N64-33624
(ACCESSION NUMBER)
(THRU)
12
(PAGES)
1
(CODE)
NASA TMX 55090
(NASA CR OR TMX OR AD NUMBER)
29
(CATEGORY)
FACILITY FORM 602

X-547-64-233

TM X-55090

# LONG-PERIOD CONTRIBUTIONS TO THE DISTURBING FUNCTIONS OF THE EARTH FROM THE SEVENTH, NINTH, AND ELEVENTH ZONAL HARMONICS

OTS PRICE

XEROX $ 1.00

MICROFILM $ .50

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AUGUST 1964

GODDARD SPACE FLIGHT CENTER

GREENBELT, MARYLAND

Long-Period Contributions to the Disturbing Functions of the Earth from the Seventh, Ninth, and Eleventh Zonal Harmonics

by

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## Introduction

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It is the purpose of this paper to present explicit formulas for the long-period terms due to the seventh, ninth, and eleventh zonal harmonics in the disturbing function of the earth in the case of an artificial earth satellite. The formulas are given for terms of the satellite's orbital elements and the Delaunay variables. G. Giacaglia (1) has given general expressions for the long-period terms due to any of the zonal harmonics, which can be expressed in terms of the orbital elements. The apparent differences between the results of this paper and those in Giacaglia's have been verified as due to the errors in the latter as it appears in the A.J.

The contributions of the long-period terms to the mean motion of the argument of perigee are also given.

author

## The Disturbing Function

The earth's gravitational potential at a distance r from the center of the earth is

$$U = \frac{\mu}{r}\left[1 - \sum_{n=2}^{\infty}\left(\frac{R}{r}\right)^{n} J_n\, P_n\,(\sin\phi)\right],$$

where

$\mu$ = GM

G = gravitational constant

M = mass of the earth

R = radius of the earth

$J_n$ = zonal harmonic coefficients (n = 2, 3, . . . )

$P_n$ = Legendre polynomials (n = 2, 3, . . . )

$\phi$ = geocentric latitude.

Here, the earth's radius is adopted as the unit of length. The seventh, ninth, and eleventh Legendre polynomials are

$$P_7(\sin\phi) = \frac{1}{16}(429\sin^7\phi - 693\sin^5\phi + 315\sin^3\phi - 35\sin\phi)$$

$$P_9(\sin\phi) = \frac{1}{128}(12155\sin^9\phi - 25740\sin^7\phi + 18018\sin^5\phi - 4620\sin^3\phi + 315\sin\phi)$$

$$P_{11}(\sin\phi) = \frac{1}{256}(88179\sin^{11}\phi - 230945\sin^9\phi + 218790\sin^7\phi - 90090\sin^5\phi + 15015\sin^3\phi - 693\sin\phi).$$

Let

a = semi-major axis of satellite's orbit

e = eccentricity of orbit

i = inclination of orbital plane to equatorial plane

$\ell$ = mean anomaly

f = true anomaly

g = argument of perigee.

The Delaunay variables L, G, and H are

$$L = \sqrt{\mu a}$$

$$G = L \sqrt{1-e^2}$$

$$H = G \cos i.$$

Use is also made of the relations

$$\sin \phi = \sin i \sin (f+g)$$

$$\frac{a}{r} = \frac{L^2}{G^2}(1 + e \cos f).$$

The long-period terms in the expansion of U as a Fourier series in $\ell$ and g are given by

$$\frac{1}{2\pi}\int_0^{2\pi} U\, d\ell, \qquad \text{(see reference 2)}$$

making use of the relation

$$d\ell = \frac{L}{G} \frac{r^2}{a^2} df.$$

Denoting the long-period parts of $U_7$, $U_9$, and $U_{11}$ by $\Delta_7 F_{2p}$, $\Delta_9 F_{2p}$, and $\Delta_{11} F_{2p}$, respectively, we have

$$\Delta_7 F_{2p} = -\frac{21\mu^9 J_7 e \sin i}{16384 L^3 G^{13}} \left[10\left(5 - 135\frac{H^2}{G^2} + 495\frac{H^4}{G^4} - 429\frac{H^6}{G^6}\right)\left(33 - 30\frac{G^2}{L^2} + 5\frac{G^4}{L^4}\right)\sin g \right.$$

$$- 15\left(3 - 69\frac{H^2}{G^2} + 209\frac{H^4}{G^4} - 143\frac{H^6}{G^6}\right)\left(11 - 14\frac{G^2}{L^2} + 3\frac{G^4}{L^4}\right)\sin 3g$$

$$\left. + 33\left(1 - 15\frac{H^2}{G^2} + 27\frac{H^4}{G^4} - 13\frac{H^6}{G^6}\right)\left(1 - 2\frac{G^2}{L^2} + \frac{G^4}{L^4}\right) \sin 5g\right]$$

$$\Delta_9 F_{2p} = -\frac{3\mu^{11} J_9 e \sin i}{524288 L^3 G^{17}} \left[210\left(7 - 308\frac{H^2}{G^2} + 2002\frac{H^4}{G^4} - 4004\frac{H^6}{G^6} + 2431\frac{H^8}{G^8}\right) \times \right.$$

$$\times \left(715 - 1001\frac{G^2}{L^2} + 385\frac{G^4}{L^4} - 35\frac{G^6}{L^6}\right)\sin g$$

$$- 10780\left(1 - 40\frac{H^2}{G^2} + 234\frac{H^4}{G^4} - 416\frac{H^6}{G^6} + 221\frac{H^8}{G^8}\right) \times$$

$$\times \left(39 - 65\frac{G^2}{L^2} + 29\frac{G^4}{L^4} - 3\frac{G^6}{L^6}\right)\sin 3g$$

$$+ 12012\left(1 - 32\frac{H^2}{G^2} + 146\frac{H^4}{G^4} - 200\frac{H^6}{G^6} + 85\frac{H^8}{G^8}\right) \times$$

$$\times \left(5 - 11\frac{G^2}{L^2} + 7\frac{G^4}{L^4} - \frac{G^6}{L^6}\right)\sin 5g$$

$$- 2145\left(1 - 20\frac{H^2}{G^2} + 54\frac{H^4}{G^4} - 52\frac{H^6}{G^6} + 17\frac{H^8}{G^8}\right) \times$$

$$\left. \times \left(1 - 3\frac{G^2}{L^2} + 3\frac{G^4}{L^4} - \frac{G^6}{L^6}\right)\sin 7g\right]$$

$$\Delta_{11} F_{2p} = -\frac{11\mu^{13} J_{11} e \sin i}{67108864 L^3 G^{21}} \left[126\left(21 - 1365\frac{H^2}{G^2} + 13650\frac{H^4}{G^4} - 46410\frac{H^6}{G^6} + \right.\right.$$

$$\left. + 62985\frac{H^8}{G^8} - 29393\frac{H^{10}}{G^{10}}\right) \times \left(41423 - 77292\frac{G^2}{L^2} + 45738\frac{G^4}{L^4} - 8652\frac{G^6}{L^6} + 63\frac{G^8}{L^8}\right)\sin g$$

$$- 163800 \left(1 - 61\frac{H^2}{G^2} + 570\frac{H^4}{G^4} - 1802\frac{H^6}{G^6} + 2261\frac{H^8}{G^8} - 969\frac{H^{10}}{G^{10}}\right) \times$$

$$\times \left(323 - 680\frac{G^2}{L^2} + 462\frac{G^4}{L^4} - 112\frac{G^6}{L^6} + 7\frac{G^8}{L^8}\right) \sin 3g$$

$$+ 7020 \left(5 - 265\frac{H^2}{G^2} + 2130\frac{H^4}{G^4} - 5746\frac{H^6}{G^6} + 6137\frac{H^8}{G^8} - 2261\frac{H^{10}}{G^{10}}\right) \times$$

$$\times \left(323 - 816\frac{G^2}{L^2} + 678\frac{G^4}{L^4} - 200\frac{G^6}{L^6} + 15\frac{G^8}{L^8}\right) \sin 5g$$

$$- 49725 \left(1 - 41\frac{H^2}{G^2} + 250\frac{H^4}{G^4} - 514\frac{H^6}{G^6} + 437\frac{H^8}{G^8} - 133\frac{H^{10}}{G^{10}}\right) \times$$

$$\times \left(19 - 60\frac{G^2}{L^2} + 66\frac{G^4}{L^4} - 28\frac{G^6}{L^6} + 3\frac{G^8}{L^8}\right) \sin 7g$$

$$+ 20995 \left(1 - 25\frac{H^2}{G^2} + 90\frac{H^4}{G^4} - 130\frac{H^6}{G^6} + 85\frac{H^8}{G^8} - 21\frac{H^{10}}{G^{10}}\right) \times$$

$$\left. \times \left(1 - 4\frac{G^2}{L^2} + 6\frac{G^4}{L^4} - 4\frac{G^6}{L^6} + \frac{G^8}{L^8}\right) \sin 9g \right]$$

<u>Contributions to $dg/dt$</u>

Since the Delaunay set of variables is canonical with respect to the Hamiltonian F, which includes $\Delta_7 F_{2p}$, $\Delta_9 F_{2p}$, and $\Delta_{11} F_{2p}$, we have

$$\frac{dg}{dt} = -\frac{\partial F}{\partial G} \quad \text{(see reference 2).}$$

Therefore, a computation of $\partial(\Delta_i F_{2p})/\partial g$ (i = 7,9,11) provides the long-period terms in dg/dt due to the seventh, ninth, and eleventh zonal harmonics.

The results are

$$\frac{\partial(\Delta_7 F_{2p})}{\partial G} = \frac{21\,\mu^9 J_7}{16384 L^3 G^{14} e \sin i} \Bigg\{ 10 \Bigg[ (\sin^2 i - e^2) \left(5 - 135\frac{H^2}{G^2} + 495\frac{H^4}{G^4} - 429\frac{H^6}{G^6}\right)\left(33 - 30\frac{G^2}{L^2} + 5\frac{G^4}{L^4}\right) + e^2 \sin^2 i \left(65 - 2025\frac{H^2}{G^2} + 8415\frac{H^4}{G^4} - 8151\frac{H^6}{G^6}\right)\left(33 - 30\frac{G^2}{L^2} + 5\frac{G^4}{L^4}\right) + 20 e^2 \sin^2 i \left(5 - 135\frac{H^2}{G^2} + 495\frac{H^4}{G^4} - 429\frac{H^6}{G^6}\right)\frac{G^2}{L^2}\left(3 - \frac{G^2}{L^2}\right)\Bigg] \sin g$$

$$-15 \Bigg[ (\sin^2 i - e^2)\left(3 - 69\frac{H^2}{G^2} + 209\frac{H^4}{G^4} - 143\frac{H^6}{G^6}\right)\left(11 - 14\frac{G^2}{L^2} + 3\frac{G^4}{L^4}\right) + e^2 \sin^2 i \left(39 - 1035\frac{H^2}{G^2} + 3553\frac{H^4}{G^4} - 2717\frac{H^6}{G^6}\right)\left(11 - 14\frac{G^2}{L^2} + 3\frac{G^4}{L^4}\right) + 4 e^2 \sin^2 i \left(3 - 69\frac{H^2}{G^2} + 209\frac{H^4}{G^4} - 143\frac{H^6}{G^6}\right)\frac{G^2}{L^2}\left(7 - 3\frac{G^2}{L^2}\right)\Bigg] \sin 3g$$

$$+33 \Bigg[ (\sin^2 i - e^2)\left(1 - 15\frac{H^2}{G^2} + 27\frac{H^4}{G^4} - 13\frac{H^6}{G^6}\right)\left(1 - 2\frac{G^2}{L^2} + \frac{G^4}{L^4}\right) + e^2 \sin^2 i \left(13 - 225\frac{H^2}{G^2} + 459\frac{H^4}{G^4} - 247\frac{H^6}{G^6}\right)\left(1 - 2\frac{G^2}{L^2} + \frac{G^4}{L^4}\right) + 4 e^2 \sin^2 i \left(1 - 15\frac{H^2}{G^2} + 27\frac{H^4}{G^4} - 13\frac{H^6}{G^6}\right)\frac{G^2}{L^2}\left(1 - \frac{G^2}{L^2}\right)\Bigg] \sin 5g \Bigg\}$$

$$\frac{\partial(\Delta_9 F_{2p})}{\partial G} = \frac{3\mu^{11} J_9}{524288\, L^3 G^{18}\, e \sin i}\Bigg\{210\left[(\sin^2 i - e^2)\left(7 - 308\frac{H^2}{G^2} + 2002\frac{H^4}{G^4}\right.\right.$$

$$\left. - 4004\frac{H^6}{G^6} + 2431\frac{H^8}{G^8}\right) \times \left(715 - 1001\frac{G^2}{L^2} + 385\frac{G^4}{L^4} - 35\frac{G^6}{L^6}\right)$$

$$+ e^2 \sin^2 i\left(119 - 5852\frac{H^2}{G^2} + 42042\frac{H^4}{G^4} - 92092\frac{H^6}{G^6} + 60775\frac{H^8}{G^8}\right)\left(715\right.$$

$$\left. - 1001\frac{G^2}{L^2} + 385\frac{G^4}{L^4} - 35\frac{G^6}{L^6}\right) + 14\, e^2 \sin^2 i\left(7 - 308\frac{H^2}{G^2} + 2002\frac{H^4}{G^4}\right.$$

$$\left.\left. - 4004\frac{H^6}{G^6} + 2431\frac{H^8}{G^8}\right)\frac{G^2}{L^2}\left(143 - 110\frac{G^2}{L^2} + 15\frac{G^4}{L^4}\right)\right] \sin g$$

$$- 10780\left[(\sin^2 i - e^2)\left(1 - 40\frac{H^2}{G^2} + 234\frac{H^4}{G^4} - 416\frac{H^6}{G^6} + 221\frac{H^8}{G^8}\right)\left(39\right.\right.$$

$$\left. - 65\frac{G^2}{L^2} + 29\frac{G^4}{L^4} - 3\frac{G^6}{L^6}\right) + e^2 \sin^2 i\left(17 - 760\frac{H^2}{G^2} + 4914\frac{H^4}{G^4} - 9568\frac{H^6}{G^6} +\right.$$

$$\left. + 5525\frac{H^8}{G^8}\right)\left(39 - 65\frac{G^2}{L^2} + 29\frac{G^4}{L^4} - 3\frac{G^6}{L^6}\right) + 2\, e^2 \sin^2 i\left(1 - 40\frac{H^2}{G^2} + 234\frac{H^4}{G^4}\right.$$

$$\left.\left. - 416\frac{H^6}{G^6} + 221\frac{H^8}{G^8}\right)\frac{G^2}{L^2}\left(65 - 58\frac{G^2}{L^2} + 9\frac{G^4}{L^4}\right)\right] \sin 3g$$

$$+ 12012\left[(\sin^2 i - e^2)\left(1 - 32\frac{H^2}{G^2} + 146\frac{H^4}{G^4} - 200\frac{H^6}{G^6} + 85\frac{H^8}{G^8}\right)\left(5 - 11\frac{G^2}{L^2} +\right.\right.$$

$$\left. + 7\frac{G^4}{L^4} - \frac{G^6}{L^6}\right) + e^2 \sin^2 i\left(17 - 608\frac{H^2}{G^2} + 3066\frac{H^4}{G^4} - 4600\frac{H^6}{G^6} +\right.$$

$$+2125\frac{H^8}{G^8}\Big)\left(5-11\frac{G^2}{L^2}+7\frac{G^4}{L^4}-\frac{G^6}{L^6}\right)+2\,e^2\sin^2 i\,\Big(1-32\frac{H^2}{G^2}+146\frac{H^4}{G^4}$$

$$-200\frac{H^6}{G^6}+85\frac{H^8}{G^8}\Big)\frac{G^2}{L^2}\left(11-14\frac{G^2}{L^2}+3\frac{G^4}{L^4}\right)\Bigg]\sin 5g$$

$$-2145\Bigg[(\sin^2 i-e^2)\left(1-20\frac{H^2}{G^2}+54\frac{H^4}{G^4}-52\frac{H^6}{G^6}+17\frac{H^8}{G^8}\right)\left(1-3\frac{G^2}{L^2}+3\frac{G^4}{L^4}-\frac{G^6}{L^6}\right)$$

$$+e^2\sin^2 i\left(17-380\frac{H^2}{G^2}+1134\frac{H^4}{G^4}-1196\frac{H^6}{G^6}+425\frac{H^8}{G^8}\right)\left(1-3\frac{G^2}{L^2}+3\frac{G^4}{L^4}-\frac{G^6}{L^6}\right)$$

$$+6\,e^2\sin^2 i\left(1-20\frac{H^2}{G^2}+54\frac{H^4}{G^4}-52\frac{H^6}{G^6}+17\frac{H^8}{G^8}\right)\frac{G^2}{L^2}\left(1-2\frac{G^2}{L^2}+\frac{G^4}{L^4}\right)\Bigg]\sin 7g\Bigg\}$$

$$\frac{\partial(\Delta_{11}F_{2p})}{\partial G}=\frac{11\,\mu^{13}J_{11}}{67108864\,L^3G^{22}\,e\sin i}\Bigg\{126\Bigg[(\sin^2 i-e^2)\Big(21-1365\frac{H^2}{G^2}+13650\frac{H^4}{G^4}$$

$$-46410\frac{H^6}{G^6}+62985\frac{H^8}{G^8}-29393\frac{H^{10}}{G^{10}}\Big)\Big(41423-77292\frac{G^2}{L^2}+45738\frac{G^4}{L^4}$$

$$-8652\frac{G^6}{L^6}+63\frac{G^8}{L^8}\Big)+e^2\sin^2 i\Big(441-31395\frac{H^2}{G^2}+341250\frac{H^4}{G^4}-1253070\frac{H^6}{G^6}+$$

$$+1826565\frac{H^8}{G^8}-911183\frac{H^{10}}{G^{10}}\Big)\times\Big(41423-77292\frac{G^2}{L^2}+45738\frac{G^4}{L^4}$$

$$-8652\frac{G^6}{L^6}+63\frac{G^8}{L^8}\Big)+72\,e^2\sin^2 i\,\Big(21-1365\frac{H^2}{G^2}+13650\frac{H^4}{G^4}-46410\frac{H^6}{G^6}+$$

$$+62985\frac{H^8}{G^8}-29393\frac{H^{10}}{G^{10}}\Big)\times\frac{G^2}{L^2}\left(2147-2541\frac{G^2}{L^2}+721\frac{G^4}{L^4}-7\frac{G^6}{L^6}\right)\Bigg]\sin g$$

$$-163800\Bigg[(\sin^2 i-e^2)\left(1-61\frac{H^2}{G^2}+570\frac{H^4}{G^4}-1802\frac{H^6}{G^6}+2261\frac{H^8}{G^8}-969\frac{H^{10}}{G^{10}}\right)\times$$

$$\times\left(323-680\frac{G^2}{L^2}+462\frac{G^4}{L^4}-112\frac{G^6}{L^6}+7\frac{G^8}{L^8}\right)$$

$$+e^2\sin^2 i\left(21-1403\frac{H^2}{G^2}+14250\frac{H^4}{G^4}-48654\frac{H^6}{G^6}+65569\frac{H^8}{G^8}-30039\frac{H^{10}}{G^{10}}\right)\times$$

$$\times\left(323-680\frac{G^2}{L^2}+462\frac{G^4}{L^4}-112\frac{G^6}{L^6}+7\frac{G^8}{L^8}\right)$$

$$+8e^2\sin^2 i\left(1-61\frac{H^2}{G^2}+570\frac{H^4}{G^4}-1802\frac{H^6}{G^6}+2261\frac{H^8}{G^8}-969\frac{H^{10}}{G^{10}}\right)\times$$

$$\times\frac{G^2}{L^2}\left(170-231\frac{G^2}{L^2}+84\frac{G^4}{L^4}-7\frac{G^6}{L^6}\right)\Bigg]\sin 3g$$

$$+7020\Bigg[(\sin^2 i-e^2)\left(5-265\frac{H^2}{G^2}+2130\frac{H^4}{G^4}-5746\frac{H^6}{G^6}+6137\frac{H^8}{G^8}-2261\frac{H^{10}}{G^{10}}\right)\times$$

$$\times\left(323-816\frac{G^2}{L^2}+678\frac{G^4}{L^4}-200\frac{G^6}{L^6}+15\frac{G^8}{L^8}\right)$$

$$+e^2\sin^2 i\left(105-6095\frac{H^2}{G^2}+53250\frac{H^4}{G^4}-155142\frac{H^6}{G^6}+177973\frac{H^8}{G^8}-70091\frac{H^{10}}{G^{10}}\right)\times$$

$$\times\left(323-816\frac{G^2}{L^2}+678\frac{G^4}{L^4}-200\frac{G^6}{L^6}+15\frac{G^8}{L^8}\right)$$

$$+24\,e^2\sin^2 i\left(5-265\frac{H^2}{G^2}+2130\frac{H^4}{G^4}-5746\frac{H^6}{G^6}+6137\frac{H^8}{G^8}-2261\frac{H^{10}}{G^{10}}\right)\times$$

$$\left.\times\frac{G^2}{L^2}\left(68-113\frac{G^2}{L^2}+50\frac{G^4}{L^4}-5\frac{G^6}{L^6}\right)\right]\sin 5g$$

$$-49725\left[(\sin^2 i-e^2)\left(1-41\frac{H^2}{G^2}+250\frac{H^4}{G^4}-514\frac{H^6}{G^6}+437\frac{H^8}{G^8}-133\frac{H^{10}}{G^{10}}\right)\times\right.$$

$$\times\left(19-60\frac{G^2}{L^2}+66\frac{G^4}{L^4}-28\frac{G^6}{L^6}+3\frac{G^8}{L^8}\right)$$

$$+e^2\sin^2 i\left(21-943\frac{H^2}{G^2}+6250\frac{H^4}{G^4}-13878\frac{H^6}{G^6}+12673\frac{H^8}{G^8}-4123\frac{H^{10}}{G^{10}}\right)\times$$

$$\times\left(19-60\frac{G^2}{L^2}+66\frac{G^4}{L^4}-28\frac{G^6}{L^6}+3\frac{G^8}{L^8}\right)$$

$$+24\,e^2\sin^2 i\left(1-41\frac{H^2}{G^2}+250\frac{H^4}{G^4}-514\frac{H^6}{G^6}+437\frac{H^8}{G^8}-133\frac{H^{10}}{G^{10}}\right)\times$$

$$\left.\times\frac{G^2}{L^2}\left(5-11\frac{G^2}{L^2}+7\frac{G^4}{L^4}-\frac{G^6}{L^6}\right)\right]\sin 7g$$

$$+20995\left[(\sin^2 i-e^2)\left(1-25\frac{H^2}{G^2}+90\frac{H^4}{G^4}-130\frac{H^6}{G^6}+85\frac{H^8}{G^8}-21\frac{H^{10}}{G^{10}}\right)\times\right.$$

$$\times\left(1-4\frac{G^2}{L^2}+6\frac{G^4}{L^4}-4\frac{G^6}{L^6}+\frac{G^8}{L^8}\right)$$

$$+ e^2 \sin^2 i \left(21 - 575\frac{H^2}{G^2} + 2250\frac{H^4}{G^4} - 3510\frac{H^6}{G^6} + 2465\frac{H^8}{G^8} - 651\frac{H^{10}}{H^{10}}\right) \times$$

$$\times \left(1 - 4\frac{G^2}{L^2} + 6\frac{G^4}{L^4} - 4\frac{G^6}{L^6} + \frac{G^8}{L^8}\right)$$

$$+ 8\, e^2 \sin^2 i \left(1 - 25\frac{H^2}{G^2} + 90\frac{H^4}{G^4} - 130\frac{H^6}{G^6} + 85\frac{H^8}{G^8} - 21\frac{H^{10}}{G^{10}}\right) \times$$

$$\times \frac{G^2}{L^2}\left(1 - 3\frac{G^2}{L^2} + 3\frac{G^4}{L^4} - \frac{G^6}{L^6}\right)\Bigg] \sin 9g \Bigg\} .$$

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