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Some Exact Solutions of the Problem of Axisymmetric Bending of Thin Spherical Shells

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NOMENCLATURE

a	radius of shell
a_{ij}	elements of matrix ($\mathbf{A}$)
a_n	coefficients of power series
$\mathbf{A}$	matrix
A_n	coefficients of power series
b_n	coefficients of power series
B_n	coefficients of power series
C_i	arbitrary constants
D	$Eh^3/12(1-\nu^2)$
E	Young's modulus
F_i	power series
h	thickness of shell
H	unit horizontal edge force
L	differential operator
M_b	meridional bending moment at edge
M_ϕ, M_θ	unit meridional, circumferential bending moment
M_n	power series function
n	power series index
N_ϕ, N_θ	unit meridional, circumferential forces
N_n	power series function
p_n	power series function
P_n	power series function
q_n	power series function
Q_n	power series function
Q_ϕ	unit transverse shear force
R	vertical resultant of external loading
s	index of power series
u_i	solutions of the differential equation
U	aQ_ϕ
v	meridional displacement
V	rotation of the meridional tangent
w	radial displacement
x, x_b	$\sin^2\phi$, $\sin^2\phi_b$
z	$U \sin\phi$
α, β	physical constants
α_n, β_n	coefficients of power series
δ	horizontal displacement at edge
$\epsilon_\phi^{(0)}, \epsilon_\theta^{(0)}$	meridional, circumferential middle surface strain
θ_n	power series function
μ	$\left\{\frac{Eh}{D} - \frac{\nu^2}{a^2}\right\}^{1/4}$
ν	Poisson's ratio
ρ	$\sqrt{\frac{a\mu^2}{2}}$
ϕ, ϕ_b	polar angle, polar angle at edge
ψ	power series

ABSTRACT

The solution of the "exact" equation of equilibrium of an axisymmetrically loaded, thin, spherical shell is presented in the form of power series. These series are computed, and the resulting theory compared with the results of shallow shell and quasi-cylindrical theories for the following two cases:

1. Influence coefficients of a complete edge-loaded shell
2. Stresses and displacements of a shell loaded at the apex with a concentrated radial force

I. INTRODUCTION

The problem of the bending of a thin spherical shell by axisymmetric loads has been formulated and formally solved according to Ref. 1 by L. Bolle in 1915. However, the range of parameters investigated ($\rho < 10$) did not correspond to what is currently regarded as a thin shell. Unfortunately, as has been remarked by many authors, the power series solution for the range of parameters corresponding to a thin shell ($\rho > 10$) is very slowly convergent. As a result, a number of approximate solutions have been formulated in terms of more easily evaluated functions. It is the purpose of this Report to extend the calculations into the thin shell range and hence develop a criterion for comparing the approximate theories. The notation used will be that of Ref. 1 and the appropriate sign convention is presented in Fig. 1.

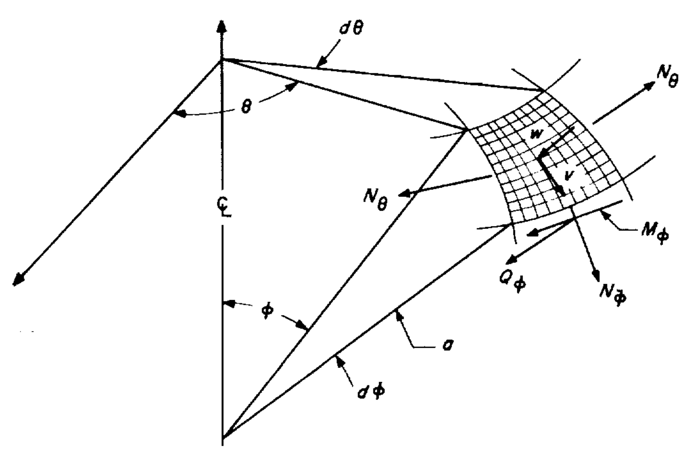

Fig. 1. Sign convention

II. GENERAL THEORY

As is well known (see Ref. 1), the axisymmetric bending of a thin spherical shell of constant thickness is described by the following set of differential equations:

$$aL(U) + \nu U = EahV \qquad (1)$$

$$aL(V) - \nu V = \frac{-Ua}{D} \qquad (2)$$

where

$$U = aQ_\phi$$

$$V = \frac{\left(v + \frac{dw}{d\phi}\right)}{a}$$

and the operator L is defined as

$$L(\ ) = \frac{1}{a}\left[\frac{d^2}{d\phi^2}(\ ) + \cot\phi \frac{d}{d\phi}(\ ) - \cot^2\phi\right]$$

With

$$\mu^4 = \frac{Eh}{D} - \frac{\nu^2}{a^2}, \quad \rho^2 = \frac{a\mu^2}{2}$$

both Eq. (1) and (2) can be reduced to the form

$$LL(U) + \mu^4 U = 0 \qquad (3)$$

Further, as

$$LL(\ ) + \mu^4 \equiv (L + i\mu^2)(L - i\mu^2) \qquad i \equiv \sqrt{-1}$$

it is apparent that

$$U = U_1 + U_2$$

where U_1, U_2 are the solutions of

$$(L + i\mu^2)\, U_1 = 0$$
$$(L - i\mu^2)\, U_2 = 0$$

As the operator L is real, it follows that U_2 is the complex conjugate of U_1. Hence, the four linearly independent solutions of Eq. (3) can be obtained by finding the real and imaginary parts of the two linearly independent solutions of

$$(L + i\mu^2)\, U = 0 \qquad (4)$$

If we restrict our attention to loadings of the type such that the vertical resultant load R is a constant, the principal quantities of interest are

$$N_\phi = -\frac{R/2\pi a}{\sin^2\phi} - \frac{U}{a}\cot\phi$$

$$N_\theta = \frac{R/2\pi a}{\sin^2\phi} - \frac{1}{a}\frac{dU}{d\phi}$$

$$M_\phi = -\frac{D}{a}\left(\frac{dV}{d\phi} + \nu V\cot\phi\right)$$

$$M_\theta = -\frac{D}{a}\left(\nu\frac{dV}{d\phi} + V\cot\phi\right)$$

$$v = \left(\frac{1+\nu}{Eh}\right)U + \frac{R}{2\pi}\frac{1+\nu}{Eh} \times\left[\cot\phi - \sin\phi \ln\left|\frac{\tan\phi}{2}\right|\right] + v_0 \sin\phi$$

$$w = \frac{1}{Eh}\left(\frac{dU}{d\phi} + U\cot\phi\right) - \frac{R}{2\pi}\frac{1+\nu}{Eh} \times\left(1 + \cos\phi\ln\left|\frac{\tan\phi}{2}\right|\right) + v_0\cos\phi$$

Finally, if a new dependent variable z, where

$$z = U\sin\phi$$

is expressed as a function of a new independent variable (x), where

$$x = \sin^2\phi$$

we find that Eq. (4) becomes

$$x(1-x)\frac{d^2z}{dx^2} - \frac{x}{2}\frac{dz}{dx} + \left(\frac{1+2i\rho^2}{4}\right)z = 0 \qquad (5)$$

If the two linearly independent solutions of Eq. (5) are

$$z_1 = \psi_1 + i\,\psi_2 \qquad z_2 = \psi_3 + i\,\psi_4$$

where ψ_1, ψ_2, ψ_3, ψ_4 are real functions of x, the principal quantities of interest become

$$EahV\sin\phi = C_1(\nu\psi_1 + 2\rho^2\psi_2) + C_2(\nu\psi_2 - 2\rho^2\psi_1) + C_3(\nu\psi_3 + 2\rho^2\psi_4) + C_4(\nu\psi_4 - 2\rho^2\psi_3)$$

$$U\sin\phi = z = C_1\psi_1 + C_2\psi_2 + C_3\psi_3 + C_4\psi_4$$

$$a\,N_\phi = -\frac{1}{x}\left(\frac{R}{2\pi} + z\cos\phi\right)$$

$$a\,N_\theta = \frac{1}{x}\left[\left(\frac{R}{2\pi} + z\cos\phi\right) - 2x\frac{dz}{dx}\cos\phi\right]$$

$$v\sin\phi = \left(\frac{1+\nu}{Eh}\right)z + \frac{R}{2\pi}\frac{1+\nu}{Eh} \times\left(\cos\phi - x\ln\left|\tan\frac{\phi}{2}\right|\right) + v_0\, x$$

$$w = \frac{2}{Eh}\frac{dz}{dx}\cos\phi - \frac{R}{2\pi}\frac{1+\nu}{Eh}\left(1 + \cos\phi \ln\left|\tan\frac{\phi}{2}\right|\right) + v_0 \cos\phi$$

$$\frac{-6(1-\nu^2)a^2}{h^2}\frac{M\phi}{\cos\phi} = (\nu C_1 - 2\rho^2 C_2)\left(\frac{d\psi_1}{dx} - \frac{1-\nu}{2}\frac{\psi_1}{x}\right) + (\nu C_2 + 2\rho^2 C_1)\left(\frac{d\psi_2}{dx} - \frac{1-\nu}{2}\frac{\psi_2}{x}\right) + (\nu C_3 - 2\rho^2 C_4)\left(\frac{d\psi_3}{dx} - \frac{1-\nu}{2}\frac{\psi_3}{x}\right) + (\nu C_4 + 2\rho^2 C_3)\left(\frac{d\psi_4}{dx} - \frac{1-\nu}{2}\frac{\psi_4}{x}\right)$$

$$\frac{-6(1-\nu^2)a^2}{h^2}\frac{M\theta}{\cos\phi} = (\nu C_1 - 2\rho^2 C_2)\left(\nu\frac{d\psi_1}{dx} + \frac{1-\nu}{2}\frac{\psi_1}{x}\right) + (\nu C_2 + 2\rho^2 C_1)\left(\nu\frac{d\psi_2}{dx} + \frac{1-\nu}{2}\frac{\psi_2}{x}\right) + (\nu C_3 - 2\rho^2 C_4)\left(\nu\frac{d\psi_3}{dx} + \frac{1-\nu}{2}\frac{\psi_3}{x}\right) + (\nu C_4 + 2\rho^2 C_3)\left(\nu\frac{d\psi_4}{dx} + \frac{1-\nu}{2}\frac{\psi_4}{x}\right)$$

However, before we can proceed with the analysis, let us formulate the solutions of Eq. (5).

III. THE SOLUTIONS OF THE HYPERGEOMETRIC EQUATION FOR $\gamma = 0$

As usually presented, the Hypergeometric Equation is written in Gauss' form

$$x(1-x)\frac{d^2y}{dx^2} + [\gamma - (\alpha + \beta + 1)x]\frac{dy}{dx} - \alpha\beta y = 0$$

In the above application, we have

$$\gamma = 0$$

$$\alpha = -\frac{1}{4} + \frac{1}{2}\sqrt{\frac{5}{4} + 2i\rho^2}$$

$$\beta = -\frac{1}{4} - \frac{1}{2}\sqrt{\frac{5}{4} + 2i\rho^2}$$

Hence, let us consider the solution of

$$x(1-x)\frac{d^2y}{dx^2} - (\alpha + \beta + 1)x\frac{dy}{dx} - \alpha\beta y = 0 \tag{6}$$

in the neighborhood of $x = 0$. If we assume

$$y = \sum_{n=0}^{\infty} A_n x^{n+s}$$

Eq. (6) becomes

$$A_0 s(s-1)x^{-1} + \sum_{n=0}^{\infty} x^n \{A_{n+1}(n+s)(n+s+1) - A_n(n+s+\alpha)(n+s+\beta)\} = 0$$

In order that $A_0 \neq 0$, we must choose $s = 0, 1$, and take

$$A_{n+1} = A_n \frac{(n+s+\alpha)(n+s+\beta)}{(n+s)(n+s+1)} \quad (n = 0, 1, 2, \cdots)$$

Apparently, as no solution exists for $s = 0$, $A_0 \neq 0$, we find $s = 1$,

$$A_{n+1} = A_n q_n \quad (n = 0, 1, 2, \cdots)$$

where

$$q_n = \frac{(n+1+\alpha)(n+1+\beta)}{(n+1)(n+2)}$$

If we arbitrarily choose $A_0 = 1$, the first solution of Eq. (6) can be written as

$$y = u_1(x)$$

$$u_1(x) = x\sum_{n=0}^{\infty} A_n x^n$$

where

$$A_n = q_0 q_1 q_2 \cdots q_{n-1} \quad (n \geqslant 1)$$

that is,

$$u_1(x) = x\{1 + q_0 x + q_0 q_1 x^2 + \cdots + q_0 q_1 \cdots q_{n-1} x^n + \cdots\} \tag{7}$$

With the first solution so defined, let the second solution be taken in the form

$$y = C\, u_1 \ln x + \sum_{n=0}^{\infty} B_n x^n$$

Hence, Eq. (6) becomes

$$(C - \alpha\beta B_0) + \sum_{n=0}^{\infty} x^{n+1} \{B_{n+2}(n+1)(n+2) - B_{n+1}(n+1+\alpha)(n+1+\beta) + C A_n [(2n+3) q_n - (n+1+\alpha) - (n+1+\beta)]\} = 0$$

As the coefficient of each term x^n must vanish, we obtain

$$B_0 = \frac{C}{\alpha\beta}$$

$$B_{n+2} = B_{n+1} q_n - C p_n A_n \quad (n = 0, 1, 2, \cdots)$$

where

$$p_n = \frac{(2n+3) q_n - (n+1+\alpha) - (n+1+\beta)}{(n+1)(n+2)}$$

Thus, it follows that

$$B_{n+2} = A_{n+1} \left\{ B_1 - C \left(\frac{p_0}{q_0} + \frac{p_1}{q_1} + \cdots + \frac{p_n}{q_n} \right) \right\} \quad (n = 0, 1, \cdots)$$

and

$$y = B_1 u_1 + C \left\{ \frac{1}{\alpha\beta} + u_1 \ln x - \sum_{n=0}^{\infty} A_{n+1} \left(\frac{p_0}{q_0} + \frac{p_1}{q_1} + \cdots + \frac{p_n}{q_n} \right) x^{n+2} \right\}$$

However, as both B_1, C are arbitrary, it is evident that we have constructed the general solution of Eq. (6) in attempting to find the second solution. Hence, let us formally set $B_1 = 0$ and write the second solution as

$$y = u_2(x)$$

where

$$u_2(x) = u_1 \ln x + \frac{1}{\alpha\beta} - x^2 \sum_{n=0}^{\infty} A_{n+1} \left(\frac{p_0}{q_0} + \frac{p_1}{q_1} + \cdots + \frac{p_n}{q_n} \right) x^n$$

In terms of α, β defined earlier, we find

$$q_n = \frac{(n^2 + 3n/2 + 1/4) - i\rho^2/2}{(n+1)(n+2)}$$

$$\frac{p_n}{q_n} = 2 \left\{ \frac{n + 3/2}{(n+1)(n+2)} - \frac{(n+3/4)(n^2 + 3n/2 + 1/4)}{(n^2 + 3n/2 + 1/4)^2 + (\rho^2/2)^2} \right\} - i\rho^2 \frac{(n+3/4)}{(n^2 + 3n/2 + 1/4)^2 + (\rho^2/2)^2}$$

Hence, if we take

$$A_n = a_n + i\alpha_n$$

$$A_{n+1} \left(\frac{p_0}{q_0} + \frac{p_1}{q_1} + \cdots + \frac{p_n}{q_n} \right) = b_n + i\beta_n$$

where $\alpha_n, a_n, \beta_n, b_n$ are real, then, writing

$$u_1 = \psi_1 + i\psi_2$$

$$u_2 = \psi_3 + i\psi_4$$

leads to

$$\psi_1 = x \sum_{n=0}^{\infty} a_n x^n \tag{9-a}$$

$$\psi_2 = x \sum_{n=0}^{\infty} \alpha_n x^n \tag{9-b}$$

$$\psi_3 = \psi_1 \ln x - \frac{1}{\rho^4 + 1/4} - x^2 \sum_{n=0}^{\infty} b_n x^n \tag{9-c}$$

$$\psi_4 = \psi_2 \ln x + \frac{2\rho^2}{\rho^4 + 1/4} - x^2 \sum_{n=0}^{\infty} \beta_n x^n \tag{9-d}$$

As u_1, u_2 satisfy Eq. (6) by definition, the functions ψ_1, ψ_2, ψ_3, ψ_4 must satisfy the following differential equations:

$$x(1-x) \frac{d^2\psi_1}{dx^2} - \frac{x}{2} \frac{d\psi_1}{dx} + \frac{1}{4} (\psi_1 - 2\rho^2 \psi_2) = 0 \tag{10-a}$$

$$x(1-x) \frac{d^2\psi_2}{dx^2} - \frac{x}{2} \frac{d\psi_2}{dx} + \frac{1}{4} (\psi_2 + 2\rho^2 \psi_1) = 0 \tag{10-b}$$

$$x(1-x) \frac{d^2\psi_3}{dx^2} - \frac{x}{2} \frac{d\psi_3}{dx} + \frac{1}{4} (\psi_3 - 2\rho^2 \psi_4) = 0 \tag{10-c}$$

$$x(1-x)\frac{d^2\psi_4}{dx^2}-\frac{x}{2}\frac{d\psi_4}{dx}+\frac{1}{4}(\psi_4+2\rho^2\psi_3)=0 \quad (10\text{-d})$$

In order to more readily evaluate the constants a_n, α_n, b_n, β_n, let us define

$$q_n = N_n + iM_n; \quad N_n, M_n\text{—real}$$

$$\frac{p_0}{q_0}+\frac{p_1}{q_1}+\cdots+\frac{p_n}{q_n}=P_n+iQ_n; \quad P_n, Q_n\text{—real}$$

Hence, the equations for determining a_n, α_n, b_n, β_n become

$$a_0 \equiv 1 \quad \alpha_0 = 0$$

$$a_{n+1} = a_n N_n - \alpha_n M_n \quad (n=0,1,2,\cdots) \quad (11\text{-a})$$

$$\alpha_{n+1} = a_n M_n + \alpha_n N_n \quad (11\text{-b})$$

$$b_n = a_{n+1} P_n - \alpha_{n+1} Q_n \quad (n=0,1,2,\cdots) \quad (12\text{-a})$$

$$\beta_n = a_{n+1} Q_n + \alpha_{n+1} P_n \quad (12\text{-b})$$

As an example of the above formulation, we have

$$N_0 = \frac{1}{8}$$

$$N_1 = \frac{11}{24}$$

$$N_2 = \frac{29}{48}$$

$$M_0 = -\frac{\rho^2}{4}$$

$$M_1 = -\frac{\rho^2}{12}$$

$$M_2 = -\frac{\rho^2}{24}$$

$$P_0 = \frac{3}{2}\left(1-\frac{1}{\rho^4+1/4}\right)$$

$$P_1 = \frac{7}{3}-\frac{3/2}{\rho^4+1/4}-\frac{77/2}{\rho^4+121/4}$$

$$Q_0 = -\frac{3\rho^2}{\rho^4+1/4}$$

$$Q_1 = -\frac{10}{\rho^2}\frac{(1+37/4\rho^4)}{(1+1/4\rho^4)(1+121/4\rho^4)}$$

$$a_1 = \frac{1}{8}$$

$$a_2 = -\frac{\rho^4}{48}\left(1-\frac{11}{4\rho^4}\right)$$

$$b_0 = -\left(\frac{3}{4}\right)^2$$

$$\alpha_1 = -\rho^2/4$$

$$\alpha_2 = -\rho^2/8$$

$$\beta_0 = -3\rho^2/8$$

As a final calculation, let us show that the power series given by Eq. (7) and (8) are absolutely, and hence uniformly, convergent in the range $0 \leqslant x \leqslant 1$). In order to do this, consider the series

$$\sum_{n=0}^{\infty}|A_n|$$

This series can be shown to be convergent by Raabe's test (Ref. 2), if a value of $\sigma > 1$ can be found such that

$$\frac{|A_n|}{|A_{n+1}|}=1+\frac{\sigma}{n}+0\left(\frac{1}{n}\right) \text{ as } n\to\infty$$

or, alternatively, that

$$\lim_{n\to\infty}\left\{n\left[\left(1+\frac{\sigma}{n}\right)-\frac{|A_n|}{|A_{n+1}|}\right]\right\}=0$$

However, as

$$\frac{|A_n|}{|A_{n+1}|}=\frac{1}{|q_n|}=\frac{(n+1)(n+2)}{\sqrt{(n^2+3n/2+1/4)^2+(\rho^2/2)^2}}$$

$$=1+\frac{3/2}{n}+\mathrm{O}\left(\frac{1}{n^2}\right)$$

it is clear that the first solution $u_i(x)$ is absolutely convergent for $x = 1$, and hence is both absolutely and uniformly convergent in the range $\mathrm{O} \leqslant x \leqslant 1$.

The absolute convergence of the power series constituent of the second solution follows from the same property of the first solution if one recognizes that the nth term in the second series is essentially that of the first series multiplied by the nth partial sequence $P_n + i\,Q_n$. If the absolute value of this sequence can be shown to be bounded, the absolute convergence of the second series follows directly. However, as

$$|P_n + i\,Q_n| \leqslant |P_n| + |Q_n|$$

and

$$|P_n| = 2\left|\sum_{r=0}^{n}\left\{\frac{(r+3/2)}{(r+1)(r+2)}-\frac{(r+3/4)(r^2+3r/2+1/4)}{(r^2+3r/2+1/4)^2+(\rho^2/2)^2}\right\}\right|$$

$$|Q_n| = \rho^2\sum_{r=0}^{n}\frac{(r+3/4)}{(r^2+3r/2+1/4)^2+(\rho^2/2)^2}$$

it can be shown that

$$\lim_{n\to\infty}\left\{n^2\left[\frac{(n+3/4)(n^2+3n/2+1/4)}{(n^2+3n/2+1/4)^2+(\rho^2/2)^2} - \frac{(n+3/2)}{(n+1)(n+2)}\right]\right\} = 3/4$$

$$\lim_{n\to\infty}\left\{n^3\left[\frac{(n+3/4)}{(n^2+3n/2+1/4)^2+(\rho^2/2)^2}\right]\right\} = 1$$

Hence, by the Comparison Test with the series $\sum 1/n^2$, $\sum 1/n^3$ respectively, the partial sequences $P_n(n), Q_n(n)$ are shown to be convergent. Plots of such sequences are given in Fig. 2 and 3.

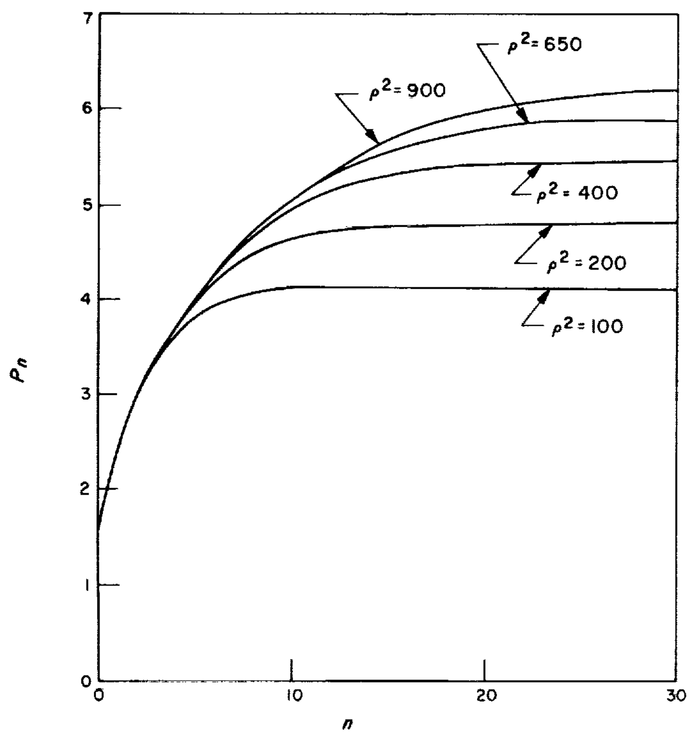

Fig. 2. Partial sequence $P_n(n)$

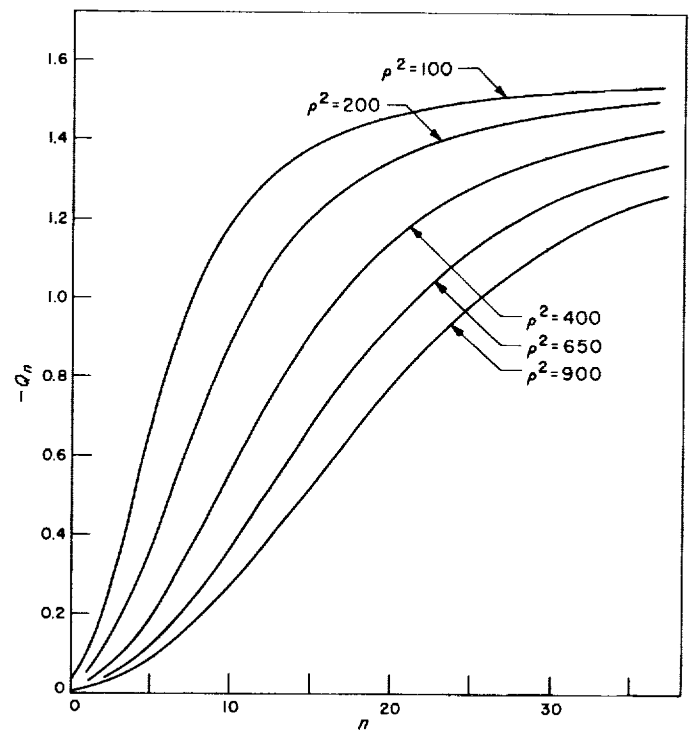

Fig. 3. Partial sequence $Q_n(n)$

IV. CALCULATION OF $\psi_1(x), \psi_2(x), \psi_3(x), \psi_4(x)$

In attempting to evaluate the above series representation for ψ_1, ψ_2, ψ_3, ψ_4, the two principal difficulties that must be remedied are truncation and round-off error. The truncation error may be overcome by simply retaining a relatively large number of terms in the expansions. This should pose no difficulty if the calculations are to be performed on a digital computer. However, this procedure contributes to the round-off error, as the nth term in the series cannot be computed directly, but must be computed from the $(n-1)$th term by means of Eq. (11) and (12). Hence, there will be a progressive error due to rounding off to the number of significant figures that can be carried.

In order to evaluate the functions $\psi_1, \cdots, \psi_4$ and, at the same time, obtain some estimate of the error in the calculations, the absolute value of A_n given by

$$|A_n| = \sqrt{a_n^2 + \alpha_n^2}$$

was computed on an IBM 7090 for several values of the parameter ρ up to $n = 49$. These values are presented in Fig. 4, and indicate that, as ρ increases, both the numerical value of $|A_n|$ and the value of n at which the value of $|A_n|$ is a maximum, increase.

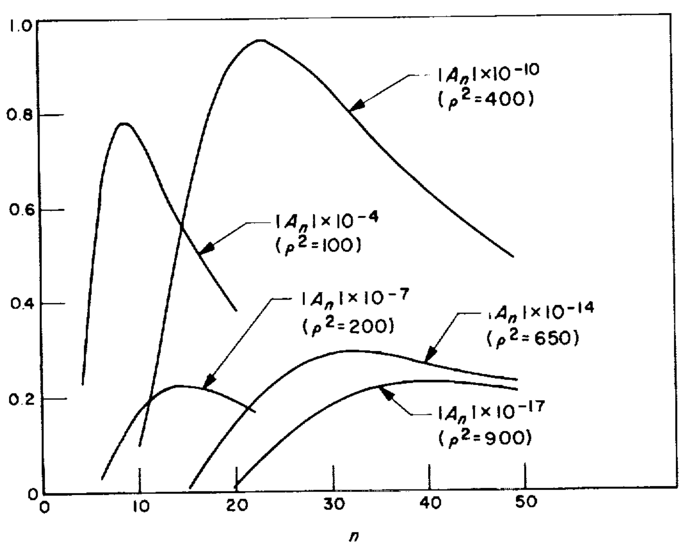

Fig. 4. Variation of power series coefficient with index (n)

From the form of $|q_n|$ it can be argued that $|q_n| < 1$ for $(\rho/n) << 1$. Hence, the function $|A_n(n)|$ is a monotone decreasing function after reaching a maximum, say $|A_N|$. This suggests that a rough estimate of error introduced by truncating the series

$$\sum_{n=0}^{\infty} |A_n| x^n$$

can be obtained by noting that, as

$$\sum_{n=N}^{\infty} |A_n| x^n < |A_N| \sum_{n=N}^{\infty} x^n = |A_N| \left\{ \sum_{n=0}^{\infty} x^n - \sum_{n=0}^{N-1} x^n \right\}$$

and

$$\sum_{n=0}^{N-1} x^n = \frac{1-x^N}{1-x}; \quad \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

one obtains

$$\sum_{n=N}^{\infty} |A_n| x^n < |A_N| \frac{x^N}{1-x}$$

Thus, if we write

$$\psi_1(x) = x \left\{ \sum_{n=0}^{N-1} a_n x^n + \sum_{n=N}^{\infty} a_n x^n \right\}$$

and note that $|a_n| \leqslant |A_n|$, it follows that

$$\left| \psi_1 - x \sum_{n=0}^{N-1} a_n x^n \right| \leqslant x \sum_{n=N}^{\infty} |a_n| x^n < x \sum_{n=N}^{\infty} |A_n| x^n$$

$$< |A_N| \frac{x^{N+1}}{1-x}$$

Similar relations can be written for the remaining functions. It is necessary only to estimate the limit of the sequence $P_n + i\,Q_n$ as $n \to \infty$.

On the other hand, the error due to round-off cannot be handled quite so easily. It is possible, however, to obtain a check on this process by comparing the functions ψ_1, ψ_2 with known functions whose nth terms can be computed directly. To this end, we note that, for $(1 - x) \approx 1$, Eq. (5) becomes a form of Bessel's equation giving rise to the Kelvin functions. Specifically, it can be shown that

$$\rho^2 \psi_1(x) \approx \sqrt{2\rho^2 x}\,\mathrm{bei}'\sqrt{2\rho^2 x}$$

$$\rho^2 \psi_2(x) \approx \sqrt{2\rho^2 x}\,\mathrm{ber}'\sqrt{2\rho^2 x}$$

for small values of x. The functions ψ_3, ψ_4 do not appear to have such a simple representation in terms of remaining Kelvin functions. The results of calculating the functions ψ_1, ψ_2 by retaining 50 terms in the series is presented in Fig. 5 and 6 along with the appropriate Kelvin function for comparison. It is apparent that the error is negligible at such small values of the argument (x). Figures 7 and 8 present the results of similar calculations for the functions F_3, F_4, where

$$F_3 = x^2 \sum_{n=0}^{\infty} b_n x^n$$

$$F_4 = x^2 \sum_{n=0}^{\infty} \beta_n x^n$$

Finally, the functions ψ_1, ψ_2, F_3, F_4 are presented over a larger range of x in Fig. 9–18 for several values of the parameter ρ, where, again, 50 terms in the series have been retained.

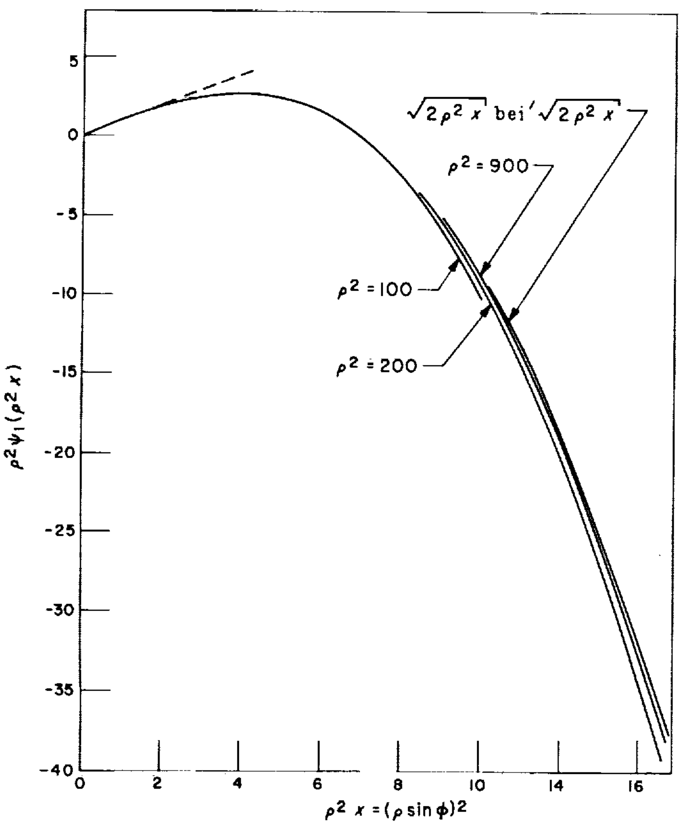

Fig. 5. Power series ψ_1

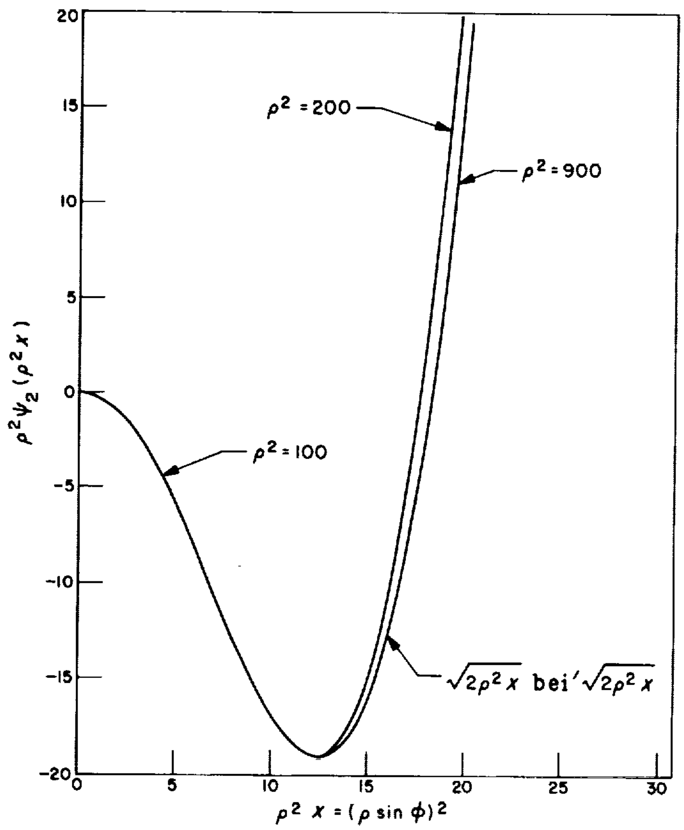

Fig. 6. Power series ψ_2

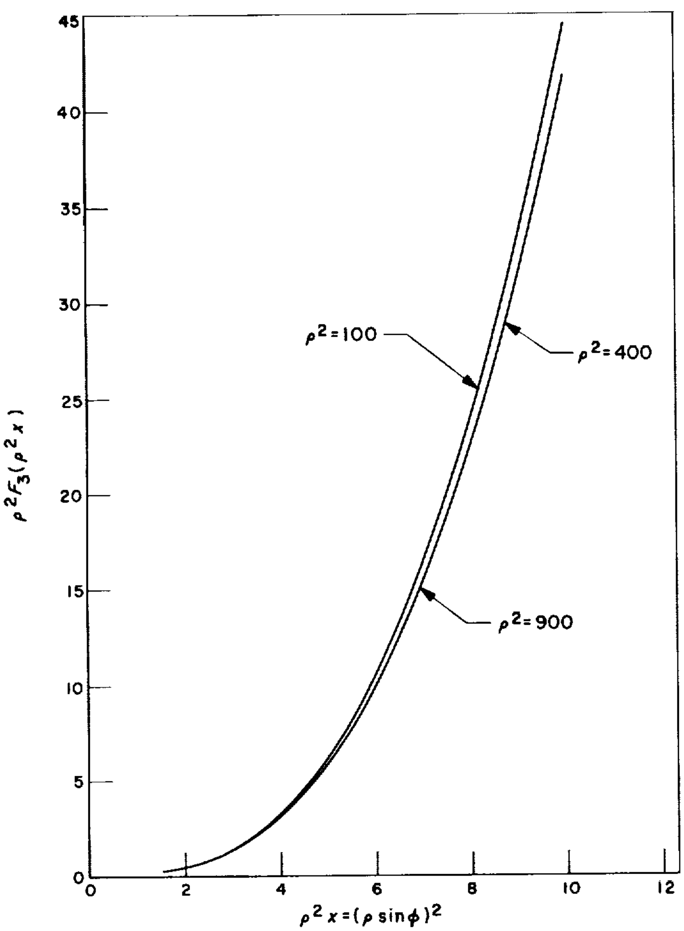

Fig. 7. Power series F_3

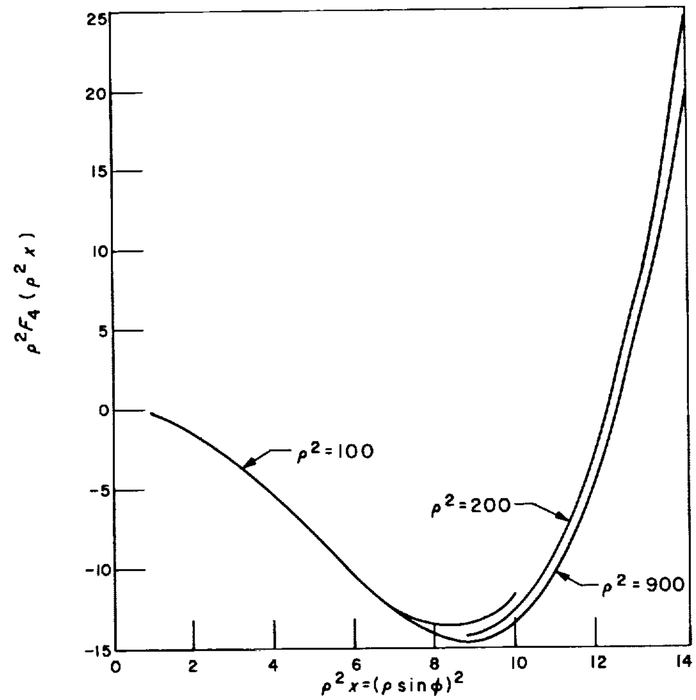

Fig. 8. Power series F_4

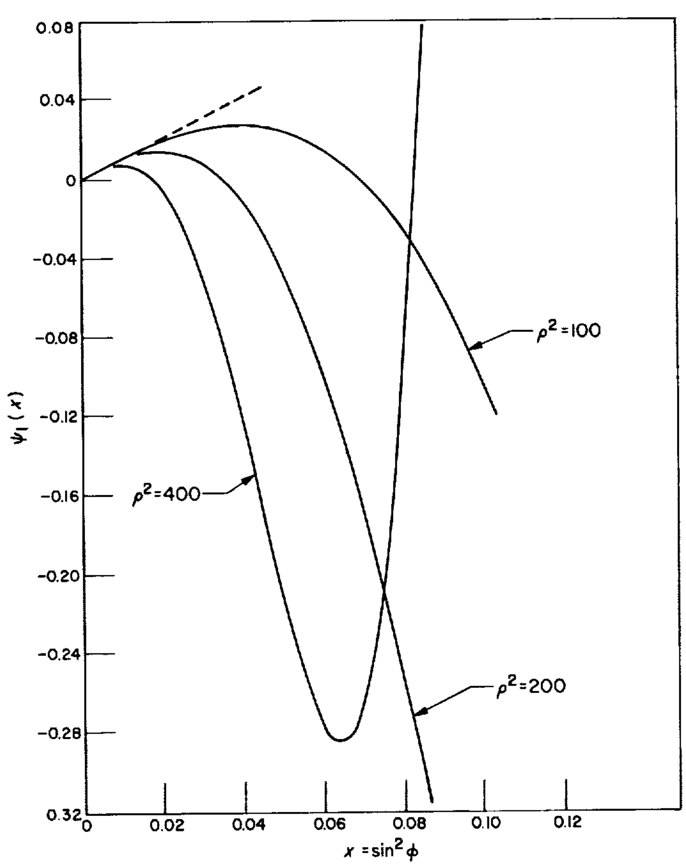

Fig. 9. Power series $\psi_1(x)$

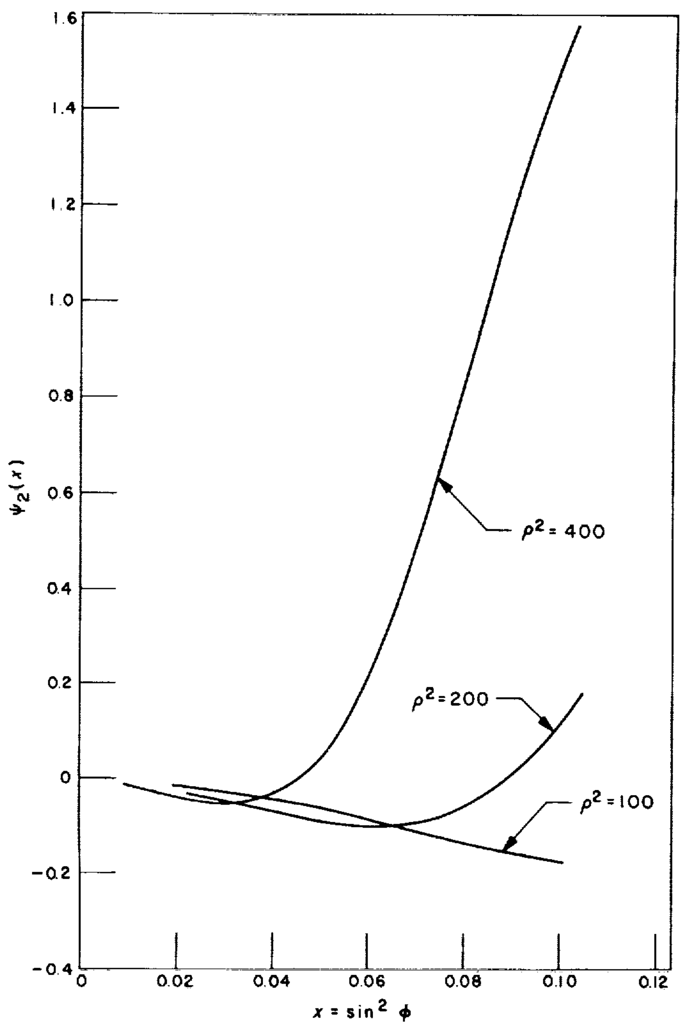

Fig. 10. Power series $\psi_2(x)$

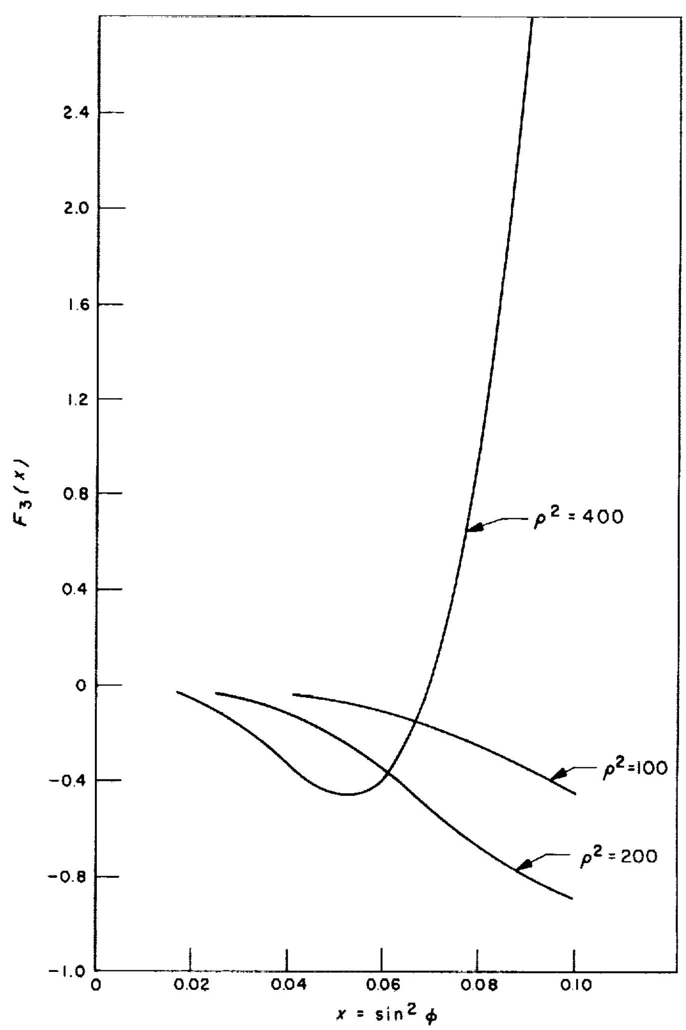

Fig. 11. Power series $F_3(x)$

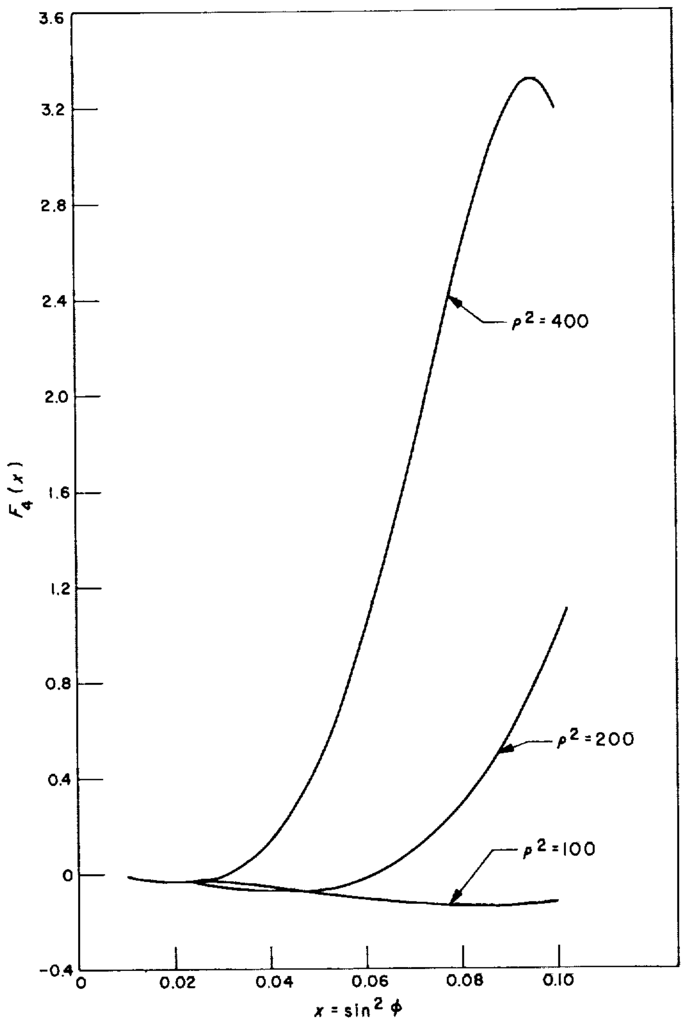

Fig. 12. Power series $F_4(x)$

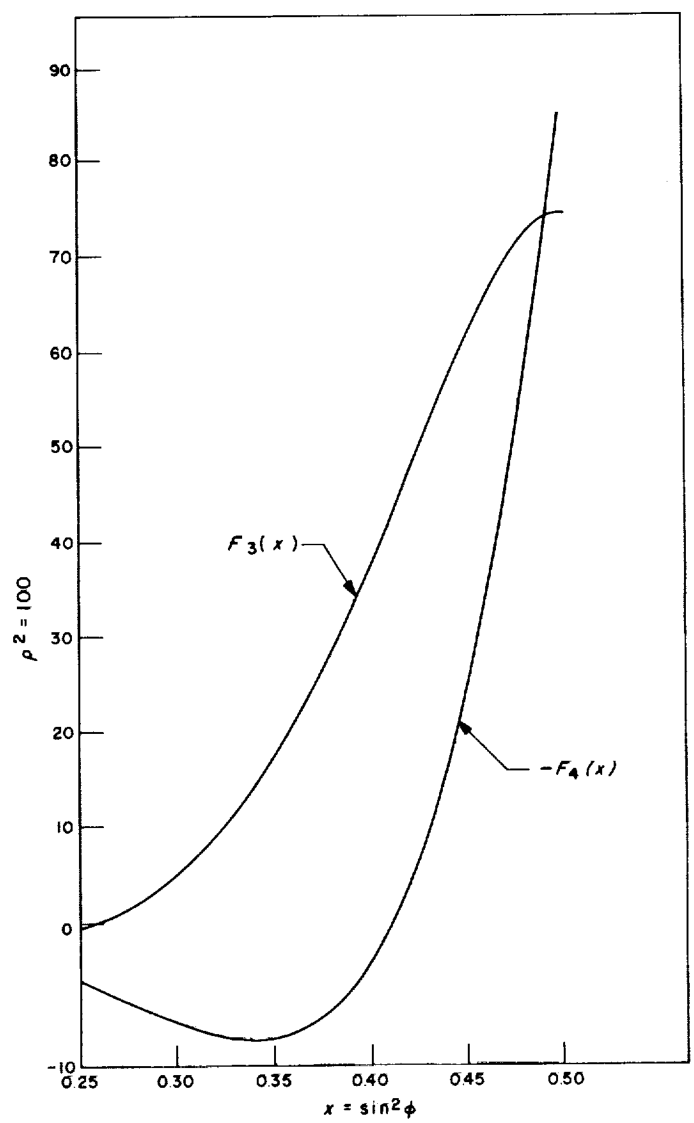

Fig. 13. Power series F_3, F_4 ($\rho^2 = 100$)

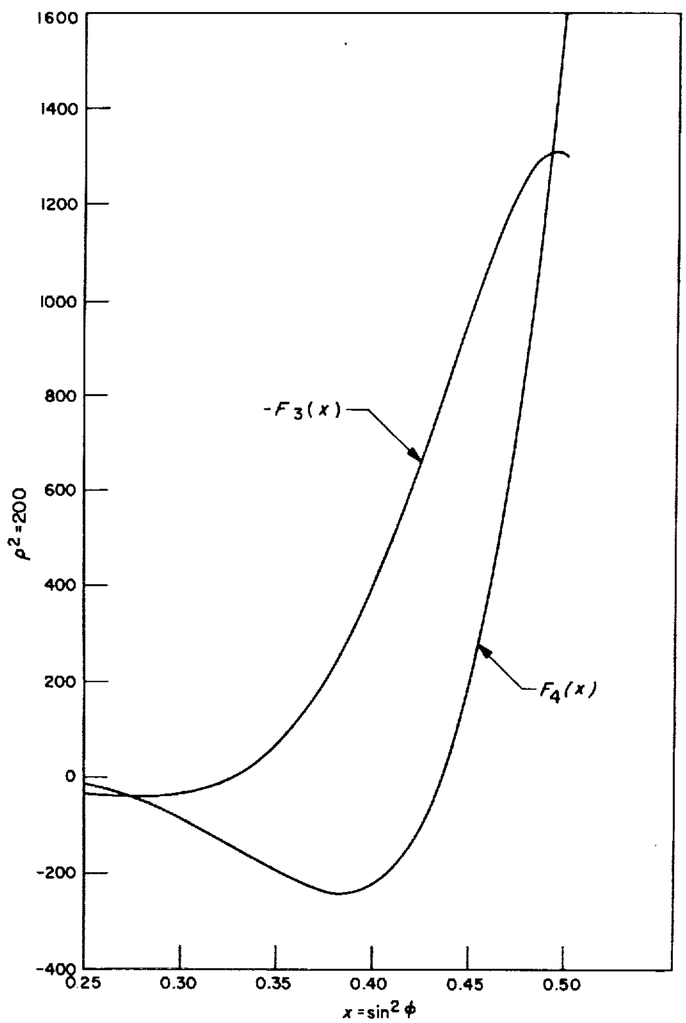

Fig. 14. Power series F_3, F_4 ($\rho^2 = 200$)

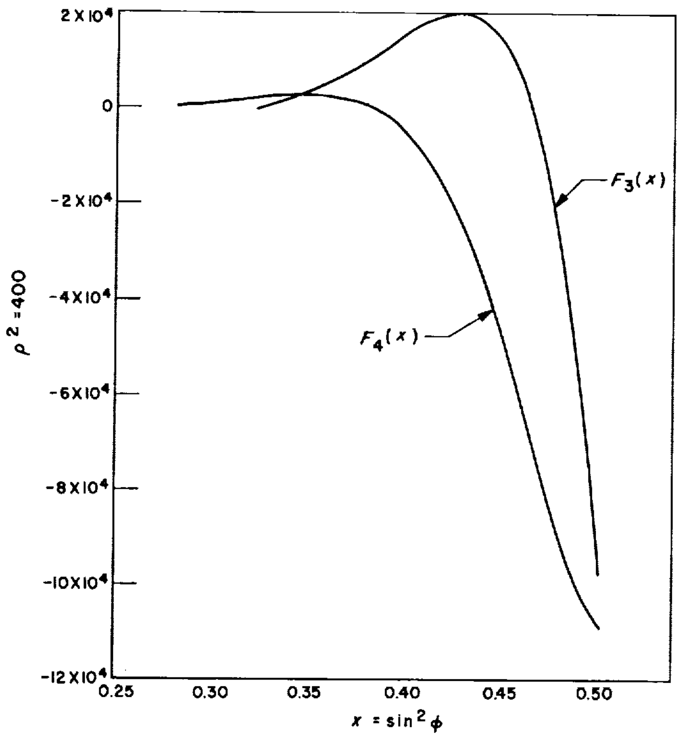

Fig. 15. Power series F_3, F_4 ($\rho^2 = 400$)

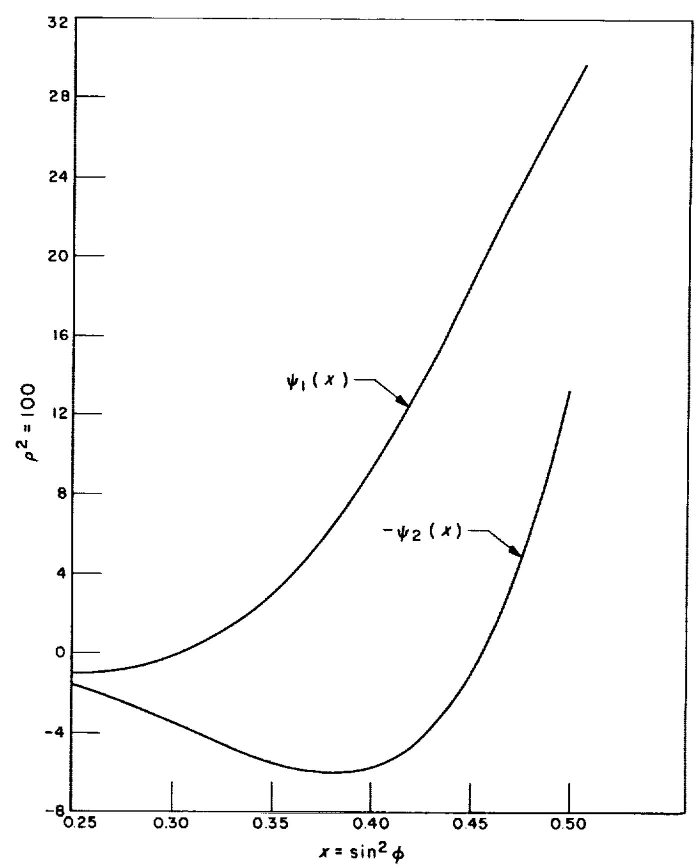

Fig. 16. Power series ψ_1, ψ_2 ($\rho^2 = 100$)

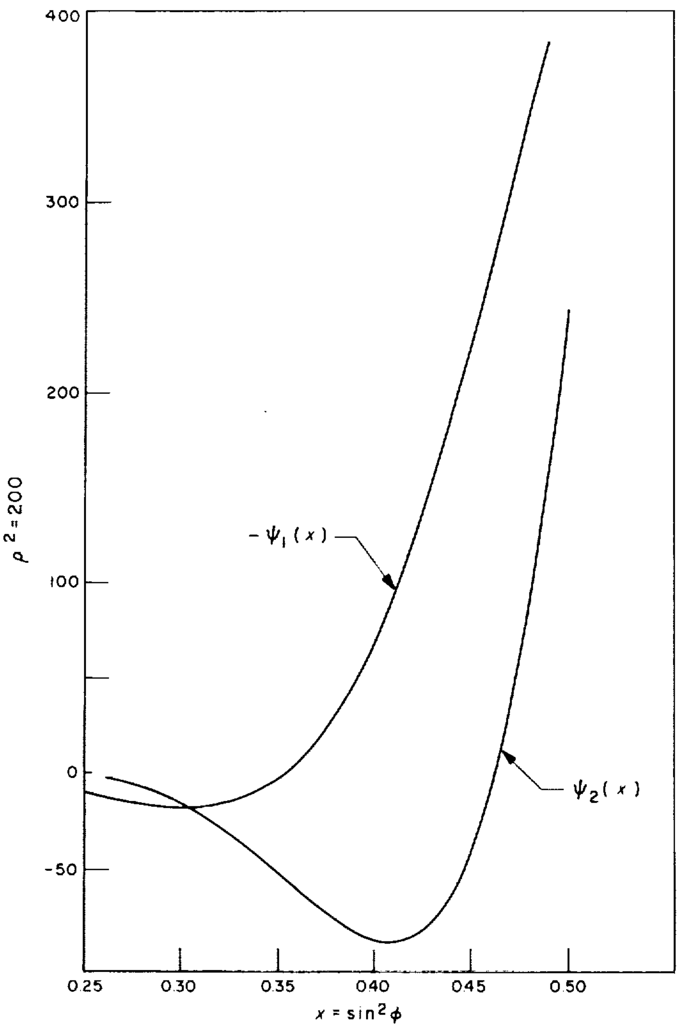

Fig. 17. Power series ψ_1, ψ_2 ($\rho^2 = 200$)

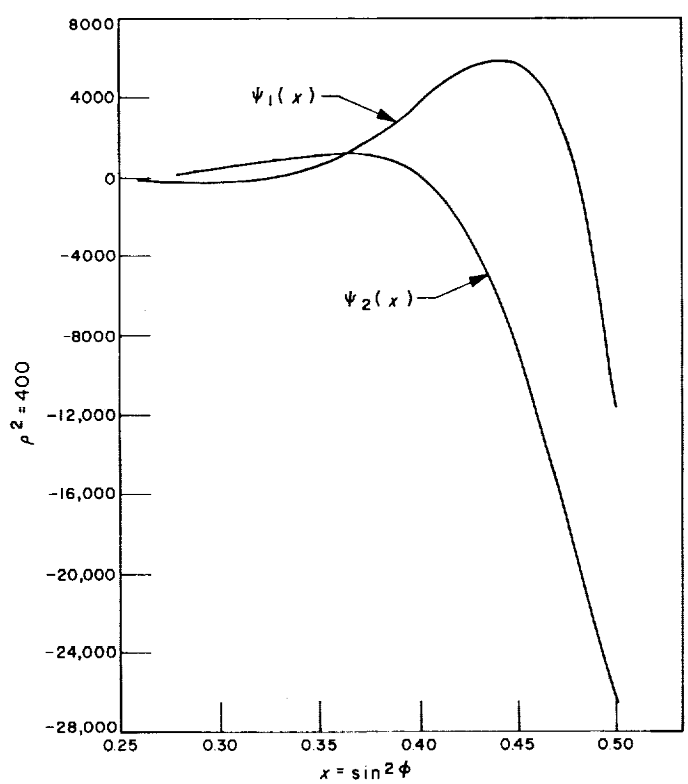

Fig. 18. Power series ψ_1, ψ_2 ($\rho^2 = 400$)

V. APPLICATION: INFLUENCE COEFFICIENTS

As an application of the above results, let us consider the problem of determining the Influence Coefficients, i.e., the deflection δ and rotation V due to a unit edge shear H or bending moment M_b applied at the edge $\phi = \phi_b$ (see Fig. 19). In order that the shear be well behaved at the apex, let us choose

$$z = C_1 \psi_1 + C_2 \psi_2 \qquad C_3, C_4 \equiv 0$$

Thus, we wish to determine δ, V, where

$$\delta = (w \sin\phi - v\cos\phi)\,|_{\phi=\phi_b}$$

$$= \frac{\sin 2\phi_b}{Eh}\left\{C_1\left[\frac{d\psi_1}{dx} - \frac{1+\nu}{2x}\psi_1\right] + C_2\left[\frac{d\psi_2}{dx} - \frac{1+\nu}{2x}\psi_2\right]\right\}\Bigg|_{\phi=\phi_b}$$

$$V = \frac{1}{Eah\sin\phi_b}\left[C_1(\nu\psi_1 + 2\rho^2\psi_2) + C_2(\nu\psi_2 - 2\rho^2\psi_1)\right]\Big|_{\phi=\phi_b}$$

subject to the condition that

$$M_\phi(\phi=\phi_b) = M_b \qquad (Q_\phi\sin\phi - N_\phi\cos\phi)\,|_{\phi=\phi_b} = H$$

That is,

$$\left(\frac{C_1\psi_1}{x} + \frac{C_2\psi_2}{x}\right)\Bigg|_{\phi=\phi_b} = aH \tag{13}$$

$$C_1\left[\left(\nu\frac{d\psi_1}{dx} + 2\rho^2\frac{d\psi_2}{dx}\right) - \frac{1-\nu}{2x}(\nu\psi_1 + 2\rho^2\psi_2)\right]\Bigg|_{\phi=\phi_b} + C_2\left[\left(\nu\frac{d\psi_2}{dx} - 2\rho^2\frac{d\psi_1}{dx}\right) - \frac{1-\nu}{2x}(\nu\psi_2 - 2\rho^2\psi_1)\right]\Bigg|_{\phi=\phi_b} = \frac{-6(1-\nu^2)a^2}{h^2}\frac{M_b}{\cos\phi_b} \tag{14}$$

If we define the matrix $\mathbf{A}$ in the usual form

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

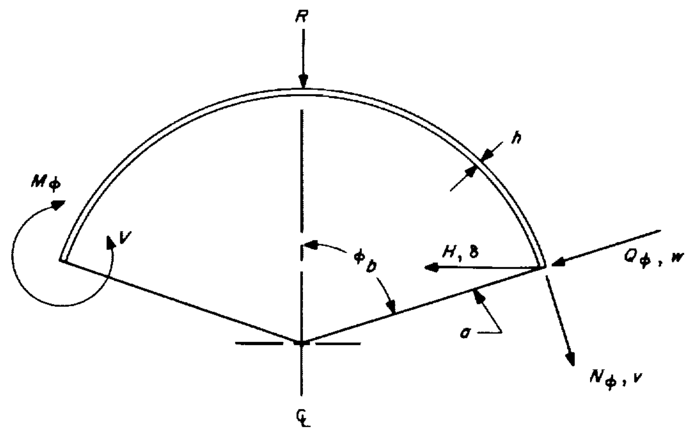

Fig. 19. Sign convention

where

$$a_{11} = \frac{\psi_1(x_b)}{x_b}$$

$$a_{12} = \frac{\psi_2(x_b)}{x_b}$$

$$a_{21} = \left[\left(\nu\frac{d\psi_1}{dx} + 2\rho^2\frac{d\psi_2}{dx}\right) - \frac{1-\nu}{2x}(\nu\psi_1 + 2\rho^2\psi_2)\right]\Bigg|_{x=x_b}$$

$$a_{22} = \left[\left(\nu\frac{d\psi_2}{dx} - 2\rho^2\frac{d\psi_1}{dx}\right) - \frac{1-\nu}{2x}(\nu\psi_2 - 2\rho^2\psi_1)\right]\Bigg|_{x=x_b}$$

Equations (13) and (14) can be represented as

$$\mathbf{A}\begin{Bmatrix} C_1 \\ C_2 \end{Bmatrix} = \begin{Bmatrix} aH \\ \dfrac{-6(1-\nu)a^2}{h^2}\dfrac{M_b}{\cos\phi_b} \end{Bmatrix}$$

In presenting the results, it is convenient to separate the loading into two parts. For the shear loading H only, we have

$$C_1 = \frac{aH}{|\mathbf{A}|}a_{22}$$

$$C_2 = -\frac{aH}{|\mathbf{A}|}a_{21}$$

$$\frac{|\mathbf{A}|}{\sin 2\phi_b}\frac{Eh\,\delta}{aH} = \left[-2\rho^2\left(\frac{d\psi_1^2}{dx} + \frac{d\psi_2^2}{dx}\right) + \frac{\nu^2}{x}\left(\psi_2\frac{d\psi_1}{dx} - \psi_1\frac{d\psi_2}{dx}\right) + \frac{2\rho^2}{x}\left(\psi_2\frac{d\psi_2}{dx} + \psi_1\frac{d\psi_1}{dx}\right) - \frac{(1-\nu^2)\rho^2}{2x^2}\left(\psi_1^2 + \psi_2^2\right)\right]\Bigg|_{x=x_b}$$

$$|\mathbf{A}| \sin \phi_b \frac{VD}{a^2 H} = \left[\psi_1 \frac{d\psi_2}{dx} - \psi_2 \frac{d\psi_1}{dx} \right]_{x=x_b}$$

while, for the bending moment M_b only, we have

$$C_1 = \frac{6(1-\nu^2)\,a^2}{h^2} \frac{M_b}{\cos \phi_b} \frac{a_{12}}{|\mathbf{A}|}$$

$$C_2 = \frac{-6(1-\nu^2)\,a^2}{h^2} \frac{M_b}{\cos \phi_b} \frac{a_{11}}{|\mathbf{A}|}$$

$$\frac{|\mathbf{A}|}{\sin \phi_b} \frac{D\,\delta}{a^2 M_b} = \left[\frac{1}{x} \left(\psi_2 \frac{d\psi_1}{dx} - \psi_1 \frac{d\psi_2}{dx} \right) \right]_{x=x_b}$$

$$|\mathbf{A}| \sin 2\phi_b \frac{DV}{2\rho^2 a M_b} = \left[\frac{1}{x} (\psi_1^2 + \psi_2^2) \right]_{x=x_b}$$

These coefficients were computed for several values of the parameter ρ and are presented in Fig. 20 and 21. The values of the opening angles that were chosen correspond to the range intermediate between shallow shell theory and quasi-cylindrical theory. The data have been normalized by dividing by the corresponding quasi-cylindrical value. A further comparison is made with the shallow shell theory of Ref. 3.

In performing the calculations, the values of the derivatives of ψ_1, ψ_2 were estimated by means of the Three Point formula (see Ref. 4):

$$\frac{d\psi}{dx} = \frac{1}{2\Delta} \left[\psi(x+\Delta) - \psi(x-\Delta) \right]$$

where Δ was taken to be 0.001. It was found that values obtained from this formula differed insignificantly from those obtained using a Five or Seven Point formula.

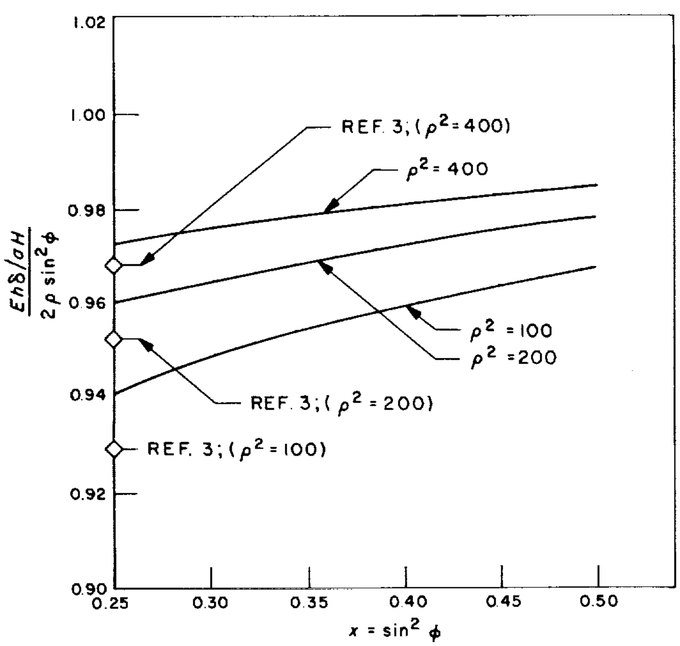

Fig. 20. Influence coefficients (ν = 0.30)

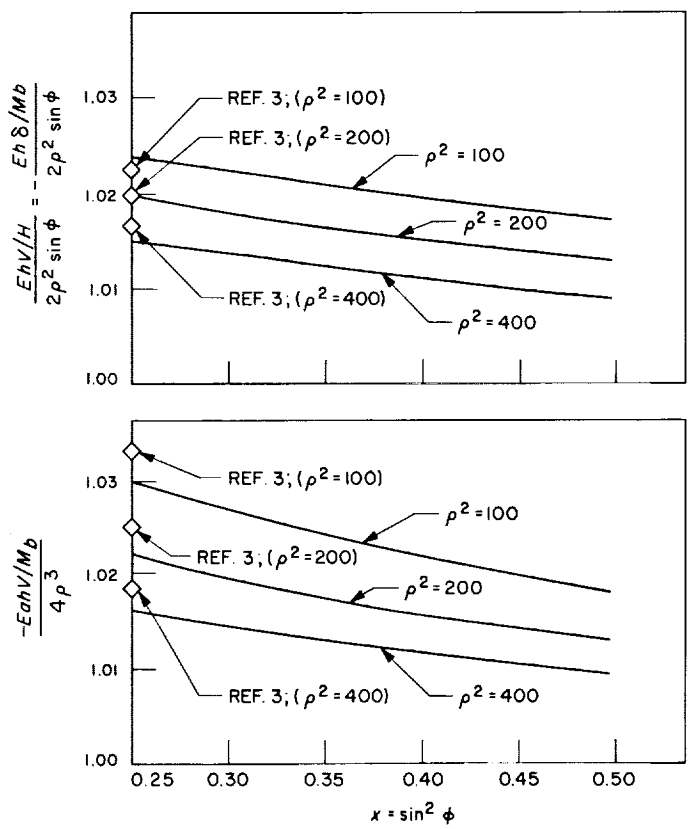

Fig. 21. Influence coefficients (ν = 0.30)

VI. APPLICATION: CONCENTRATED LOAD

As a further application of the above results, let us consider the problem of determining the stresses and deflections due to a concentrated force R applied at the apex (see Fig. 19). The boundary condition at the outer edge $\phi = \phi_b$ will be left arbitrary for the present. With

$$z = C_1 \psi_1 + C_2 \psi_2 + C_3 \psi_3 + C_4 \psi_4$$

the constants C_3, C_4 can be determined independently of C_1, C_2 by analyzing the behavior of the stresses, bending moments and deflections in the neighborhood of the apex. As

$$\psi_1(x) = x + \frac{x^2}{8} + O(x^3)$$

$$\frac{d\psi_1}{dx}(x) = 1 + \frac{x}{4} + O(x^2)$$

$$\psi_2(x) = -\frac{\rho^2 x^2}{4} - \frac{\rho^2 x^3}{8} + O(x^4)$$

$$\frac{d\psi_2}{dx}(x) = -\frac{\rho^2 x}{2} - \frac{3\rho^2 x^2}{8} + O(x^3)$$

$$\psi_3(x) = -\frac{1}{\rho^4 + 1/4} + \alpha_1 x^2 \ln x - \beta_0 x^2 - b_0 x^2 + O(x^3; x^3 \ln x)$$

$$\frac{d\psi_3}{dx}(x) = \ln x + 1 + (a_1 - 2b_0)x + 2a_1 x \ln x + O(x^2; x^2 \ln x)$$

$$\psi_4(x) = \frac{2\rho^2}{\rho^4 + 1/4} + \alpha_1 x^2 \ln x - \beta_0 x^2 + O(x^3; x^3 \ln x)$$

$$\frac{d\psi_4}{dx}(x) = (\alpha_1 - 2\beta_0)x + 2\alpha_1 x \ln x + O(x^2; x^2 \ln x)$$

it follows that

$$EahV = \sqrt{x}\left\{\frac{1/x}{\rho^4 + 1/4}\left[(4\rho^4 - \nu)C_3 + 2\rho^2(1+\nu)C_4\right] + (\nu C_3 - 2\rho^2 C_4)\ln x + (\nu C_1 - 2\rho^2 C_2) + O(x; x\ln x)\right\}$$

$$aN\phi = -\frac{1}{x}\left(\frac{R}{2\pi} + \frac{2\rho^2 C_4 - C_3}{\rho^4 + 1/4}\right) - C_3 \ln x - \left[C_1 + \frac{1/2}{\rho^4 + 1/4}(C_3 - 2\rho^2 C_4)\right] + O(x; x\ln x)$$

$$aN_\theta = \frac{1}{x}\left(\frac{R}{2\pi} + \frac{2\rho^2 C_4 - C_3}{\rho^4 + 1/4}\right) - C_3 \ln x + \left[\frac{1/2}{\rho^4 + 1/4}(C_3 - 2\rho^2 C_4) - 2C_3 - C_1\right] + O(x; x\ln x)$$

$$v = \frac{1+\nu}{Eh}\left\{\frac{1}{\sqrt{x}}\left[\frac{R}{2\pi} + \frac{2\rho^2 C_4 - C_3}{\rho^4 + 1/4}\right] + \sqrt{x}\ln x\left(C_3 - \frac{R}{4\pi}\right) + \sqrt{x}\left[\frac{Ehv_0}{1+\nu} + \frac{R}{2\pi}\left(\ln 2 - \frac{1}{2}\right) + C_1\right]\right\} + O(x^{3/2}; x^{3/2}\ln x)$$

$$Ehw = \left(2C_3 - \frac{1+\nu}{2}\frac{R}{2\pi}\right)\ln x + \left[2(C_1 + C_3)\frac{R(1+\nu)}{2\pi}(1 - \ln 2) + Ehv_0\right] + O(x; x\ln x)$$

$$\frac{-6(1-\nu^2)a^2}{h^2}\frac{M\phi}{\cos\phi} = -\frac{1-\nu}{2}\frac{1/x}{\rho^4 + 1/4}\left[C_3(4\rho^4 - \nu) + 2\rho^2 C_4(1+\nu)\right] + \frac{1+\nu}{2}(\nu C_3 - 2\rho^2 C_4)\ln x + \left[(\nu C_3 - 2\rho^2 C_4) + \frac{1+\nu}{2}(\nu C_1 - 2\rho^2 C_2)\right] + O(x; x\ln x)$$

$$\frac{-6(1-\nu^2)a^2}{h^2}\frac{M\theta}{\cos\phi} = \frac{1-\nu}{2}\frac{1/x}{\rho^4 + 1/4}\left[C_3(4\rho^4 - \nu) + 2\rho^2 C_4(1+\nu)\right] + \frac{1+\nu}{2}(\nu C_3 - 2\rho^2 C_4)\ln x + \left[\nu(\nu C_3 - 2\rho^2 C_4) + \frac{1+\nu}{2}(\nu C_1 - 2\rho^2 C_2)\right] + O(x; x\ln x)$$

Apparently, one may modify the singularity at the apex by choosing

$$C_3 (4 \rho^2 - \nu) + 2 \rho^2 C_4 (1 + \nu) = 0$$

$$\frac{R}{2\pi} + \frac{1}{\rho^4 + 1/4} (2 \rho^2 C_4 - C_3) = 0$$

that is, by requiring

$$C_3 = \frac{R (1 + \nu)}{8\pi}$$

$$C_4 = - \frac{\rho^4 - \nu/4}{\rho^2} \frac{R}{4\pi}$$

Thus, the form of the solution in the neighborhood of the apex is

$$N_\phi \rightarrow - \frac{R (1 + \nu)}{8\pi a} \ln x \approx - \frac{R (1 + \nu)}{4\pi a} \ln \phi$$

$$N_\theta \rightarrow - \frac{R (1 + \nu)}{8\pi a} \ln x \approx - \frac{R (1 + \nu)}{4\pi a} \ln \phi$$

$$w \rightarrow \frac{R\rho^2}{Eh} \left[\frac{2C_1}{R\rho^2} + \frac{Ehv_0}{R\rho^2} - \frac{1 + \nu}{4\pi\rho^2} (1 - \ln 4) \right]$$

$$M_\phi \rightarrow - \frac{R (1 + \nu)}{8\pi} \ln x \approx - \frac{R (1 + \nu)}{4\pi} \ln \phi$$

$$M_\theta \rightarrow - \frac{R (1 + \nu)}{8\pi} \ln x \approx - \frac{R (1 + \nu)}{4\pi} \ln \phi$$

Finally, let us determine the constants C_1, C_2 to correspond to an unrestrained edge at $\phi = \phi_b$; i.e., such that $H, M\phi \equiv 0$. This set of conditions requires that

$$\begin{Bmatrix} C_1 \\ \\ C_2 \end{Bmatrix} = - \frac{R}{2\pi} \mathbf{A}^{-1} \begin{Bmatrix} \frac{1}{x_b} \left[\cos \phi_b + \frac{1 + \nu}{4} \psi_3 - \left(\frac{\rho^4 - \nu/4}{2\rho^2} \right) \psi_4 \right]_{x=x_b} \\ \frac{3(1 - \nu^2)a^2}{h^2} \left[\left(\frac{d\psi_3}{dx} - \frac{1 - \nu}{2x} \psi_3 \right) + \frac{1}{2\rho^2} \left(\frac{d\psi_4}{dx} - \frac{1 - \nu}{2x} \psi_4 \right) \right]_{x=x_b} \end{Bmatrix}$$

where the elements of the matrix $\mathbf{A}$ are given in Section V.

Calculation of the constants C_1, C_2 was performed for the same values of the parameters that were used in Section V. The values of the derivatives of the functions $\psi_1, \cdots, \psi_4$ were estimated, as before, by means of the Three Point formula. The resulting values for C_1, C_2 are presented in Table 1.

Of particular significance is the value of the radial deflection under the load. From the expansion in the neighborhood of the apex, it follows that

$$w(0) \approx \frac{Ra}{Eh^2} \sqrt{3(1 - \nu^2)} \left(\frac{2C_1}{R\rho^2} \right) \qquad v_0 \equiv 0$$

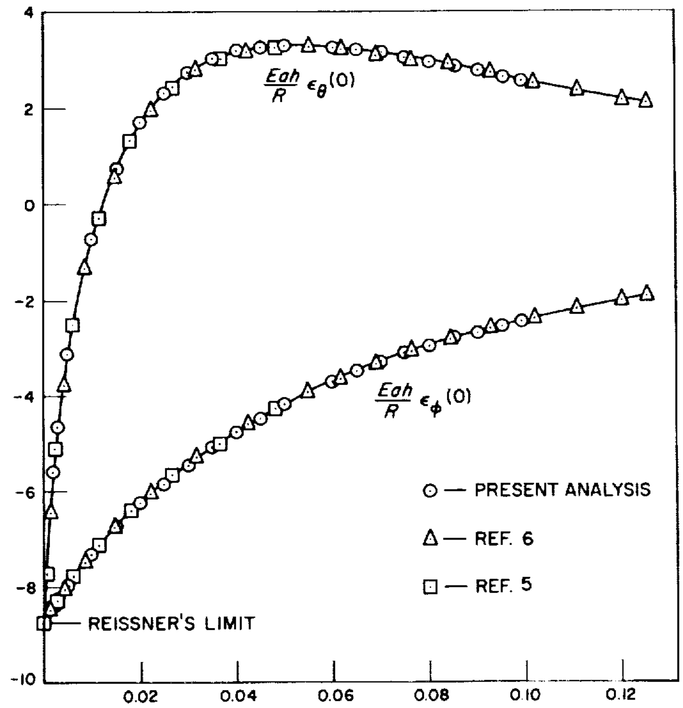

Fig. 22. Direct strains (ρ^2 = 100) (ν = 0.30)

The values of $2C_1/R\rho^2$ are to be compared with the shallow shell theory of Reissner (Ref. 5) in which the value of the coefficient is exactly 1/4.

Table 1. Concentrated load constants

ρ^2	$x(\phi_b)$	$2C_1/R\rho^2$	$2C_2/R\rho^2$
100	0.250	0.255	—0.648
	0.330	0.255	—0.646
	0.410	0.255	—0.645
	0.499	0.255	—0.645
200	0.250	0.253	—0.757
	0.330	0.253	—0.757
	0.410	0.253	—0.757
	0.499	0.253	—0.757
400	0.250	0.252	—0.868
	0.330	0.252	—0.868
	0.410	0.252	—0.868
	0.499	0.252	—0.868

As a further comparison, the direct component of the meridional and circumferential strain given by

$$\epsilon_\phi^{(0)} = \frac{(N_\phi - \nu N_\theta)}{Eh}$$

$$\epsilon_\theta^{(0)} = \frac{(N_\theta - \nu N_\phi)}{Eh}$$

is presented in Fig. 22, along with similar relations from Ref. 5 and 6. For both of the latter references, the strain relations being presented for camparison were computed for the case of no edge restraint.

VII. DISCUSSION

Of particular importance in the above work is the numerical evaluation of the functions ψ_1, ψ_2, ψ_3, ψ_4, as these form the basis on which to evaluate the relative merits of the various approximate theories. The only major difficulty seems to be the round-off error. Apparently, if a computation procedure could be constructed wherein the nth term in the series could be evaluated directly, this problem could be partially avoided. Unfortunately, there seems to be no solution to this difficulty. However, an alternate scheme can be proposed, and might be pursued at a later date, for comparison with the one used above. If one defines

$$q_n = |q_n| e^{-i\theta_n}$$

where

$$\tan\theta_n = \frac{(\rho^2/2)}{(n^2 + 3n/2 + 1/4)}$$

then, it follows that

$$A_n = |q_0||q_1||q_2|\cdots\cdots|q_{n-1}| \times \exp\left[-i(\theta_0 + \theta_1 + \cdots\cdots + \theta_{n-1})\right]$$

and hence

$$a_n = |q_0||q_1|\cdots\cdots|q_{n-1}|\cos(\theta_0 + \theta_1 + \cdots + \theta_{n-1}) \qquad (n \geqslant 1)$$

$$\alpha_n = -|q_0||q_1|\cdots\cdots|q_{n-1}|\sin(\theta_0 + \theta_1 + \cdots\cdots + \theta_{n-1}) \qquad (n \geqslant 1)$$

It can be argued that this method is not devoid of round-off error, but it does offer a more direct computational procedure.

Ignoring the possibility of numerical error, it seems evident from the examples presented in this Report that the approximate shell theories are certainly adequate to describe the stresses and displacements of thin shells. However, Fig. 20 indicates that a better approximation than an equivalent cylindrical shell should be made for engineering analyses for the horizontal deflection due to an edge shear H. Fortunately, shallow shell theory seems to give adequate agreement for the range of parameters investigated, so that one does not have to look too far.

Finally, it should be noted that the solution of Reissner (Ref. 5) in the neighborhood of a concentrated load shows excellent agreement with the analysis in Section VI. This is of particular importance, since shallow shell theory is considerably easier to use than the analysis in this Report. The only discrepancy exists in the immediate neighborhood of the apex where the stresses derived from the present analysis have a logarithmic singularity. This is convenient, as one might argue intuitively that the stress under the load R should be tensile.

As a final comparison, the nature of the singularity is presented in Table 2 for the present analysis and Ref. 5 and 6. It can be seen from Fig. 22 that the discrepancy exists over a negligibly small region of space near the apex where the solution would not be expected to be valid in any case.

Table 2. Comparison of several theories concerning the nature of the singularity under a concentrated load

	N_ϕ	N_θ	M_ϕ	w
Present analysis	$-\frac{R(1+\nu)}{4\pi a}\ln\phi$	$-\frac{R(1+\nu)}{4\pi a}\ln\phi$	$-\frac{R(1+\nu)}{4\pi}\ln\phi$	$\frac{Ra}{Eh^2}\sqrt{3(1-\nu^2)}\left[\frac{2C_1}{R\rho^2}+0\left(\frac{1}{\rho^2}\right)\right]$
Ref. 5 (Reissner)	$-\frac{R}{8h}\sqrt{3(1-\nu^2)}$	$-\frac{R}{8h}\sqrt{3(1-\nu^2)}$	$-\frac{R(1+\nu)}{4\pi}\ln\phi$	$\frac{Ra}{Eh^2}\sqrt{3(1-\nu^2)}\left(\frac{1}{4}\right)$
Ref. 6 (Author)	$-\frac{\nu R}{4\pi a}\ln\phi$	$-\frac{\nu R}{4\pi a}\ln\phi$	$-\frac{R(1+\nu)}{4\pi}\ln\phi$	$-\frac{R}{2\pi Eh}\ln\phi$

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