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PERIODIC SOLUTIONS OF THE RESTRICTED THREE BODY PROBLEM REPRESENTING ANALYTIC CONTINUATIONS OF KEPLERIAN ELLIPTIC MOTIONS

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SUMMARY

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A detailed mathematical proof is given in this report for the following new result: In the Restricted Three Body Problem with small mass ratio there exist one-parametric analytic families of synodically closed solution curves, which are near rotating Keplerian ellipses with rational sidereal frequencies and appropriate positive eccentricities.

I. INTRODUCTION

The equations of motion for the plane Restricted Three Body Problem can be written in the form

$$x''+2ix'-x = -(1-\mu)(x+\mu)|x+\mu|^{-3} - \mu(x+\mu-1)|x+\mu-1|^{-3}, \quad (' = d/dt) \tag{1}$$

where $x = x_1+ix_2$ is the complex position vector of the infinitesimal body P referred to a co-system rotating with angular velocity 1 about the center of gravity S of the two attracting bodies E and M of masses $1-\mu$ and $\mu(0\leq \mu \leq 1)$ as origin.

When $\mu = 0$, the solutions of (1) are well known and can be represented as $x(t) = e^{-it}z(t)$, where the complex position vector $z(t)$ describes the Keplerian motion: that is a solution of $z'' = -z|z|^{-3}$. Under suitable initial conditions this latter motion will be periodic. For instance, with

$$z(0) = a(1+\epsilon), z'(0) = ic^*/z(0), \ c^{*2} = a(1-\epsilon^2), \ (a>0, \ 0<\epsilon<1), \tag{2}$$

$z(t)$ moves along an ellipse with major half axis a and eccentricity ϵ, having $z = 0$ as focus and $z(0)$ at maximum distance from 0. Its sidereal period is

$T_0 = 2\pi|a^{3/2}|$. The corresponding $x(t)$ will be periodic, iff. T_0 is commensurable with 2π, or $a^{3/2} = m/k$, where k and m are relatively prime integers, $m > 0$ and k is chosen positive respectively negative, if $z(t)$ is direct respectively retrograde, that is, sign k = sign c^*. The synodical period on the rotating ellipse then is $T^* = 2\pi m$ and the curve $x = x(t)$, $(0 \leqq t \leqq T^*)$ is closed after k-m positive revolutions around the origin. We denote this solution of (1) with $\mu = 0$ from now on by $x^*(t)$ and obtain from (2) for its initial values

$$x^* = a(1+\epsilon), \quad dx^*/dt = i(c^* - x^{*2})/x^* \quad \text{at } t = 0. \tag{3}$$

We shall show the existence of periodic solutions $x(t)$ of (1) for small $\mu > 0$, which are near the generating solutions $x^*(t)$ belonging to arbitrary integers $k,m \neq 0$ and properly restricted ϵ: Namely, there are for fixed $a = (m/k)^{2/3}$ at most finitely many ϵ in $0 < \epsilon < 1$ with $\epsilon = \sqrt{(1-a^{-3})}$ or with $x^*(t) = 1$ at least once in $0 \leqq t \leqq T^*$. For every closed ϵ-interval I, containing none of these exceptional values, there exists a positive μ^* such that (1) possesses for every fixed μ in $0 \leqq \mu < \mu^* \leqq 1$ a family of periodic solutions depending analytically upon the parameter ϵ in I. These solutions are holomorphic also in μ and transfer into $x^*(t)$ for $\mu = 0$. Their synodical periods and Jacobi constants are holomorphic in ϵ and μ and depend both actually upon μ.

This result includes especially the existence of the periodic solutions of the so-called second kind for the Restricted Three Body Problem. Their existence had been claimed with supposed proofs by H. Poincaré [5], K. Schwarzschild [6] and C. L. Charlier [2], whose invalidity was shown by P. Staeckel [8] and A. Wintner [9], [10], however. In these attempts the continuation method of Poincaré was employed in an isoperiodic or an isoenergetic manner. We too apply this continuation method, but replace the general periodicity condition of Poincaré with a more special one which is based on the symmetry of the dynamical problem (1), and has already been used by G. D. Birkhoff [1]. This condition is not only simpler and more natural, it also points out and reduces the redundancy in the classical periodicity conditions, which caused the critical functional determinants to vanish (Ref. [9] and [7]). Otherwise, we achieve our goal by employing appropriate variables, which render the dependence of the Keplerian motion upon its initial values in a most simple form.

In this regard, it is to be mentioned that G. D. Birkhoff [1] showed, among others, the existence of periodic solutions of (1) for small μ, which close in the rotating co-system only after sufficiently many revolutions about the mass $1-\mu$, if their Jacobi constant determines a simple closed zero-velocity curve containing the orbits in its interior. B. O. Koopmann [3] established the analogon for the exterior case of a zero-velocity curve with forbidden bounded interior. More recently, J. Moser [4] showed the existence of periodic solutions of (1) for small μ, which close after many revolutions and are near solutions of the existing first kind of Poincaré (generated from circular motions for $\mu = 0$). All these solutions correspond for $\mu = 0$ to periodic motions along rotating ellipses with rational $a^{3/2}$. Presently, it is not known if these solutions for $\mu > 0$ coincide with certain of our above solutions $x(t)$ generated from $x^{*}(t)$ with large $|k-m|$ and suitable, small $\epsilon > 0$.

Finally, we remark that several of these solutions for different m/k, whose existence is shown here for small $\mu > 0$, have been numerically calculated by us for increasing μ on high speed electronic computers. They are particularly of interest when $a(1-\epsilon) < \delta$ and $1 < a(1+\epsilon) < 1+\delta$ with small $\delta > 0$, since then they pass repeatedly near both masses of the Restricted Three Body Problem. The calculations indicate their existence for values of μ at least as large as that for the case $1-\mu/\mu$ equal to mass of the Earth/mass of the Moon. Thus, their practical significance for astronautics is apparent and was one of the incentives for the investigation presented here.

II. EXISTENCE PROOF

Let $x(t)$ be a solution of (1), which is holomorphic on an interval $0 \leqq t \leqq T$, $T > 0$; that is, free of collisions. If (with a bar denoting the conjugate complex number)

$$x(\tfrac{1}{2}T) = \bar{x}(\tfrac{1}{2}T) \;, \; x'(\tfrac{1}{2}T) = -\bar{x}'(\tfrac{1}{2}T), \tag{4}$$

then the function $\bar{x}(T-t)$ of t is identical with $x(t)$, since it also satisfies (1) and the two functions and their first derivatives coincide respectively at $t = \frac{1}{2}T$. This implies, if additionally

$$x(0) = \bar{x}(0) \;, \; x'(0) = -\bar{x}'(0), \tag{5}$$

that $x(T) = x(0)$, $x'(T) = x'(0)$, so that $x(t)$ will be periodic with period T, since (1) is autonomous. Then the closed curve $x = x(t)$, $(0 \leqq t \leqq T)$ is symmetric over the x_1-axis, since $x(-t) = \bar{x}(t)$. Especially $x^*(t)$ satisfies (5) by (3), and also (4) with $T = T^* = |k|T_0$, since as a consequence of (2) $z(k\cdot T_0/2)$ is real and $z'(k\cdot T_0/2)$ is pure imaginary for every integer k.

We introduce new real variables F, H, U, V instead of $x = x_1 + ix_2$ and $y = x' + ix = y_1 + iy_2$ by

$$\begin{cases} F = \operatorname{arctg} x_2/x_1 \;, & H = \tfrac{1}{2}(y_1^2 + y_2^2) - r^{-1} - c, \; r = (x_1^2 + x_2^2)^{\frac{1}{2}}, \\ U = x_1/r - cy_2 \;, & V = x_2/r + cy_1 \qquad\qquad , \; c = x_1y_2 - x_2y_1 \,. \end{cases} \tag{6}$$

The functional determinant of this transformation is $D = -(2H+3c)\cdot c^2/r^2$. Now c and H are first integrals of (1) for $\mu = 0$, since with $z(t) = e^{it}x(t)$ then

$$c = \mathrm{Im}\,\bar{x}y = \mathrm{Im}\,\bar{z}z' = \text{const. of area},$$

$$H + c = \tfrac{1}{2}|z'|^2 - |z|^{-1} = \text{const. of energy},$$

as well known for the Keplerian motion. By (2) then

$$c = c^*,\ H = -c^* - 1/2a\ ,\ r = |x^*(t)| \geqq a(1-\epsilon) > 0 \text{ on } x^*(t).$$

Thus

$$D = -(c^* - 1/a)c^{*2}/r^2 \quad \text{on } x^*(t), \tag{7}$$

and the transformation (6) is analytic and locally 1-to-1 in a neighborhood of every point on the trajectory $x^*(t)$, $y^*(t) = dx^*(t)/dt + ix^*(t)$, $(0 \leqq t \leqq T^*)$ if $D \neq 0$ or $ac^* \neq 1$. This holds for $0 < \epsilon < 1$ always, when $a \leqq 1$ or when $\epsilon \neq \sqrt{(1-a^{-3})}$ for $a > 1$, $c^* > 0$, and this assumption will be made from here on. Now (6) transforms (1) in case $\mu = 0$ into

$$\begin{cases} F' + 1 = c/r^2 = c^{-3}(1-U\cos F-V\sin F)^2, \\ H' = 0,\ U' = V,\ V' = -U, \end{cases} \tag{8}$$

since from (6) $c = r(1-U\cos F-V\sin F)/c$. If we denote initial values by the corresponding small letter, (8) can be integrated up to

$$\begin{cases} H = h,\ U + iV = (u+iv)e^{-it}, \\ F' + 1 = c^{-3}(1-u\cos(F+t) - v\sin(F+t))^2\ . \end{cases} \tag{9}$$

This and (6) yield

$$u^2 + v^2 = U^2 + V^2 = 1-2c^2/r + c^2(y_1^2 + y_2^2) = 1 + 2hc^2 + 2c^3, \tag{10}$$

so that c depends upon the initial values h, u, v. Substituting for these the special values

$$f^* = 0,\ h^* = -c^*-1/2a\ ,\ u^* = \epsilon,\ v^* = 0 \tag{11}$$

derived from (3) and (6) gives $c = c^*$ and by (9) the original solution $x^*(t)$, but now represented in the new variables.

In general (6) transforms (1) into a system

$$F' = g_1 \ , \ H' = g_2 \ , \ U' = g_3 \ , \ V' = g_4 \ , \tag{12}$$

where the $g_n = g_n(F,H,U,V,\mu)$ are holomorphic functions of all variables in a neighborhood of the special solution determined by (9), (11) and for sufficiently small $\mu \geqq 0$, if $x^*(t)$ is free of collisions for $\mu = 0$. Given $a = (m/k)^{2/3}$, such a collision, that is, $x^*(t) = 1$ for some t in $0 \leqq t \leqq T^*$, can happen only for finitely many values of ϵ in $0 < \epsilon < 1$. We return to this condition later and assume here only that these ϵ are omitted. Then, according to Poincaré's extension of Cauchy's existence theorem for ordinary differential equations (for a modern proof see Ref. [7]) the solutions of (12) are holomorphic functions of $t,f,h,u,v,$ and μ for $0 \leqq t \leqq 2T^*$, say, and sufficiently small $|f| + |h-h^*| + |u-\epsilon| + |v| + \mu$, $\mu \geqq 0$, using (11). For $\mu = 0$ (12) becomes (8).

We consider now the solutions of (12) with small $\mu > 0$ and initial values near (11). These solutions can be assumed remaining near the original solution given by (9), (11) for $0 \leqq t \leqq 2T^*$, so that especially (6) is applicable. In order that such a solution will be periodic in the former coordinates x_1, x_2 with period $T > 0$, it is sufficient according to (4), (5), and (6) that

$$\begin{cases} F(\tfrac{1}{2}T, f,h,u,v,\mu) = \pi(k-m) \ , & f = 0 \ , \\ V(\tfrac{1}{2}T, f,h,u,v,\mu) = 0 & , \ v = 0 \ , \end{cases} \tag{13}$$

since for the considered solutions of (12) c remains near c^*, and therefore the assumption $c \neq 0$ at $t = \frac{1}{2}T$ can be made, if $T < 2T^*$. These equations are actually satisfied for the original solution $x^*(t)$ and $\mu = 0$, or with (11) for

$$T = T^* = 2m\pi, \ f = f^*, \ h = h^*, \ u = u^*, \ v = v^*, \ \mu = 0 \ , \tag{14}$$

since

$$\arc x^*(\tfrac{1}{2}T^*) = \arc\left[e^{-im\pi} z(|k|T_o/2)\right] = \arc\left[e^{-im\pi} e^{ik\pi}\right] = (k-m)\pi$$

by (2). Hence (13) can be satisfied for small $\mu \geqslant 0$ and initial values near (14), if for instance the functional determinant

$$D^* = F_t V_h - F_h V_t \neq 0 \quad \text{for } t = \tfrac{1}{2}T \text{ at (14).} \tag{15}$$

By the holomorphy it suffices putting $\mu = 0$ in F, V first and then calculate the partial derivatives, which therefore can be found from (9) and (10). Denoting partial derivatives by a corresponding index, (9) gives $V_h = 0$ and $V_t = -U$, hence with (11) and (14)

$$D^* = \epsilon \cos m\pi \cdot F_h(m\pi, 0, h^*, \epsilon, 0), \quad (\mu=0). \tag{16}$$

We put

$$\phi = t + F(t,0,h,\epsilon,0) = \phi(t,h) \;. \tag{17}$$

Then it follows from (9) that ϕ is uniquely determined by inversion of the integral

$$t = c^3 \int_0^{\phi} (1-\epsilon \cos\psi)^{-2}\, d\psi \;. \tag{18}$$

The value of c here is determined by $\epsilon^2 = 1+2hc^2+2c^3$ from (10). This implies

$$c_h = -c(2h+3c)^{-1} \;. \tag{19}$$

Differentiating (18) partially with respect to h gives

$$0 = 3c^2 c_h \int_0^{\phi} (1-\epsilon \cos\psi)^{-2} d\psi + c^3 (1-\epsilon \cos \phi)^{-2} \phi_h \;,$$

hence with (19)

$$\phi_h(m\pi, h^*) = 3(1-\epsilon \cos \phi^*)^2 \, (2h^* + 3c^*)^{-1} \int_0^{\phi^*} (1-\epsilon \cos \psi)^{-2} d\psi \;, \tag{20}$$

where $\phi^* = \phi(m\pi, h^*)$. But

$$2c^{*3} \int_0^{\pi} (1-\epsilon \cos \psi)^{-2} d\psi = \pm T_0 = 2\pi m/k$$

for the period on the original Keplerian ellipse as well known, thus by (18) $\phi^* = k\pi$ and by (16), (17), and (20) finally

$$D^* = 3\epsilon(-1)^m(1-\epsilon(-1)^k)^2 m\pi / \; (c^*-a^{-1})c^{*3} \neq 0 \;, \tag{21}$$

so that (15) actually holds, since $0 < \epsilon < 1$ and $c^* \neq a^{-1}$ as required for (7).

From the implicit function theorem follows now that (13) can be solved for T and h in a neighborhood of (14), and that $T-T^*$ and $h-h^*$ for $u=u^*=\epsilon$ result as power series in μ without constant terms having positive radius of convergence. This implies especially $0 < T < 2T^*$ for small $\mu > 0$ as assumed before, so that by (13) the existence of the desired periodic solutions $x(t)$ is now actually shown. Their initial values $x(0)$, $x'(0)$ are determined by (6) and

$$f = 0, \; h = h(\mu,\epsilon), \; u = \epsilon, \; v = 0 \tag{22}$$

as functions of μ, ϵ and a.

Since for any given $a = (m/k)^{2/3}$ the foregoing is valid as long as ϵ is not one of the previously excepted values, it is a consequence of the local existence theorem for implicit functions and of the covering theorem that $T = T(\mu,\epsilon)$ and $h = h(\mu,\epsilon)$ are holomorphic functions of μ and ϵ on every closed ϵ-interval not containing one of the exceptional values and on $0 \leqq \mu < \mu^*$ with corresponding sufficiently small $\mu^* > 0$. The solutions of (12) belonging to (22) are holomorphic functions of their initial values and thus the corresponding periodic solutions $x(t)$ are holomorphic on $0 \leqq t \leqq T$ and in ϵ and μ as stated in the introduction.

If one introduces in the transformation (6) instead of the variable H the Jacobi integral

$$J = \tfrac{1}{2}|y|^2 - c - (1-\mu)|x+\mu|^{-1} - \mu|x+\mu-1|^{-1},$$

our whole consideration can be carried through in the same way. Thus also $J = J(\mu,\epsilon)$ is a holomorphic function of both variables, which clearly follows from

$$J = h + |x_o|^{-1} - (1-\mu)|x_o+\mu|^{-1} - \mu|x_o+\mu-1|^{-1}, \quad x_o = x(0)$$

and the foregoing, too. In (13) then h is to be replaced by the initial value j of J, but J = j now. It is then of interest to consider besides (15) the other two functional determinants suggested by (13): namely,

$$D_1^* = F_t V_u - F_u V_t \quad \text{and} \quad D_2^* = F_j V_u - F_u V_j$$

for $t = \frac{1}{2}T$ at (14), where $j = h^*$. Since here $V_u = -\sin m\pi = 0$, $V_j = 0$ and $V_t = -U$ by (9), it follows $D_2^* = 0$ and $D_1^* = \epsilon \cos m\pi \cdot F_u(m\pi, 0, h^*, \epsilon, 0)$.

Putting

$$\psi = t + F(t, 0, h^*, u, 0) = \psi(t, u) ,$$

it follows from (9) $\psi_t = c^{-3}(1-u \cos \psi)^2$ and thus similar as from (18) for Φ now $\psi(m\pi, u) = k\pi$ identically in $0 < u < 1$. Hence also $D_1^* = 0$. Thus, solvability of (13) with respect to T,u for fixed $j = h^*$, or with respect to j,u for fixed $T = T^*$ remains at least doubtful, if at all possible. In fact, A. Wintner [9] has shown that for sufficiently small $\epsilon > 0$ isoperiodic solutions do not exist.

It is interesting to note that the treatment presented here is also effective for the periodic solutions of the first kind. In this case, $\epsilon = 0$ and $a^{3/2} = \pm T_o/2\pi = \omega^{-1}$ can be taken arbitrarily, especially not rational, with the exceptions $\omega \neq -1, 1$ and $\omega \neq 1 \pm m^{-1}$, m natural. Then it follows $D_1^* \neq 0$, $D_2^* \neq 0$, but $D^* = 0$ in (15), which is the reverse of our present situation for $\epsilon > 0$. Of course, an existence proof for the solutions of the first kind is long known.

Finally, we consider the restrictions placed upon ϵ for our existence proof. These are

$$0 < \epsilon < 1, \quad \epsilon \neq \sqrt{(1-a^{-3})} \quad \text{for } a > 1, \quad c^* > 0,$$

$$x^*(t) \neq 1 \quad \text{in } 0 \leqq t \leqq m\pi \quad \text{for } a = (m/k)^{2/3} . \tag{23}$$

The first of them is equivalent to $ac^* \neq 1$ or $2h^* + 3c^* \neq 0$ and was required for (7) and (21). The dynamical meaning of this condition becomes clear, when the sidereal frequency $\omega = 2\pi/T_o = |a^{-3/2}|$ is used instead of a to characterize the generating elliptic motion together with its eccentricity. Since here $\mu = 0$, H coincides with the Jacobi-integral J, and then (19) shows that the Jacobi-constant $j = h$ of the generating motion $x^*(t)$, for which now

$$j^* = -c^* - 1/2a = -\tfrac{1}{2}\omega^{2/3} - \text{sign } c^* \cdot \omega^{-1/3} \sqrt{(1-\epsilon^2)} \;, \quad (\omega > 0)$$

has for given ϵ as function of c^* or ω a relative extremum (absolute maximum) for $ac^* = 1$ or $\omega = \sqrt{(1-\epsilon^2)}$, in the direct case only. The inverse function and thus the sidereal period T_o then cannot be defined uniquely as functions of the Jacobi constant and the eccentricity in a neighborhood of the branch points $(j^* = -(1-\epsilon^2)^{2/3} \cdot 3/2, \epsilon)$. The dynamical meaning of (23) is to exclude collisions with the perturbing body.

We shall show that for fixed k and m (23) excludes at most finitely many values of ϵ. To see this, we represent the Keplerian motion $z = z(t) = e^{it}x^*(t)$ with the help of the eccentric anomaly w, namely with (2) in the well known form

$$\begin{cases} z = a(\epsilon + \cos w + i\sqrt{(1-\epsilon^2)} \sin w) \;, \\ t = a^{3/2} (w + \epsilon \sin w) \end{cases} \tag{24}$$

If (23) does not hold, then with appropriate ϵ between 0 and 1

$$z = e^{it} \;, \quad |z| = a\,(1 + \epsilon \cos w) = 1 \tag{25}$$

for some w between 0 and $k\,\pi$ inclusively, using (24). The last equation implies $|1-1/a| \leqq \epsilon < 1$, thus $a > \frac{1}{2}$. Hence (23) holds always, when $2a \leqq 1$. If $a = 1$, then by (25) $\cos w = 0$ and thus with (24)

$$\epsilon = \cos t = \cos(w \pm \epsilon) = - \sin \epsilon,$$

which is impossible for $0 < \epsilon < 1$. Let now $a > \frac{1}{2}$, $a \neq 1$ and (25) be satisfied for appropriate ϵ and w. Then with $-A = 1-1/a$ and (24)

$$\epsilon \cos w = A \neq 0, \; a(\epsilon + \cos w) = \text{cost},$$

$$a(\cos w + A/\cos w) - \cos\left[a^{3/2}(w + A \operatorname{tg} w)\right] = 0. \tag{26}$$

Since $|A| \leqq \epsilon < 1$ and with w^* from $\cos w^* = |A|$, $0 < w^* \leqq \pi/2$, we have here

$$|w - n\pi| < w^* , \quad n = \text{integer}, \; |n| \leqq |k|. \tag{27}$$

Now the left side in (26) is a holomorphic function of w in (27) and even in all of the finitely-many closed circles $|w-n\pi| \leqq w^*$. Thus it has at most finitely-many zeros in (27), and these correspond to at most finitely-many $\epsilon = A/\cos w$ satisfying (25). This proves our statement about (23).

One can easily see that our whole derivation remains valid, if we begin in (2) with $z(0) = a(1-\epsilon)$ at minimum distance from 0. This merely replaces ϵ by $-\epsilon$ in the subsequent equations. But it leads to new periodic solutions of (1) for small μ, which are actually different from the previous ones, even when considering solution curves only, if k-m is odd. Additional such solutions are found starting with $x^* = -a(1+\epsilon)$ in (2), if k-m is even. All of our periodic solution curves are symmetric over the x_1-axis, and they can be readily visualized in the rotating co-system.

Added are a few illustrations calculated for the case $\mu = 1/82$: Figure 1 and Figure 3 show synodically closed solution curves of (1) for m=1, k=2 and m=2, k=5 respectively, in the rotating co-system. Figure 2 shows for the first case the circular paths of E and M and the path of P in an inertial co-system with origin at S. Figure 4 shows for the second case the paths of P and M in a co-system with origin at E and space fixed orientation. Note the short duration capture of P by M with subsequent rejection, which happens after elapse of every T units of time; i.e., every time when P has completed nearly k Keplerian elliptic orbits with focus at E.

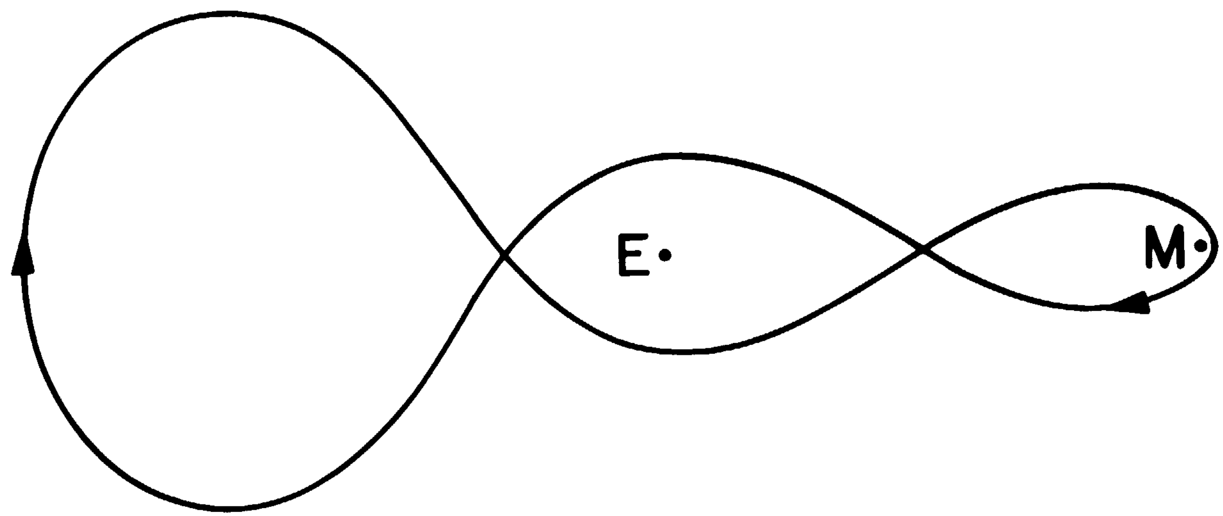

FIGURE 1. CLOSED PATH OF P IN ROTATING CO.-SYSTEM WITH m=1, k=2.

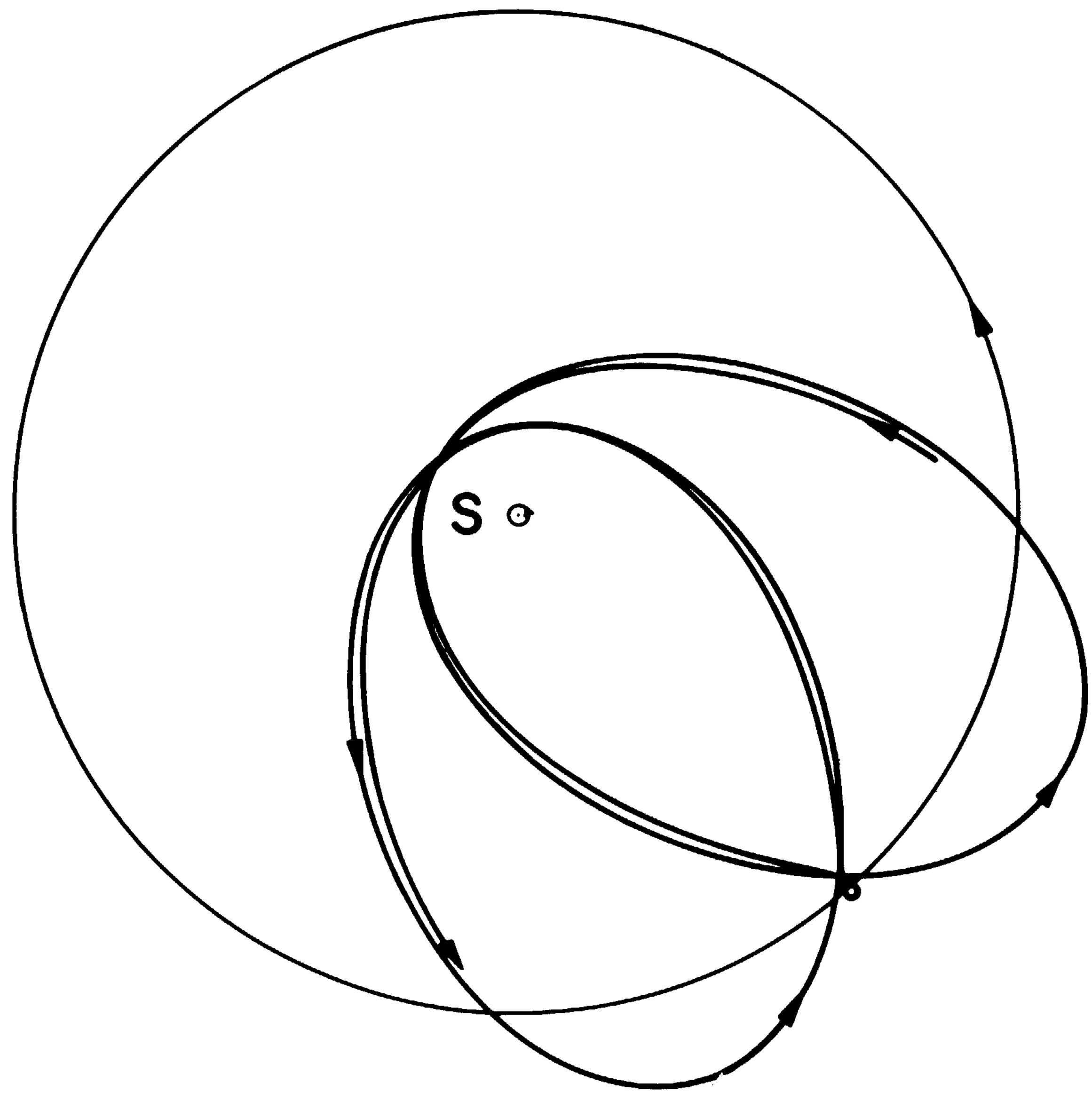

FIGURE 2. CIRCULAR PATHS OF E, M AND THE PATH FROM FIGURE 1. IN INERTIAL CO.-SYSTEM FROM t=0 TO NEARLY t=2T. CAPTURE AND REJECTION OF P BY M.

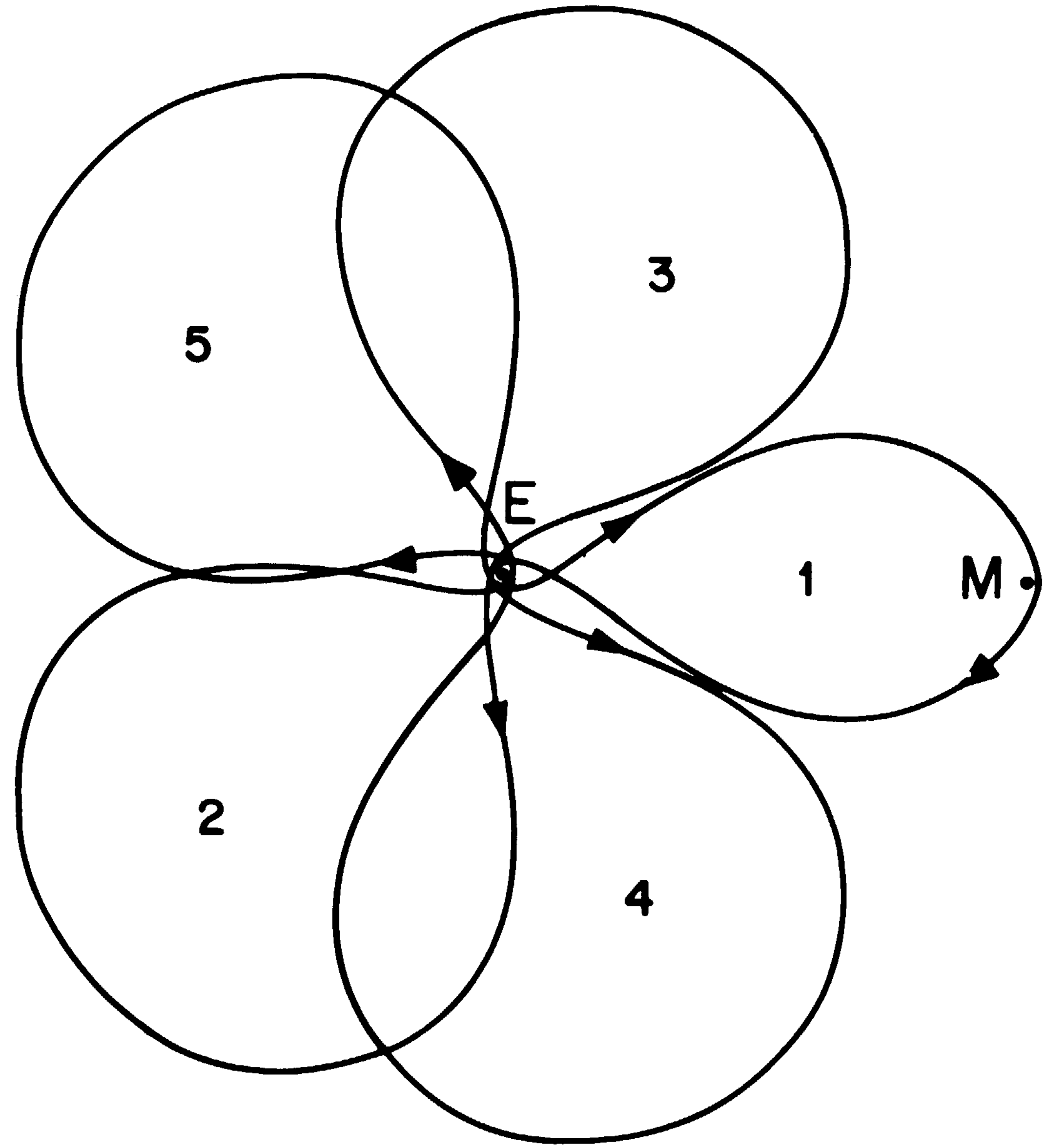

FIGURE 3. CLOSED PATH OF P IN ROTATING CO.-SYSTEM WITH $m=2$, $k=5$. NUMBERS SHOW LOOPS MADE BY P IN SUCCESSION.

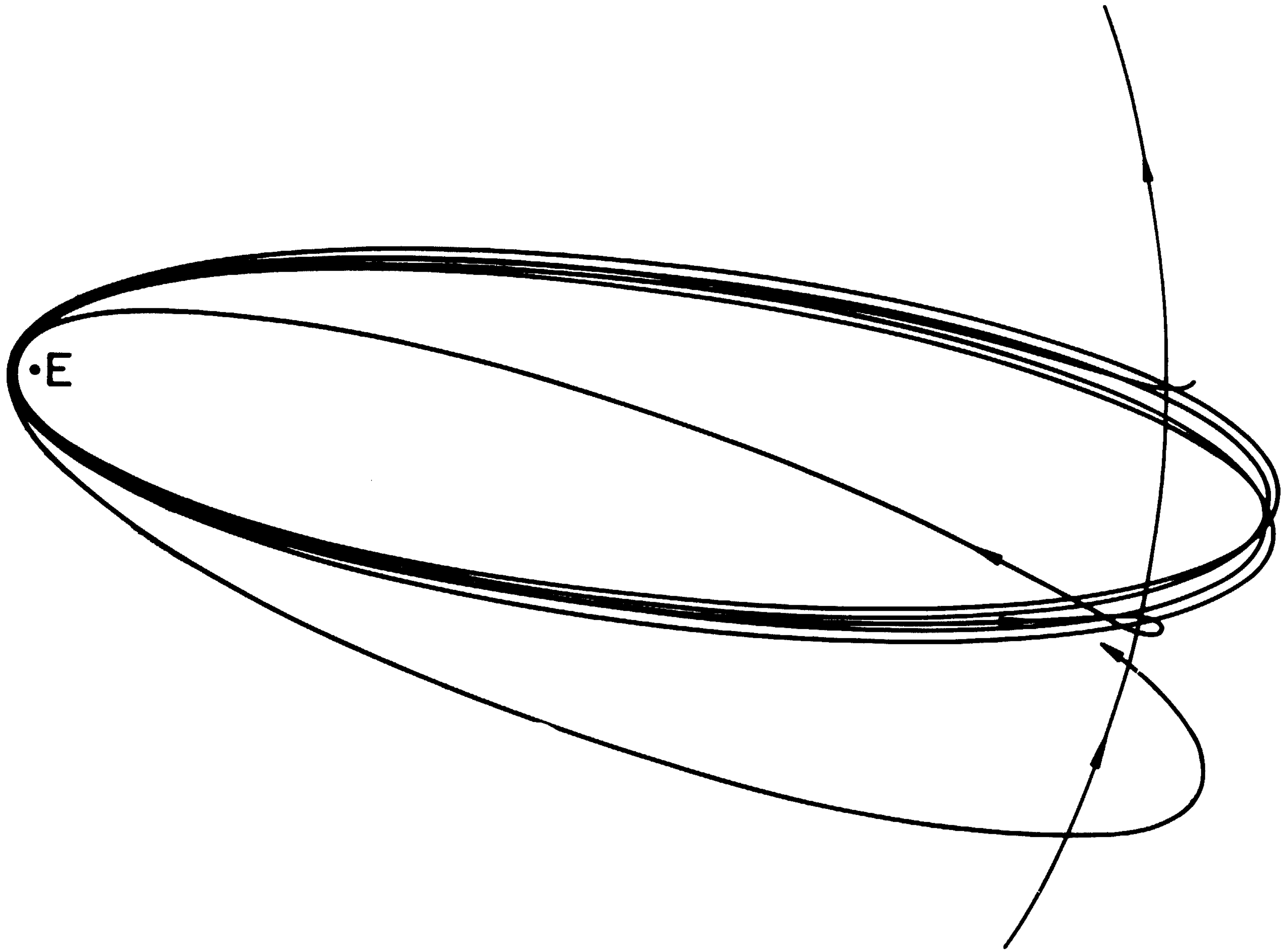

FIGURE 4. CIRCULAR ARC OF M, AND THE PATH FROM FIGURE 3 AS SEEN FROM E UNDER SPACE FIXED ORIENTATION NEARLY FROM $t=0$ TO $t=6/5$ T. CAPTURE AND REJECTION OF P BY M.

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