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A SIMPLIFIED METHOD OF DETERMINING THE ELASTIC STATE OF THERMAL STRESS IN A THIN, FLAT PLATE OF FINITE DIMENSIONS

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SUMMARY

An integrodifferential equation that defines the behavior of the shear stress in a thermally loaded, thin, flat plate of finite dimensions is derived. The equation is reduced to a system of linear algebraic equations by a method that employs polynomial approximations for the shear stresses. The boundary conditions are satisfied identically. Several examples of the method, presented in detail, show that solutions of high accuracy can be produced on a desk calculator with a minimum of labor. A calculation of the displacements shows that the strain compatibility conditions are closely satisfied everywhere.

INTRODUCTION

Several methods now exist for determining the elastic stress distribution in a thermally loaded, thin, flat plate of finite dimensions (refs. 1 to 6). All these methods are numerical procedures that eventually produce a system of simultaneous ordinary differential equations or algebraic equations that require considerable labor to effect a solution.

Reference 6 provides intermediate information from a collocation procedure in tabular form that minimizes the effort required to determine the spanwise and chordwise stresses for a large variety of plate geometries and temperature distributions. However, for span to chord ratios other than those tabulated in that reference, an interpolation procedure must be used that requires extreme care because the functions vary quite rapidly. A relatively simple method for obtaining a solution to this problem is therefore desirable.

A similar problem for a finite cylinder, rather than for a plate, has recently been treated in reference 7. In that reference, an integrodifferential equation in the shear stresses is derived that lends itself to solution by a two-dimensional collocation procedure requiring relatively little labor.

In a similar fashion, this report derives the basic integrodifferential equation that defines the behavior of the elastic shear stresses in a finite, thin, flat plate subjected to an arbitrary thermal gradient and gives the functions necessary to approximate the solution in a collocation procedure.

Examples that are given and compared with other solutions show good agreement. As an additional check, it is shown that the compatibility conditions are closely satisfied throughout the region.

SYMBOLS

C^{ij}_{kl} dimensionless constant

E Young's Modulus

P_i polynomial in x associated with i^{th} station

Q_j polynomial in y associated with j^{th} station

T temperature function

T_o reference temperature

t_{ij} constant, equal to dimensionless shear stress at the point (x_i,y_j)

u spanwise displacement

v chordwise displacement

x transformed spanwise coordinate

y transformed chordwise coordinate

α linear coefficient of thermal expansion

β span to chord ratio

γ shear strain

ϵ normal strain

η chordwise coordinate

ν Poisson's ratio

ξ spanwise coordinate

σ normal stress

τ shear stress

$\prod_{k \neq l}$ product for all values of k except $k = l$

Superscripts:

* dimensionless quantity

' derivative

ANALYSIS

Integrodifferential Equation

With reference to the coordinate system shown in figure 1(a), the relations that define the elastic state of stress and strain for the case of plane stress are given as follows in reference 8. (Note that the stress-strain relations have been modified to include thermal strain terms.)

$$\epsilon_\xi = \frac{1}{E}(\sigma_\xi - \nu\sigma_\eta) + \alpha T \tag{1a}$$

$$\epsilon_\eta = \frac{1}{E}(\sigma_\eta - \nu\sigma_\xi) + \alpha T \tag{1b}$$

$$\gamma = \frac{1 + \nu}{E}\tau \tag{1c}$$

$$\frac{\partial}{\partial\xi}\sigma_\xi + \frac{\partial}{\partial\eta}\tau = 0 \tag{2a}$$

$$\frac{\partial}{\partial\eta}\sigma_\eta + \frac{\partial}{\partial\xi}\tau = 0 \tag{2b}$$

$$\frac{\partial^2}{\partial\eta^2}\epsilon_\xi + \frac{\partial^2}{\partial\xi^2}\epsilon_\eta - 2\frac{\partial^2}{\partial_\xi\,\partial_\eta}\gamma = 0 \tag{3}$$

Making all quantities dimensionless by dividing equations (1) and (3) by αT_o and equation (2) by $E\alpha T_o$ and transforming ξ and η to x and y (fig. 1(b)) yield:

$$\epsilon_x^* = \sigma_x^* - \nu\sigma_y^* + T^* \tag{4a}$$

$$\epsilon_y^* = \sigma_y^* - \nu\sigma_x^* + T^* \tag{4b}$$

$$\gamma^* = (1 + \nu)\tau^* \tag{4c}$$

$$\frac{1}{\beta} \frac{\partial}{\partial x} \sigma_x^* + \frac{\partial}{\partial y} \tau^* = 0 \tag{5a}$$

$$\frac{\partial}{\partial y} \sigma_y^* + \frac{1}{\beta} \frac{\partial}{\partial x} \tau^* = 0 \tag{5b}$$

$$\frac{\partial^2}{\partial y^2} \epsilon_x^* + \frac{1}{\beta^2} \frac{\partial^2}{\partial x^2} \epsilon_y^* - \frac{2}{\beta} \frac{\partial^2}{\partial x \, \partial y} \gamma^* = 0 \tag{6}$$

Expressing the compatibility equation, (6), in terms of the stresses, first by substituting the stress-strain equations, (4), into equation (6) and then eliminating the shear terms by differentiating equation (5a) with respect to x and (5b) with respect to y and substituting yield:

$$\left(\frac{1}{\beta^2} \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(\sigma_x^* + \sigma_y^*\right) = -\left(\frac{1}{\beta^2} \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) T^* \tag{7}$$

Integrating equations (5) produces the following expressions for σ_x^* and σ_y^* in the first quadrant:

$$\sigma_x^* = \beta \int_x^1 \frac{\partial}{\partial y} \tau^* \, dx + \sigma_x^*(1,y) \tag{8a}$$

$$\sigma_y^* = \frac{1}{\beta} \int_y^1 \frac{\partial}{\partial x} \tau^* \, dy + \sigma_y^*(x,1) \tag{8b}$$

But the boundary conditions on this system require that all normal and shear stresses on the free boundaries vanish. Hence,

$$\sigma_x^*(1,y) \equiv \sigma_y^*(x,1) \equiv 0 \tag{9}$$

Therefore,

$$\sigma_x^* = \beta \int_x^1 \frac{\partial}{\partial y} \tau^* \, dx \tag{10a}$$

$$\sigma_y^* = \frac{1}{\beta} \int_y^1 \frac{\partial}{\partial x} \tau^* \, dy \tag{10b}$$

Substituting equations (10) into equation (7) produces a single integrodifferential equation that defines the shear stresses in the first quadrant of figure 1(b):

$$- \frac{2}{\beta} \frac{\partial^2}{\partial x\, \partial y} \tau^* + \frac{1}{\beta^3} \int_y^1 \frac{\partial^3}{\partial x^3} \tau^* \, dy + \beta \int_x^1 \frac{\partial^3}{\partial y^3} \tau^* \, dx = -\left(\frac{1}{\beta^2} \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) T^* \tag{11}$$

with the boundary conditions

$$\tau^*(x,1) \equiv \tau^*(1,y) \equiv 0 \tag{12}$$

Reference 6 shows that the solution in one quadrant is all that is necessary to define the solution everywhere. This is discussed further in the following sections.

Solution by Collocation

Equation (11) will be solved approximately by the method of collocation in two dimensions. It is assumed that there exists a sequence of functions τ^*_{mn} that converge to the solution τ^* of equation (11) as m and n approach infinity and that are given by

$$\tau^*_{mn}(x,y) = \sum_{i=1}^{m} \sum_{j=1}^{n} P_i(x) Q_j(y) t_{ij} \tag{13}$$

where the P_i and Q_j are known polynomials selected in such a manner that the boundary conditions and certain other conditions to be defined are satisfied. The constants t_{ij} are as yet unknown. The collocation method now requires that the error in replacing τ^* by τ^*_{mn} in equation (11) vanish at m by n specified points as shown in figure 1(b). To do this, equation (13) is substituted into equation (11) and the t_{ij} determined so that equation (11) is satisfied at each of these m by n stations. This will result in m by n linear algebraic equations for the unknown t_{ij}.

In addition to satisfying the boundary conditions the polynomials P_i and Q_j are chosen to satisfy the following conditions:

$$P_i(x_i) = 1 \tag{14a}$$

$$P_i(x_k) = 0, \; k \neq i \tag{14b}$$

The quantity $Q_j(y)$ is similarly a polynomial in y associated with the j^{th} station and satisfies similar conditions:

$$Q_j(y_j) = 1 \tag{15a}$$

$$Q_j(y_k) = 0,\ k \neq j \tag{15b}$$

Hence, as seen from equation (13), the boundary conditions, equations (12), are identically satisfied, and $t_{ij} = \tau^*(x_i,y_j)$.

Polynomials that have the desired properties are easily obtained. For example,

$$P_i(x) = \frac{(x^2 - 1)}{\left(x_i^2 - 1\right)} \prod_{k\neq i} (x - x_k) \Bigg/ \prod_{k\neq i} (x_i - x_k), \qquad x_i \neq 1 \tag{16}$$

where $\prod_{k\neq i}$ is the product for all values of k except $k = i$. Polynomials of the form of equation (16) are of the lowest possible degree that will satisfy conditions (15).

Equation (13) is now substituted into equation (11) and evaluated at each of the m stations in x and n stations in y to produce a system of m by n simultaneous linear algebraic equations in t_{ij} of the following form:

$$\sum_{i=1}^{m} \sum_{j=1}^{n} C_{kl}^{ij}\, t_{ij} = -\left(\frac{1}{\beta^2}\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)T^*(x_k,y_l) \qquad \begin{array}{l} k = 1,2,\ldots,m \\ l = 1,2,\ldots,n \end{array} \tag{17}$$

where

$$C_{kl}^{ij} = -\frac{2}{\beta} P_i'Q_j' + \frac{1}{\beta^3} P_i''' \int_y^1 Q_j\, dy + \beta Q_j''' \int_x^1 P_i\, dx \qquad \begin{array}{l} x = x_k \\ y = y_l \end{array} \tag{18}$$

The shear stresses at any point can now be determined from equation (13). The spanwise and chordwise stresses are given by

$$\sigma_x^* = \beta \sum_{i=1}^{m} \sum_{j=1}^{n} Q_j' \int_x^1 P_i\, dx\ t_{ij} \tag{19a}$$

$$\sigma_y^* = \frac{1}{\beta} \sum_{i=1}^{m} \sum_{j=1}^{n} P_i' \int_y^1 Q_j\, dy\ t_{ij} \tag{19b}$$

obtained by substitution of equation (13) into equations (10).

Special Cases of Temperature Variation

If the temperature function is an even or odd function of only one of the independent variables, the solution need be determined in only one quadrant of the coordinate system shown in figure 1(b). The solution everywhere else is known by symmetry, or asymmetry, depending on the function. Writing the equations in only one quadrant reduces the number of stations required for a given degree of accuracy, significantly reducing the amount of labor required to produce a solution. Inasmuch as any function of a single variable can be separated into the sum of an even function and an odd function, it is possible to solve the problem of an arbitrary temperature distribution in one variable in two steps and add the two solutions. For the case in which the temperature function depends on two variables, it is frequently possible to separate the function into the sum of two functions of a single variable. This procedure is covered more fully in reference 6. Separating the problem into several simpler problems will always result in a reduction in labor. For example, if it is decided to use four stations along each axis to represent the solution, 16 simultaneous equations of the form of equation (17) will be produced, and 16 terms will appear in each double summation. But there are only two stations along each axis in each quadrant that produce only four simultaneous equations and only four terms in each double sum. Hence, even if as many as four solutions were required, an even and odd solution in each variable, less labor is required. An example of an even polynomial that has the desired properties is

$$P_i(x) = \frac{(x^2 - 1)}{(x_i^2 - 1)} \prod_{k \neq i} \left(x^2 - x_k^2\right) \Bigg/ \prod_{k \neq i} \left(x_i^2 - x_k^2\right) \tag{20}$$

An example of an odd polynomial that has the desired properties is

$$P_i(x) = \frac{x(x^2 - 1)}{x_i(x^2 - 1)} \prod_{k \neq i} \left(x^2 - x_k^2\right) \Bigg/ \prod_{k \neq i} \left(x_i^2 - x_k^2\right) \tag{21}$$

EXAMPLES

Two by Two Collocation on a Square Plate

As a first example, consider a square plate subjected to a thermal gradient in the y-direction independent of x:

$$T^* = y^2$$

This temperature distribution will produce a normal stress distribution symmetrical about both axes and a shear stress distribution that is asymmetrical about both axes. Therefore, polynomials of the form of equation (21) will be chosen. Two equally spaced stations in each direction will be used; and, therefore, the P polynomials are identical to the Q polynomials. Writing the polynomials at equally spaced points, in this case at 1/4 and 3/4, produces

$$P_1 = Q_1 = 8.533\ r^5 - 13.33\ r^3 + 4.800\ r$$

$$P_2 = Q_2 = -6.095\ r^5 + 6.476\ r^3 - 0.3810\ r$$

where r is a dummy variable representing x for the P_i and y for the Q_j polynomials. Substituting these polynomials into equations (17) and (18) produces the following system:

$$-45.92\ t_{11} - 17.99\ t_{12} - 17.99\ t_{21} + 12.37\ t_{22} = -2.0$$

$$96.70\ t_{11} - 72.17\ t_{12} + 92.10\ t_{21} - 68.11\ t_{22} = -2.0$$

$$96.70\ t_{11} + 92.10\ t_{12} - 72.17\ t_{21} - 68.11\ t_{22} = -2.0$$

$$-60.05\ t_{11} + 56.72\ t_{12} + 56.72\ t_{21} - 64.50\ t_{22} = -2.0$$

The solution to the system is:

$$t_{11} = 0.02991$$

$$t_{12} = 0.04683$$

$$t_{21} = 0.04683$$

$$t_{22} = 0.08553$$

Substituting these values and the P's and Q's evaluated at the desired x's and y's into equations (13) and (19) produces the stresses everywhere. The stress distribution in the first quadrant is tabulated in columns 4, 7, and 10 of table I and is shown in figure 2.

Compared with the distribution produced by equations (13) and (19) of this report is the elastic stress distribution in unpublished NASA data given in columns 3, 6, and 9 of table I and shown in figure 2. These stresses are calculated from Airy's stress function, which is produced as the solution to the biharmonic equation. The solution was obtained by a finite-difference method, which required the solution of 441 simultaneous linear algebraic equations. An inspection of table I and figure 2 discloses that the agreement between the two methods is excellent.

Three by Two Collocation on a Square Plate

As a second example, consider the square plate of the first example subjected to the same one-dimensional thermal gradient. For this example, however, take three stations in the x-direction. The P polynomials become

$$P_1 = -41.66\ x^7 + 81.00\ x^5 - 46.58\ x^3 + 7.232\ x$$

$$P_2 = 27.00\ x^7 - 46.50\ x^5 + 20.02\ x^3 - 0.5208\ x$$

$$P_3 = 15.90\ x^7 + 20.32\ x^5 - 4.529\ x^3 + 0.1104\ x$$

written at the equally spaced stations $x_1 = 1/6$, $x_2 = 1/2$, $x_3 = 5/6$. Substituting into equations (17) and (18), solving the resultant 6 by 6 matrix, and evaluating the stresses by using equations (13) and (19) produce the solution tabulated in columns 5, 8, and 11 of table I and shown in figure 2. It is evident that the additional stations produced no improvement in the solution.

Two by Two and Three by Two Collocation on a Rectangular Plate

Next, consider a plate that has a span three times its chord and is subjected to the same thermal gradient as the square plate of the preceding discussion. Using the 2 by 2 station collocation produces the solution tabulated in columns 4, 7, and 10 of table II and shown in figure 3. Comparing that solution with the finite-difference solution tabulated in columns 3, 6, and 9 of table II and shown in figure 3, shows poor agreement. However, if another station is chosen in the x-direction, the solution is considerably improved. This is also shown in figure 3 and columns 5, 8, and 11 of table II.

Comparison of Displacements

As an additional check, the compatibility of strains was determined from the strain-displacement relations. Relations were derived for both spanwise and chordwise displacements in two different ways. First, the appropriate strain-displacement equation was integrated directly (e.g., eq. (22b) for the spanwise displacement). Then the shear-displacement equation (22c) was integrated with respect to the appropriate variable, and the normal displacement was eliminated by making use of the remaining equation (e.g., for the spanwise displacment eq. (22c) was integrated over y, and the chordwise displacement was eliminated by using eq. (22a)). The displacements were then calculated by the two different relations and compared.

Proceeding in this manner, from reference 8,

$$\epsilon_y^* = \frac{\partial v^*}{\partial y} \tag{22a}$$

$$\epsilon_x^* = \frac{1}{\beta} \frac{\partial u^*}{\partial x} \tag{22b}$$

$$\gamma^* = \frac{1}{2}\left(\frac{\partial u^*}{\partial y} + \frac{1}{\beta} \frac{\partial v^*}{\partial x}\right) \tag{22c}$$

Integrating equation (22b) yields

$$u_1^* = \beta \int_0^x \epsilon_x^* \, dx \tag{23}$$

The spanwise displacement may be calculated a different way by using equations (22a) and (22c):

$$u_2^* = 2 \int_0^y \gamma^* \, dy - \frac{1}{\beta} \int_0^y \frac{\partial v^*}{\partial x} \, dy + f_1(x)$$

From equation (22a),

$$v^* = \int_0^y \epsilon_y^* \, dy$$

Therefore,

$$u_2^* = 2 \int_0^y \gamma^* \, dy - \frac{1}{\beta} \int_0^y \int_0^y \frac{\partial \epsilon_y^*}{\partial x} \, dy \, dy + f_1(x)$$

or integrating by parts yields

$$u_2^* = 2 \int_0^y \gamma^* \, dy - \frac{y}{\beta} \int_0^y \frac{\partial \epsilon_y^*}{\partial x} \, dy + \frac{1}{\beta} \int_0^y y \, \frac{\partial \epsilon_y^*}{\partial x} \, dy + f_1(x)$$

where

$$f_1(x) = u^*(x,0)$$

To make $u_1^* = u_2^*$ at $y = 0$, $f_1(x)$ is chosen to equal $u_1^*(x,0)$. Hence,

$$u_1^* = \beta \int_0^x \epsilon_x^* \, dx \tag{23}$$

$$u_2^* = 2 \int_0^y \gamma^* \, dy - \frac{y}{\beta} \int_0^y \frac{\partial \epsilon_y^*}{\partial x} \, dy + \frac{1}{\beta} \int_0^y y \frac{\partial \epsilon_y^*}{\partial x} \, dy + u_1^*(x,0) \tag{24}$$

Similarly,

$$v_1^* = \int_0^y \epsilon_y^* dy \tag{25}$$

$$v_2^* = 2\beta \int_0^x \gamma^* \, dx - \beta^2 x \int_0^x \frac{\partial \epsilon_x^*}{\partial y} \, dx + \beta^2 \int_0^x x \frac{\partial \epsilon_x^*}{\partial y} \, dx + v_1^*(0,y) \tag{26}$$

For a parabolic thermal gradient, $T^* = y^2$ and $\nu = 1/3$, from equations (4), (13), and (19) the result is

$$\epsilon_y^* = \left(\frac{1}{\beta} \sum_{i=1}^{m} \sum_{j=1}^{n} P_i' \int_y^1 Q_j \, dy - \frac{\beta}{3} \sum_{i=1}^{m} \sum_{j=1}^{n} Q_j' \int_x^1 P_i \, dx \right) t_{ij} + y^2 \tag{27a}$$

$$\epsilon_x^* = \left(\beta \sum_{i=1}^{m} \sum_{j=1}^{n} Q_j' \int_x^1 P_i \, dx - \frac{1}{3\beta} \sum_{i=1}^{m} \sum_{j=1}^{n} P_i' \right) \int_y^1 Q_j \, dy \quad t_{ij} + y^2 \tag{27b}$$

$$\gamma^* = \frac{4}{3} \sum P_i \sum Q_j t_{ij} \tag{27c}$$

Substituting equations (27) into equations (23), (24), (25), and (26) produces a check on the compatibility of the strains. Figure 4 and columns 3 to 6 of table III show values of u_1^*, u_2^*, v_1^*, and v_2^* throughout the first quadrant for 2 by 2 collocation on the square plate. It is evident that there is excellent agreement between the displacements.

DISCUSSION

The examples of the preceding section demonstrate the simplicity and accuracy of this method. Reference 6 points out that for a one-dimensional temperature function, it is unnecessary to use values of β greater than 3, which produces the solution for the semi-infinite plate at the center. For most problems, therefore, it is unnecessary to use more than three stations in the x-direction, indicating that the largest system of equations to be solved will be 6 by 6. This permits the use of a desk calculator instead of an electronic data processing machine, as required by most of the other methods discussed.

Despite the fact that the entire stress distribution is written in terms of the shear stresses at only four or six points within the region of definition and those stresses are calculated from an equation that is approximated only at those same points, a solution of remarkable accuracy is obtained. At this point, it is appropriate to mention that previous uses of the double collocation method (ref. 9) required more stations, because the solution was obtained in Airy's stress function, which required the solution of a higher order equation and also required differentiation to obtain the stresses. The solutions presented in references 5 and 6 are compared with those in references 1 to 4 and with each other. Therefore, it was necessary to compare the solutions produced herein with only one of the others. The finite-difference solution was chosen because the only approximations used there were the standard ones used in numerical quadrature and differentiation. It is to be noted that the solutions produced by the various references and the solution of this report are virtually identical. Where any differences occur, the check of the compatibility condition produced by the method reported herein places greatest confidence in this solution.

Because of the small number of stations used in the approximation, the temperature function must not change rapidly. This restriction was also pointed out in reference 6. However, it is impossible to state quantitatively how rapidly the function may vary. For the case of a rapidly varying function, more stations must be taken. One way to assure that a sufficient number of stations has been chosen is to solve the problem several times for increasing numbers of stations. When increasing the number of stations produces no change in the solutions, it can be assumed that maximum accuracy has been obtained from the method. References 5 and 6 state that for the problems considered, up to a ninth-power temperature function, no improvement was noted for more than three stations in one direction. It should be noted that in those references, collocation was in one direction only, producing a system of simultaneous linear ordinary differential equations in Airy's stress function.

CONCLUSIONS

A method of determining the elastic state of stress in a thermally loaded, thin, flat plate of finite dimensions has been presented, which, for the class of temperature distributions investigated, produces a very accurate solution with a minimum of labor. Its principal advantage is that it produces a solution of the same order of accuracy of other methods by a technique that requires the solution of only four or six simultaneous linear algebraic equations instead of a system of many algebraic equations or a system of differential equations. Its limitations are that the temperature function may not be a rapidly varying function and must be separable into the sum of functions of a single variable.

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National Aeronautics and Space Administration
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TABLE I. - DIMENSIONLESS STRESS IN SQUARE PLATE SUBJECTED TO CHORDWISE PARABOLIC THERMAL GRADIENT

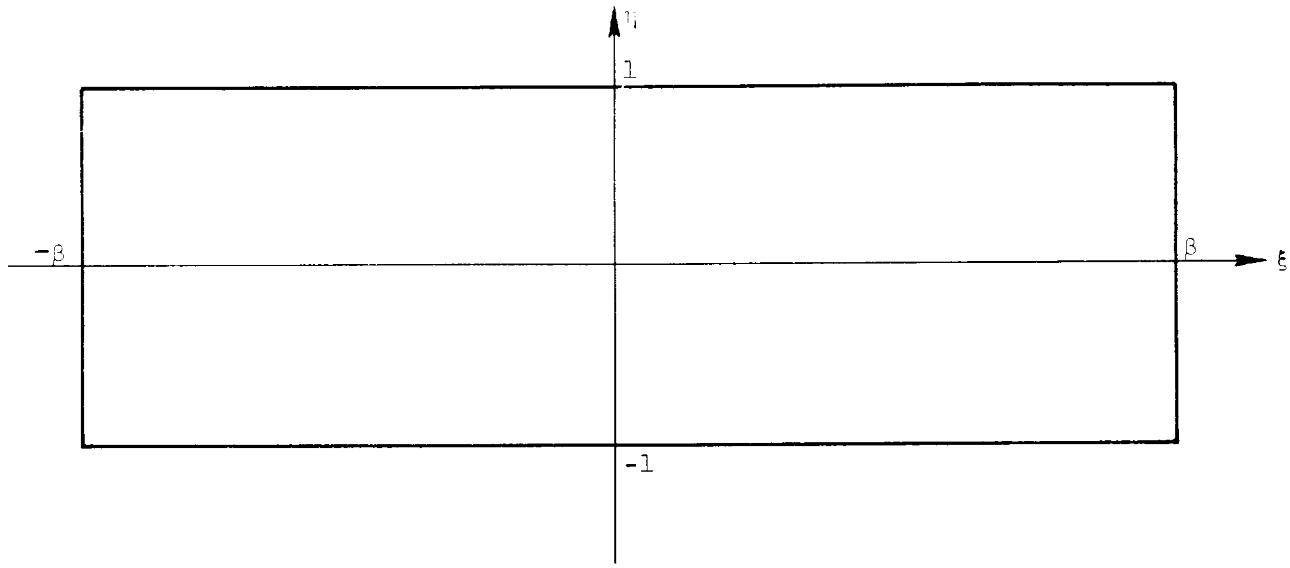
Coordinate		Stress distribution						Shear stress, τ^*		
Span, x	Chord, y	Spanwise, σ_x^*			Chordwise, τ_y^*			Finite-difference solution	2 by 2 colloca-tion	3 by 2 colloca-tion
		Finite-difference solution	2 by 2 colloca-tion	3 by 2 colloca-tion	Finite-difference solution	2 by 2 colloca-tion	3 by 2 colloca-tion			
Column										
1	2	3	4	5	6	7	8	9	10	11
0	0	0.141	0.141	0.141	0.141	0.141	0.142	0	0	0
	.2	.131	.131	.131	.131	.130	.131	0	0	0
	.4	.095	.096	.096	.102	.101	.102	0	0	0
	.6	.016	.017	.017	.061	.060	.060	0	0	0
	.8	-.136	-.138	-.138	.020	.020	.020	0	0	0
	1.0	-.409	-.412	-.411	0	0	0	0	0	0
0.2	0	0.131	0.130	0.130	0.131	0.131	0.131	0	0	0
	.2	.122	.122	.121	.122	.122	.122	.020	.020	.020
	.4	.089	.090	.090	.095	.095	.095	.035	.036	.036
	.6	.016	.017	.017	.057	.057	.057	.042	.043	.043
	.8	-.126	-.128	-.128	.019	.019	.019	.034	.034	.034
	1.0	-.384	-.386	-.386	0	0	0	0	0	0
0.4	0	0.102	0.101	0.101	0.095	0.096	0.095	0	0	0
	.2	.095	.095	.095	.089	.090	.089	.035	.036	.036
	.4	.071	.072	.072	.071	.072	.072	.064	.065	.065
	.6	.015	.016	.016	.044	.045	.045	.078	.079	.079
	.8	-.099	-.100	-.100	.016	.016	.016	.063	.064	.064
	1.0	-.311	-.312	-.312	0	0	0	0	0	0
0.6	0	0.061	0.060	0.060	0.016	0.017	0.016	0	0	0
	.2	.057	.057	.057	.016	.017	.016	.042	.043	.042
	.4	.044	.045	.045	.015	.016	.015	.078	.079	.078
	.6	.011	.012	.012	.011	.012	.011	.096	.098	.098
	.8	-.059	-.060	-.060	.005	.005	.005	.081	.082	.082
	1.0	-.196	-.197	-.196	0	0	0	0	0	0
0.8	0	0.020	0.020	0.020	-0.136	-0.138	-0.137	0	0	0
	.2	.019	.019	.019	-.126	-.128	-.127	.034	.034	.034
	.4	.016	.016	.015	-.099	-.100	-.100	.063	.064	.064
	.6	.005	.005	.005	-.060	-.060	-.061	.081	.082	.082
	.8	-.021	-.020	-.020	-.021	-.020	-.020	.069	.071	.071
	1.0	-.065	-.069	-.069	0	0	0	0	0	0
1.0	0	0	0	0	-0.409	-0.412	-0.412	0	0	0
	.2	0	0	0	-.384	-.386	-.387	0	0	0
	.4	0	0	0	-.311	-.312	-.311	0	0	0
	.6	0	0	0	-.196	-.197	-.195	0	0	0
	.8	0	0	0	-.065	-.069	-.068	0	0	0
	1.0	0	0	0	0	0	0	0	0	0

TABLE II. - DIMENSIONLESS STRESS IN RECTANGULAR PLATE SUBJECTED TO CHORDWISE PARABOLIC THERMAL GRADIENT

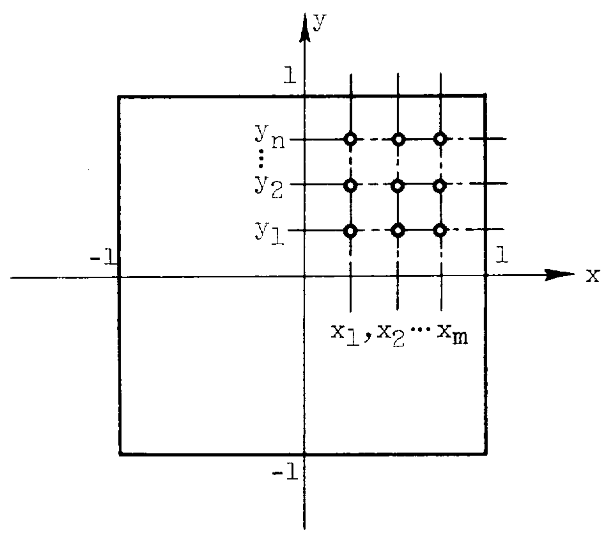
Coordinate		Stress distribution						Shear stress, τ^*		
Span, x	Chord, y	Spanwise, σ_x^*			Chordwise, σ_y^*			Finite-difference solution	2 by 2 colloca-tion	3 by 2 colloca-tion
		Finite-difference solution	2 by 2 colloca-tion	3 by 2 colloca-tion	Finite-difference solution	2 by 2 colloca-tion	3 by 2 colloca-tion			
Column										
1	2	3	4	5	6	7	8	9	10	11
0	0	0.335	0.310	0.335	0.001	-0.042	0.004	0	0	0
	.2	.295	.270	.295	.001	-.039	.004	0	0	0
	.4	.175	.154	.175	.001	-.031	.004	0	0	0
	.6	-.026	-.034	-.025	0	-.019	.003	0	0	0
	.8	-.308	-.283	-.308	0	-.007	.001	0	0	0
	1.0	-.670	-.578	-.678	0	0	0	0	0	0
0.2	0	0.333	0.331	0.333	0.005	-0.011	0.003	0	0	0
	.2	.294	.291	.293	.005	-.011	.003	.002	-.012	.001
	.4	.174	.170	.173	.003	-.009	.002	.002	-.023	.003
	.6	-.025	-.029	-.027	.002	-.006	.001	.002	-.029	.004
	.8	-.306	-.304	-.306	0	-.002	0	.001	-.026	.003
	1.0	-.669	-.650	-.666	0	0	0	0	0	0
0.4	0	0.321	0.359	0.322	0.023	0.058	0.019	0	0	0
	.2	.283	.318	.284	.021	.054	.017	.008	-.001	.007
	.4	.170	.195	.169	.016	.042	.012	.013	-.005	.011
	.6	-.021	-.018	-.024	.009	.025	.006	.014	-.009	.011
	.8	-.296	-.333	-.297	.003	.008	.001	.009	-.011	.006
	1.0	-.658	-.767	-.653	0	0	0	0	0	0
0.6	0	0.269	0.310	0.272	0.060	0.101	0.068	0	0	0
	.2	.240	.277	.243	.055	.094	.063	.027	.035	.028
	.4	.149	.176	.153	.042	.073	.049	.046	.062	.046
	.6	-.010	-.006	-.008	.025	.044	.029	.051	.070	.049
	.8	-.250	-.291	-.254	.008	.015	.009	.035	.052	.032
	1.0	-.589	-.706	-.608	0	0	0	0	0	0
0.8	0	0.139	0.143	0.137	0.048	0.009	0.050	0	0	0
	.2	.126	.129	.125	.046	.009	.047	.054	.065	.057
	.4	.084	.084	.085	.038	.008	.039	.096	.116	.101
	.6	.004	0	.007	.025	.005	.026	.111	.137	.119
	.8	-.130	-.135	-.131	.009	.002	.010	.085	.107	.093
	1.0	-.350	-.339	-.354	0	0	0	0	0	0
1.0	0	0	0	0	-0.436	-0.369	-0.446	0	0	0
	.2	0	0	0	-.407	-.342	-.416	0	0	0
	.4	0	0	0	-.322	-.266	-.329	0	0	0
	.6	0	0	0	-.196	-.158	-.201	0	0	0
	.8	0	0	0	-.065	-.052	-.068	0	0	0
	1.0	0	0	0	0	0	0	0	0	0

TABLE III. - DIMENSIONLESS DISPLACEMENT IN SQUARE PLATE SUBJECTED TO CHORDWISE PARABOLIC THERMAL GRADIENT

Coordinate		Displacement			
Span, x	Chord, y	Spanwise		Chordwise	
		u_1^*	u_2^*	v_1^*	v_2^*
Column					
1	2	3	4	5	6
0	0	0	0	0	0
	.2	0	0	.021	.021
	.4	0	0	.055	.055
	.6	0	0	.118	.118
	.8	0	0	.228	.228
	1.0	0	0	.410	.410
0.2	0	0.018	0.018	0	0
	.2	.025	.025	.020	.020
	.4	.044	.044	.053	.053
	.6	.071	.071	.115	.115
	.8	.100	.099	.225	.224
	1.0	.119	.117	.405	.404
0.4	0	0.034	0.034	0	0
	.2	.048	.048	.015	.014
	.4	.087	.086	.044	.044
	.6	.143	.143	.103	.103
	.8	.204	.203	.211	.210
	1.0	.249	.248	.387	.385
0.6	0	0.046	0.046	0	0
	.2	.067	.067	.002	.001
	.4	.127	.127	.021	.020
	.6	.216	.216	.072	.072
	.8	.315	.317	.174	.175
	1.0	.397	.400	.345	.340
0.8	0	0.057	0.057	0	0
	.2	.086	.086	-.026	-.027
	.4	.168	.167	-.031	-.031
	.6	.291	.292	.003	.005
	.8	.435	.439	.094	.097
	1.0	.571	.575	.258	.251
1.0	0	0.076	0.076	0	0
	.2	.112	.111	-.078	-.080
	.4	.214	.212	-.130	-.131
	.6	.372	.367	-.131	-.128
	.8	.565	.558	-.059	-.054
	1.0	.766	.762	.099	.095

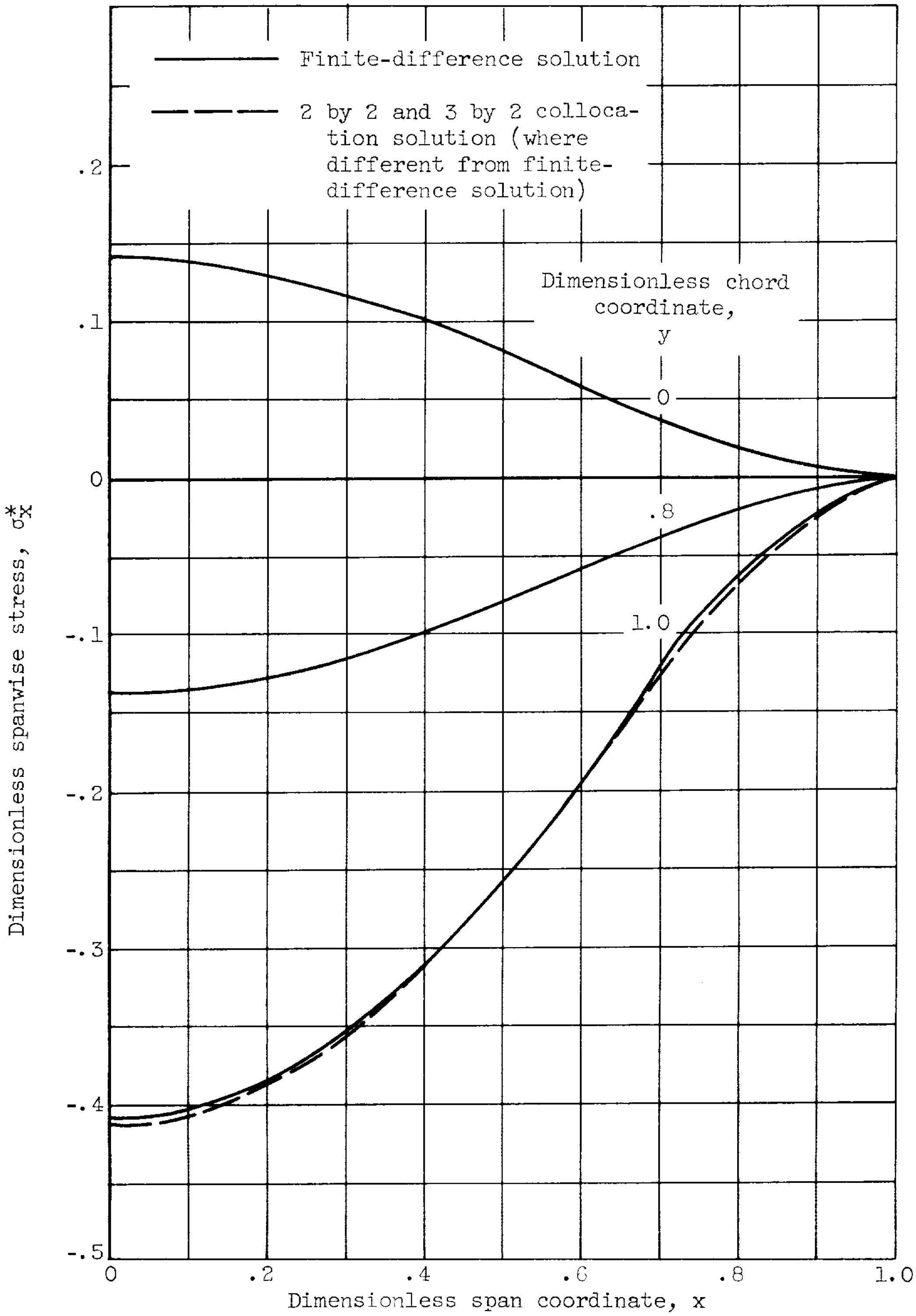


(a) ξ, η Coordinate system.



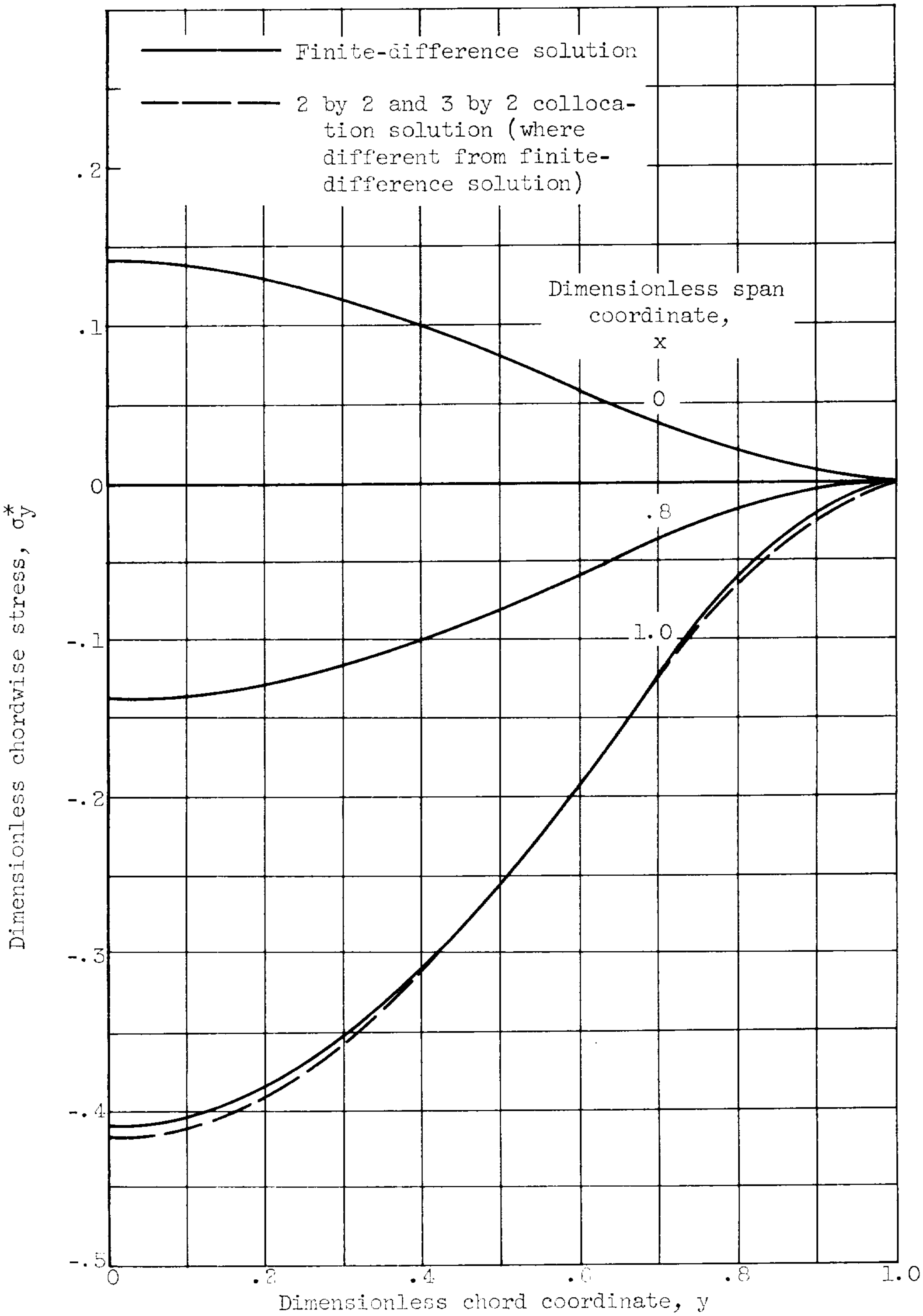
(b) x, y Coordinate system.

Figure 1. - Coordinate systems used to calculate stress distribution in thermally loaded, thin, flat plate.



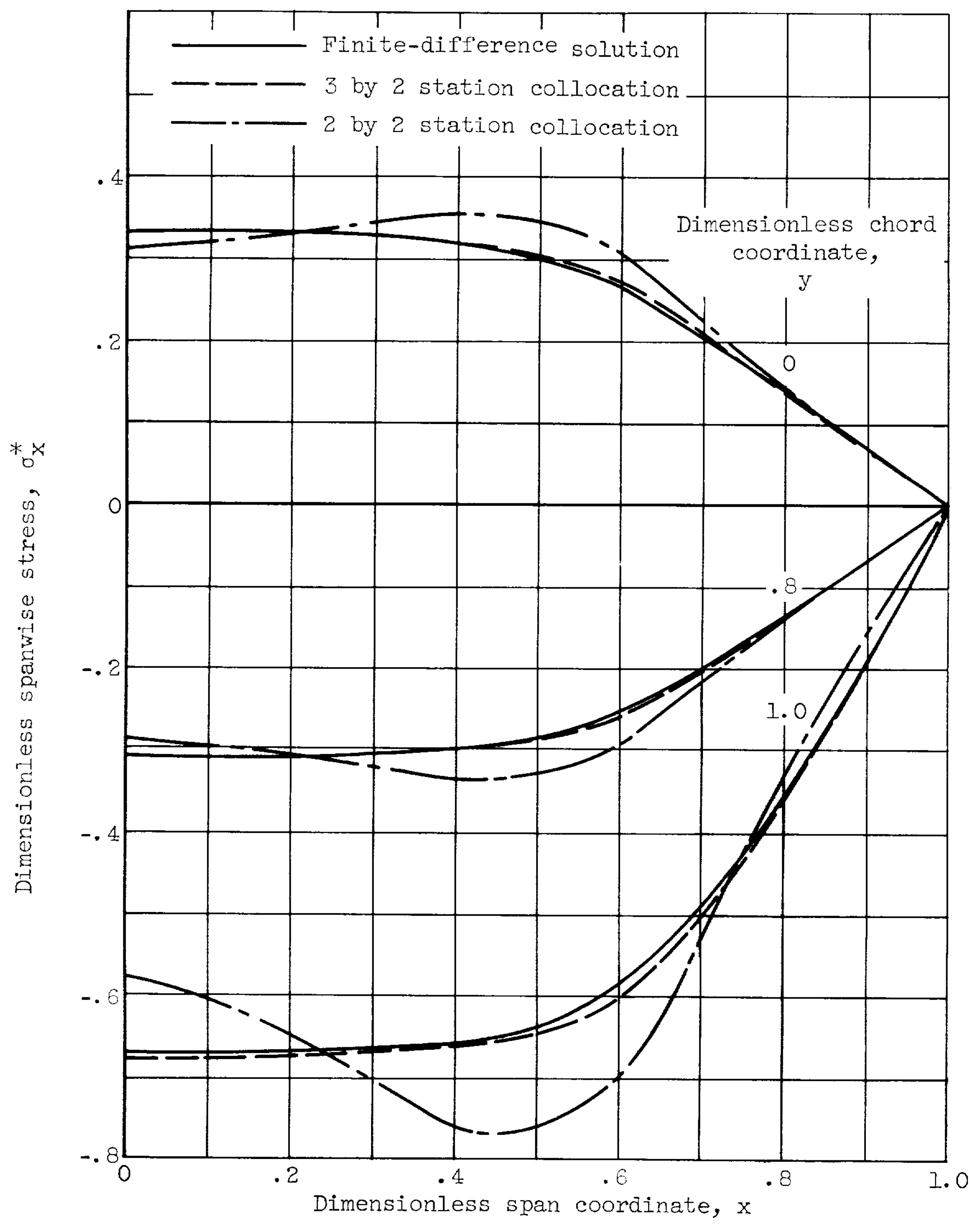
(a) Spanwise stress.

Figure 2. - Stress in square plate subjected to chordwise parabolic thermal gradient.



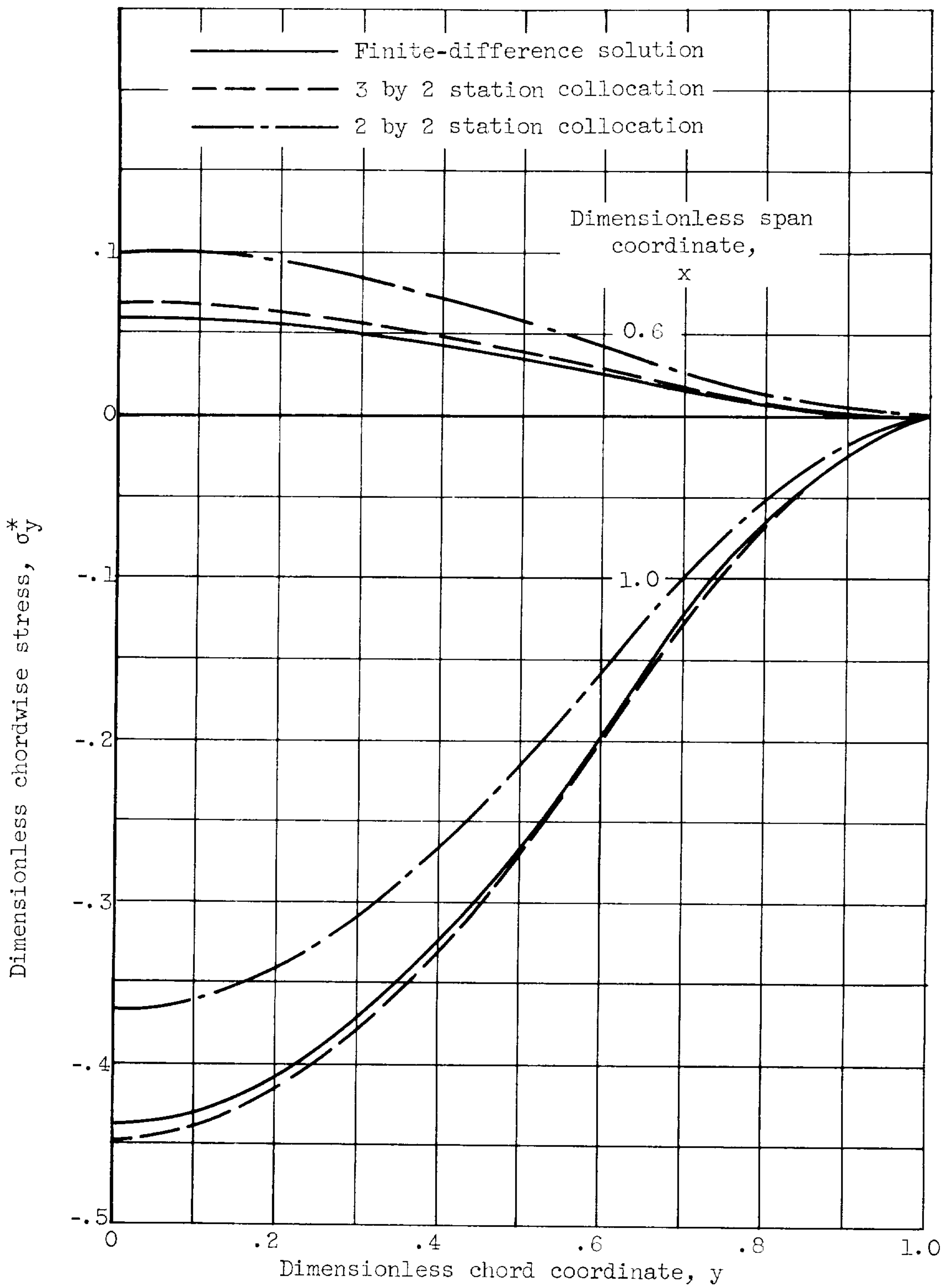
(b) Chordwise stress.

Figure 2. - Concluded. Stress in square plate subjected to chordwise parabolic thermal gradient.



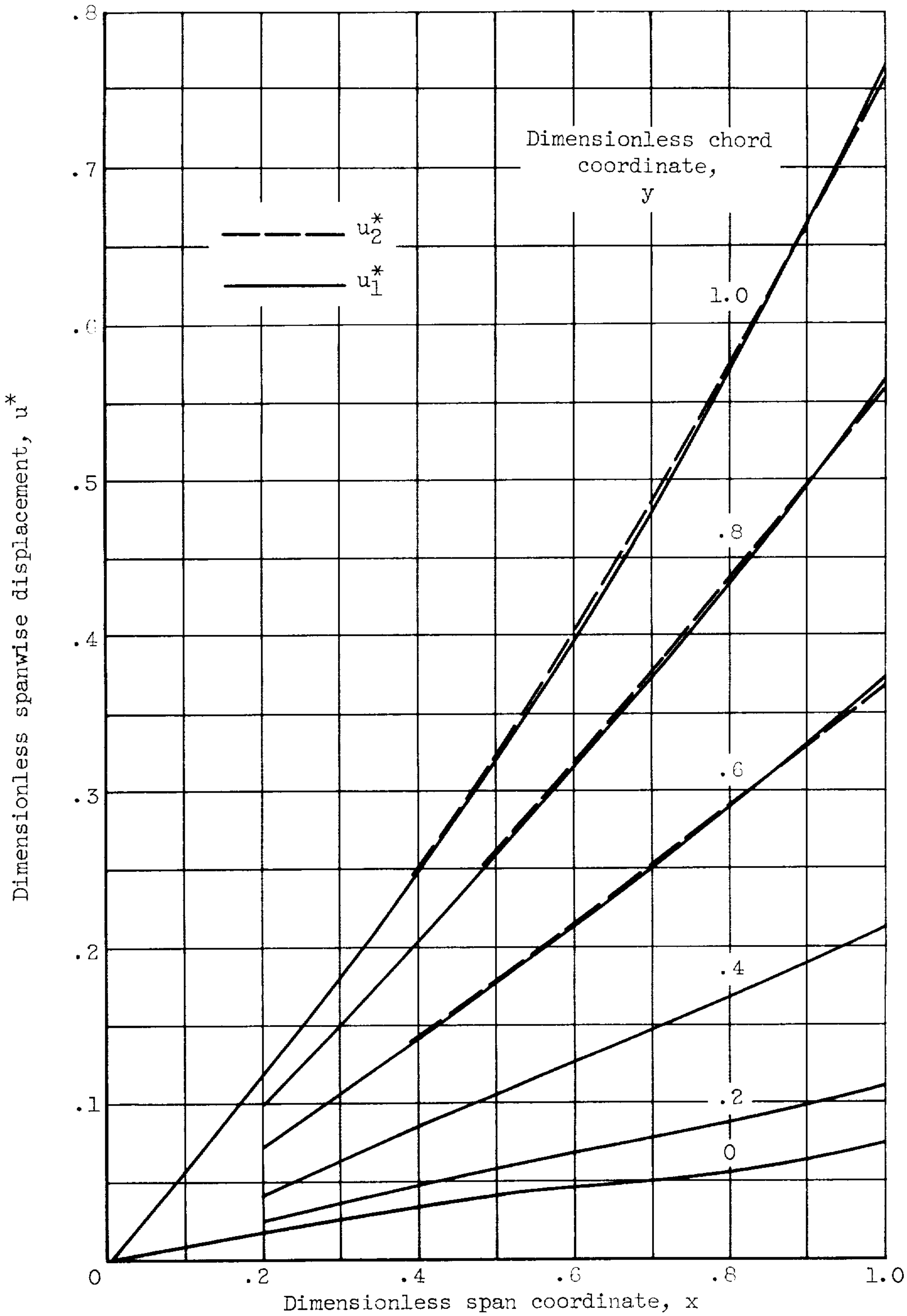
(a) Spanwise stress.

Figure 3. - Stress in 3 by 1 plate subject to chordwise parabolic thermal gradient.



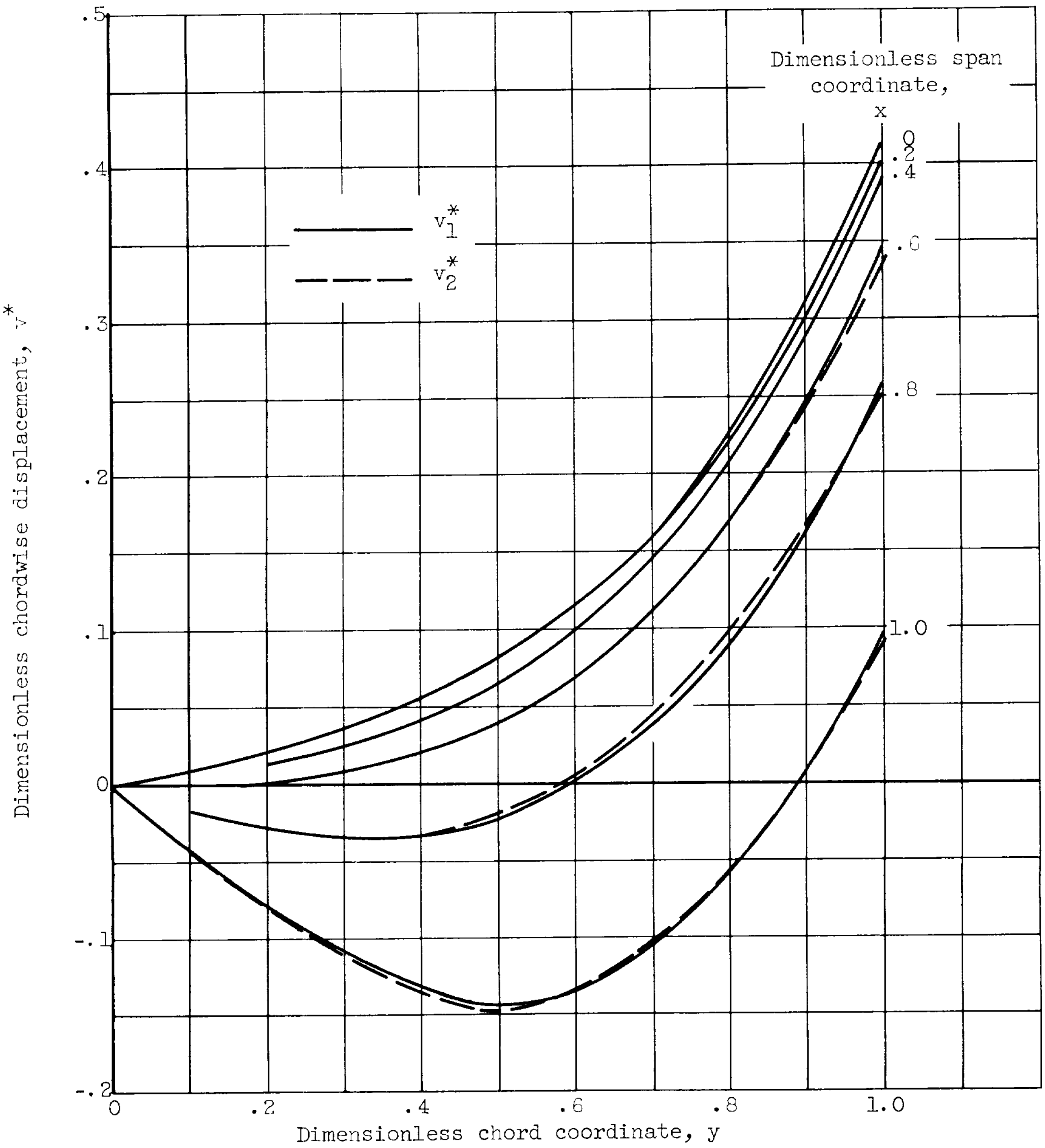
(b) Chordwise stress.

Figure 3. - Concluded. Stress in 3 by 1 plate subjected to chordwise parabolic thermal gradient.



(a) Spanwise displacement.

Figure 4. - Displacements in square plate subjected to chordwise parabolic thermal gradient.



(b) Chordwise displacement.

Figure 4. - Concluded. Displacements in square plate subjected to chordwise parabolic thermal gradient.