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# TECHNICAL NOTE

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## REMARKS ON HILL'S LUNAR THEORY. PART II

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# REMARKS ON HILL'S LUNAR THEORY. PART II

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## SUMMARY

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In 1879 G. W. Hill, using his differential equation system, found a series of direct lunar orbits, which represent a simplification of the restricted three-body problem (neglecting solar parallax and eccentricity). If  $T$  is the period of revolution time, and  $T = 2\pi m$  is fixed, the initial conditions,  $\xi_0(m)$ ,  $\dot{\eta}_0(m)$ , and  $(\dot{\xi}_0 = \eta_0 = 0)$ , are dependent upon  $m$  and give periodic solutions. Hill obtained these functions in the form of power series, with respect to  $m$ , which converges rapidly for small distances of the satellite; and he gave their first coefficients.

In the present discussion, Hill's coefficients are successively determined from linear equations and are in the form of closed trigonometric expressions. Although this method—which does not extend beyond the third order—is only an interesting attempt at achieving the same goal via a different mathematical route, it does not extend as far as Hill's method (which may be carried out to the fifth or sixth order without difficulty).



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# REMARKS ON HILL'S LUNAR THEORY.

## PART II

by

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### INTRODUCTION

In his famous writing of 1878, G. W. Hill<sup>†</sup> found a series of direct lunar orbits as solutions of his differential equation system. This series constitutes a simplification of the *problème restreint* (by discounting the solar parallax). Hill found that if  $T$  is the (synodic) circuit time of the satellite and  $T = 2\pi m$  is fixed, the initial conditions are  $\xi_0(m)$ ,  $\dot{\eta}_0(m)$ , ( $\dot{\xi}_0 = \eta_0 = 0$ ), and are dependent upon  $m$  as a parameter and give such periodic solutions. He obtained these functions in a power series with respect to  $m$ , which converge rapidly for small distances of the satellite (short circuit times, small  $m$ ), and gave their first coefficients. In the following examination, another method will be presented which leads to the same goal, but which permits these coefficients to appear as finite trigonometric expressions.

### HILL'S DIFFERENTIAL EQUATIONS

Hill's differential equations

$$\left. \begin{aligned} \ddot{\xi} - 2\dot{\eta} &= \xi \left( 3 - \frac{1}{r^3} \right), \\ \ddot{\eta} + 2\dot{\xi} &= -\frac{\eta}{r^3}, \end{aligned} \right\} \quad (1)$$

where  $r^2 = \xi^2 + \eta^2$ , are based upon the rectangular coordinates of a massless satellite in reference to its central body (planet) and in a coordinate system which revolves at the same angular velocity,  $n = 1$ , as the planet which orbits the sun (Figure 1). The location of the sun  $S$  is therefore assumed to be in the direction of the negative  $\xi$ -axis, at any desired distance, so that the parallax

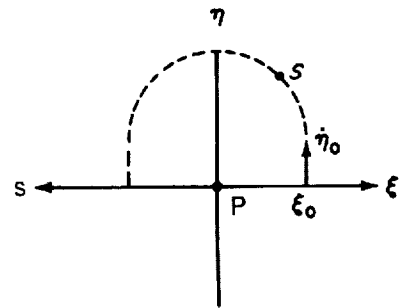


Figure 1—Geometric representation of a massless satellite  $s$  of a planet  $P$ .

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†Hill, G. W., "Researches in the Lunar Theory," in *The Collected Works of George William Hill, Vol. I*, Washington, D. C.: Carnegie Institution, 1905.

of the sun can be discounted. Under these circumstances, Equations 1 apply for the satellite S. Periodic solutions can be expected if we let the satellite start from a point  $(\xi_0, 0)$  on the  $\xi$ -axis with a velocity  $(0, \dot{\eta}_0)$ , whereby the initial (positive) velocity must occur perpendicular to the  $\xi$ -axis and  $\dot{\eta}_0$  will depend to a certain extent upon  $\xi_0$ . Every periodic solution of this sort is symmetric to both axes, since Equations 1 do not change when  $\xi$  is exchanged with  $-\xi$  and  $\eta$  with  $-\eta$ . If  $T = 2\pi m$  is the circuit time in such a periodic orbit, then the total of these orbits also can be characterized by determining  $\xi_0 = \xi_0(m)$ ,  $\dot{\eta}_0 = \dot{\eta}_0(m)$  as functions of the parameter  $m$ .

If  $\tau = 0$  is the moment at which the satellite crosses the positive  $\xi$ -axis at right angles, then the periodic motion has the form

$$\left. \begin{aligned} \xi(\tau) &= A_1 \cos \nu\tau + A_3 \cos 3\nu\tau + A_5 \cos 5\nu\tau + \dots, \\ \eta(\tau) &= B_1 \sin \nu\tau + B_3 \sin 3\nu\tau + B_5 \sin 5\nu\tau + \dots, \end{aligned} \right\} \quad (2)$$

where  $\nu = 1/m = 2\pi/T$ . For circuit times as brief as desired ( $m \rightarrow 0$ ) — hence for very close orbits — the perturbations by the sun become as small as desired. The above equations (Equation 2) now change into

$$\xi(\tau) = a \cos \nu\tau, \quad \eta(\tau) = a \sin \nu\tau, \quad (3)$$

or into the form of uniform circular motion, for circles are the only undisturbed orbits which are symmetric to both axes. If we differentiate Equation 3 any number of times, we find, for this boundary case and for  $\tau = 0$ ,

$$m^{2n} \xi_0^{(2n)} = (-1)^n \xi_0,$$

$$m^{2n-1} \eta_0^{(2n-1)} = (-1)^{n-1} \xi_0.$$

The above equations mean that, for  $m \rightarrow 0$ , the quantities

$$\left. \begin{aligned} \alpha_2 &= -m^2 \frac{\ddot{\xi}_0}{\xi_0} \rightarrow 1, & \beta_1 &= +m \frac{\dot{\eta}_0}{\xi_0} \rightarrow 1, \\ \alpha_4 &= +m^4 \frac{\xi_0^{(4)}}{\xi_0} \rightarrow 1, & \beta_3 &= -m^3 \frac{\ddot{\eta}_0}{\xi_0} \rightarrow 1, \\ \alpha_6 &= -m^6 \frac{\xi_0^{(6)}}{\xi_0} \rightarrow 1, & \beta_5 &= +m^5 \frac{\eta_0^{(5)}}{\xi_0} \rightarrow 1, \end{aligned} \right\} \quad (4)$$

all approach unity.



If for the sake of abbreviation we set  $1/r^3 = \mu$ , then the differential equations (Equation 1) can be written:

$$\ddot{\xi} = 2\dot{\eta} - \xi(\mu - 3) ; \quad \text{and} \quad \ddot{\eta} = -2\dot{\xi} - \eta\mu .$$

If we differentiate these equations any number of times and if we consider that uneven derivatives of  $\xi$  and  $\mu$  and even derivatives of  $\eta$  disappear for  $\tau = 0$ , we then obtain the system:

$$\frac{-m^2}{\xi_0} \quad \ddot{\xi}_0 = 2\dot{\eta}_0 - \xi_0(\mu_0 - 3) ,$$

$$\frac{+m^4}{\xi_0} \quad \xi_0^{(4)} = 2\ddot{\eta}_0 - [\ddot{\xi}_0(\mu_0 - 3) + \xi_0 \ddot{\mu}_0] ,$$

$$\frac{-m^6}{\xi_0} \quad \xi_0^{(6)} = 2\eta_0^{(5)} - [\xi_0^{(4)}(\mu_0 - 3) + 6\ddot{\xi}_0 \ddot{\mu}_0 + \xi_0 \mu_0^{(4)}] ,$$

$$\frac{-m^3}{\xi_0} \quad \ddot{\eta}_0 = -2\dot{\xi}_0 - \dot{\eta}_0 \mu ,$$

$$\frac{+m^5}{\xi_0} \quad \eta_0^{(5)} = -2\xi_0^{(4)} - (\ddot{\eta}_0 \mu + 3\dot{\eta}_0 \dot{\mu}) ,$$

$$\frac{-m^7}{\xi_0} \quad \eta_0^{(7)} = -2\xi_0^{(6)} - (\eta_0^{(5)} \mu + 10 \ddot{\eta}_0 \ddot{\mu} + 5\dot{\eta}_0 \mu^{(4)}) ,$$

If these equations are multiplied by the factors to their left and if Equation 4 is substituted they become:

$$\left. \begin{aligned} a_2 &= -2m\beta_1 - 3m^2 + \phi , & \beta_1 &= \psi , \\ a_4 &= -2m\beta_3 - 3m^2 a_2 + \phi\nu_2 , & \beta_3 &= -2ma_2 + \phi\nu_1 , \\ a_6 &= -2m\beta_5 - 3m^2 a_4 + \phi\nu_4 , & \beta_5 &= -2ma_4 + \phi\nu_3 , \\ a_8 &= -2m\beta_7 - 3m^2 a_6 + \phi\nu_6 , & \beta_7 &= -2ma_6 + \phi\nu_5 , \end{aligned} \right\} \quad (5)$$

where

$$\phi = m^2 \mu_0 \text{ and } \psi = \beta_1 = m \frac{\dot{\eta}_0}{\xi_0} ; \quad (6)$$

and, with

$$\mu_2 = -m^2 \frac{\ddot{\mu}_0}{\mu_0} , \quad \mu_4 = +m^4 \frac{\mu_0^{(4)}}{\mu_0} , \quad \mu_6 = -m^6 \frac{\mu_0^{(6)}}{\mu_0} , \quad \dots , \quad (7)$$

$$\left. \begin{aligned} \nu_1 &= \beta_1 , & \nu_2 &= a_2 + \mu_2 , \\ \nu_3 &= \beta_3 + 3\mu_2 \beta_1 , & \nu_4 &= a_4 + 6\mu_2 a_2 + \mu_4 , \\ \nu_5 &= \beta_5 + 10\mu_2 \beta_3 + 5\mu_4 \beta_1 , & \nu_6 &= a_6 + 15\mu_2 a_4 + 15\mu_4 a_2 + \mu_6 . \end{aligned} \right\} \quad (8)$$

If the quantities  $\mu_2, \mu_4, \dots$  are eliminated from Equation 8 by forming the even derivatives of  $\mu$  and substituting their values for  $\tau = 0$  into Equation 7, the following system of recurrence formulas is obtained for  $\nu_i$ :

$$\left. \begin{aligned} \nu_1 &= \beta_1 , \\ \nu_3 &= \beta_3 - 3a_2^2 \nu_1 + 3\beta_1 \nu_2 , \\ \nu_5 &= \beta_5 - 5a_4 \nu_1 + 10\beta_3 \nu_2 - 10a_2 \nu_3 + 5\beta_1 \nu_4 , \\ \nu_2 &= -(1+1) a_2 + (1+2) \beta_1 \nu_1 , \\ \nu_4 &= -(1+1) a_4 + (3+4) \beta_3 \nu_1 - (3+6) a_2 \nu_2 + (1+4) \beta_1 \nu_3 , \\ \nu_6 &= -(1+1) a_6 + (5+6) \beta_5 \nu_1 - (10+15) a_4 \nu_2 + (10+20) \beta_3 \nu_3 - (5+15) a_2 \nu_4 + (1+6) \beta_1 \nu_5 , \end{aligned} \right\} \quad (9)$$

In these formulas we again recognize the binomial coefficients of uneven order in the coefficients of  $\nu_i$  with uneven indices, while the coefficients of  $\nu_i$  with even indices are made up of the binomial coefficients of uneven and subsequent even orders. In all cases, the signs are alternating and the sum of the coefficients is unity. However, this means that all  $\nu_i$ , just as the  $a_i$  and  $\beta_i$ , approach unity for  $m \rightarrow 0$ . The same also applies for  $\phi$  and  $\psi$ , as can be seen from Equation 5. With the help of Equations 5 and 9,  $a_i$  and  $\beta_i$  can be determined one after another, in the order of increasing indices ( $\beta_1, a_2, \beta_3, a_4, \dots$ ), as functions of  $m, \phi, \psi$ .

The orbit of a satellite which originates from  $(\xi_0, 0)$  at time  $\tau = 0$  with the velocity  $(0, \dot{\eta}_0)$  will be periodic if it reaches the point  $(-\xi_0, 0)$  at  $\tau = (1/2)T = \pi m$ . Therefore the periodicity conditions can be written:

$$\xi(\pi m) = -\xi_0, \quad \eta(\pi m) = 0, \quad (10)$$

or, if we set up the Taylor series,

$$\begin{aligned} \xi(\pi m) &= -\xi_0 = \xi_0 + \frac{(\pi m)^2}{2!} \ddot{\xi}_0 + \frac{(\pi m)^4}{4!} \xi_0^{(4)} + \dots, \\ \eta(\pi m) &= 0 = \pi m \dot{\eta}_0 + \frac{(\pi m)^3}{3!} \ddot{\eta}_0 + \frac{(\pi m)^5}{5!} \eta_0^{(5)} + \dots. \end{aligned}$$

If we divide by  $\xi_0$  and substitute the quantities of Equation 4, we can then write

$$\begin{aligned} f(m) &= 0 = 2 - \alpha_2 \frac{\pi^2}{2!} + \alpha_4 \frac{\pi^4}{4!} - \alpha_6 \frac{\pi^6}{6!} + \dots \\ g(m) &= 0 = \beta_1 \pi - \beta_3 \frac{\pi^3}{3!} + \beta_5 \frac{\pi^5}{5!} - \dots \end{aligned} \quad (11)$$

On the other hand,

$$\begin{aligned} f(m) &= f_0 + f'_0 m + f''_0 \frac{m^2}{2!} + f'''_0 \frac{m^3}{3!} + \dots = 0, \\ g(m) &= g_0 + g'_0 m + g''_0 \frac{m^2}{2!} + g'''_0 \frac{m^3}{3!} + \dots = 0, \end{aligned} \quad (12)$$

if the index 0 is based upon  $m = 0$  and the derivatives according to  $m$  indicated by primes. The Equations 12 are identities, since they are fulfilled for any given value of  $m$ . Thus the requirement of periodic orbits is given by

$$\begin{aligned} f_0 &= f'_0 = f''_0 = f'''_0 = \dots = 0, \\ g_0 &= g'_0 = g''_0 = g'''_0 = \dots = 0. \end{aligned} \quad (13)$$

In general, then, if  $n$  is any positive whole number or zero and if  $\alpha_1^{(n)}, \beta_1^{(n)}$  are understood to be the  $n^{\text{th}}$  derivatives of the quantities  $\alpha_1, \beta_1$  with respect to  $m$  for  $m = 0$ , then

$$\left. \begin{aligned}
 f_0 &= 2 - \frac{\pi^2}{2!} + \frac{\pi^4}{4!} - \frac{\pi^6}{6!} + \dots = 1 + \cos \pi = 0, \\
 g_0 &= \pi - \frac{\pi^3}{3!} + \frac{\pi^5}{5!} - \frac{\pi^7}{7!} + \dots = \sin \pi = 0,
 \end{aligned} \right\} \quad (14)$$

since all  $\alpha_i, \beta_i$  assume the value 1 for  $m \rightarrow 0$ , and furthermore

$$\left. \begin{aligned}
 f_0^{(n)} &= 0 = -\alpha_2^{(n)} \frac{\pi^2}{2!} + \alpha_4^{(n)} \frac{\pi^4}{4!} - \alpha_6^{(n)} \frac{\pi^6}{6!} + \dots \\
 g_0^{(n)} &= 0 = \beta_1^{(n)} \pi - \beta_3^{(n)} \frac{\pi^3}{3!} + \beta_5^{(n)} \frac{\pi^5}{5!} - \dots
 \end{aligned} \right\} \quad (15)$$

The Equations 14 are identically fulfilled and only show that the arrangements were correct. For additional computations we shall make an analysis of Equations 15 for  $n = 1$  explicitly; i.e.,

$$\left. \begin{aligned}
 f_0' &= 0 = -\alpha_2' \frac{\pi^2}{2!} + \alpha_4' \frac{\pi^4}{4!} - \alpha_6' \frac{\pi^6}{6!} + \dots \\
 g_0' &= 0 = \beta_1' \pi - \beta_3' \frac{\pi^3}{3!} + \beta_5' \frac{\pi^5}{5!} - \dots
 \end{aligned} \right\} \quad (16)$$

We have shown above that it is possible to represent the quantities  $\alpha_i, \beta_i$  as functions of  $m, \phi$  and  $\psi$ . Since  $\phi$  and  $\psi$  assume the value 1 for  $m = 0$ , and if we set

$$\left. \begin{aligned}
 \phi(m) &= 1 + a_1 m + a_2 \frac{m^2}{2!} + a_3 \frac{m^3}{3!} + \dots \\
 \psi(m) &= 1 + b_1 m + b_2 \frac{m^2}{2!} + b_3 \frac{m^3}{3!} + \dots
 \end{aligned} \right\} \quad (17)$$

it follows, for  $m = 0$ , that

$$\phi_0^{(n)} = a_n, \quad \psi_0^{(n)} = b_n. \quad (18)$$

From Equations 5 and 9 we obtain, one after another, the equations:

$$\begin{aligned}
 \beta_1' &= \psi', \\
 \alpha_2' &= -2\beta_1 - 2m\beta_1' - 6m + \phi',
 \end{aligned}$$

$$\beta_3' = -2\alpha_2 - 2m\alpha_2' + \phi'\beta_1 + \phi\beta_1' ,$$

$$\alpha_4' = -2\beta_3 - 2m\beta_3' - 6m\alpha_2 - 3m^2\alpha_2' + \phi'\nu_2 + \phi(-2\alpha_2' + 6\beta_1\beta_1') , \text{ etc.}$$

If, in these and the following formulas, we set  $m = 0$  and consider that the  $\alpha_i$ ,  $\beta_i$ ,  $\nu_i$ ,  $\phi$ , and  $\psi$  go towards 1 and the derivations  $\phi'$ ,  $\psi'$  towards  $a_1$ ,  $b_1$ , then, after some brief calculations, we get

$$\alpha_2' \rightarrow a_1 - 2 ,$$

$$\beta_1' \rightarrow b_1 ,$$

$$\alpha_4' \rightarrow 6b_1 - (a_1 - 2) ,$$

$$\beta_3' \rightarrow b_1 + (a_1 - 2) ,$$

$$\alpha_6' \rightarrow 48b_1 - 21(a_1 - 2) ,$$

$$\beta_5' \rightarrow 19b_1 - 7(a_1 - 2) ,$$

$$\alpha_8' \rightarrow 234b_1 - 113(a_1 - 2) ,$$

$$\beta_7' \rightarrow 109b_1 - 51(a_1 - 2) .$$

If we substitute these expressions into Equation 16, we get two linear equations:

$$Ab_1 - B(a_1 - 2) = 0 ,$$

$$Cb_1 - D(a_1 - 2) = 0 ,$$

$$\left. \vphantom{\begin{matrix} Ab_1 - B(a_1 - 2) = 0 \\ Cb_1 - D(a_1 - 2) = 0 \end{matrix}} \right\}$$

(19)

the coefficients of which are represented by the  $\pi$ -series:

$$A = 6 \frac{\pi^4}{4!} - 48 \frac{\pi^6}{6!} + 234 \frac{\pi^8}{8!} - \dots ,$$

$$B = \frac{\pi^2}{2!} + \frac{\pi^4}{4!} - 21 \frac{\pi^6}{6!} + 113 \frac{\pi^8}{8!} - \dots ,$$

$$C = \pi - \frac{\pi^3}{3!} + 19 \frac{\pi^5}{5!} - 109 \frac{\pi^7}{7!} + \dots ,$$

$$D = \frac{\pi^3}{3!} + 7 \frac{\pi^5}{5!} - 51 \frac{\pi^7}{7!} + \dots .$$

This series can be easily summed, for from

$$\cos \pi = 1 - \frac{\pi^2}{2!} + \frac{\pi^4}{4!} - \frac{\pi^6}{6!} + \frac{\pi^8}{8!} - \dots ,$$

$$\frac{1}{2} \cos 2\pi = \frac{1}{2} - 2 \frac{\pi^2}{2!} + 8 \frac{\pi^4}{4!} - 32 \frac{\pi^6}{6!} + 128 \frac{\pi^8}{8!} - \dots ,$$

$$2\pi \sin \pi = 4 \frac{\pi^2}{2!} - 8 \frac{\pi^4}{4!} + 12 \frac{\pi^6}{6!} - 16 \frac{\pi^8}{8!} + \dots$$

we obtain by summation:

$$\begin{aligned} \cos \pi + \frac{1}{2} \cos 2\pi + 2\pi \sin \pi &= \frac{3}{2} + \left( \frac{\pi^2}{2!} + \frac{\pi^4}{4!} - 21 \frac{\pi^6}{6!} + 113 \frac{\pi^8}{8!} - \dots \right) \\ &= \frac{3}{2} + B = -2 . \end{aligned}$$

Similarly we represent A, C, D by closed expressions: specifically,

$$\left. \begin{aligned} A &= -3 + 2 \cos \pi + \cos 2\pi + 3\pi \sin \pi = -4 , \\ B &= -\frac{3}{2} + \cos \pi + \frac{1}{2} \cos 2\pi + 2\pi \sin \pi = -2 , \\ C &= 2 \sin \pi + \sin 2\pi - 3\pi \cos \pi = 3\pi , \\ D &= \sin \pi + \frac{1}{2} \sin 2\pi - 2\pi \cos \pi = 2\pi . \end{aligned} \right\} \quad (20)$$

Since the determinant of the homogeneous system (Equation 19),  $BC - AD = 2\pi$ , differs from zero, it follows that

$$a_1 = 2, \quad b_1 = 0 . \quad (21)$$

After  $a_1, b_1$  are known, we can determine  $a_2, b_2$  in a corresponding manner from the pair of equations  $f_0'' = 0, g_0'' = 0$ . From these a pair of linear equations results:

$$\begin{aligned} Ab_2 - Ba_2 &= X_2 , \\ Cb_2 - Da_2 &= Y_2 , \end{aligned} \quad (22)$$

in which A, B, C, D are defined as in Equation 20, and in which  $X_2, Y_2$  represent new  $\pi$ -series. The coefficients of this  $\pi$ -series can be determined numerically by using Equation 21. With the values (Equation 20) for A, ... D, we find the general expressions

$$\left. \begin{aligned} Ab_n - Ba_n &= X_n, & a_n &= -\frac{3}{2} X_n - \frac{2}{\pi} Y_n, \\ Cb_n - Da_n &= Y_n, & b_n &= -X_n - \frac{1}{\pi} Y_n, \end{aligned} \right\} \quad (23)$$

where  $X_n, Y_n$  are known, if all  $a_i, b_i$  up to  $i = n - 1$  are present numerically.

By carrying out this relatively easy analysis to the order  $n = 3$ , we obtain:

$$\left. \begin{aligned} X_1 &= 3 - 2 \cos \pi - \cos 2\pi - 4\pi \sin \pi = -2B = +4, \\ X_2 &= -3 + \frac{19}{8} \cos \pi + \cos 2\pi - \frac{3}{8} \cos 3\pi - \frac{3}{2} \pi \sin \pi = -4, \\ X_3 &= \frac{15}{2} - 2 \cos \pi - \frac{5}{2} \cos 2\pi - 3 \cos 3\pi - 36\pi \sin \pi = +10, \\ Y_1 &= -2 \sin \pi - \sin 2\pi + 4\pi \cos \pi = -2D = -4\pi, \\ Y_2 &= -\frac{19}{8} \sin \pi + \sin 2\pi - \frac{3}{8} \sin 3\pi + \frac{3}{2} \cos \pi = \frac{3}{2} \pi, \\ Y_3 &= -22 \sin \pi - \frac{5}{2} \sin 2\pi - 3 \sin 3\pi + 36\pi \cos \pi = -36\pi. \end{aligned} \right\} \quad (24)$$

The first coefficients of the development (Equation 17) are therefore known, and we obtain

$$\phi(m) = 1 + 2m + 9 \frac{m^2}{2!} + 57 \frac{m^3}{3!} + \dots,$$

and

$$\psi(m) = 1 + \frac{11}{2} \frac{m^2}{2!} + 26 \frac{m^3}{3!} + \dots.$$

But according to Equation 6, we have

$$\phi = \frac{m^2}{r_0^3} = \frac{m^2}{\xi_0^3}, \quad \psi = m \frac{\dot{\eta}_0}{\xi_0},$$

so that we can also calculate  $\phi$  and  $\psi$  from Hill's power series for  $\xi_0$  and  $\eta_0$ . This calculation yields:

$$\phi(m) = 1 + 2m + 9 \frac{m^2}{2!} + 57 \frac{m^3}{3!} + \frac{1649}{4} \frac{m^4}{4!} + \frac{1235611}{324} \frac{m^5}{5!} + \dots, \quad (25a)$$

$$\psi(m) = 1 + \frac{11}{2} \frac{m^2}{2!} + 26 \frac{m^3}{3!} + \frac{3857}{24} \frac{m^4}{4!} + \frac{23153}{18} \frac{m^5}{5!} + \dots, \quad (25b)$$

and completely verifies our analysis.

## CONCLUDING REMARKS

The foregoing method of obtaining Hill's coefficients is not intended to be more than an interesting attempt to achieve the same goal via a completely different mathematical route. This attempt is interesting for two reasons: (1) Because the coefficients of the series (Equations 25a and 25b) are determined successively from linear equations (Equation 23); and (2) Because they are in the form of closed trigonometric expressions. It must not be overlooked, however, that this method – elegant though it may be – does not lead as far as Hill's, the developments of which can be carried out to the fifth or sixth order without difficulty. Here we could hardly pass beyond the third order – at least not with the method used, which is, however, probably not the only possible one. The reason for this is as follows:

1. The method of determining the coefficients of the trigonometric formulas (Equation 24) may be demonstrated by using the example of  $X_2$ , for which we obtain the  $\pi$ -series

$$X_2 = -6 \frac{\pi^2}{2!} - 6 \frac{\pi^4}{4!} + 198 \frac{\pi^6}{6!} - 2190 \frac{\pi^8}{8!} + \dots \quad (26)$$

Then the arrangement

$$\left. \begin{aligned} X_2 &= a + b \cos \pi + c \cos 2\pi + d \cos 3\pi + e\pi \sin \pi \\ &= a + b \left( 1 - \frac{\pi^2}{2!} + \frac{\pi^4}{4!} - \dots \right) \\ &\quad + c \left( 1 - 4 \frac{\pi^2}{2!} + 16 \frac{\pi^4}{4!} - \dots \right) \\ &\quad + d \left( 1 - 9 \frac{\pi^2}{2!} + 81 \frac{\pi^4}{4!} - \dots \right) \\ &\quad + e \left( 2 \frac{\pi^2}{2!} - 4 \frac{\pi^4}{4!} - \dots \right) \end{aligned} \right\} \quad (27)$$



leads to the equations:

$$\begin{aligned}
 a + b + c + d &= 0 , \\
 -b - 4c - 9d + 2e &= -6 , \\
 b + 16c + 81d - 4e &= -6 , \\
 -b - 64c - 729d + 6e &= 198 , \\
 b + 256c + 6561d - 8e &= -2190 ,
 \end{aligned}
 \tag{28}$$

the number of which can be expanded as far as the coefficients of the  $\pi$ -series are known. Only if the first five of these equations yield the coefficients of the trigonometric arrangement (Equation 27), will this formula solve the problem exactly; however, when the coefficients —  $a, \dots, e$  — calculated by means of it also fulfill the other equations (Equation 28), a demand which could only be satisfied if it were possible to find a law (and then to prove it) by which the terms on the right side of Equation 28 — the coefficients of Equation 26 — would increase.

In the case of the developments (Equation 24), it was possible to check the validity of the formula with at least *one* of the supernumerary equations and to thereby prove it. Furthermore, in this case, it is not difficult to find the law of the progression of the coefficients. The agreement with the results of Hill also vouch for their correctness.

2. The difficulties of extending this method begin with the determination of  $X_4, Y_4$ , for the numerical values of the coefficients of the  $\pi$ -series (Equation 26) increase extraordinarily, and because until now it has remained uncertain which formula (Equation 27) would achieve the right goal. A trial of the formula

$$X_4 = a + b \cos \pi + c \cos 2\pi + d \cos 3\pi + e \cos 4\pi + f \pi \sin \pi$$

and a corresponding one for  $Y_4$  has not succeeded, probably because the formula requires still more terms — at least one with  $\cos 5\pi$ .

## RECOMMENDATIONS

Further experiments with this mathematical problem could be useful and interesting, and we shall conclude this report with some suggestions:

1. The expressions of Equation 24 can be given in a simpler form. For instance, we could write

$$X_1 = 2(1 - \cos \pi) + 2 \sin^2 \pi - 4\pi \sin \pi ,$$

and by the substitution of the "c-functions" known from the two-body motion,\*

$$c_0 = \cos \lambda, \quad c_1 = (\sin \lambda)/\lambda, \quad c_2 = (1 - \cos \lambda)/\lambda^2, \quad c_3 = (\lambda - \sin \lambda)/\lambda^3, \quad \dots,$$

we then obtain, for  $\lambda = \pi$ :

$$\begin{aligned} X_1 &= \pi^2 \left[ 2(c_2 + c_1^2) - 4c_1 \right], \\ X_2 &= \pi^2 \left[ -2(c_2 + c_1^2) + \frac{3}{2} c_1 (c_0 c_1 - 1) \right], \\ X_3 &= \pi^2 \left[ 5(c_2 + c_1^2) + 12c_1 (c_0 c_1 - 3) \right], \end{aligned}$$

and

$$\begin{aligned} Y_1 &= \pi^3 \left[ 2(c_3 - c_2) + 2c_0 c_3 \right], \\ Y_2 &= \pi^3 \left[ \frac{3}{2} (c_3 - c_2) + c_1 \left( \frac{3}{2} c_1^2 - 2c_2 \right) \right], \\ Y_3 &= \pi^3 \left[ 36(c_3 - c_2) + c_1 (12c_1^2 + 5c_2) \right]. \end{aligned}$$

2. Noticeable and displeasing is the fact that the expressions of Equation 24 contain a rather large number of disappearing terms, which are necessarily created by the algorithm which leads to them. It should be possible to modify the process in such a way that these superfluous and complicating terms can be avoided at the beginning. Here is a mathematical problem that is not only challenging, but its solution would be very useful in celestial mechanics wherever the problem of seeking initial conditions for periodic orbits exists.

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\*Stumpff, K., "Calculation of Ephemerides from Initial Values," NASA Technical Note D-1415, 1962.