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THE EQUILIBRIUM FREE SURFACE OF A CONTAINED LIQUID UNDER LOW GRAVITY AND CENTRIFUGAL FORCES

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SUMMARY

The equilibrium free surface of a liquid contained in a tank which is being simultaneously subjected to a translational acceleration and a rotation about an axis fixed with respect to the tank has been investigated. A general differential equation has been derived which governs the three-dimensional equilibrium free surface as a function of an inertial field due to translational acceleration, angular velocity, and surface tension. Because of the nonlinear character of this equation, a complete closed form solution is not presently possible. However, a closed form solution is presented which is valid when the capillary forces acting on the liquid may be neglected.

When the capillary forces are of the same order of magnitude as the inertial forces acting on the liquid, attention has been restricted to two dimensions and solutions were obtained by numerical integration in an effort to display the effects of various parameters on the free surface shape. Representative two-dimensional solution curves are presented and discussed for various magnitudes of translational acceleration, rotational rate, liquid-to-solid contact angle, and vapor volume. By obtaining a lattice of two-dimensional surfaces, the possibility of determining a three-dimensional surface is indicated.

INTRODUCTION

Liquid fuel systems and fluid subsystems for space vehicles must be designed for an environment where the forces acting on the liquid due to translational and rotational accelerations are of the same order of magnitude as the capillary forces acting on the liquid. When a liquid free surface exists, the motions and eventual equilibrium position of the liquid will be determined by these small forces during some part of almost any space mission.

A knowledge of the liquid equilibrium configuration in a low force environment will be needed in order to design systems such as those requiring restart of liquid fuel rockets after periods of coasting flight, evaporative temperature control devices, and effective liquid fuel tank venting. The determination of liquid equilibrium configurations is also a necessary first step in the application of pertubation analysis to the problem of fuel sloshing.

Although some experimental and analytical work bearing on some aspects of the problem (refs. 1 to 6) has been previously carried out, none of these investigations have included the effect of rotations about an axis perpendicular to an axis of symmetry of the liquid and removed from the surface of the liquid.

The purpose of this investigation is to determine the equilibrium free surface shape of a contained liquid experiencing inertial forces due to thrust and rotation that are of the same order of magnitude as the capillary forces acting on the liquid. A differential equation is derived which governs the three-dimensional free surface of the liquid as a function of an inertial field, angular velocity, and surface tension. Under certain restricted conditions this equation can be solved and the surface determined in closed form. These cases are discussed. In an effort to display the effects of various parameters on the free surface shape, attention is then restricted to two dimensions. The resulting equation is solved by numerical integration subject to constraints of an invariant contact angle at the interface of the liquid, liquid vapor, and tank wall, and an invariant vapor area. Representative solution curves are shown for various force loadings. In the solution of the equation an unknown constant is required in order that the solution satisfy all boundary conditions. The constant must be obtained by iterative trial-and-error techniques on an electronic computer. An empirical equation for this constant is presented for the range of interest of the equations. A method for obtaining a three-dimensional free surface by the superposition of a lattice of normal two-dimensional surfaces is indicated.

SYMBOLS

A	constant defined in equation (11)
a	acceleration due to thrust
a_x, a_y, a_z	components of acceleration relative to tank-fixed coordinates
$\frac{d}{dt}$	total time derivative
F	body forces acting on fluid particle
$\vec{i}, \vec{j}, \vec{k}$	base vectors for x,y,z coordinate system
K	mean curvature of liquid free surface
N_{Bo}	Bond number, $\frac{\rho a R^2}{\sigma}$
N_{cu}	centrifugal number, $\frac{\rho \omega^2 R^3}{2\sigma}$

p pressure in fluid

p_v pressure in vapor

q velocity of fluid particle

R half-width of tank

r position vector

V vapor area ratio

X,Y,Z inertial coordinates

x,y,z tank-fixed coordinates

θ_c liquid-to-solid contact angle

ξ,η nondimensionalized tank-fixed coordinates

ρ fluid density

σ surface-tension coefficient of fluid

Ω scalar potential of body force

ω angular velocity

$\omega_x,\omega_y,\omega_z$ components of angular velocity relative to inertial axes

∇ vector differential operator, $\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k$

$|_I$ differentiation with respect to inertially fixed coordinates

$|_T$ differentiation with respect to tank-fixed coordinates

$(\vec{\ }) \times (\vec{\ })$ vector cross product

$(\vec{\ }) \cdot (\vec{\ })$ vector dot product

Subscripts:

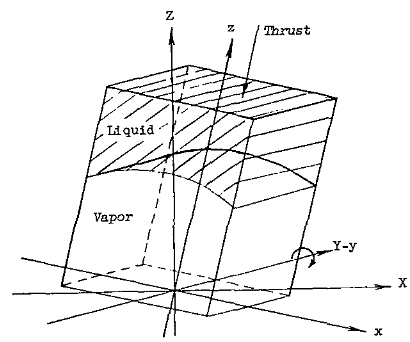
o initial value

n nth value

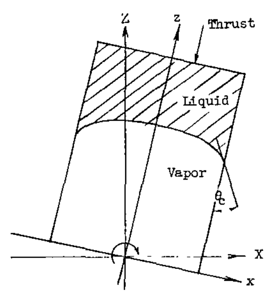
Primes denote partial differentiation with ξ. An asterisk denotes coordinate transformation. Arrows over symbols indicate vectors. Dots over symbols indicate derivatives with respect to time.

DERIVATION OF FREE SURFACE EQUATION

A tank partially filled with an incompressible fluid is subjected to a thrust which is constant in magnitude and the direction of which is constant with respect to the tank. This tank is also experiencing a constant rate of rotation about some axis the position of which with respect to the tank remains fixed. Two systems of coordinates centered on the axis of rotation are defined, one inertial and one fixed with respect to the tank. These coordinate systems are shown in figure 1.



(a) Three-dimensional view.



(b) Two-dimensional view.

Figure 1.- Liquid fuel tank under the influence of a constant thrust in the negative z-direction and a constant rate of rotation about the Y-y axis.

The equation of motion of a particle of the fluid may be written as (ref. 7):

$$\left.\frac{d\vec{q}}{dt}\right|_I = \vec{F} - \frac{\nabla p}{\rho} \tag{1}$$

where

$\big|_I$ denotes differentiation with respect to inertially fixed coordinates

$\vec{q}$ particle velocity relative to inertially fixed coordinates

$\vec{F}$ body forces

p fluid pressure

ρ fluid density

The acceleration of the particle in the tank-fixed coordinate system is (from ref. 8):

$$\left.\frac{d\vec{q}}{dt}\right|_T = \left.\frac{d\vec{q}}{dt}\right|_I - \vec{\omega} \times \vec{q} \tag{2}$$

where $\big|_T$ denotes differentiation with respect to tank-fixed coordinates.

In order to find the equilibrium surface, the rate of rotation of the tank is considered to be constant and the motion of the fluid relative to the tank is zero. Therefore, equation (2) is simply

$$\left.\frac{d\vec{q}}{dt}\right|_I = \vec{\omega} \times \left(\vec{\omega} \times \vec{r}\right) = -\nabla \frac{\left(\vec{\omega} \times \vec{r}\right) \cdot \left(\vec{\omega} \times \vec{r}\right)}{2} \tag{3}$$

While the tank rotates, it is subjected to a thrust which is constant with respect to the tank. The thrust is assumed to create a conservative force field $\vec{a}$ which may be represented as

$$\vec{F} = -\nabla\Omega$$

so that

$$\Omega = -\vec{a} \cdot \vec{r} \tag{4}$$

Substituting equations (3) and (4) into equation (1) yields:

$$\nabla\left[\vec{a} \cdot \vec{r} + \frac{\left(\vec{\omega} \times \vec{r}\right) \cdot \left(\vec{\omega} \times \vec{r}\right)}{2} - \frac{p}{\rho}\right] = \vec{0} \tag{5}$$

where $\vec{0}$ is the null vector.

Integration of equation (5) yields

$$\vec{a} \cdot \vec{r} + \frac{\left(\vec{\omega} \times \vec{r}\right)^2}{2} = C + \frac{p}{\rho} \tag{6}$$

where C is a constant of integration.

At the liquid-vapor interface, a pressure discontinuity exists. This pressure difference is represented by the Young-LaPlace equation (ref. 2) as

$$p = p_v + \sigma K$$

The substitution of this value of p into equation (6) yields the following equation for the equilibrium free surface:

$$\vec{a} \cdot \vec{r} + \frac{\left|\vec{\omega} \times \vec{r}\right|^2}{2} = C + \frac{\sigma K}{\rho} + \frac{p_v}{\rho}$$

The mean curvature of the free surface is therefore

$$K = \frac{\rho}{\sigma}\left(\vec{a} \cdot \vec{r} + \frac{\left|\vec{\omega} \times \vec{r}\right|^2}{2} - \frac{p_v}{\rho} - C\right) \tag{7}$$

Equation (7) is the most general equation for the equilibrium free surface of a liquid in a partially filled container subjected to a combined rotation and a carried acceleration field. It is this equation for which a solution is desired.

The mean curvature of a three-dimensional surface is a nonlinear function of partial derivatives of the surface which is of such complexity that, in general, a complete closed form solution for the free surface is not possible at the present time. In addition, the actual boundary conditions are not completely understood. In an attempt to provide some answers as to the effects of the combinations of capillary, acceleration, and rotation forces, two cases which permit the simplification of equation (6) are investigated.

SOLUTIONS FOR FREE SURFACE

Inertial Forces Dominant

Under the assumption that the inertial force due to thrust and/or the centrifugal force is sufficiently large to merit disregarding the effects of capillary forces (such as surface tension), multiplying both sides of equation (7) by σ, and setting σ equal to zero, the following is obtained:

$$\frac{\left(\omega_y^2 + \omega_z^2\right)x^2}{2} + \frac{\left(\omega_x^2 + \omega_z^2\right)y^2}{2} + \frac{\left(\omega_x^2 + \omega_y^2\right)z^2}{2} - \omega_x\omega_y xy - \omega_x\omega_z zx$$

$$- \omega_z\omega_y yz + a_x x + a_y y + a_z z - D = 0 \tag{8}$$

where

$$D = \frac{p_v}{\rho} + C$$

No generality is lost by selecting the coordinate system such that the y-axis lies along the axis of rotation. In this case equation (8) reduces to

$$\frac{\omega_y^2 x^2}{2} + \frac{\omega_y^2 z^2}{2} + a_x x + a_y y + a_z z - D = 0 \tag{9}$$

By expressing equation (9) in canonical form by the coordinate transformations:

$$x^* = x + \frac{a_x}{2\omega_y^2}$$

$$y^* = y - \frac{D}{a_y} - \frac{a_x^2}{4a_y\omega_y^2} - \frac{a_z^2}{4a_y\omega_y^2}$$

$$z^* = z + \frac{a_z}{2\omega_y^2}$$

$$a_y \neq 0$$

there is obtained:

$$x^{*2} + z^{*2} + \frac{a_y}{\omega_y^2} y^* = 0 \tag{10}$$

which is the equation of an elliptic paraboloid with vertex at the origin of the x^*,y^*,z^* coordinate system, as illustrated in figure 2.

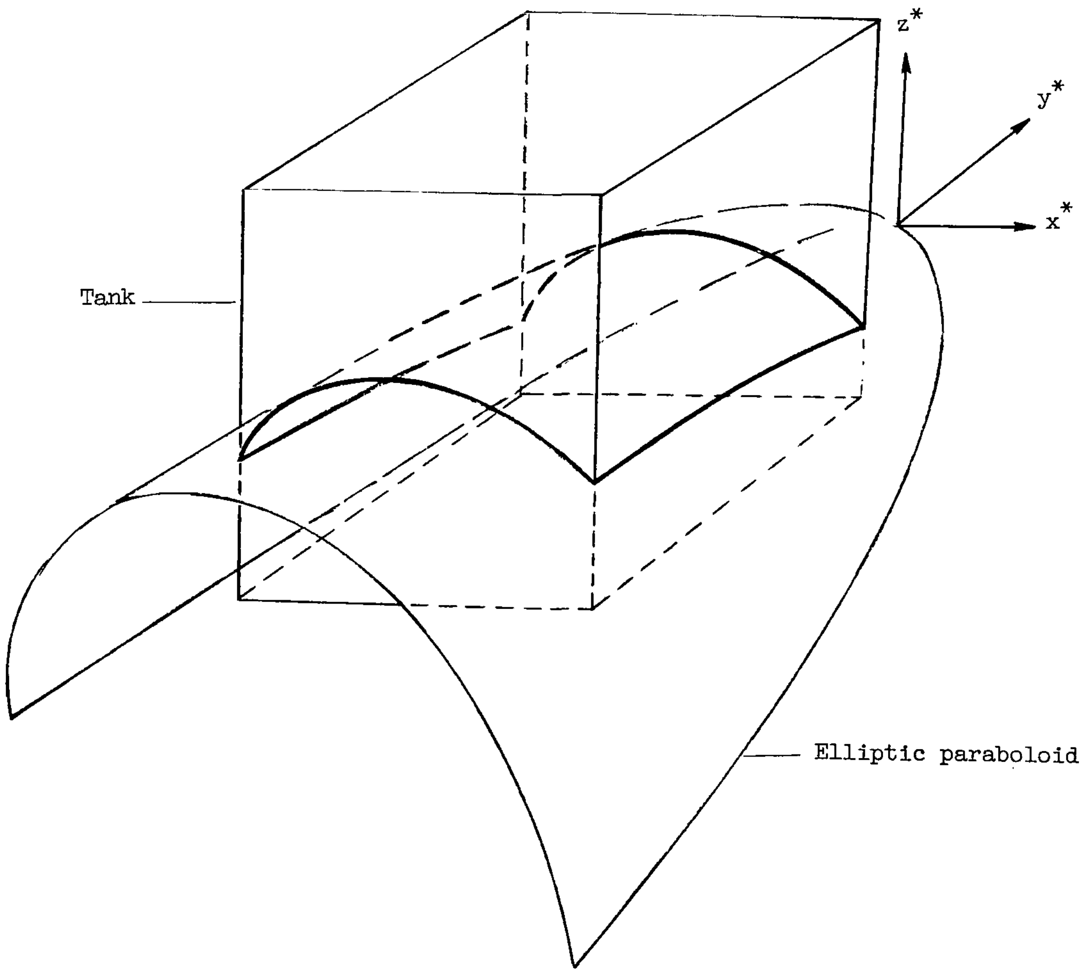


Figure 2.- Equilibrium liquid free surface determined by intersection of tank and elliptic paraboloid.

For the case $a_y = 0$, equation (9) reduces to

$$x^2 + z^2 + \frac{a_x}{\omega_y^2} x + \frac{a_z}{\omega_y^2} z = \frac{D}{\omega_y^2} \tag{11}$$

which is the equation of a right circular cylinder with radius

$$\left(\frac{D}{\omega_y^2} + \frac{a_x^2}{4\omega_y^4} + \frac{a_z^2}{4\omega_y^4}\right)^{1/2}$$

and the axis of which is the line xz where $x = -\frac{a_x}{2\omega_y^2}$ and $z = -\frac{a_z}{2\omega_y^2}$. If the equation of the tank is known in the appropriate coordinate system, either equation (10) or (11) depending upon the existence of a_y may be solved to yield the liquid free surface. An example of the surface is shown in figure 2. This solution, however, is subject to the constraint that the vapor volume which is the portion of the tank not occupied by the liquid must remain constant.

Solutions With Significant Capillary Forces

When the capillary forces are significant, the problem is simplified by restricting the investigation to two dimensions. It is further assumed that $\vec{a}$ is parallel to the z-axis and $\vec{\omega}$ lies along the y-axis so that equation (7) reduces to

$$K = \frac{\rho\omega^2}{2\sigma}\left(x^2 + z^2\right) + \frac{\rho a}{\sigma} z - \frac{p_v + \rho C}{\sigma} \tag{12}$$

At this point the space coordinates are nondimensionalized by the transformation:

$$x = \xi R$$

$$z = \eta R$$

By using the expression for curvature in two dimensions, equation (12) becomes

$$\frac{\eta''}{\left[1 + (\eta')^2\right]^{3/2}} = \frac{\rho\omega^2 R^3}{2\sigma}\left(\xi^2 + \eta^2\right) + \frac{\rho a R^2}{\sigma} \eta - \frac{R}{\sigma} \quad p_v + \rho C$$

where

$$\eta' = \frac{d\eta}{d\xi}$$

The following substitutions are now made:

$$N_{Bo} = \frac{\rho a R^2}{\sigma}$$

$$A = -\frac{R}{\sigma}\left(p_v + \rho C\right)$$

$$N_{cu} = \frac{\rho \omega^2 R^3}{2\sigma}$$

Where N_{Bo} is the familiar Bond number and N_{cu} is hereby defined as centrifugal number. Therefore

$$\frac{\eta''}{\left[1 + (\eta')^2\right]^{3/2}} = N_{cu}\left(\xi^2 + \eta^2\right) + N_{Bo}\eta + A \tag{13}$$

Equation (13) is second order so that its general solution will contain two arbitrary constants. (See ref. 9.) In addition, A contains a constant of integration so that altogether three independent boundary conditions are required to determine a unique solution for the free surface. Three such conditions are as follows:

(1) Because of the symmetry of the forces acting on the free surface about the η-axis, the slope of the surface at $\xi = 0$ is zero; that is, $\left.\frac{d\eta}{d\xi}\right|_{\xi=0} = 0$.

(2) At the interface of three different mediums such as liquid, gas, and solid, a contact angle is formed. Experimental work on these contact angles indicates a characteristic angle is formed by a given liquid in contact with a given tank wall material and surface condition. In reference 2 it is shown that this contact angle is invariant with inertial loading. In references 4, 5, and 6 the influence of the contact angle on the equilibrium liquid surface is shown. Therefore, for this problem various contact angles are assumed by requiring that the slope of the free surface at the tank wall be equal to the cotangent of the contact angle; that is, $\left.\frac{d\eta}{d\xi}\right|_{\xi=1} = -\cot \theta_c$.

(3) The vapor area bounded by the tank base, walls, and liquid free surface must be maintained at a constant value.

In all cases it was assumed that the vapor area was such that a continuous liquid free surface intersecting opposite tank walls would be formed.

In reference 6, the two-dimensional equilibrium free surface was investigated in the absence of angular velocity. The free surface equation derived in reference 6 corresponds to equation (13) with N_{cu} equal to zero. In this form of the equation ξ does not appear explicitly. Therefore it is possible, by separating variables, to obtain an analytic solution and to determine the unknown quantity A appearing in equation (13). With the inclusion of angular velocity the variables cannot be separated and arriving at an analytical solution is not presently possible.

Procedure for Obtaining Two-Dimensional Solution Curves

The procedure used to obtain solutions in the present study was to transform equation (13) into a system of two first-order equations in the following manner:

Let

$$\frac{d\eta}{d\xi} = \varphi \tag{14}$$

then

$$\frac{d\varphi}{d\xi} = \left(1 + \varphi^2\right)^{3/2}\left[N_{cu}\left(\xi^2 + \eta^2\right) + N_{Bo}\eta + A\right] \tag{15}$$

Equations (14) and (15) were then solved simultaneously by applying the fourth-order Runge-Kutta method of numerical integration on a digital computer. (See ref. 10.)

The difficulty in applying the method of numerical integration is the selection of input data that will yield a solution curve which satisfies all the boundary conditions. As part of the data, a point on the free surface and the free surface slope at that point must be specified. If the intercept point of the free surface with either the tank wall or the η-axis is known, the question of initial free surface point and slope is answered. However, in general, neither intercept is known. It is necessary, therefore, to guess at a value, however refined the technique of guessing may be. In addition, a value for the constant A must be assigned. The initial value of A was based on an analytic expression in reference 6 for the case of zero rotation and a value of η some place along the η-axis where the slope is known to be zero.

Thus sufficient information is known to specify the computer input data for a given case. However, the solution curve obtained for this set of input data does not, in general, satisfy either the contact angle or the vapor area constraint. Therefore, an iterative procedure was incorporated in the computer program to converge upon values of A and the η-axis intercept point with the free surface that would yield the solution curve which satisfied the prescribed boundary conditions.

The iterative procedure used was to compute the solution curve corresponding to the first guesses for A and the η-intercept denoted as A_o,η_o. For each such computation the contact angle θ_c and the vapor area ratio V bounded by the curve were received as computer output data. Then A_o and η_o were changed slightly and cases were computed corresponding to $\left(A_o+\Delta A,\eta_o\right)$ and $\left(A_o,\eta_o+\Delta\eta_o\right)$. By using the information from these cases, the necessary changes in A_o and η_o were approximated by the simultaneous solution of the equations:

$$d\theta_c = \frac{\theta_c\left(A_o+\Delta A_o,\eta_o\right) - \theta_c\left(A_o,\eta_o\right)}{\Delta A_o}\,dA + \frac{\theta_c\left(A_o,\eta_o+\Delta\eta_o\right) - \theta_c\left(A_o,\eta_o\right)}{\Delta\eta_o}\,d\eta_o$$

$$dV = \frac{V\left(A_o+\left(A_o,\eta_o\right) - \left(A_o,\eta_o\right)\right)}{\Delta A_o}\,dA + \frac{V\left(A_o,\eta_o+\Delta\eta_o\right) - V\left(A_o,\eta_o\right)}{\Delta\eta_o}\,d\eta_o$$

where

$d\theta_c$ desired change in computed contact angle

dV desired change in computed vapor area

A solution curve was then computed corresponding to $\left(A_o+dA,\eta_o+d\eta_o\right)$ and the procedure using the new values of $\left(A_o,\eta_o\right)$ repeated until a solution curve was obtained which satisfied the prescribed boundary conditions. An example of the curves utilized in obtaining the solution is seen in figure 3.

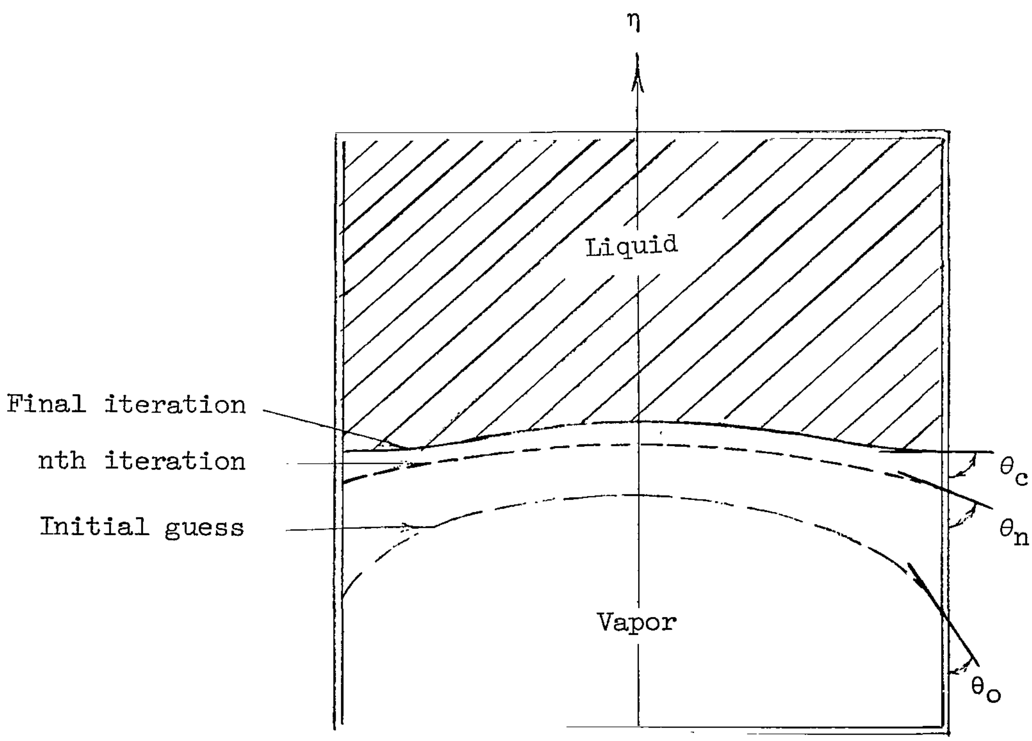


Figure 3.- Representative results of iterative procedure for converging to appropriate A and η.

Determination of an Empirical Formula for A

In utilizing this technique it was found that solutions to equations (14) and (15) were highly sensitive to variations in A and the η-intercept. Usually a deviation of a few percent in one of these quantities from the appropriate value would cause a divergence in the solution, and thereby terminate the computation because of demands for numerical values outside the range of computer capability. Because of the sensitivity of the solution curves to A and η_o, it was felt that it would be desirable to obtain empirical equations for A and η_o as functions of the input data, that is, θ_c, N_{Bo}, N_{cu}, and V. Such data would provide for a more accurate selection and could lead to a better physical insight into the free surface shape.

In order to investigate variations of A with θ_c, N_{Bo}, N_{cu}, and V, each of these parameters was varied independently to determine its effect. These variations were then superimposed to obtain a trial function $A\left(\theta_c, N_{Bo}, N_{cu}, V\right)$.

By setting N_{Bo} and N_{cu} equal to zero, equation (13) may be reduced to

$$\frac{\eta''}{\left(1 + \eta'^2\right)^{3/2}} = A \tag{16}$$

Equation (16) describes the free surface as a surface of constant curvature; a condition satisfied, in two dimensions, by a circle. The variation of A with contact angle can be determined by integrating equation (14) and evaluating at the tank wall intercept. Thus,

$$\int_0^\xi \frac{\eta''}{\left(1 + \eta'^2\right)^{3/2}} d\xi = \frac{\eta'}{\left(1 + \eta'^2\right)^{1/2}} = A\xi$$

If this equation is evaluated at the tank wall intercept $\xi = 1$,

$$A = \frac{-\cot\theta_c}{\csc\theta_c} = -\cos\theta \qquad \left(N_{Bo} = N_{cu} = 0\right)$$

Computer solutions showed the free surface to be flat for $N_{cu} = 0$ and $\theta_c = 90^\circ$, and hence a surface of zero curvature. For this case equation (13) reduces to

$$A = -N_{Bo}\eta_o$$

where η_o is the η-intercept of the flat surface bounding the appropriate vapor area. Since the surface is flat, $\eta_o = \frac{V}{2}$ and $A = -\frac{N_{Bo}V}{2}$ where the vapor area ratio $V = 4\frac{\text{Vapor area}}{\text{Total area}}$.

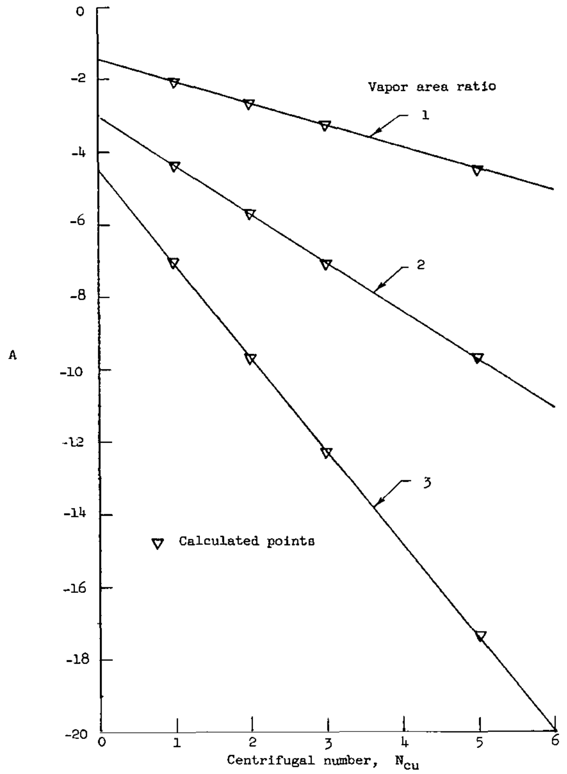


Figure 4.- Variation of A with centrifugal number for Bond number of 3 and contact angle of 90°.

Computer solutions also showed the variation of A with N_{cu} to be linear when θ_c, N_{Bo}, and V were held constant. However, the slope of this linear variation was seen to be a function of V denoted by F(V). Plots of the variation of A with N_{cu} for various values of V are shown in figure 4. Figure 5 illustrates the nearly exponential change in the slopes of these lines with vapor area ratio. The solid curve in figure 5 represents the least squares exponential function which best fits the data obtained.

A linear combination of the foregoing results was formed as a trial solution for A:

$$A(\theta_c, N_{Bo}, N_{cu}, V) = -\left[\frac{N_{Bo}V}{2} + \cos\,\theta_c + F(V)\;N_{cu}\right] \qquad (17)$$

This trial solution for A was found to agree with the computer results, in a

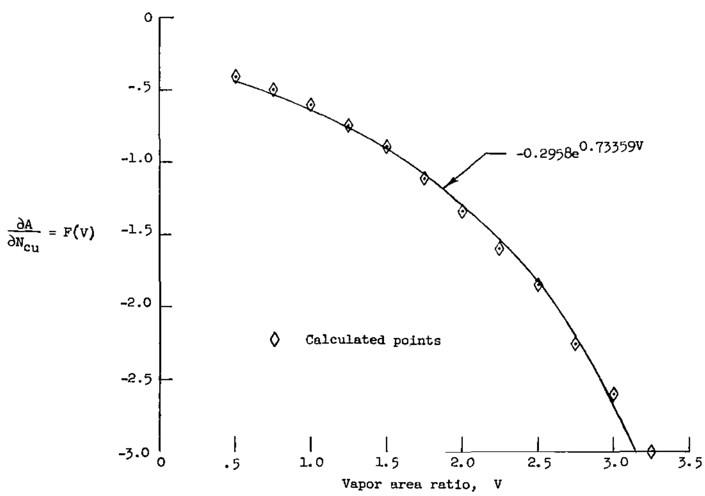


Figure 5.- Slope of A as a function of centrifugal number lines for N_{Bo} and θ_c constant.

$$A = -\left[\frac{N_{Bo}V}{2} + \cos\,\theta_c + F(V)\;N_{cu}\right].$$

number of cases to within 0.1 percent, and hence is considered to be a sufficiently accurate empirical formula for estimating an initial value of A within the following region of investigation:

$$0 \leqq N_{Bo} \leqq 6$$

$$0 \leqq N_{cu} \leqq 6$$

$$30^{\circ} \leqq \theta_c \leqq 90^{\circ}$$

$$0.5 \leqq V \leqq 3.5$$

Bond numbers and centrifugal numbers were chosen in order to cover a large range of tanks with liquids exposed to significant capillary forces. The contact angle was chosen to correspond with those for wetting fluids. However, below a contact angle of 30° limitations of the computer technique prohibited any further solutions.

Determination of Bounds of Free Surface η-Intercept

As stated previously, the solutions are sensitive to a variation of the η-intercept. Thus, η had to be determined as a function of the boundary conditions in order to obtain solutions.

As shown in the preceding section, the liquid free surface is circular for $N_{Bo} = N_{cu} = 0$. Its curvature is given by

$$K = -\cos \theta_c$$

For this case the free surface may be represented by an equation of the form:

$$\xi^2 + (\eta + d)^2 = \frac{1}{\cos^2 \theta_c} \tag{18}$$

where d is the displacement of the center of curvature from the origin. Equation (18) was solved to yield the variation of the η-intercept η_o from a flat surface intercept with contact angle as:

$$\eta_o(\theta_c) - \frac{V}{2} = \sec \theta_c - 0.5 \tan \theta_c - \frac{\frac{\pi}{2} - \theta_c}{2 \cos^2 \theta_c}$$

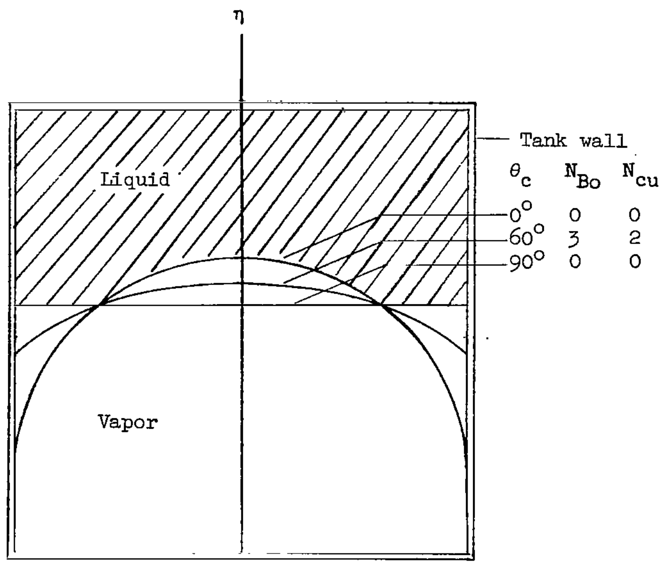


Figure 6.- Examples of η-intercept variation with contact angle. Total tank area, 4.

An example of the variation of the η-intercept with contact angle is seen in figure 6. Since the free surface is flat for the case of 90° contact angle, its η-intercept is given by

$$\eta_o = \frac{V}{2}$$

This solution for $\eta_o(\theta_c)$ is plotted in figure 7 for $0 \leqq \theta_c \leqq 90^\circ$. The effect of N_{Bo} on $\eta_o(\theta_c)$ is also shown. It is seen that with the introduction of N_{Bo}, the variation of η_o with contact angle is decreased. Figure 8 shows the increase in the η-intercept with N_{cu} for constant N_{Bo} and various values of θ_c and V.

Sufficient data were not obtained to determine an accurate empirical formula for $\Delta\eta_o(\theta_c, N_{Bo}, N_{cu}, V)$. However, figures 7 and 8 indicate the effects of these parameters. For the region investigated the η-intercepts of the free surfaces always lay between that of the flat surface corresponding to $\theta_c = 90^\circ$, $N_{Bo} = N_{cu} = 0$; and the circular surface corresponding to $\theta_c = 0^\circ$, $N_{Bo} = N_{cu} = 0$. Therefore, the analytical equation for $\eta_o(\theta_c)$ shown in figure 7 gives the bounds on the η-intercept variation.

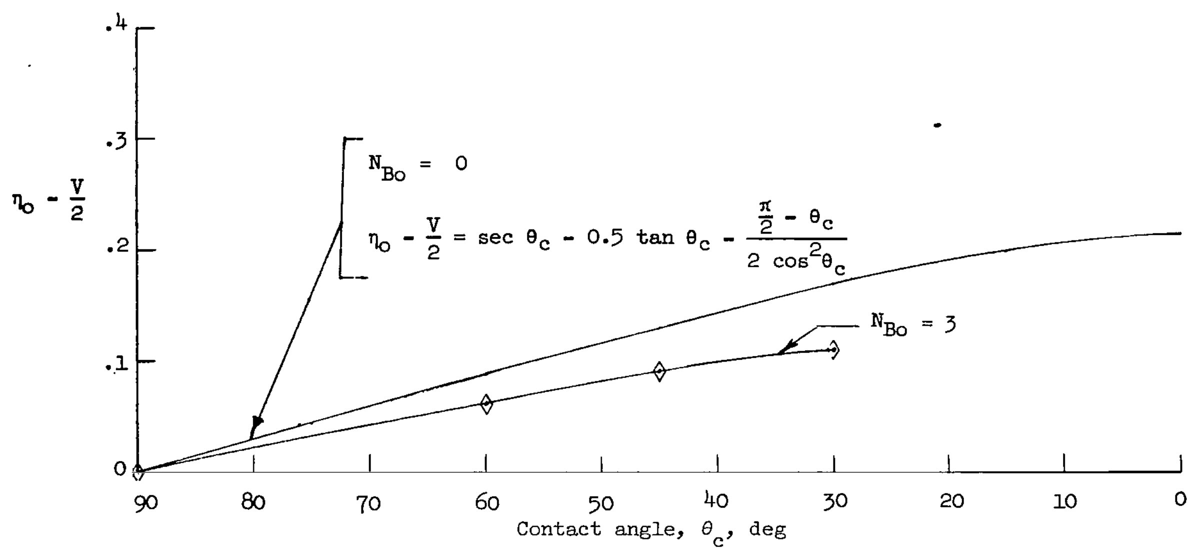
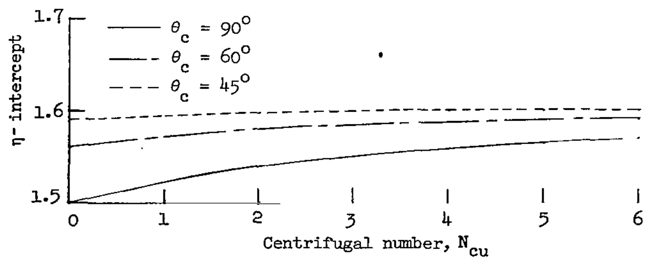
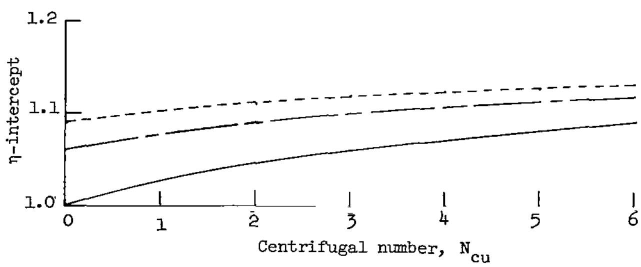


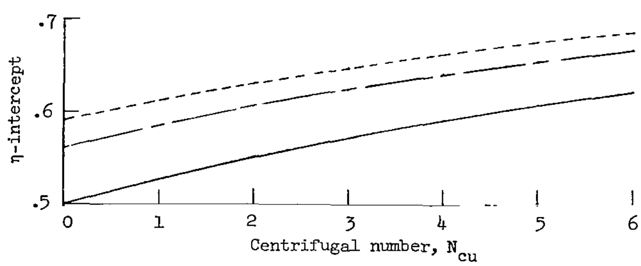
Figure 7.- Increase in free surface η-intercept with decreasing contact angle. $N_{cu} = 0$.



(a) Vapor area ratio of 3.



(b) Vapor area ratio of 2.



(c) Vapor area ratio of 1.

Figure 8.- Variation of the η-intercept with centrifugal number for Bond number equal to 3.

Possible Application of Two-Dimensional Methods To Obtain the Three-Dimensional Free Surface

It is believed that the three-dimensional free surface to at least a first-order approximation can be determined by applying the two-dimensional technique simultaneously in two normal directions and defining the three-dimensional surface by a lattice of such normal two-dimensional solutions. (See fig. 9.)

A possible method of approximating the three-dimensional free surface would be to obtain an initial two-dimensional curve by the method already presented. The second step would be to obtain a number of two-dimensional curves normal to and intercepting the initial curve. This second step presents a new problem in specifying A in that the computed curve is required to pass through some specific point in the ξ,η space. An investigation was conducted to determine an

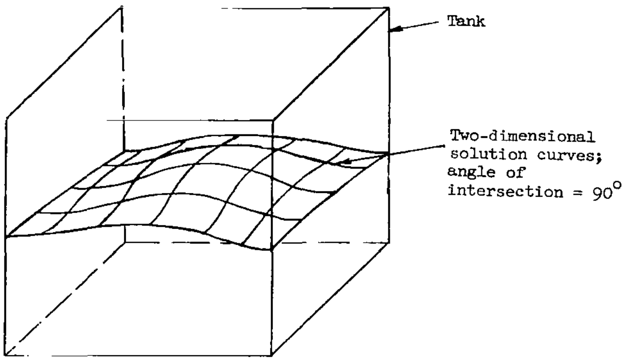


Figure 9.- Definition of the three-dimensional liquid free surface by a lattice of two-dimensional solution curves.

empirical formula for A under this new constraint; however, because of the accumulation of considerable computer time, this investigation was conducted only for the following region:

$$0 \leqq N_{Bo} \leqq 2$$

$$0 \leqq N_{cu} \leqq 2$$

$$30^\circ \leqq \theta_c \leqq 90^\circ$$

$$\eta_o = 1$$

$$\xi_o = 0$$

The "vapor area" constraint was removed in order not to overspecify the problem.

The empirical formula for A was obtained by varying one parameter at a time and superimposing these results to form a trial formula. The following equation is considered sufficiently accurate to approximate A over the region of investigation:

$$A\left(\theta_c, N_{Bo}, N_{cu}\right) = -N_{Bo} - 1.22N_{cu} - \left[1 + F_1\left(N_{Bo}\right) + F_2\left(N_{cu}\right) - F_1\left(N_{Bo}\right) F_2\left(N_{cu}\right)\right]\cos\theta_c$$

where

$$F_1\left(N_{Bo}\right) \approx \frac{2}{\pi} \text{ arc } \tan\left(0.25N_{Bo}\right)$$

$$F_2\left(N_{cu}\right) \approx 0.02168 + 0.24614N_{cu} - 0.01523N_{cu}^{2}$$

RESULTS AND DISCUSSION

An equation has been derived that governs the three-dimensional equilibrium free surface of a contained liquid and which is valid for all magnitudes of inertial forces due to translational and rotational accelerations. It has been shown that when the inertial forces are sufficiently large to merit disregarding the effect of the capillary forces, the equilibrium liquid free surface is defined by the intersection of a quadratic equation (in the space coordinates) and the region bounded by the tank walls. However, the primary purpose of this report is to illustrate qualitatively the effects of various combinations of translational and rotational accelerations and capillary forces on the free surface shape in a region in which the inertial forces due to the accelerations are of the same order of magnitude as the capillary forces.

Figure 10 presents examples of the change in the free surface due to changes in contact angle for a Bond number of 3, a centrifugal number of 2, and a vapor area ratio of 2. The tank is illustrated with a specific height to simplify the presentation, but the free surface shape is independent of the amount of vapor area added to the bottom of the tank or the amount of liquid area added to the top of the tank. It is seen that when the forces due to thrust and rotation are sufficiently small with respect to the capillary forces, the contact angle has a significant effect on the free surface shape.

Figure 11 illustrates the flattening effect of increasing N_{Bo} for constant values of N_{cu}, θ_c, and V. Actually, the effect of N_{Bo} is small in the range of interest. It was found that in the absence of rotation, the free surface shape was independent of vapor area. This effect is a result of the assumption of constant pressure throughout the vapor.

Figure 12 shows the variation of the free surface with changes in the values of N_{cu}. The tank is rotating in the plane of the diagram about the center of the tank base. As stated before, the free surface shape is independent of any increase in either fluid or vapor area as long as the position of the axis of rotation with respect to the fluid is not changed.

It is seen that with an increase in rotational rate the free surface tends to become circular about the axis of rotation. Figure 12 again illustrates the pronounced effect of contact angle on the free surface shape since the deflection of the surfaces shown away from the circular trend is due to a required 90° contact angle.

Figure 13 presents the variation of the free surface with changes in vapor area ratio. In this figure the axis of rotation is again at the center of the tank base. The increases in vapor area shown are completely analogous to shifting the axis of rotation vertically away from the fluid.

For the values of N_{Bo} and θ_c corresponding to the cases shown in figure 13, the free surface would be flat for $N_{cu} = 0$. Thus for this combination of N_{Bo} and θ_c, it is seen that the effect of rotation on the free surface shape decreases as the distance to the axis of rotation increases although the centrifugal force on the free surface increases. This effect is due to the fact seen before that the effect of N_{cu} is to force the free surface to become circular about the axis of rotation. Therefore the farther the free surface is away from the axis of rotation, the greater the radius of curvature and hence the flatter the surface which will satisfy the effect of N_{cu}. Equations (8) and (9) represent the extreme case in which the effects of N_{Bo} and N_{cu} dominate.

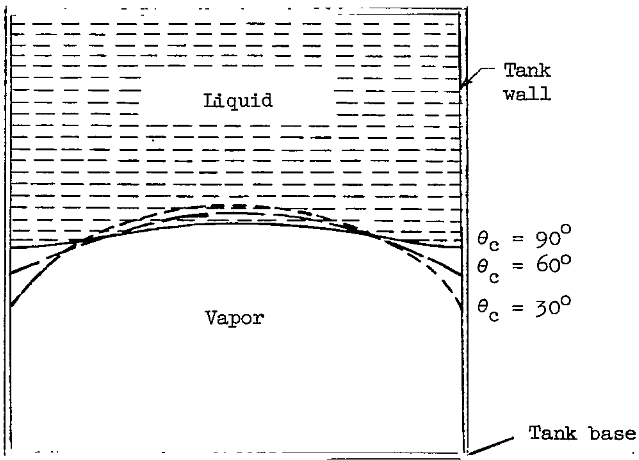


Figure 10.- Variations in free surface due to changes in contact angle. Bond number, 3; centrifugal number, 2; vapor area ratio, 2; total tank area, 4.

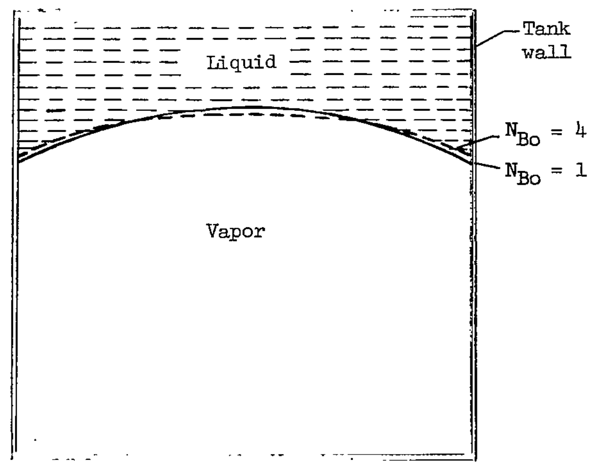


Figure 11.- Variations in free surface due to changes in Bond number. Centrifugal number, 0; contact angle, 60°; vapor area ratio, 3; total tank area, 4.

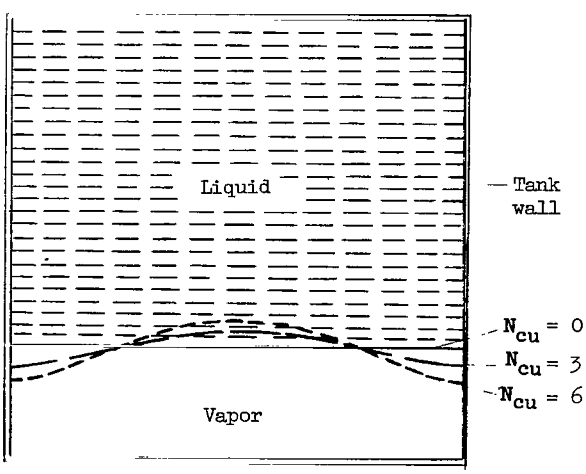


Figure 12.- Variations in free surface due to changes in centrifugal number. Bond number, 3; contact angle, 90°; vapor area ratio, 1; total tank area, 4.

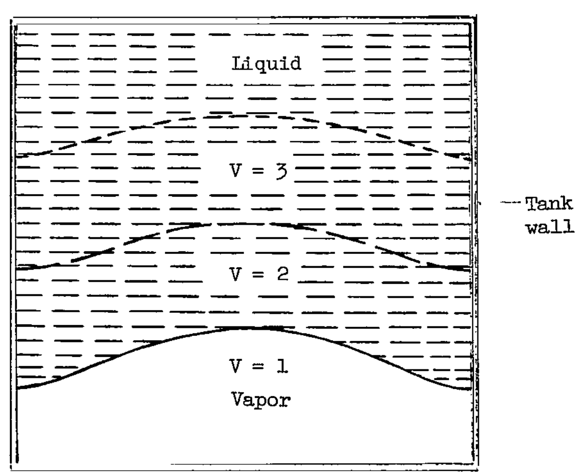


Figure 13.- Variations in free surface due to changes in vapor area ratio, V. Bond number, 3; centrifugal number, 6; contact angle, 90°; total tank area, 4.

CONCLUSIONS

This report has presented a general equation for the three-dimensional equilibrium liquid free surface under the influence of thrust, rotation, and liquid-to-solid contact angle. A closed form solution of this equation has been obtained which is valid when capillary forces may be neglected. A method has been presented for determining solutions to the two-dimensional form of this equation by numerical integration on a digital computer. Methods have been presented for estimation of the input data required to perform this computer solution. Trends in the two-dimensional free surface shape with changes in the imposed forces have been shown.

This investigation was carried out for a specific direction of tank acceleration, position of axis of rotation, and tank geometry. However, the free surface equation is valid for the general case. Therefore, various directions of thrust, positions of axis of rotation, and tank geometries may be investigated by appropriate selection of the boundary conditions under which solutions to the free surface equation are obtained.

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