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SPINOR SOLUTION OF THE SOUND WAVE PROBLEM

BY

LAWRENCE A. SCHMID

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GODDARD SPACE FLIGHT CENTER
GREENBELT, MARYLAND

Abstract

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The spinor formulation of magnetogas dynamics that was developed by the author in a previous paper is applied to the problem of a sound wave in an electrically neutral gas. This simple, one-dimensional problem serves to illustrate the essential features of the formalism. The solution is completely relativistic and, for sound waves of macroscopic wavelengths, satisfies Euler's equation, the continuity equation, and the adiabatic condition. For wavelengths of the order of atomic dimensions, spin-dependent terms, which are completely negligible for macroscopic waves, become important and drastically alter the form of the solution. Thus the spinor solution of the sound wave problem has the desirable feature that it automatically breaks down at the point where classical mechanics breaks down, namely when the characteristic length of the problem becomes of atomic dimensions.

Author

Introduction

This paper is a supplement to the author's earlier paper "Spinor Formulation of Magnetogas Dynamics" (Goddard Space Flight Center X-640-64-55). In order to illustrate the formalism developed in that paper (which shall henceforth be referred to as "I"), the spinor formulation of fluid dynamics is applied to the one-dimensional problem of a sound wave in an electrically neutral perfect gas.

The solution is carried out by means of a perturbation on a simple zero-order solution which corresponds to a fluid of constant density moving in the positive z direction with constant velocity. By choosing the particle spins, whose role in any macroscopic problem is always insignificant, to be aligned in the z direction, it is possible to reduce two of the four spinor components to zero, thereby reducing the calculational burden. The fluid enthalpy, which is regarded as the driving function of the problem, is next assigned a sinusoidal time-independent variation in the z direction, and the spinor equations are solved to find the spinor functions consistent with this form of variation in the enthalpy. The solution is carried out using the first-order perturbation approximation in which the perturbation parameter is the ratio of the maximum change in the specific enthalpy to the particle rest-mass. For practical problems, this ratio is always very small.

Once the spinor equations have been solved for a sinusoidal variation in enthalpy, spinor relations derived in I are applied to calculate the fluid flux density, which turns out to be constant, and the particle density, which turns out to have a sinusoidal variation of the same wavelength as the assigned variation in the enthalpy. It is shown that the solution maintains conservation of particle energy, which is just the condition required by Euler's equation. That this is so can be seen from the fact that for this problem eq. (2-37) of I (Euler's equation), reduces to

$$0 = \vec{\nabla}(\breve{m} u^{\circ} c^{2}) \approx \vec{\nabla}\left(mc^{2} + \tfrac{1}{2} m v^{2} + m h\right) \tag{1}$$

which states that the sum of the kinetic and thermal energies per particle must remain constant.

The spinor equations, like Euler's equation, must be supplemented by the adiabatic condition (eq. (2-31) of I):

$$\rho\,\delta h = (\gamma - 1)\, h\,\delta\rho \tag{2}$$

where $\delta\rho$ and δh are the changes in the density and enthalpy respectively as one moves along the wave, which appears to be stationary because the fluid is streaming in the +z direction with exactly the same speed as that with which the wave is propagating in the -z direction. It is shown that (2) yields a condition on the fluid velocity which is just the usual expression for the speed of sound in terms of the absolute temperature of the gas.

Finally, it is shown that, although the effects of particle spin are completely negligible for macroscopic wavelengths, they become important when the wavelengths become of atomic dimensions, with the result that the classical solution is no longer valid.

Necessary Spinor Relations

The spinor relations that will be needed are recapitulated below. The numbers to the left are the formula references in I. Since in the present problem two of the spinor components (ρ^2 and χ^1) are zero, the spinor equations take the following simplified form:

(8-1a,b)

$$\begin{aligned} i\left(\frac{1}{c}\frac{\partial \rho^1}{\partial t} + \frac{\partial \rho^1}{\partial z}\right) &= -\check{\lambda}^{-1}\,\overline{\chi^2} \\ i\left(\frac{1}{c}\frac{\partial \overline{\chi^2}}{\partial t} - \frac{\partial \overline{\chi^2}}{\partial z}\right) &= -\check{\lambda}^{-1}\,\rho^1 \end{aligned} \tag{3}$$

where the overhead bar indicates complex conjugation and

$$\check{\lambda} = \frac{\hbar}{\check{m}c} \tag{4}$$

where

(8-1c) $$\breve{m} = m\left(1+\frac{\delta h}{c^2}\right) \qquad (5)$$

where δh is the variable part of the enthalpy, the constant part h_0 having been absorbed into the particle rest-mass m.

Once the spinors are known, the flux density 4-vector ρu^j is calculated from the following relations:

(B-15) $$\begin{aligned} \rho u^0 &= \frac{1}{\sqrt{2}}\left[|\xi^1|^2+|\xi^2|^2+|\chi^1|^2+|\chi^2|^2\right] = \frac{1}{\sqrt{2}}\left[|\xi^1|^2+|\chi^2|^2\right] \\ \rho u^1 &= \sqrt{2}\,\mathcal{R}\left[\xi^1\overline{\xi^2}+\chi^1\overline{\chi^2}\right] = 0 \\ \rho u^2 &= -\sqrt{2}\,\mathcal{I}\left[\xi^1\overline{\xi^2}+\chi^1\overline{\chi^2}\right] = 0 \\ \rho u^3 &= \frac{1}{\sqrt{2}}\left[|\xi^1|^2-|\xi^2|^2+|\chi^1|^2-|\chi^2|^2\right] = \frac{1}{\sqrt{2}}\left[|\xi^1|^2-|\chi^2|^2\right] \end{aligned} \qquad (6)$$

where the simplifications that result when $\xi^2 = \chi^1 = 0$ have been carried out. The invariant density ρ is found from the relation

(B-9) $$\rho = \left[(\rho u^j)(\rho u_j)\right]^{1/2} = \sqrt{2}\,|\xi^1\chi^2-\xi^2\chi^1| = \sqrt{2}\,|\xi^1\chi^2| \qquad (7)$$

The particle 3-velocity v^j is given by the following well-known relation:

$$v^j = c\left(\frac{\rho u^j}{\rho u^0}\right) \qquad j = 1, 2, 3 \qquad (8)$$

Finally, the particle 4-momentum p^j is found as follows:

(C-5b)
(C-79) $$\begin{aligned} p^j &= \frac{\hbar}{2}\,\mathcal{I}\left[(\xi_\alpha\chi^\alpha)^{-1}(\chi^\beta\partial^j\xi_\beta - \xi_\beta\partial^j\chi^\beta)\right] \\ &= \frac{\hbar}{2}\,\mathcal{I}\left[(\xi^1\chi^2)^{-1}(\chi^2\partial^j\xi^1 - \xi^1\partial^j\chi^2)\right] \end{aligned} \qquad (9)$$

Zero-Order Solution

The zero-order function is taken to have the following form:

$$
\begin{aligned}
\varphi_0^1 &= A_0 e^{i(Et - Pz)/\hbar} \\
\varphi_0^2 &= 0 \\
\chi_0^1 &= 0 \\
\chi_0^2 &= B_0 e^{-i(Et - Pz)/\hbar}
\end{aligned}
\tag{10}
$$

where A_0, B_0, E, and P are constants. (10) is a solution of (3) for $\delta \hbar = 0$ if the following relations are satisfied:

$$B_0 = \frac{E - Pc}{mc^2} A_0 \tag{11a}$$

$$A_0 = \frac{E + Pc}{mc^2} B_0 \tag{11b}$$

These can be satisfied only if

$$E^2 = (mc^2)^2 + (Pc)^2 \tag{12}$$

which is the familiar relativistic relation between the energy E and the momentum P of a particle having rest-mass m.

It will be convenient to work with the dimensionless quantities $\hat{E}$ and $\hat{P}$ defined as follows:

$$\hat{E} = \frac{E}{mc^2} = \frac{1}{\sqrt{1-(V/c)^2}} \tag{13a}$$

$$\hat{P} = \frac{P}{mc} = \frac{(V/c)}{\sqrt{1-(V/c)^2}} \tag{13b}$$

Thus (12) assumes the following form:

$$\hat{E}^2 = 1 + \hat{P}^2 \tag{14}$$

For convenience we introduce the ratio $R = \frac{B_0}{A_0}$:

$$R = \frac{B_0}{A_0} = \hat{E} - \hat{P} = \frac{1}{\hat{E} + \hat{P}} \tag{15}$$

The following identities will be useful:

$$1+R^2 = 2R\hat{E} \tag{16a}$$

$$1-R^2 = 2R\hat{P} \tag{16b}$$

Substituting (10) into (7) and (6), we find

$$\rho_0 = \sqrt{2}\, A_0^2 R \tag{17}$$

$$(\rho u^0)_0 = \frac{A_0^2}{\sqrt{2}}(1+R^2) = \rho_0 \hat{E} \tag{18a}$$

$$(\rho u^3)_0 = \frac{A_0^2}{\sqrt{2}}(1-R^2) = \rho_0 \hat{P} \tag{18b}$$

Substitution of (18) into (8) yields

$$v^3 = v_z = c\,\frac{\hat{P}}{\hat{E}} = \frac{P}{(E/c^2)} = V \tag{19}$$

This equation is taken as the definition of the constant velocity V, which will later be identified with the velocity of sound.

We note that the nonrelativistic case is defined by the following conditions:

$$\hat{P} = \frac{P}{mc} \ll 1, \qquad \hat{E} = \frac{E}{mc^2} \approx 1 \qquad \text{N.R. case} \tag{20}$$

In this case

$$v_z = V \approx \frac{P}{m} \qquad \text{N.R. case} \tag{21}$$

Solution in the Presence of a Sound Wave

In order to generate a sound wave, we regard the change in specific enthalpy δh as the driving function, and assign it the following functional form:

$$\delta h = \Delta \cos kz \tag{22}$$

where k is the wave number of the sound wave. (For an ordinary sound wave in hydrogen at S.T.P. $k \sim 10^{-2}\,\mathrm{cm}^{-1}$.) δh appears in the spinor equations as a contribution to $\check{m}$:

$$\check{m} = m\left(1+\frac{\delta h}{c^2}\right) = m\left[1+\left(\frac{\Delta}{c^2}\right)\cos kz\right] \tag{23}$$

The dimensionless constant $\frac{\Delta}{c^2}$ plays the role of the perturbation parameter. For a strong sound wave at S.T.P., for which the variation in pressure produced by the wave is of the order 10^2dyne/cm^2, the constant $\frac{\Delta}{c^2}$ is of order 10^{-15}.

Because we wish to satisfy the spinor equations (3) only to first order in $\frac{\Delta}{c^2}$, we can approximate a term like $\varsigma^1(\delta \hbar)$ by $\varsigma^1_0(\delta \hbar)$.

We seek a solution of (3) having the form

$$\begin{aligned}
\varsigma^1 &= A_0\left[1 + \alpha \cos kz + i(\xi - \eta)\sin kz\right] e^{i(Et - Pz)/\hbar} \\
\varsigma^2 &= 0 \\
\chi^1 &= 0 \\
\chi^2 &= B_0\left[1 + \beta \cos kz - i(\xi + \eta)\sin kz\right] e^{-i(Et - Pz)/\hbar}
\end{aligned} \tag{24}$$

where α, β, ξ, and η are constants.

The constant ξ has the effect of producing phase changes in ς^1 and χ^2 that are equal in magnitude and opposite in sign. When (24) is substituted into (6) and (7), ξ does not appear to first order in $\frac{\Delta}{c^2}$. It is only in (9) that it makes itself felt, and even there the effect is negligible for macroscopic waves. In fact, as an alternative procedure, we could omit ξ entirely and compensate for the omission by replacing Pz in the phase by an arbitrary function of z. We shall later see what this function would have to be.

The constant η is a measure of the degree to which the constraint $\mathcal{J}(\varsigma_\alpha \chi^\alpha) = \mathcal{J}(\varsigma^1 \chi^2) = 0$ is violated. It was pointed out in Section 8 of I that, although in principle this constraint should be imposed, the effect of neglecting it in any macroscopic problem is completely negligible. We shall see that this is true in the case of a sound wave.

It is to be noted that the two constants ξ and η, being coefficients of the sin kz term, specify the part of ρ^1 and χ^2 that is out of phase with δh, whose variation is given by $\Delta \cos kz$. We shall see that these constants have no influence on macroscopic solutions. Only the in-phase constants α and β are important for such solutions.

The spinor specification given in (24) is a solution of (3) (to first order in $\frac{\Delta}{c^2}$) if the constants have the following values:

$$\alpha = \frac{1}{2}\left(\frac{\Delta}{c^2}\right)\left[\frac{1-\hat{P}(\hat{E}-\hat{P})}{\hat{P}^2-(\Theta/2)^2}\right] \tag{25a}$$

$$\beta = \frac{1}{2}\left(\frac{\Delta}{c^2}\right)\left[\frac{1+\hat{P}(\hat{E}+\hat{P})}{\hat{P}^2-(\Theta/2)^2}\right] \tag{25b}$$

$$\xi = \frac{1}{\Theta}\left(\frac{\Delta}{c^2}\right)\hat{P}\left[\frac{1+(\Theta/2)^2}{\hat{P}^2-(\Theta/2)^2}\right] \tag{25c}$$

$$\eta = \frac{\Theta}{4}\ \frac{\hat{E}(\Delta/c^2)}{\hat{P}^2-(\Theta/2)^2} \tag{25d}$$

where

$$\Theta = k\ \lambda \tag{25e}$$

Using (24), (25), and the relations (6)-(8), we arrive at the following expressions:

$$\begin{aligned}\rho &= \sqrt{2}\, A_o^2 R\left[1 + (\alpha+\beta)\cos kz\right] \\ &= \rho_o\left\{1 + \left(\frac{\Delta}{c^2}\right)\left[\frac{\hat{E}^2}{\hat{P}^2-(\Theta/2)^2}\right]\cos kz\right\}\end{aligned} \tag{26}$$

$$\begin{aligned}\rho u^0 &= \frac{A_o^2}{\sqrt{2}}\left\{(1+R^2)\left[1+2(\alpha+R^2\beta)\cos kz\right]\right\} \\ &= \rho_o \hat{E}\left[1 + \frac{(\Delta/c^2)\cos kz}{\hat{P}^2-(\Theta/2)^2}\right]\end{aligned} \tag{27}$$

$$\begin{aligned}\rho u^3 &= \frac{A_o^2}{\sqrt{2}}\left\{(1-R^2)\left[1+2(\alpha-R^2\beta)\cos kz\right]\right\} \\ &= \rho_o \hat{P}\end{aligned} \tag{28}$$

$$v_z = c\left(\frac{\rho u^3}{\rho u^0}\right) = V\left[1 - \frac{(\Delta/c^2)\cos kz}{\hat{P}^2-(\Theta/2)^2}\right] \tag{29}$$

$$u^0 = \frac{\rho u^0}{\rho} = \frac{1}{2R}\left[(1+R^2) + (1-R^2)(\alpha-\beta)\cos kz\right]$$
$$= \hat{E}\left\{1 - \left(\frac{\Delta}{c^2}\right)\left[\frac{\hat{p}^2}{\hat{p}^2-(\Theta/2)^2}\right]\cos kz\right\} \tag{30}$$

From (23) and (30) we find

$$\breve{m}c^2u^0 + \left(\frac{\Delta}{c^2}\right)\left[\frac{E(\Theta/2)^2}{\hat{p}^2-(\Theta/2)^2}\right]\cos kz = E \tag{31}$$

All of the above results are, of course, valid only to first order in $\frac{\Delta}{c^2}$.

For all macroscopic sound waves $\Theta \ll 1$. In fact, for an ordinary sound wave in hydrogen $k \sim 10^{-2}\,\mathrm{cm}^{-1}$ and $\lambda\!\!\!^{-} \sim 10^{-14}\,\mathrm{cm}$, so in this case $\Theta \sim 10^{-16}$. For comparison, in the same case

$$\hat{p} = \frac{p}{mc} \approx \frac{V}{c} \sim 10^{-5} \tag{32}$$

Thus $(\Theta/2)^2$ is completely negligible compared with $\hat{p}^2$. Using this fact in (26)-(31), we have

$$\rho = \rho_0\left[1 + \frac{\delta h}{V^2}\right] \tag{33}$$

$$\rho u^0 = \rho_0\left(\frac{E}{mc^2}\right)\left[1 + \frac{\delta h}{(P/m)^2}\right] \tag{34}$$

$$\rho u^3 = \rho_0\left(\frac{p}{mc}\right) \tag{35}$$

$$v_z = V\left[1 - \frac{\delta h}{(P/m)^2}\right] \tag{36}$$

$$u^0 = \frac{E}{mc^2}\left[1 - \frac{\delta h}{c^2}\right] \tag{37}$$

$$\breve{m}c^2u^0 = E \tag{38}$$

These expressions are, of course, completely relativistic.

From (27) and (28) we see that

$$\partial_j(\rho u^j) = \frac{1}{c}\frac{\partial(\rho u^0)}{\partial t} + \frac{\partial(\rho u^3)}{\partial z} = 0 \tag{39}$$

Thus the continuity condition is always fulfilled, even when θ is not negligible. (38) is obviously a solution to (1), which is just Euler's equation for the case of a sound wave. In (31), however, we find another term appearing in the energy equation. For a strong sound wave in hydrogen at S.T.P. for which $k \sim 10^{-2}\,\text{cm}^{-1}$, this term has a magnitude of the order 10^{-40} erg as compared with a magnitude of order 10^{-3} erg for the term $\check{m}c^2u^0$. This small extra energy contribution results from the presence of particle spin. For macroscopic sound waves it is completely negligible, but for extremely short wavelengths it becomes important.

We shall return to th question of short wavelengths, but first we shall impose the adiabatic condition (2) in order to arrive at the familiar expression for the speed of sound. We note that, to first order in $\frac{\Delta}{c^2}$, it is permissible to write (2) as follows:

$$\rho_0\,\delta h = (\gamma - 1)\,h_0\,\delta\rho = (\gamma-1)\kappa_p T_0\,\delta\rho \tag{40}$$

From (33) we have

$$\delta\rho = \rho_0\,\frac{\delta h}{V^2} \tag{41}$$

Substituting (41) into (40) we have

$$V = \sqrt{(\gamma-1)\kappa_p T_0} \tag{42}$$

which is the well-known expression for the speed of sound in a perfect gas, where κ_p is the constant-pressure specific heat per unit mass and γ is the ratio of specific heats.

To complete the solution, we apply (9) to arrive at the following expression for particle momentum:

$$p_z = P - \hbar k \xi \cos kz \qquad (43a)$$

$$= \left(\frac{P}{V}\right) v_z - \frac{P(\theta/2)^2(\delta \hbar)}{\hat{P}^2 - (\theta/2)^2} \qquad (43b)$$

For the macroscopic case, the second term on the right side of (43b) is completely negligible, and so p_z and v_z stand in just the linear relation to each other that we would expect.

It is obvious that (43a) could be rewritten as follows:

$$p_z = \frac{d}{dz}\left[Pz - \hbar \xi \sin kz\right] \qquad (44)$$

This indicates that if we had replaced Pz in the phase factor in (24) by $Pz - \hbar\xi \sin kz$ where ξ has the value given in (25c), then it would have been unnecessary to include ξ in the square brackets multiplying A_0 and B_0 in (24).

Spin-Velocity Resonance

The expressions (25) have the interesting feature that for $\theta/2 = \hat{P}$ or

$$P = \tfrac{1}{2}\hbar k \qquad (45)$$

the denominators vanish, so that even for $A = 0$ it would be possible to have a wave of finite magnitude. The physical explanation for this is that for very short wavelengths, the spin-dependent forces that are negligible for macroscopic waves become large enough to play the role that is played by the pressure in a macroscopic sound wave. This phenomenon could be called spin-velocity resonance. It occurs only for wavelengths of the order of atomic dimensions, for which classical mechanics loses its validity. For example, for waves in hydrogen at S.T.P., resonance would occur at a wavelength of the order 10^{-9}cm.

When resonance occurs, the solution given above is no longer valid (even if the ratio $\frac{(A/c^2)}{\hat{P}^2 - (\theta/2)^2}$ were held constant) because the perturbation procedure is no longer valid. Moreover, before any physical reality could be claimed for such a resonance, it would be necessary to find a solution that satisfied the constraint $\mathfrak{J}(\mathfrak{L}_\alpha \chi^\alpha)=0$, because otherwise the forces arising from the neglect of this constraint, which are completely spurious and without physical meaning, would play an important role.

The occurrence of resonance does, however, represent a breakdown of the classical solution and so serves to answer the question "Why, if the constant $\hbar$ plays no role in macroscopic problems, must it be chosen to have the value 1.05×10^{-27} erg-sec if our only interest in the spinor alternative to Euler's equation is to solve macroscopic problems?" The answer is that, in order to give the correct breakdown point, namely the point at which the characteristic length of the problem becomes of atomic dimensions, the value assigned to $\hbar$ in the classical spinor theory must be at least approximately equal to the value given above.

Note Concerning Speed of Sound

It should be noted that c_p in the expression (42) for the speed of sound is a function of the temperature T_o. This is most directly seen from the expression for c_p in terms of Boltzmann's constant k , particle mass m, and number of degrees of freedom f:

$$c_p = \frac{k}{m}\left(1+\frac{f}{2}\right) \tag{46}$$

But the mass m contains a contribution from the enthalpy h_o, and so is a function of T_o. Now let c_p^o be the specific heat referred to the particle mass m_o that does not contain the enthalpy contribution:

$$c_p^o = \frac{k}{m_o}\left(1+\frac{f}{2}\right) \tag{47}$$

Since

$$mc^2 = m_o c^2 + m_o h_o = m_o(c^2 + c_p^o T_o) \tag{48}$$

we have

$$c_p = c_p^o\left(\frac{m_o}{m}\right) = \frac{c^2 c_p^o}{c^2 + c_p^o T_o} \tag{49}$$

which when substituted into (45) yields

$$V = \left[(\gamma-1)\left(\frac{c^2 c_p^o T_o}{c^2 + c_p^o T_o}\right)\right]^{1/2} \tag{50}$$

According to (49)

$$\lim_{T_o \to \infty} V = c\sqrt{(\gamma-1)} = c\sqrt{\frac{2}{f}} \tag{51}$$

This limit cannot, however, be taken very seriously since, as Synge* points out on p.58 of his book on the relativistic gas, equation (2) for the adiabatic condition is not valid at temperatures for which the thermal energy is comparable with the rest energy. The limiting speed of sound for a monatomic gas (f = 3) that Synge gives (p.77) is

$$V_{lim} = \frac{c}{\sqrt{3}} \tag{52}$$

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*J.L.Synge, The Relativistic Gas (Interscience Publishers, New York, 1957)