynomials in the Laplace variable s.

To write equation (13) in standard form free from derivatives of the forcing function f(t), the following technique can be used (ref. 3). Let

$$x = x_1 + G_o(t)\,f$$

$$\frac{dx_1}{dt} = x_2 + G_1(t)\,f \tag{14}$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$\frac{dx_{n-1}}{dt} = x_n + G_{n-1}(t)\, f$$

$$\frac{dx_n}{dt} = -a_n(t)\, x_1 - a_{n-1}(t)\, x_2 - \dots - a_1(t)\, x_n + G_n(t)\, f$$

Now the G_i's can be determined by successively eliminating $x_1, x_2, \dots, x_n$ from (14) and requiring that the resulting differential equation be identical with (13). This leads to the following recurrence formula for determining the G_i's.

$$G_o(t) = b_o(t)$$

$$G_i(t) = b_i(t) - \sum_{k=o}^{i-1} \sum_{s=o}^{i-k} \binom{n+s-i}{n-i} a_{i-k-s}(t) \frac{d^s G_k}{dt^s} \qquad (15)$$

where $\binom{n+s-i}{n-i}$ is the $(n-i+1)$ th binomial coefficient of order $n+s-i$. These binomial coefficients are defined by the following formula

$$\binom{p}{q} = \frac{p!}{[q!\,(p-q)!]}$$

As an example of the use of the recurrence relation (15), consider the system:

$$\dddot{e} + 2\zeta\omega\ddot{e} + \omega^2\dot{e} = -au - b\dot{u}$$

where a, b, ζ, ω, are constants.

Comparing with equation (13),

$$n = 3$$

$a_1 = 2\zeta\omega$	$b_o = 0$
$a_2 = \omega^2$	$b_1 = 0$
$a_3 = 0$	$b_2 = -b$
$f(t) = u(t)$	$b_3 = -a$

Equations (14) become

$$e = x_1 + G_o u$$

$$x_1 = x_2 + G_1 u$$

$$\dot{x}_2 = x_3 + G_2 u$$

$$\dot{x}_3 = -\omega^2 x_2 - 2\zeta\omega x_3 + G_3 u$$

According to (15)

$$G_o = b_o = 0$$

$$G_1(t) = b_1(t) - \sum_{s=o}^{1} \binom{2+s}{2} a_{1-s} \frac{d^s G_o}{dt^s} = 0$$

$$G_2(t) = b_2 - \sum_{k=o}^{1} \sum_{s=o}^{2-k} \binom{1+s}{1} a_{2-k-s} \frac{d^s G_k}{dt^s}$$

$$= b_2 - \left[\sum_{s=o}^{2} \binom{1+s}{1} a_{2-s} \cancelto{0}{\frac{d^s G_o}{dt^s}} \right]$$

$$- \left[\sum_{s=o}^{1} \binom{1+s}{1} a_{1-s} \cancelto{0}{\frac{d^s G_1}{dt^s}} \right] = b_2 = -b$$

$$G_3(t) = b_3 - \sum_{k=o}^{2} \sum_{s=o}^{3-k} \binom{s}{o} a_{3-k-s} \frac{d^s G_k}{dt^s} = -a - a_1 G_2$$

$$= -a + 2\zeta\omega b$$

APPLICATION TO THE PITCH AXIS OF AN AEROELASTIC VEHICLE

If the pitch axis of an aeroelastic vehicle is to be time-optimally controlled, it is necessary to be able to measure with standard instrumentation the body-bending displacements and the angle of attack. Techniques for doing this are presented in the following subsections.

Measurement of Body-Bending Displacements

To permit the measurement of the body-bending displacements, with standard instrumentation, it is necessary to make three assumptions:

1. The location of the center of gravity is known.
2. The mode shapes are known.
3. Only a finite number, N, of modes exists.

The first two of these are naturally compatible with a basic assumption of the report, i. e., the characteristics of the vehicle are known. The third assumption must be made for any analysis or synthesis technique that uses the eigenfunctions of the partial differential equations that describe the flexure of the free-free body under accelerated or unaccelerated motion. Since the eigenfunctions present the best approximation to the characteristics of a large complex structure that is likely to be found in the near future, it appears that the third assumption is not restrictive.

Consider figure 1 where θ_G is the pitch attitude sensed at the position x_G. Then

$$\theta(t) = \theta_G(t) + \tan^{-1} \frac{\partial z}{\partial x}(x_G, t) \tag{16}$$

For $\partial z/\partial x$ small, $\tan^{-1} \partial z/\partial x \cong \partial z/\partial x$. Assuming this approximation is satisfactory, equation (16) can be rewritten as

$$\theta_G(t) = \theta(t) - \frac{\partial z}{\partial x}(x_G, t) \tag{17}$$

Assumption 3 is mathematically $z(x,t) = \sum_{i=1}^{N} (z_i)_x z_i(t)$.

The $(z_i)_x$ is the normalized z coordinate of the ith flexure mode at station x.

Then $\partial z/\partial x = \sum_{i=1}^{N} (z_i)'_x \, z_i(t)$ so that equation (17) is by assumption 3

$$\theta_G(t) = \theta(t) - \sum_{i=1}^{N} (z_i)'_{x_G} \, z_i(t) \tag{18}$$

Assumption 2 states that $(z_i)_x$ is known and hence also $(z_i)'_x \overset{\Delta}{=} d/dx\,(z_i)_x$ $i = 1, 2, \ldots, N$. If $(N + 1)$ altitude sensors are placed at $(N + 1)$ different positions along the fuselage, an algebraic system of $(N + 1)$ linear equations for the $(N + 1)$ unknowns, θ, z_1, z_2, $\ldots$, z_N, is obtained (i.e., if the Kth sensor is placed at x_{GK} and its output denoted by θ_{GK}, the following system results):

$$\theta_{G_K}(t) = \theta(t) - \sum_{i=1}^{N} (z_i)'_{x_{GK}} \, z_i(t), \quad K = 1, 2, \ldots, N + 1 \tag{19}$$

This system may be written in vector form as

$$\theta_G = Az \tag{20}$$

where

θ_G is a column vector with K<u>th</u> component $= \theta_{GK}$

z is a column vector with K<u>th</u> component $= \left\{ \begin{matrix} \theta \text{ for } K = 1 \\ z_{K-1} \text{ for } K > 1 \end{matrix} \right\}$

and

$$A = (a_{ij}) \text{ with } a_{i1} = 1 \quad \text{and} \quad a_{ij+1} = -(z_j)'_{x_{Gi}} \quad \text{for } j \geq 1$$

Note that A is a function of the mode shapes only. Thus, locations for the sensors can be chosen such that A is nonsingular. Then

$$z = A^{-1}\theta_G \tag{21}$$

and differentiating with respect to time,

$$\dot{z} = A^{-1}\dot{\theta}_G \tag{22}$$

Thus if (N + 1) rate sensors are used, $\dot{\theta}_G$ is obtained, and since A is known, $\dot{z}$ may be computed.

The entire procedure has been developed on the knowledge that each sensor in essence measures one unknown and that if sufficient sensors are used, sufficient equations can be written to determine the variables of the equations of motion. The only problem is to place the sensors so the A matrix is nonsingular and well set. This need not be a matter of great concern in that the sensors can be first arbitrarily placed for convenience and the resulting A matrix checked for singularity and to see that it is not ill-conditioned.

Measurement of Angle of Attack

Since the typical angle-of-attack sensor is unacceptable for flight control purposes and is likely to remain so for many important applications, it is desirable to be able to derive a high-quality substitute. The angle of attack that can be measured without an external probe is that between the relative wind and the longitudinal reference line. This proves adequate for flight control purposes.

Now consider figure 2 where the x and z axes are shown at time t. At time (t + Δ t) they are shown as x' and z' axes.

Let

$$P_z = \text{the z coordinate of P and } P_z' = \text{z coordinate of P'}$$

Then

$$P_z = z(x_A, t)$$

and

$$P_z' = -U_o\Delta t \sin\alpha - (x_A - x_{CG}) \sin\left[\theta(t+\Delta t) - \theta(t)\right]$$
$$+ \left[z(x_A, t+\Delta t) \cos\left[\theta(t+\Delta t) - \theta(t)\right]\right]$$

Since

$$\theta(t+\Delta t) - \theta(t) = \dot{\theta}(t)\Delta t + 0\left[(\Delta t)^2\right],$$

$$\cos\left[\theta(t+\Delta t) - \theta(t)\right] = 1 + 0\left[(\Delta t)^2\right]$$

and

$$\sin\left[\theta(t+\Delta t) - \theta(t)\right] = \dot{\theta}(t)\,\Delta t + 0\left[(\Delta t)^2\right],$$

then

$$\frac{P_z' - P_z}{\Delta t} = -U_o \sin\alpha(t) - (x_A - x_{CG})\,\dot{\theta}(t) +$$

$$\frac{z(x_A, t+\Delta t) - z(x_A, t)}{\Delta t} + 0(\Delta t)$$

Therefore,

$$w(x_A, t) = -U_o \sin\alpha(t) - (x_A - x_{CG})\,\dot{\theta}(t) + \frac{\partial z(x_A, t)}{\partial t} \tag{23}$$

where $w(x_A, t)$ = velocity of point at x_A in + z direction (i. e., the integral of normal acceleration at the point x_A). Solving for $\alpha(t)$ gives

$$\alpha(t) = \sin^{-1}\left\{\frac{1}{U_o}\left[\frac{\partial z}{\partial t}(x_A, t) - w(x_A, t) - (x_A - x_{CG})\,\dot{\theta}(t)\right]\right\} \tag{24}$$

Using assumption 3, equation (24) becomes

$$\alpha(t) = \sin^{-1}\left\{\frac{1}{U_o}\left[\sum_{i=1}^{N} (z_i)_{x_A}\, \dot{z}_i(t) - w(x_A, t) - (x_A - x_{CG})\,\dot{\theta}(t)\right]\right\} \tag{25}$$

By assumption 1, $x_A - x_{CG}$ is known and by assumption 2, $(z_i)_{x_A}$ is known for $i = 1, 2, \ldots, N$. Thus since $w(x_A, t)$ can be measured and $\dot{\theta}(t)$, $\dot{z}(t)$, ... $\dot{z}_N(t)$ can be found from (22), the right side of (25) is known when U_o is known. In most cases $\alpha(t)$ is small so that $\sin\alpha \cong \alpha$. Then (25) simplifies to

$$\alpha(t) = \frac{1}{U_o}\left[\sum_{i=1}^{N} (z_i)_{x_A}\, \dot{z}_i(t) - w(x_A, t) - (x_A - x_{CG})\,\dot{\theta}(t)\right] \tag{26}$$

The state variables necessary for the equations of motion are α, $\dot{\theta}$, $\dot{z}_1$, ..., $\dot{z}_N$, z_1, ..., z_N. Hence if (N + 1) pitch rate sensors are used, $\dot{\theta}$, $\dot{z}_1$, ..., z_N can be obtained from (22). Then z_1, ..., z_N can be obtained by integration. The variable α may be computed from (25) [or (26) when applicable] if $w(x_A, t)$ is obtained from a sensor located at x_A.

The transformation may be written compactly as

$$\begin{bmatrix} \sin\alpha \\ \dot{\theta} \\ \dot{z}_1 \\ \cdot \\ \cdot \\ \cdot \\ \dot{z}_N \end{bmatrix} = C \begin{bmatrix} \dot{\theta}_{G_1} \\ \cdot \\ \cdot \\ \dot{\theta}_{G_{N+1}} \\ w(x_A, t) \end{bmatrix} \tag{27}$$

where

$$C_{1j} = \begin{cases} \dfrac{1}{U_o}\left[\displaystyle\sum_{i=2}^{N+2} (z_{i-1})_{x_A} \; p_{ij} - (x_A - x_{CG})\, p_{ij}\right], & j = 1, 2, \ldots, N+1 \\ -\dfrac{1}{U_o} & , j = N+2 \end{cases}$$

$$C_{ij} = \left\{ \begin{array}{ll} p_{i-1j}, & j = 1, 2, \ldots, N+1 \\ 0 & , j = N+2 \end{array} \right\} \quad i = 2, 3, \ldots, N+2$$

and $p_{ij} = (A^{-1})_{ij}$.

Standard Form for Pitch Equations for an Aeroelastic Vehicle

The equations of motion can be written as

$$D \begin{bmatrix} \dot{\alpha} \\ \ddot{\theta} \\ \ddot{z}_1 \\ \cdot \\ \cdot \\ \ddot{z}_N \end{bmatrix} + E \begin{bmatrix} \alpha \\ \dot{\theta} \\ \dot{z}_1 \\ \cdot \\ \cdot \\ \dot{z}_N \end{bmatrix} \div F \begin{bmatrix} z_1 \\ z_2 \\ \cdot \\ \cdot \\ \cdot \\ z_N \end{bmatrix} = \begin{bmatrix} \dfrac{Z_{\delta e}}{U_o} \\ M_{\delta e} \\ Z_{1\delta e} \\ \cdot \\ \cdot \\ Z_{N\delta e} \end{bmatrix} (-\delta e) \tag{28}$$

where

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & \cdots & 0 \\ M_{\dot\alpha} & -1 & M_{\ddot z_1} & M_{\ddot z_2} & \cdots & M_{\ddot z_N} \\ Z_{1\dot\alpha} & 0 & -1+Z_{1\ddot z_1} & Z_{1\ddot z_2} & \cdots & Z_{1\ddot z_N} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ Z_{N\dot\alpha} & 0 & Z_{N\ddot z_1} & Z_{N\ddot z_2} & \cdots & -1+Z_{N\ddot z_N} \end{bmatrix}$$

$$E = \begin{bmatrix} Z_w & 1+\frac{Z_{\dot\theta}}{U_o} & \frac{Z_{\dot z_1}}{U_o} & \cdots & \frac{Z_{\dot z_N}}{U_o} \\ M_\alpha & M_{\dot\theta} & M_{\dot z_1} & \cdots & M_{\dot z_N} \\ Z_{1_\alpha} & Z_{1\dot\theta} & Z_{1\dot z_1} & \cdots & Z_{1\dot z_N} \\ \vdots & \vdots & \vdots & & \vdots \\ Z_{N_\alpha} & Z_{N\dot\theta} & Z_{N\dot z_1} & \cdots & Z_{N\dot z_N} \end{bmatrix} \text{ and } F = \begin{bmatrix} \frac{Z_{z_1}}{U_o} & \cdots & \frac{Z_{z_N}}{U_o} \\ M_{z_1} & \cdots & M_{z_N} \\ Z_{1_{z_1}} & \cdots & Z_{1_{z_N}} \\ \vdots & & \\ Z_{N_{z_1}} & \cdots & Z_{N_{z_N}} \end{bmatrix}$$

Now let

$$x_i = z_i \qquad i = 1, 2, \ldots, N$$

$$x_{N+1} = \alpha$$

$$x_{N+2} = \dot\theta$$

$$x_i = \dot x_{i-N-2} = \dot z_{i-N-2} \qquad i = N+3, \ldots, 2N+2$$

and

$$x = \begin{bmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_{2N+2} \end{bmatrix}$$

Then

$$H\dot{x} = Kx + \begin{bmatrix} \left.\begin{matrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{matrix}\right\} (N\ 0\text{'s}) \\ \dfrac{Z_{\delta e}}{U_o} \\ \cdot \\ \cdot \\ \cdot \\ Z_{N_{\delta e}} \end{bmatrix} (-\delta e) \tag{29}$$

where

$$H = \begin{array}{c} \left[\begin{array}{cccc|c} 1 & & & 0 & \\ & 1 & & & \\ & & \ddots & & 0 \\ 0 & & & 1 & \\ \hline & & & & \\ & & 0 & & D \\ & & & & \end{array}\right] \begin{array}{l} \left.\vphantom{\begin{matrix}1\\1\\1\\1\end{matrix}}\right\} N \text{ rows} \\ \left.\vphantom{\begin{matrix}1\\1\\1\end{matrix}}\right\} N+2 \text{ rows} \end{array} \\ \underbrace{\hspace{6em}}_{N \text{ columns}} \underbrace{\hspace{8em}}_{N+2 \text{ columns}} \end{array}$$

and

$$K = \left[\begin{array}{c|c} \overset{N+2\text{ columns}}{0} & \begin{matrix}1 & & & 0\\ & 1 & & \\ & & \ddots & \\ & & & 1\end{matrix} \\ \hline \underbrace{-F}_{N\text{ columns}} & \underbrace{-E}_{N+2\text{ columns}}\end{array}\right] \begin{array}{l} \left.\vphantom{\begin{matrix}1\\1\\1\\1\end{matrix}}\right\} N\text{ rows} \\ \left.\vphantom{\begin{matrix}1\\1\end{matrix}}\right\} N+2\text{ rows}\end{array}$$

To add the θ equations to the system (29), let $\hat{x} = \begin{bmatrix}\theta \\ x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_{2N+2}\end{bmatrix}$

Then

$$\hat{H}\dot{\hat{x}} = \hat{K}\hat{x} + \begin{bmatrix} \left.\begin{matrix}0\\ \vdots \\ 0\end{matrix}\right\} N+1 \text{ 0's} \\ \dfrac{Z_{\delta e}}{U_o} \\ \vdots \\ Z_{N_{\delta e}} \end{bmatrix} (-\delta e) \qquad (30)$$

with

$$\hat{H} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & & & & \\ 0 & & H & & \\ \vdots & & & & \\ 0 & & & & \end{bmatrix} \text{ and } \hat{K} = \begin{array}{c} \hphantom{0\ 0 \cdots 0\ }(N+3)^{rd} \text{ column} \\ \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & & & & & & & \\ 0 & & & & K & & & \\ \vdots & & & & & & & \\ 0 & & & & & & & \end{bmatrix}\end{array}$$

To put the equations in terms of the sensed variables let

$$y = \begin{bmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ y_{2N+3} \end{bmatrix} \qquad \begin{aligned} y_i &= \theta_{G_i}, \quad i = 1, 2, \ldots, N+1 \\ \text{with } y_i &= \dot{\theta}_{G_{i-N-1}}, \quad i = N+2, N+3, \ldots, 2N+2 \\ y_{2N+3} &= w(x_A, T) \end{aligned}$$

Then if $\sin\alpha \cong \alpha$, using (21) and (27),

$$\hat{x} = Ly$$

where

$$L = \left[\begin{array}{c|c} A^{-1} & 0 \\ \hline 0 & C \end{array}\right] \tag{31}$$

Finally from (30) and (31),

$$\hat{H} L \dot{y} = \hat{K} L y + \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ \dfrac{Z_{\delta e}}{U_o} \\ \cdot \\ \cdot \\ \cdot \\ Z_{N\delta e} \end{bmatrix} (-\delta e) \tag{32}$$

or

$$y = L^{-1}\hat{H}^{-1}\hat{K}\,L\,y + L^{-1}\hat{H}^{-1}\begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ \dfrac{Z_{\delta e}}{U_o} \\ \cdot \\ \cdot \\ \cdot \\ Z_{N_{\delta e}} \end{bmatrix}(-\delta e) \tag{33}$$

BLENDING TWO RATE GYRO OUTPUTS

Two rate gyros are placed on the vehicle in such a way that one is located forward and the other aft of the first bending mode antinode. The output of the two gyros is automatically blended. The blender is designed so that it will not affect the rigid body output but may be chosen to give a positive, negative, or zero pickup for the first mode, which ever is desired.

The block diagram of the blender and logic circuit involved is shown in figure 3. The logic circuit consists of two band-pass filters with the pass band centered at the first bending frequency, two absolute value units, and a servo or electronic multiplier. The output of the blender is denoted by G_o. The outputs of the forward and aft gyros are denoted by G_F and G_A, respectively. Then

$$G_F = \dot{\theta} - \sum_{i=1}^{N} (z_i)'_{x_{G_F}}\,\dot{z}_i, \quad G_A = \dot{\theta} - \sum_{i=1}^{N} (z_i)'_{x_{G_A}}\,\dot{z}_i \tag{34}$$

From figure 3, it is clear that

$$G_o = KG_F + (1-K)G_A = \dot{\theta} - \sum_{i=1}^{N} \lambda_i \dot{z}_i \tag{35}$$

where

$$\lambda_i = K (z_i)'_{x_{G_F}} + (1-K)(z_i)'_{x_{G_A}}, \quad i = 1, 2, \ldots, N.$$

Two band-pass filters are used to discriminate all other frequencies from that of the first mode. In other words, the blender logic is primarily sensitive to $K (z_1)'_{x_{G_F}} \dot{z}_1$ and $(1 - K)(z_1)'_{x_{G_A}} \dot{z}_1$. The outputs of the band-pass filters are then rectified and compared. The difference drives a servo or electronic multiplier which will change K until the difference becomes zero, i.e. $\lambda_1 = 0$. If it is desired to have λ_1 small but always of one sign, then an attenuator K_1 could be used as shown in figure 3.

CONCLUSIONS

It is shown that, if a finite number of flexure modes are sufficient to represent the aeroelastic distortions of a flexible vehicle, the state of the vehicle can be measured with commonly available instruments. The method requires a number of sensors of the same order of magnitude as the number of flexure modes necessary to sufficiently described the aeroelasticity of the vehicle. Also it is shown that, by blending the signals from two rate gyros, the effects of the first flexure mode can be eliminated.

REFERENCES

1. Stone, Johnson, Smith, Lee, and Harvey: Time-Optimal Control of Linear Systems. Minneapolis-Honeywell Aero Document R-ED 6134, 1 September 1959.

2. Lee, R. C. K. and Falkner, V. L.: Adaptive Control System for Large Elastic Boosters. Presented at 1960 IRE Symposium on Adaptive Control Systems, Garden City, Long Island, October 17-19, 1960.

3. Laning and Battin: Random Processes in Automatic Control. McGraw-Hill Book Company, New York, 1956.

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Minneapolis-Honeywell Regulator Company
Minneapolis, Minnesota

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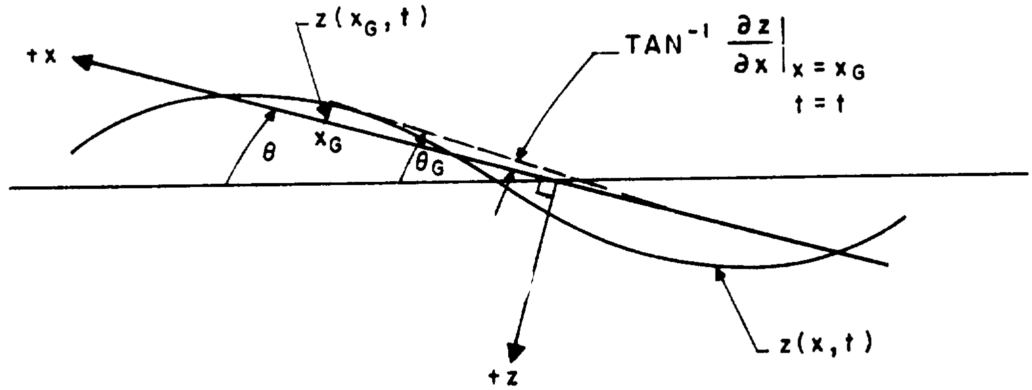

Figure 1. - The flexible vehicle

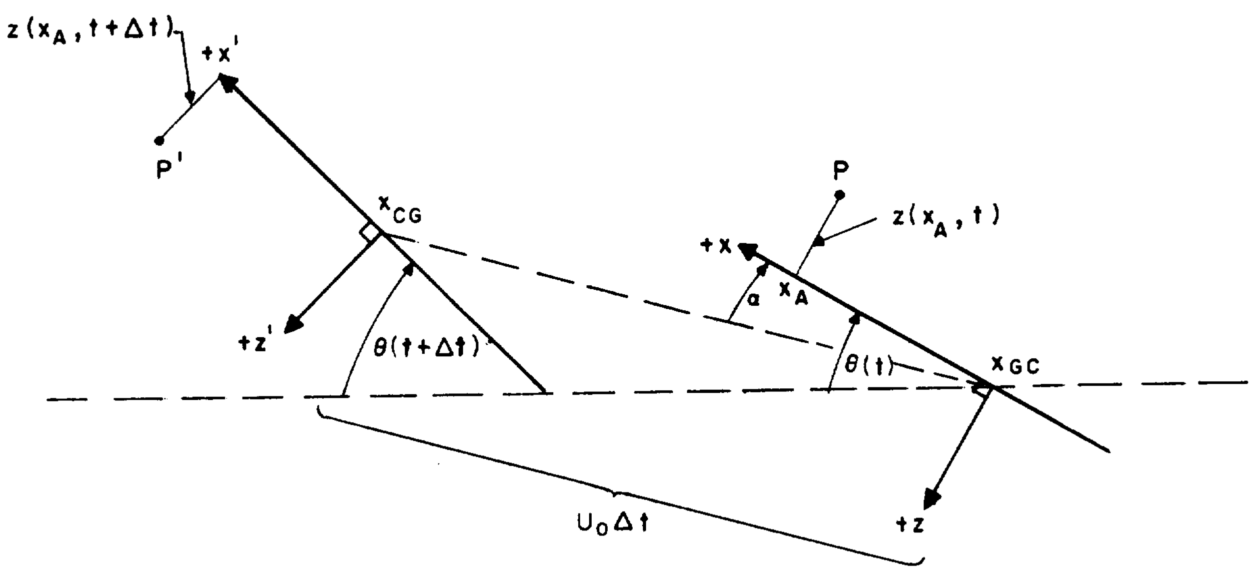

Figure 2. - Effects of time

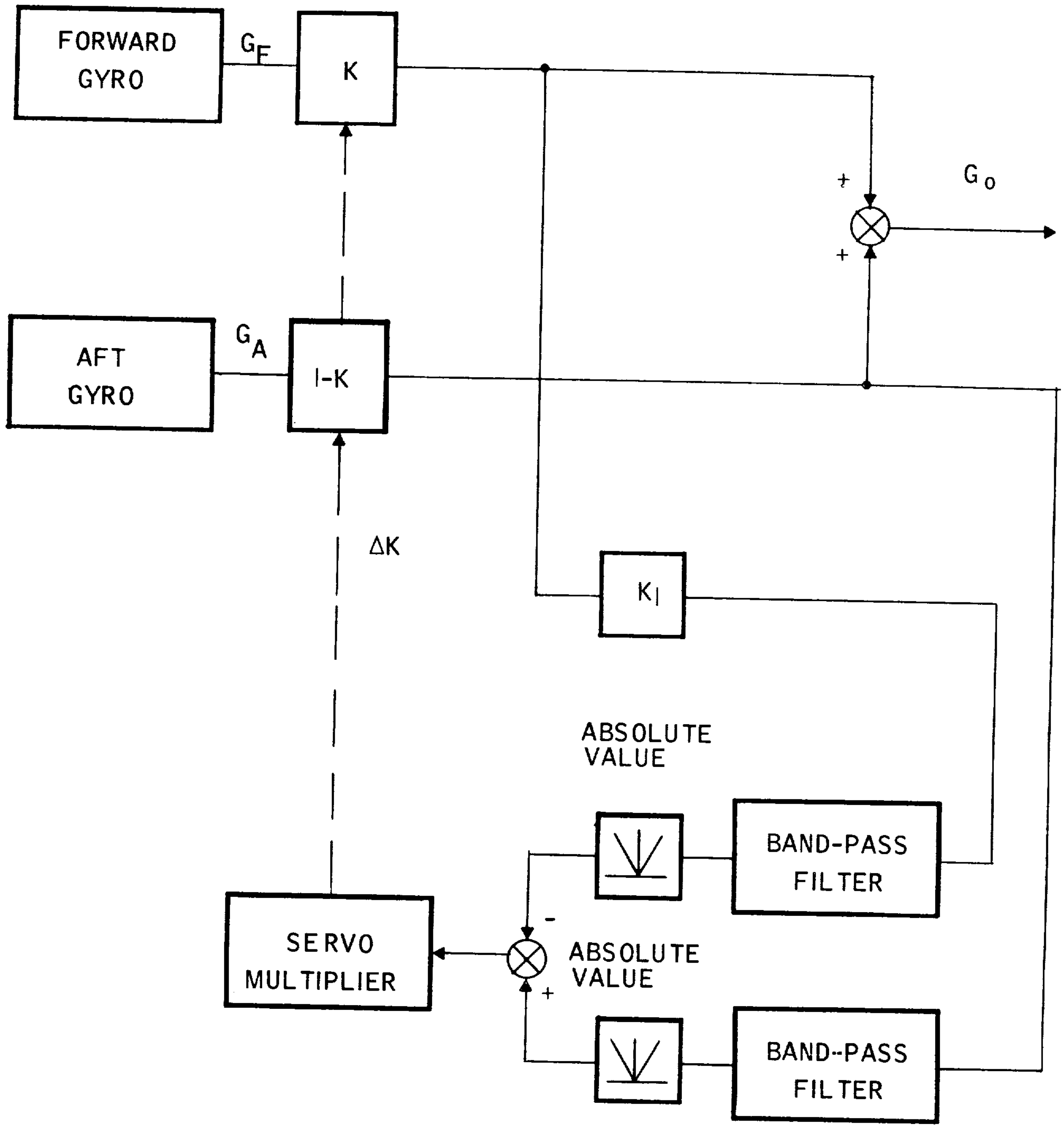

Figure 3. - Block diagram of blender and logic circuit

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