NASA TECHNICAL NOTE

NASA TN D-2961

NASA TN D-2961

N65-30699
(ACCESSION NUMBER) (THRU)
16 /
(PAGES) (CODE)
30
(NASA CR OR TMX OR AD NUMBER) (CATEGORY)

FACILITY FORM 602

GPO PRICE $

CSFTI PRICE(S) $ 1.00

Hard copy (HC)

Microfiche (MF) .50

ff 653 July 65

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by Donald J. Jezewski

Manned Spacecraft Center

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NATIONAL AERONAUTICS AND SPACE ADMINISTRATION • WASHINGTON, D. C. • AUGUST 1965

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SUMMARY

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A method is presented for obtaining numerical solutions of nonlinear, optimum trajectory problems by using an approximate analytical solution in a convergence process. Convergence to specified terminal conditions is based on an analytic solution representing a reasonable approximation to the nonlinear system of equations.

A numerical example of three-dimensional, time-optimal trajectory solutions in vacuum with a constant thrust is used to illustrate the technique. A gravitational force varying as the inverse square of the radius is considered. The approximate analytic solution is obtained from a system of quasi-optimum transcendental equations. This analytic solution is used to obtain the initial and succeeding approximations to the initial values of the Lagrange multipliers. Generally, only a few iterations are required to obtain solutions which satisfy specified boundary conditions.

Author

INTRODUCTION

The requirement for solutions of nonlinear, optimum trajectory problems has steadily increased in recent years. The nonlinear differential equations are difficult to solve because of the deficiency of a process which would converge rapidly to the specified boundary conditions. The first order perturbation techniques used in convergence processes normally work well in the vicinity of the solution where the linearity assumption is valid, but they are usually unpredictable when large errors are present.

The present paper deals with a technique which uses an approximate analytical solution in a convergence process for the corresponding nonlinear system of equations (refs. 1 and 2). The basis for convergence is that the solution curves of both the approximate and nonlinear equations are similar so that an identical change in the initial values of the Lagrange multipliers in each will produce approximately the same change in the terminal boundary conditions. These terminal boundary errors can be nullified by generating corrections to the initial values of Lagrange multipliers from the analytic solution.

SYMBOLS

$1_r, 1_\theta, 1_\varphi$	unit vectors in r, θ, and φ directions, respectively
A_z	azimuth angle measured from 1_θ, deg
$\vec{B}$	terminal boundary vector
$C_1, C_2, \ldots, C_6$	initial values of Lagrange multipliers
F	analytic solution
$f_1, f_2, \ldots, f_6$	general nonlinear functions
H	function defined by equation (4)
h	altitude, ft
I_{sp}	specific impulse, sec
k	logical control switch
m	mass of vehicle, slugs
r	radius to vehicle from center of attracting body, ft
T	thrust, lb
t	time, sec
u,v,w	components of total velocity in 1_φ, 1_r, and 1_θ directions, respectively, ft/sec
V	total velocity, ft/sec
W_O	initial weight of vehicle, lb
X,Y,Z	inertial reference frame, ft
$\vec{x}$	state vector
$\vec{\alpha}$	control vector
β_χ	function defined by equation (A10)
β_ψ	function defined by equation (A12)

γ	flight-path angle, deg
θ	latitude angle, deg
$\lambda_1,\lambda_2,\ldots,\lambda_6$	Lagrange multipliers
μ	gravitational constant, ft^3/sec^2
φ	longitude angle, deg
χ	thrust pitch angle measured from local horizontal plane, deg
Ψ	thrust yaw angle measured from 1_φ, deg

Subscripts:

0,1	initial and final values of state and control variables
c	referring to a corrected solution
I	referring to an integrated solution
R	reference solution

Superscripts:

$\rightarrow$	vector quantity
$\hat{}$	approximate value of variables
T	transpose matrix

Operators:

$(\dot{\ })$	time differentiation

ANALYSIS

Statement of Problem

Consider a system of first-order differential equations of the form

$$f\left(\dot{\vec{x}},\vec{x},\vec{\alpha},t\right) = 0 \qquad \vec{x}\left(t_0\right)\text{given} \qquad t_0 \leq t \leq t_1 \tag{1}$$

where $\vec{x}$ and $\vec{\alpha}$ are respectively the state and control vectors. Certain terminal boundary conditions on the state vector are desired of the form

$$\vec{B}\left[\vec{x}\left(t_1\right), t_1\right] = 0 \tag{2}$$

The problem is to determine the control vector $\vec{\alpha}(t)$ which forces the state from its given initial values $\vec{x}\left(t_0\right)$ to the boundary conditions defined by equation (2).

In optimization problems, the control vector is obtained from an optimality condition given by

$$\partial H/\partial \alpha = 0 \tag{3}$$

where

$$H = \vec{\lambda}^{T} \cdot f\left(\dot{\vec{x}}, \vec{x}, \vec{\alpha}, t\right) \tag{4}$$

The superscript T refers to the transpose matrix. The λ's are Lagrange multipliers governed by the differential equation

$$\frac{d}{dt}\left(\frac{\partial H}{\partial \dot{x}}\right) - \frac{\partial H}{\partial x} = 0 \tag{5}$$

This system of nonlinear equation (equations (1) to (5)) has a solution $\vec{x}(t)$ for any initial values of the Lagrange multipliers $\vec{\lambda}\left(t_0\right)$. The initial control vector is obtained from equation (3) by the choice of the multipliers $\vec{\lambda}\left(t_0\right)$. Thus, the problem is reduced to one of determining the initial vector $\vec{\lambda}\left(t_0\right)$.

Approximate Solution

Consider an approximate analytic solution of the described system of nonlinear equations such that the solution evaluated at the terminal boundary conditions is

$$F\left(\vec{x}_1, \vec{x}_0, \hat{\vec{\lambda}}_R, t_1, t_0\right) = 0 \qquad \begin{aligned} \hat{\vec{\lambda}}_R &= \hat{\vec{\lambda}}_R\left(t_0\right) \\ \vec{x}_1 &= \vec{x}\left(t_1\right) \\ \vec{x}_0 &= \vec{x}\left(t_0\right) \end{aligned} \tag{6}$$

where the circumflex over the variable indicates an approximate solution and the subscript R refers to a reference value. This solution can generally be derived by applying some simplifying assumptions to the nonlinear equations. The initial values of the Lagrange multiplier from the analytic solution $\hat{\vec{\lambda}}_R$ represent a first approximation to the Lagrange multipliers of the nonlinear system of equations. Ideally, they can be solved explicitly although in practice the equations are usually transcendental, and iterative techniques must be applied.

Iterative Technique

Figure 1 is a logical flow chart of the iterative technique used in the convergence process. A switch k governs the various phases of the solution.

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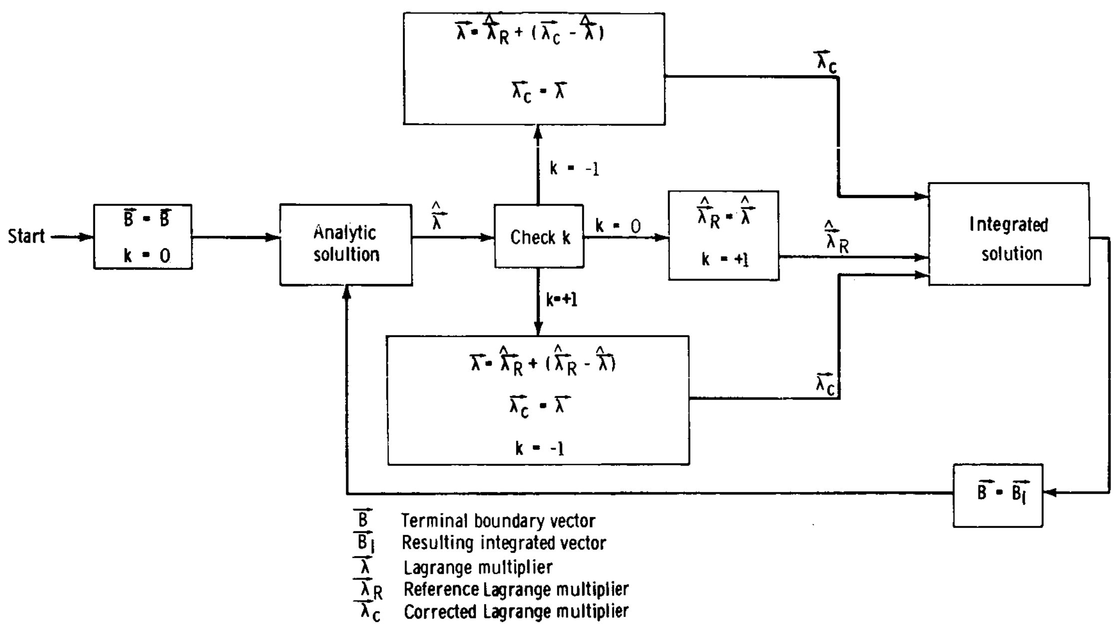

Figure 1. - The use of an analytic solution in a convergence technique

With the desired terminal boundary conditions and the switch k equal to 0, the analytic solution is solved for the initial values of the Lagrange multipliers $\hat{\vec{\lambda}}_R$ denoted as the reference solution. If the nonlinear system of equations is integrated to a terminal cutoff using the initial values $\vec{x}_0$ and $\hat{\vec{\lambda}}_R$, the resulting terminal vector $\vec{B}_I$ should resemble the desired terminal boundary conditions. The error in these conditions is indicative of the accuracy of the approximate solution. Replacing the desired boundary conditions by those obtained from the integrated solution $\vec{B} = \vec{B}_I$, the analytic solution is resolved for the initial vector $\hat{\vec{\lambda}}$. The switch k is set equal to +1. In the analytic solution, a change in the initial vector of

$$\Delta\hat{\vec{\lambda}} = \hat{\vec{\lambda}} - \hat{\vec{\lambda}}_R \tag{7}$$

has produced a change in the terminal conditions of

$$\Delta\vec{B} = \vec{B}_I - \vec{B} \tag{8}$$

Since the analytic solution is an approximation to the nonlinear system of equations, it is assumed that the negative of this correction should produce an approximate change in the terminal boundary conditions of the integrated solution. Therefore, the corrected initial value of the Lagrange multipliers for the nonlinear system of equations is

$$\vec{\lambda}_c = \hat{\vec{\lambda}}_R - \Delta\hat{\vec{\lambda}} \tag{9}$$

The switch k for the next and all future passes is set equal to -1. The procedure is repeated with one exception. The correction to the reference vector $\hat{\vec{\lambda}}_R$ is obtained by subtracting the multiplier from the analytic solution $\hat{\vec{\lambda}}$ and the previous integrated solution $\vec{\lambda}_c$. This correction is given by

$$\Delta\hat{\vec{\lambda}} = \hat{\vec{\lambda}} - \vec{\lambda}_c \tag{10}$$

The solution is continued in this manner until the terminal boundary conditions approach the desired conditions within some arbitrary tolerance. Battin and Gibson (refs. 3 and 4) have used a similar technique in solving circumlunar trajectories using a matched conic as the analytic solution.

DISCUSSION OF RESULTS

The example problem that is used to illustrate the solution is a time-optimum descent from lunar orbit. The problem is typical of a number of examples which were solved. Refer to appendix A for a description of the numerical problem and the results of the derivation of the analytic solution.

An IBM 7040 electronic data processing machine was used for the computations. The solution time varied with the individual problem, depending mainly on the number of iterations required. For the longitude-free solution presented, each iteration required approximately 6 seconds.

The iteration data and terminal boundary conditions are listed in table I for a time-optimum descent from a lunar orbit to an altitude of 5000 feet. The initial multipliers – those listed in the first iteration – were obtained by evaluating equations (A13) to (A18) of appendix A from results obtained by the analytic, quasi-optimum solution. The equations given in appendix B were integrated forward using these initial conditions. The independent variable, time, was chosen as a stopping condition for the integration, since a value of this quantity is obtained from the analytic solution. For each subsequent iteration, the terminal time was corrected in the same manner as the initial values of the Lagrange multipliers. Listed below the multipliers in table I are the terminal conditions on the boundary vector. A comparison of these values with the desired terminal boundary conditions shows good agreement, with the largest apparent error being in altitude. (This error is misleading, for the problem is actually solved in terms of the magnitude of the radius vector and not the altitude as the data imply. The error in this variable is less than 1 percent for this iteration.) The solution can be observed to converge very rapidly; each iteration, which required only 5 or 6 seconds to compute, reduced the errors in the terminal boundary vector and the initial values of the Lagrange multipliers by approximately 1 magnitude. A comparison of the multipliers in the first and last iteration indicates the excellent approximation of the analytic solution and the acceptability of the convergence technique. Total solution time for this problem was less than 30 seconds.

TABLE I.- TIME OPTIMUM DESCENT FROM LUNAR ORBIT

(a) Iterations

	1	2	3	4
λ_1	0.0	0.0	0.0	0.0
$\lambda_2 \cdot 10^4$	-17.16130	-22.23967	-22.73874	-22.73913
$\lambda_3 \cdot 10^6$	11.65346	10.65851	-.26852	-.17464
λ_4	1.0	1.0	1.0	1.0
$\lambda_5 \cdot 10^1$	-1.219591	-.880386	-.943033	-.943845
$\lambda_6 \cdot 10^1$	-1.174105	-1.153440	-1.173056	-1.173146
v_1, ft/sec . .	678.5133	89.3276	99.7368	100.0205
γ_1, deg	-86.16649	-7.18572	.26271	.00499
A_{z1}, deg . . .	-108.6837	-87.3814	-86.9300	-86.9981
h_1, ft	-71 477.5	5411.5	5062.5	5000.4
θ_1, deg	-.366338	-.346616	-.351411	-.351574
φ_1, deg	-28.02573	-28.03312	-28.03926	-28.03928
t_1, sec	326.0405	327.6104	327.4008	327.3864

(b) Boundary conditions

$\left[T/W_0 = 0.4172;\ I_{sp} = 314 \text{ sec}\right]$

Boundary conditions	v, ft/sec	γ, deg	A_z, deg	h, ft	θ, deg	φ, deg
Initial	5685	0	-83.3807	50 000	-1.5321	-17.93
Terminal	100	0	-86.9994	5 000	-0.3516	Free

CONCLUDING REMARKS

A technique has been presented for obtaining solutions of optimum trajectory problems by using an approximate analytic solution in a convergence process. Convergence is based on the analytic solution representing a reasonable approximation to the nonlinear equations. A numerical example of time-optimum, constant-thrust trajectory solution in three dimensions has been used to illustrate the technique. The analytic equations were obtained from a quasi-optimum solution in which the initial values of Lagrange multipliers were an excellent approximation. Generally, only a few iterations were required to converge to the exact solution.

Numerical results for a longitude-free solution are presented for a descent from lunar orbit. Solution time on an IBM 7040 digital computer was approximately 30 seconds for this case.

Manned Spacecraft Center
National Aeronautics and Space Administration
Houston, Texas, April 5, 1965

APPENDIX A

DERIVATION OF APPROXIMATE LAGRANGE MULTIPLIERS

The mathematical model employed is that of a mass particle with three degrees of freedom referred to a set of rotating coordinates. The axis system and the associated notation are illustrated in figure 2. The nonlinear differential equations for this model and their associated Euler-Lagrange equations are given in appendix B. The vehicle is assumed to have a constant thrust and mass flow rate. No aerodynamic forces are considered. The angles χ and Ψ are control variables and may be chosen arbitrarily. The final time t_1 was selected to be minimized since, under the above assumptions, this will yield a minimum fuel consumption.

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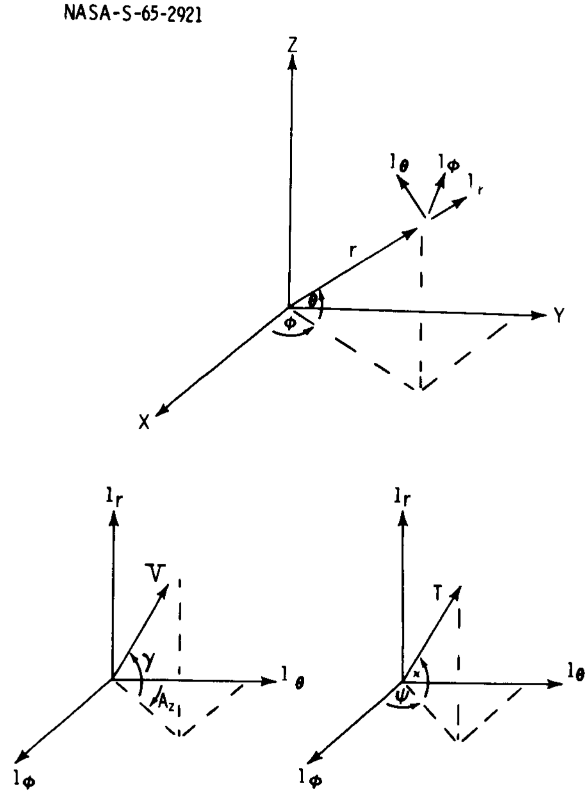

Figure 2. - Coordinate system and angle definition

If, in the equations of motion, the nonlinear terms are replaced by power series in time and the calculus of variation is applied to this linearized set of equations, the following optimality conditions result (ref. 2).

$$\tan \chi = \lambda_5/\lambda_4 \tag{A1}$$

$$\sin \Psi = \lambda_6/\lambda_4 \tag{A2}$$

where Ψ has been assumed to be small. The multipliers are of the form

$$\lambda_1 = C_1 \tag{A3}$$

$$\lambda_2 = C_2 \tag{A4}$$

$$\lambda_3 = C_3 \tag{A5}$$

$$\lambda_4 = C_4 - C_1 t \tag{A6}$$

$$\lambda_5 = C_5 - C_2 t \tag{A7}$$

$$\lambda_6 = C_6 - C_3 t \tag{A8}$$

where C_1 to C_6 are constants. If the terminal boundary condition on the longitude is allowed to be free, the multiplier λ_1 is 0. This result also facilitates integrating the linearized equations of motion.

Evaluating equations (A1) and (A2) at the initial and final time and making use of equations (A3) to (A8), the following relationships among the constants C will result:

$$C_5/C_4 = \tan \chi_0 \tag{A9}$$

$$C_2/C_4 = \beta_\chi = \left(\tan \chi_0 - \tan \chi_1\right)/t_1 \tag{A10}$$

$$C_6/C_4 = \sin \Psi_0 \tag{A11}$$

$$C_3/C_4 = \beta_\Psi = \left(\sin \Psi_0 - \sin \Psi_1\right)/t_1 \tag{A12}$$

Using these relationships, the Lagrange multipliers take the form

$$\lambda_1 = 0 \tag{A13}$$

$$\lambda_2 = \beta_\chi \tag{A14}$$

$$\lambda_3 = \beta_\Psi \tag{A15}$$

$$\lambda_4 = 1 \tag{A16}$$

$$\lambda_5 = \tan \chi_0 - \beta_\chi t \tag{A17}$$

$$\lambda_6 = \sin \Psi_0 - \beta_\Psi t \tag{A18}$$

where the constant C_4 has been arbitrarily set equal to unity since the Euler-Lagrange equations are homogenous in the multipliers.

From the solution of the analytic equations in reference 2, the initial and final values of the controls χ and Ψ along with the value of the final time t_1 are obtained. With these results, equations (A13) to (A18) are evaluated at the initial time to obtain a first approximation to the initial values of the Lagrange multipliers for the nonlinear system of equations.

A result which is immediately apparent from equations (A13) to (A18) is that a reasonably good approximation to the multipliers can be obtained by simply guessing five new parameters: χ_0, χ_1, Ψ_0, Ψ_1, and t_1. A knowledge of this set of variables for a particular problem would certainly be more extensive than the set of multipliers. Except for cases in which the multipliers are extremely sensitive to the terminal conditions, as in a deceleration problem, this method should prove fruitful.

APPENDIX B

CONSTRAINT AND EULER-LAGRANGE EQUATIONS

The mathematical model is a mass particle with three degrees of freedom referred to a set of rotating coordinates. The axis system and the associated notation are illustrated in figure 2. In spherical coordinates, the equations of motion are:

$$\ddot{r} - r\left(\dot{\theta}^2 + \dot{\varphi}^2 \cos^2 \theta\right) = T/m \sin \chi - \mu/r^2 \tag{B1}$$

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} + r\dot{\varphi}^2 \sin\theta \cos\theta = T/m \cos\chi \sin\Psi \tag{B2}$$

$$r\ddot{\varphi}\cos\theta - 2r\dot{\theta}\dot{\varphi}\sin\theta + 2\dot{r}\dot{\varphi}\cos\theta = T/m \cos\chi \cos\Psi \tag{B3}$$

$$\left.\begin{aligned} m &= m_0 - \dot{m}t \\ \dot{m} &= \text{constant} \\ t_0 &\leq t \leq t_1 \end{aligned}\right\} \tag{B4}$$

where T/m is the acceleration due to the thrust and χ and Ψ are arbitrarily chosen control variables. Let

$$v = \dot{r} \tag{B5}$$

$$u = r\dot{\varphi}\cos\theta \tag{B6}$$

$$w = r\dot{\theta} \tag{B7}$$

The constraint functions are

$$f_1 = r\dot{\varphi}\cos\theta - u = 0 \tag{B8}$$

$$f_2 = \dot{r} - v = 0 \tag{B9}$$

$$f_3 = r\dot{\theta} - w = 0 \tag{B10}$$

$$f_4 = \dot{u} - u(v - w\tan\theta)/r - T/m \cos\chi \cos\Psi = 0 \tag{B11}$$

$$f_5 = \dot{v} + \left(u^2 + w^2\right)/r - T/m \sin\chi + \mu/r^2 = 0 \tag{B12}$$

$$f_6 = \dot{w} - \left(vw + u^2 \tan\theta\right)/r - T/m \cos\chi \sin\Psi = 0 \tag{B13}$$

The Euler-Lagrange equations corresponding to the six first-order differential equations described in equations (B8) to (B13) are

$$\dot{\lambda}_1 = -\lambda_1(v - w\tan\theta)/r \tag{B14}$$

$$\dot{\lambda}_2 = \left(\lambda_1 u + \lambda_3 w\right)/r - \Big[\lambda_4 u(v - w\tan\theta) - \lambda_5\left(u^2 + w^2 - 2\mu/r\right) + \lambda_6\left(vw + u^2\tan\theta\right)\Big]\Big/r^2 \tag{B15}$$

$$\dot{\lambda}_3 = -\left(u\lambda_1\tan\theta + v\lambda_3\right)/r - u\left(w\lambda_4 - u\lambda_6\right)\frac{\sec^2\theta}{r^2} \tag{B16}$$

$$\dot{\lambda}_4 = -\lambda_1 + \Big[\lambda_4(v - w\tan\theta) - 2u\left(\lambda_5 - \lambda_6\tan\theta\right)\Big]\Big/r \tag{B17}$$

$$\dot{\lambda}_5 = -\lambda_2 + \left(\lambda_4 u + \lambda_6 w\right)/r \tag{B18}$$

$$\dot{\lambda}_6 = -\lambda_3 - \left(\lambda_4 u\tan\theta + 2\lambda_5 w - \lambda_6 v\right)/r \tag{B19}$$

The optimality conditions are

$$\tan\chi = \frac{\lambda_5\cos\Psi}{\lambda_4} \tag{B20}$$

$$\tan\Psi = \lambda_6/\lambda_4 \tag{B21}$$

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