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RELATIVISTIC CHARGED FLUID FLOW III: GENERALIZED HAMILTON-JACOBI EQUATION

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January 1966

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ABSTRACT

When the forces resulting from viscosity and heat injection are described in terms of the thermal 4-potential introduced in the preceding paper, it is possible to derive a scalar equation of motion that has the form of a generalized Hamilton-Jacobi equation. This equation involves not only the electromagnetic and gravitational fields, as in the single-particle case, but also the thermodynamic properties of the fluid as characterized by the specific enthalpy, whose role is analogous to that of the gravitational potential, and by the thermal 4-potential, whose dynamical effects are analogous to those of the electromagnetic 4-potential.

The formalism employs a generalization of the canonical particle momentum that includes the thermal 4-potential as well as the electromagnetic 4-potential. This generalized canonical momentum can be represented in terms of three scalar functions by means of a generalized Clebsch Transformation. One of these scalar functions is Hamilton's Characteristic Function, which is the unknown in the generalized Hamilton-Jacobi equation. The other two functions, which are called the vorticity invariants, determine the intrinsic vorticity of the fluid, which is defined as the curl of the generalized canonical momentum and which, according to the generalized Larmor Theorem derived in the preceding paper, is to be associated with that part of the fluid rotation that is a residual of the initial conditions of the fluid. The vorticity invariants are both constants of the fluid motion.

In the case of adiabatic flow, it is possible to express the thermal 4-potential in terms of the specific entropy and the temperature integral, which is defined as the scalar function whose substantial time derivative is equal to the temperature. This allows a simple interpretation of the generalized canonical momentum in terms of the heat reservoir model introduced in the first paper of this series.

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RELATIVISTIC CHARGED FLUID FLOW

III. GENERALIZED HAMILTON-JACOBI EQUATION

I. INTRODUCTION

In the preceding paper,* henceforth referred to as II, it was shown that in the relativistic dynamics of a charged fluid, a fundamental role is played by the intrinsic vorticity tensor $2\mu\omega^{jk}$ where μ is the variable (but relativistically invariant) particle mass defined in terms of the constant particle mass m, the gravitational potential G, and the specific enthalpy h by the relation

$$\mu \equiv m\,(1 + G/c^2 + h/c^2). \tag{1.1}$$

From the generalized Larmor Theorem that was derived in II, it was shown that the antisymmetric tensor ω^{jk} may be regarded as describing the intrinsic rotation of the fluid, i.e. that part of the total fluid rotation Ω^{jk} that is not produced by the action of external fields, but rather is to be associated with the starting conditions of the fluid at some instant of past time.

The physical importance of the intrinsic vorticity tensor $2\mu\omega^{jk}$ lies in the fact that the 3-vector given by its space-space components is frozen into the fluid and carried along with it. This tensor is defined in terms of the generalized canonical momentum $\not{p}^j$ by the relation

$$2\mu\omega^{jk} \equiv -(\partial^j \not{p}^k - \partial^k \not{p}^j), \tag{1.2}$$

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where $\not{p}^j \equiv (\not{p}^0, \not{\mathbf{p}})$ is defined by

$$\not{p}^j \equiv \mu v^j + (q/c) A^j + (m/c) a^j; \tag{1.3a}$$

or

$$c \not{p}^0 \equiv \mu^* c^2 + q A^0 + m a^0, \tag{1.3b}$$

$$\not{\mathbf{p}} \equiv \mu^* \mathbf{v} + (q/c) \mathbf{A} + (m/c) \mathbf{a}, \tag{1.3c}$$

where $a^j \equiv (a^0, \mathbf{a})$ is a thermal 4-potential that is analogous in its dynamical effects to the electromagnetic 4-potential $A^j \equiv (A^0, \mathbf{A})$, except that the particle mass m rather than the particle charge q plays the role of coupling constant. As in the preceding paper, the local fluid velocity is designated by $v^j \equiv \Gamma(c, \mathbf{v})$ where c is the speed of light and $\Gamma \equiv (1 - v^2/c^2)^{-1/2}$ where v is the magnitude of the 3-velocity $\mathbf{v}$. The mass $\mu^* = \Gamma\mu$ is the mass as seen in the observer's frame of reference, as opposed to the invariant mass μ which is the particle mass in the fluid rest-frame.

The equation of motion of the thermal 4-potential a^j is given by (4.9) of II:

$$d a^j/d\tau = v_k \partial^j a^k + (c/m)(m T \partial^j s - \pi^j - \eta^j) \tag{1.4}$$

where T and s are respectively the temperature and specific entropy in the fluid rest-frame, π^j is the energy-momentum per particle that is injected into the fluid because of heat absorption, and η^j is the viscous 4-force.

It was shown in the preceding paper that ω^{jk} must be orthogonal to v_k, that is

$$\omega^{jk} v_k = 0. \tag{1.5}$$

Contraction of (1.2) with v_k yields Euler's equation:

$$d(\mu v^j)/d\tau = m\,\partial^j G + m(\partial^j h - T\,\partial^j s) + (q/c)F^{jk}v_k + \pi^j + \eta^j \qquad (1.6)$$

where

$$F^{jk} = \partial^j A^k - \partial^k A^j \qquad (1.7)$$

is the electromagnetic field tensor. It was pointed out in (5.2) of the first paper of this series* (henceforth referred to as I) that

$$m(\partial^j h - T\,\partial^j s) = \rho^{-1}\,\partial^j \mathbb{P} \qquad (1.8)$$

where ρ is the invariant particle (not mass) density and $\mathbb{P}$ is the partial pressure of the charged fluid. Substitution of (1.8) into (1.6) yields the more usual form of Euler's equation. The form given in (1.6), however, has the advantage that it does not explicitly involve ρ, and so has the form of a single-particle equation. Once v^j has been found by solving (1.6), ρ can be found from the continuity equation:

$$d\ln\rho/d\tau \equiv v^j\,\partial_j \ln\rho = -\,\partial_j v^j. \qquad (1.9)$$

If we regard the fields G and F^{jk} that appear in (1.6) as given space-time functions, then the independent variables of the problem may be taken to be the three components of the 3-velocity $\mathbf{v}$, the particle density ρ, the enthalpy h, and the entropy s – a total of six degrees of freedom. (From the thermodynamic properties of the fluid the temperature T is regarded as a known function of the variables ρ, h, and s, and from the physical nature of the heat injection and

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viscosity π^j and η^j are regarded as known functions of ρ, h, s, and **v**.) The corresponding six equations necessary for the solution are the three independent space-like components of (1.6) (the time-like component being derivable from these), the continuity equation, the thermodynamic equation of state of the fluid in terms of ρ, h, and s, and an equation that specifies $ds/d\tau$ in terms of the six independent variables.

We could, however, use (1.4) to eliminate s, π^j, and η^j from (1.6):

$$d(\mu v^j)/d\tau = m\,\partial^j(G + h) + (q/c)\,F^{jk}v_k + (m/c)\,f^{jk}v_k, \tag{1.10}$$

where

$$f^{jk} = \partial^j a^k - \partial^k a^j \tag{1.11}$$

is a thermal field that is analogous in its dynamical effects to F^{jk}. This thermal field is specified in terms of the 4-potential a^j whose "field equations" are given by (1.4). Our independent variables could then be taken to be the three components of **v**, the density ρ, the enthalpy h, and the three significant degrees of freedom of a^j. (One of the four degrees of freedom of a^j corresponds to the specification of its gauge, which has no physical significance.) Thus we are now involved with a total of eight independent variables (**v**, ρ, h, and **a**). The corresponding eight equations are the three independent space-like components of (1.10), the continuity equation, the thermodynamic equation of state (which now involves **a** and **v** as well as ρ and h), and the three space-like components of (1.4) (the time-like component simply specifying the gauge of a^j).

Comparing these two approaches, we see that the second one, which utilizes **a**, has the disadvantage that it adds two extra variables to the problem. This disadvantage can be removed, however, by taking the fluid equation of motion to be a generalized Hamilton-Jacobi equation, rather than the Euler equation. The velocity $\mathbf{v}$ is then replaced by the scalar Characteristic Function S, so we again have a total of only six variables (S, ρ, h, and $\mathbf{a}$), even though we continue to use $\mathbf{a}$. The desired Hamilton-Jacobi equation is derived in Section II.

In Section III it is shown that it is always possible to describe the intrinsic vorticity $2\mu\omega^{jk}$ in terms of two scalars M and ϕ, called vorticity invariants, that are constants of the fluid motion. The introduction of these two scalars leads to a generalized Clebsch Transformation according to which the generalized canonical momentum $\mathfrak{p}^j$ is expressed in terms of three scalar functions, the Characteristic Function S and the vorticity invariants M and ϕ.

In Section IV the foregoing results are specialized to the case of adiabatic (but not necessarily isentropic) flow of an inviscid charged fluid. In this case it is possible to express the thermal 4-potential a^j in terms of two scalar functions, one of which is the specific entropy s. The other is a temperature integral $\mathfrak{Z}$ whose substantial time derivative $d\mathfrak{Z}/d\tau$ is just the local fluid temperature T. If $c\,\partial^j \mathfrak{Z}$ is identified with the temperature 4-vector T^j_R of a heat reservoir which coexists in space with the fluid but which, in the adiabatic case, does not interact with it, then it can be shown that the entropy-dependent contribution to the generalized canonical momentum $\mathfrak{p}^j$ may be interpreted in terms of this heat reservoir. Because the temperature 4-vector T^j_R of this reservoir is expressible as the gradient of the scalar $c\mathfrak{Z}$, it is called a scalar reservoir.

The general problem of nonadiabatic flow discussed in Section II is formulated in terms of the six independent variables S, ρ, h, and **a**, the last three of which are the space-like components of the 4-vector a^j. In Section IV, however, it is shown that the adiabatic case admits a <u>scalar formulation</u> in that it can be formulated in terms of the variables S, ρ, $\mathfrak{I}$, s, **M**, and ϕ, all six of which are scalar functions.

In Section V it is shown that, at the expense of increasing the number of independent variables, a scalar formulation is also possible even in the most general case of nonadiabatic flow. This formulation can be given a natural interpretation in terms of four imaginary scalar hear reservoirs that exchange heat with the fluid. An alternative form of this scalar formulation is also given that introduces into the formalism three constants of motion that are related to viscosity and heat exchange with the fluid.

II. GENERALIZED HAMILTON-JACOBI EQUATION

From (1.2) it is evident that $2\mu\omega^{jk}$ is expressible as the curl of a 4-vector, which we shall designate $(m/c)\,b^j$. It is also evident that this 4-vector must differ from $-\mathfrak{p}^j$ by no more than the gradient of some scalar function. Thus

$$2\mu\omega^{jk} = (m/c)\,(\partial^j b^k - \partial^k b^j) \tag{2.1}$$

where

$$-(m/c)\,b^j = \mathfrak{p}^j + \partial^j S, \tag{2.2}$$

where S is an as yet unspecified scalar function. Using (1.3) we have

$$-\partial^j S = \mu v^j + (q/c)\,A^j + (m/c)\,\alpha^j; \tag{2.3a}$$

or

$$-\partial S/\partial t = \mu^* c^2 + q A^0 + m \alpha^0, \tag{2.3b}$$

$$\nabla S = \mu^* \mathbf{v} + (q/c)\,\mathbf{A} + (m/c)\,\boldsymbol{\alpha}, \tag{2.3c}$$

where

$$\alpha^j \equiv (\alpha^0, \boldsymbol{\alpha}) \equiv a^j + b^j. \tag{2.4}$$

Notation introduced in the preceding two papers has been used in writing (2.3b) and (2.3c). In particular $\partial^j \equiv (\partial/c\,\partial t, -\nabla)$; $v^j \equiv \Gamma(c, \mathbf{v})$ where $\Gamma = (1 - v^2/c^2)^{-1/2}$; $\mu^* \equiv \Gamma\mu$; and $A^j \equiv (A^0, \mathbf{A})$.

Contracting the curl of α^j with v_k, and using (1.4), (1.5), and (2.1), we arrive at the equation of motion of α^j:

$$d\alpha^j/d\tau = v_k\,\partial^j \alpha^k + (c/m)\,(m T \partial^j s - \pi^j - \eta^j). \tag{2.5}$$

This has exactly the same form as the equation of motion of a^j given in (1.4). From (1.5) and (2.1) it follows that the equation of motion of b^j is

$$d b^j/d\tau = v_k\,\partial^j b^k. \tag{2.6}$$

Thus from (2.4) and the forms of the equations of motion of α^j, a^j, and b^j, it follows that a^j may be regarded as the "driven" part of α^j, i.e. the inhomogeneous part of the solution of (2.5), that responds to the "driving force" $(c/m)(mT\partial^j - \pi^j - \eta^j)$, whereas b^j is the homogeneous part of the solution which is insensitive to the driving force. According to (2.1), it is this homogeneous part of α^j whose curl is the intrinsic vorticity $2\mu\omega^{jk}$. Thus the description of the intrinsic vorticity and the dynamical effects of the viscosity and the entropy 4-gradient are combined in the single 4-potential α^j, which will be called the fluid 4-potential to distinguish it from the thermal 4-potential a^j, and from b^j, which will be called the vorticity 4-potential.

In this way we can give a physical significance to the homogeneous part of the solution of (2.5), which is not uniquely determined by (2.5) alone, but is also dependent upon the initial conditions of the fluid specified at some past instant of time. The fact that this homogeneous part of α^j determines the intrinsic vorticity $2\mu\omega^{jk}$ is consistent with the fact, emphasized in the preceding paper, that in the generalized Larmor Theorem ω^{jk} is that part of the fluid rotation that is determined by the initial conditions of the fluid. This dependence on initial conditions will be made very transparent in the next section when it will be shown that $2\mu\omega^{jk}$ can be expressed in terms of two scalar functions that are constants of the fluid motion.

It is our objective to arrive at a scalar fluid equation of motion which does not involve v^j, and which has the form of a generalized Hamilton-Jacobi equation. Such an equation follows directly from (2.3a) and the velocity normalization condition

$$v_j v^j = c^2. \tag{2.7}$$

Taking the terms $(q/c)\,A^j$ and $(m/c)\,\alpha^j$ to the left side of (2.3a) and equating the norms of both sides, we arrive at the desired equation:

$$(\partial_j S + qA_j/c + m\alpha_j/c)(\partial^j S + qA^j/c + m\alpha^j/c) = (\mu c)^2. \tag{2.8}$$

An alternative form of this equation that makes the effect of α^j more explicit can be derived by taking only $(q/c)\,A^j$ to the left side of (2.3a) and equating norms of both sides:

$$(\partial_j S + q A_j/c)(\partial^j S + qA^j/c) = (\mu c)^2 + (m/c)^2 \alpha_j \alpha^j + 2(\mu m/c) v_j \alpha^j . \quad (2.9)$$

The gauge indeterminacy of α^j may be used to simplify this equation by choosing the gauge so that the condition

$$v_j \tilde{\alpha}^j = 0 \quad (2.10a)$$

is satisfied, which implies that

$$\tilde{\alpha}^j = \left[(\mathbf{v} \cdot \tilde{\boldsymbol{\alpha}} / c), \tilde{\boldsymbol{\alpha}}\right]. \quad (2.10b)$$

This choice of gauge (which is indicated by the overhead tilde) will be called the "space-like gauge." Given any α^j, the corresponding $\tilde{\alpha}^j$ is

$$\tilde{\alpha}^j = \alpha^j + \partial^j \psi \quad (2.11a)$$

where ψ is the solution of the scalar equation

$$d\psi/d\tau = - v_j \alpha^j . \quad (2.11b)$$

Note that from (2.10b) it follows that

$$\tilde{\alpha}_j \tilde{\alpha}^j = \left[(\mathbf{v} \cdot \tilde{\boldsymbol{\alpha}})^2/c^2 - \tilde{\boldsymbol{\alpha}} \cdot \tilde{\boldsymbol{\alpha}}\right] \leq 0. \quad (2.12)$$

That is, the norm of $\tilde{\alpha}^j$ is always negative.

When we choose the space-like gauge for α^j, the Hamilton-Jacobi equation in the form (2.9) becomes

$$(\partial_j \tilde{S} + q A_j/c)(\partial^j \tilde{S} + q A^j/c) = (\mu c)^2 + (m/c)^2 \tilde{\alpha}_j \tilde{\alpha}^j \quad (2.13)$$

where the tilde over the $\tilde{S}$ gives explicit recognition to the fact that the Characteristic Function $\tilde{S}$ which corresponds to $\tilde{\alpha}^j$ is in general different from the S which corresponds to α^j.

From (2.12) it is evident that the effect of $\tilde{\alpha}^j$ is to diminish the right side of (2.13), regardless of the nature of the nonisentropy, viscosity, or intrinsic vorticity described by $\tilde{\alpha}^j$.

As a preliminary to finding the nonrelativistic limit of (2.8), we introduce the Principal Function $\mathcal{S}$ defined as

$$\mathcal{S} = S + m V^j x_j \tag{2.14}$$

where x^j is the position 4-vector and V^j is an arbitrary constant 4-velocity which, if the system under study is closed and so conserves its total 4-momentum, may be identified with the velocity of the center of mass of the system. From (2.3a) and (2.14) it follows that

$$-\partial^j \mathcal{S} = (\mu v^j - m V^j) + (q/c) A^j + (m/c) \alpha^j \tag{2.15}$$

which, together with the normalization condition $V_j V^j = c^2$, yields the following form of the Hamilton-Jacobi equation:

$$\begin{gathered} V^j \partial_j \mathcal{S} - (\partial_j \mathcal{S} + q A_j/c + m\alpha_j/c)(\partial^j \mathcal{S} + q A^j/c + m\alpha^j/c)/2m \\ + q A_j V^j/c + m\alpha_j V^j/c + m(G + h)[1 + (G + h)/2c^2] = 0. \end{gathered} \tag{2.16}$$

Now we note that in the special frame of reference for which

$$V^j = (c, 0, 0, 0) \tag{2.17}$$

(2.16) may be written as follows:

$$\partial \mathcal{S}/\partial t + (\nabla \mathcal{S} + q\mathbf{A}/c + m\boldsymbol{\alpha}/c) \cdot (\nabla \mathcal{S} + q\mathbf{A}/c + m\boldsymbol{\alpha}/c)/2m$$
$$+ qA^0 + m\alpha^0 + m(G + h) = [(\partial \mathcal{S}/\partial t + qA^0 + m\alpha^0) - (mG + mh)^2]/2mc^2. \tag{2.18}$$

From (1.1) and the time-like component of (2.15) for the frame in which V^j has the form (2.17), we find

$$[(\partial \mathcal{S}/\partial t + qA^0 + m\alpha^0) - (mG + mh)^2]/2mc^2$$
$$= c^4[(\mu^* - m)^2 - (\mu - m)^2]/2mc^2 \tag{2.19}$$
$$\approx (1/2)mv^2[(1/2)mv^2 + 2(mG + mh)]/2mc^2.$$

Because this is obviously zero in the nonrelativistic limit, the right side of (2.18) vanishes in this limit, and we are left with an equation having the form of the familiar nonrelativistic single-particle Hamilton-Jacobi equation, except for the additional terms involving h, α^0, and $\boldsymbol{\alpha}$, which play roles analogous to G, A^0, and $\mathbf{A}$ respectively.

III. GENERALIZED CLEBSCH TRANSFORMATION

It was remarked in the preceding section that the vorticity 4-vector b^j must be regarded as a function of the initial conditions of the fluid, and that this could be made most evident by showing that b^j can be expressed, quite generally, in terms of two scalar functions that are constants of the fluid motion. Thus the specification of these scalars everywhere at some instant of past time suffices to determine them, and hence b^j, for all time.

As a preliminary to proving this statement, we first recall that in (4.12) of II it was pointed out that the orthogonality condition $\omega^{jk} v_k = 0$ requires that ω^{jk} have the following form:

$$(\omega^{10}, \omega^{20}, \omega^{30}) = -\Gamma \boldsymbol{\omega} \times \mathbf{v} / c, \tag{3.1a}$$

$$(\omega^{23}, \omega^{31}, \omega^{12}) \equiv \Gamma \boldsymbol{\omega}, \tag{3.1b}$$

where (3.1b) constitutes the definition of the 3-vector $\boldsymbol{\omega}$. It was further pointed out in (4.18) of II that a necessary consequence of the requirement that ω^{jk} have the form given in (3.1), and the requirement that $2\mu\omega^{jk}$ be expressible as the curl of a 4-vector (cf. (2.1) above), is that the 3-vector intrinsic vorticity $2\mu^*\boldsymbol{\omega}$ must satisfy the following two equations:

$$\nabla \cdot (2\mu^*\boldsymbol{\omega}) = 0; \tag{3.2a}$$

$$\partial(2\mu^*\boldsymbol{\omega})/\partial t = \nabla \times [\mathbf{v} \times (2\mu^*\boldsymbol{\omega})]. \tag{3.2b}$$

For the moment let us confine ourselves to a fixed frame of reference, and let us assume that $2\mu^*\boldsymbol{\omega}$ is continuously differentiable. It is well known[1] that such a vector, which by (3.2a) must be solenoidal, can always be expressed as follows in terms of two scalar (in the three-dimensional sense) functions, which we shall denote M and ϕ:

$$2\mu^*\boldsymbol{\omega} = (\nabla M) \times (\nabla \phi). \tag{3.3}$$

For given $2\mu^*\boldsymbol{\omega}$, this equation which follows from (3.2a), may be regarded as a condition on the spatial dependence of M and ϕ. Eq. (3.2b) represents a condition on both the spatial and time dependence of M and ϕ. This condition is equivalent to the following requirement:

$$\mathbf{v} \times (2\mu^*\boldsymbol{\omega}) = (\partial M/\partial t)\nabla\phi - (\partial\phi/\partial t)\nabla M. \tag{3.4}$$

That (3.4) is equivalent to (3.2b) is easily verified by taking the curl of (3.4) and using (3.3). The gradient of an arbitrary scalar function could have been added to the right side of (3.4) without affecting its equivalence to (3.2b), but because this scalar function is arbitrary, we are free to equate it to zero with no loss of generality.

We have shown that, because $2\mu\omega^{jk}$ is the curl of a 4-vector, $2\mu^*\boldsymbol{\omega}$ must be expressible in terms of two scalars M and ϕ that satisfy the condition (3.4). So far we have restricted ourselves to a single frame of reference, and have said nothing about the Lorentz transformation properties of $2\mu^*\boldsymbol{\omega}$. The fact that this 3-vector is the space-space part of an antisymmetric world tensor represents an additional restriction on the space-time dependence of $2\mu^*\boldsymbol{\omega}$, and hence of M and ϕ. This requirement on M and ϕ turns out to be just that they both be scalars not only in the three-dimensional sense, but also in the four-dimensional sense as well. That is, they must be Lorentz invariant. The validity of this statement follows from the fact that, making use of (3.1), we can write the six component equations of (3.3) and (3.4) in the following way, which has the formal appearance of a tensor equation:

$$\begin{aligned} 2\mu\omega^{jk} &= (\partial^j M)(\partial^k \phi) - (\partial^k M)(\partial^j \phi) \\ &= \partial^j (M\partial^k \phi) - \partial^k (M\partial^j \phi). \end{aligned} \tag{3.5}$$

If now we require that M and ϕ be Lorentz invariant, then the right side of (3.5) is a genuine world tensor, and consequently so is $2\mu\omega^{jk}$ on the left side, which guarantees that $2\mu^*\boldsymbol{\omega}$ has the correct Lorentz transformation properties.

There is a more abstract, but much more direct, way of arriving at (3.5). Referring to (4.18) of II, we note that the four component equations of (3.2) can be written in the following form:

$$\partial^j (2\mu\omega^{k\ell}) + \partial^k (2\mu\omega^{\ell j}) + \partial^\ell (2\mu\omega^{jk}) = 0. \tag{3.6}$$

The tensor $2\mu\omega^{jk}$ has six degrees of freedom (before we impose the orthogonality requirement (1.5)). But (3.6), which is equivalent to the four component equations of (3.2), removes four of these degrees of freedom. Thus any antisymmetric world-tensor with two degrees of freedom that automatically satisfies (3.6) is a perfectly acceptable and general way of representing $2\mu\omega^{jk}$. It is obvious that (3.5) satisfies these requirements, which is all the justification it needs.

Having derived (3.5) from the requirement that $2\mu\omega^{jk}$ be the curl of a 4-vector, we may now impose the orthogonality condition (1.5) in order to derive the equations of motion of M and ϕ. Contracting (3.5) with v_k and using (1.5) we find

$$(\partial^j \phi)\, dM/d\tau = (\partial^j M)\, d\phi/d\tau. \tag{3.7}$$

Constructing the four-dimensional cross-product of this equation with $\partial^k M$ and using (3.5), we have

$$2\mu\omega^{jk}\, dM/d\tau = 0. \tag{3.8}$$

If $2\mu\omega^{jk} = 0$, then from (3.5) it is evident that M and ϕ may be set equal to constants, and so may be considered constants of the motion. In the non-trivial case in which $2\mu\omega^{jk} \neq 0$, it follows from (3.8) that

$$dM/d\tau = 0, \tag{3.9}$$

which says that M is a constant of motion. Similarly, by constructing the cross-product of (3.7) with $\partial^k \phi$, we arrive at the conclusion that

$$d\phi/d\tau = 0. \tag{3.10}$$

Thus both M and ϕ are constants of motion.

Comparing (2.1) with (3.5), we arrive at the following expression for the vorticity 4-vector b^j in terms of the vorticity invariants M and ϕ:

$$(m/c)\, b^j = M \partial^j \phi, \tag{3.11}$$

where we note that the gradient of an arbitrary scalar function might have been added to the right side of (3.11), but because only the curl of b^j is of physical significance, this scalar function has been set equal to zero, with no loss of generality. It is easily verified that the equations (3.9)-(3.11) guarantee that the equation of motion for b^j given in (2.6) is automatically ~~verified~~ satisfied.

Substituting (3.11) into (2.2), we arrive at the generalized Clebsch Transformation:

$$\wp^j = -(\partial^j S + M \partial^j \phi); \tag{3.12a}$$

or

$$c\wp^0 = -(\partial S/\partial t + M \partial \phi/\partial t), \tag{3.12b}$$

$$\boldsymbol{\wp} = \nabla S + M \nabla \phi, \tag{3.12c}$$

where the definition of the generalized canonical momentum $\wp^j = (\wp^0, \boldsymbol{\wp})$ is given by (1.3). Unlike the familiar form[2] of the Clebsch Transformation, which expresses the fluid velocity in terms of three scalars, the generalized transformation refers to the generalized canonical momentum.

An important special case arises if $2\mu^*\omega = 0$. From (3.2b) it is evident that, if this condition holds everywhere in 3-space at any instant of time, then it will propagate itself for all time. From (3.3) or (3.5), it is evident that in terms of M and ϕ this zero-vorticity case corresponds to one of the following conditions: Either one or both of the scalars M and ϕ is everywhere constant for all time, or else M can be expressed as a function of ϕ. When any of these conditions holds, it is evident that the term $-M\partial^j\phi$ in (3.12a) can be expressed as the gradient of a scalar function, which function could then be absorbed into the function S. Thus the zero-vorticity case can be characterized by the condition

$$\text{(Potential Flow)} \qquad \mathfrak{p}^j = -\partial^j S, \qquad (3.13)$$

and so will be referred to as generalized potential flow. It is the self-perpetuating feature of this special type of flow that makes it important. This self-perpetuation is intuitively obvious in terms of M and ϕ: Because these are constant along every particle trajectory, if either of them is the same for all trajectories at any instant of time, then it must be the same for all time. The formal expression of this remark for the case of M, for example, follows from (3.9) which can be written

$$\partial M/\partial t = -\mathbf{v} \cdot \nabla M. \qquad (3.14)$$

It is obvious that if $\nabla M = 0$, then $\partial M/\partial t = 0$ and the spatial uniformity of M perpetuates itself for all time.

Our intuitive feeling for the physical meaning of M and ϕ can be helped by noting that, because ω is an angular velocity, it follows from (3.3) that the product $M\phi$ must have the dimensions of angular momentum. With no loss of generality we may assert that ϕ is dimensionless and that M has the dimensions of angular momentum. Thus M may be regarded as an angular momentum (per particle) – in some way related to the intrinsic vorticity – which is conserved along the particle trajectory. We may think of ϕ as the initial value at $t = 0$ of one of the particle coordinates (in dimensionless form), whose memory is retained by the particle for all time.

A simple example that illustrates these points is provided by the case of nonrelativistic rigid rotation of the fluid in the absence of any external fields. The rotation is thus characterized by a single angular velocity vector of magnitude Ω that is everywhere the same. It is easily shown[3] that in this case M turns out to be equal to the angular momentum of a particle about the rotation axis, i.e. $M = m\Omega r^2$ where r is the particle distance from the axis. ϕ turns out to be the value φ_0 of the particle's azimuthal coordinate at $t = 0$, i.e. $\phi = \varphi - \Omega t$, where the azimuthal angle φ is given by $\varphi = \Omega t + \varphi_0$.

If there were a uniform magnetic field parallel to the rotation axis, then the expression for ϕ would remain unchanged, but M would become $M = m(\Omega - \Omega_L) r^2 = \omega r^2$ where Ω_L is the Larmor rotation velocity produced by the magnetic field, and $\omega = \Omega - \Omega_L$ is just the magnitude of the intrinsic angular velocity. Thus M would be that part of the total particle angular momentum that is produced by the intrinsic angular velocity ω.

Note that in these examples the existence of a nonzero intrinsic angular velocity is associated with an M that is not spatially uniform. This is in fact always the case because, if $\nabla M = 0$, then from (3.3) it follows that $\boldsymbol{\omega} = 0$ and, according to (3.14), once $\nabla M = 0$, this condition will perpetuate itself. It is interesting to speculate [3] that there might exist a natural mechanism, namely turbulence, that would tend to make M uniform even if this were not originally the case. The argument goes as follows: Inasmuch as M is a constant of motion, it may be regarded as an intrinsic property of every small sample of the fluid, which is carried along with the fluid and characterizes it for all time. Because turbulence tends to mix the fluid, it is reasonable to expect that it might produce a diffusion of the nonuniformities in M, with the result that M would tend to become uniform throughout the fluid (to the extent that it is not impeded from doing this by imposed constraints and boundary conditions[3]).

Because uniform M implies the existence of generalized potential flow as defined by (3.13), the above speculation suggests that potential flow may be encountered very frequently in situations characterized by strong turbulence, i.e. by a very large Reynolds number.

IV. ADIABATIC FLOW

In the preceding section it has been shown that it is always possible to express the vorticity 4-potential b^j in terms of two scalar functions. It will now be shown that, in the case of adiabatic flow, it is possible to express the thermal 4-potential a^j in terms of two scalar functions, one of which is the specific entropy s.

Adiabatic flow is characterized by the fulfillment of the following three conditions:

$$\dot{s} \equiv ds/d\tau = 0; \qquad (4.1a)$$

$$\pi^j = 0; \tag{4.1b}$$

$$\eta^j = 0. \tag{4.1c}$$

From the physical point of view these conditions are not independent, because the first implies the other two. Thus, if its local entropy cannot change, the fluid cannot (reversibly) gain or lose heat, which implies that $\pi^j = 0$. As to (4.1c), which says that the viscous force must be zero, this follows from the observation that viscosity always causes entropy generation.

From the formal point of view, however, if we are not given the functional dependence of π^j and η^j on $\dot{s}$, the conditions (4.1b) and (4.1c) must be regarded as independent of (4.1a). Such a functional relation for π^j was in fact given in (5.7) of I:

$$\pi^j = m T_R^j \dot{s}/c \tag{4.2}$$

where T_R^j is the temperature 4-vector of the heat reservoir from which heat is reversibly transferred to the fluid. The formal condition guaranteeing reversibility of the transfer was given in (4.27a) of I:

$$T = v_j T_R^j/c. \tag{4.3}$$

(Following the notation change noted in II, we designate the fluid temperature in the fluid rest-frame by T rather than by $\overset{0}{T}$ as in I.) It is obvious that, given (4.2), the condition (4.1b) follows from (4.1a). Lacking an expression for η^j analogous to the one for π^j given in (4.2), we must regard (4.1c) as being formally independent of (4.1a). In any case, we assume that all three conditions of (4.1) are fulfilled.

Thus (1.4) may be written in the form

$$(\partial^j a^k - \partial^k a^j) v_k = -c T \partial^j s. \tag{4.4}$$

The solution of this equation can be expressed directly in terms of s and the temperature integral[4] $\mathfrak{I}$ which is defined by the relation

$$T \equiv d\mathfrak{I}/d\tau \equiv v_j \partial^j \mathfrak{I}. \tag{4.5}$$

It is easily verified that when (4.1a) holds, either of the following expressions for a^j satisfies (4.4):

$$a^j = -c s \partial^j \mathfrak{I}; \tag{4.6}$$

$$\tilde{a}^j = c \mathfrak{I} \partial^j s; \tag{4.7}$$

which proves our assertion that for adiabatic flow it is possible to express a^j in terms of two scalar functions, one of which is s.

Note that, because these two expressions for the thermal 4-potential differ only by the gradient of $c s \mathfrak{I}$, they are actually the same 4-potential with different choices for the gauge. Because of (4.1a), $\tilde{a}^j$ is obviously orthogonal to v_j:

$$v_j \tilde{a}^j = c \mathfrak{I} \dot{s} = 0. \tag{4.8}$$

Although this orthogonality is a formal advantage, the choice of gauge made in writing (4.6) has the intuitive advantage that it lends itself to an explanation of the physical significance of a^j, or more particularly, of the term $(m/c)\, a^j$ that appears in the expression for the generalized canonical momentum $\mathfrak{p}^j$ that is given in (1.3a).

Intuitive Interpretation of Generalized Canonical Momentum

This intuitive interpretation is based on an identification of $c\,\partial^j \mathfrak{Z}$ with the heat reservoir temperature 4-vector T^j_R, which is suggested by a comparison of (4.5) with (4.3). Thus we assert that T^j_R is defined by the relation

$$T^j_R \equiv c\,\partial^j \mathfrak{Z}. \tag{4.9a}$$

Because $T^j_R = T^0_R\,(1, \mathbf{v}_R/c)$ where T^0_R, the time-like component of T^j_R, is the reservoir temperature as seen in the observer's frame, and $\mathbf{v}_R$ is the reservoir 3-velocity, it is possible to write (4.9a) in the following form:

$$T^0_R \equiv \partial \mathfrak{Z}/\partial t; \tag{4.9b}$$

$$T^0_R\,\mathbf{v}_R/c^2 \equiv -\nabla \mathfrak{Z}. \tag{4.9c}$$

Whether or not the heat reservoir specified by T^j_R may actually be identified with whatever physically real reservoir that may happen to coexist in space with the fluid (but not interact with it for the case of adiabatic flow) depends on whether or not the temperature 4-vector of this real reservoir can be expressed as the gradient of a scalar. (Such a reservoir will henceforth be called a scalar reservoir.) A restriction of this kind can be expected to exclude many physically realizable reservoirs. Thus the intuitive argument given below, which is based on the identification made in (4.9), has physical significance only for this rather narrow class of scalar reservoirs.

Using (1.1), (4.6), and (4.9) in (1.3), we find

$$\begin{aligned} \not{p}^j &= \mu v^j + (q/c) A^j - (m/c) s T^j_R \\ &= m v^j + (m G/c^2) v^j + (q/c) A^j + m [(h/c^2) v^j - s T^j_R/c]; \end{aligned} \tag{4.10a}$$

or

$$\begin{aligned} c \not{p}^0 &= \mu^* c^2 + q A^0 - m s T^0_R \\ &= m^* c^2 + m^* G + q A^0 + (m^* h - m s T^0_R), \end{aligned} \tag{4.10b}$$

$$\begin{aligned} \boldsymbol{\not{p}} &= \mu^* \mathbf{v} + (q/c) \mathbf{A} - (m s T^0_R/c^2) \mathbf{v}_R \\ &= m^* \mathbf{v} + (m^* G/c^2) \mathbf{v} + (q/c) \mathbf{A} + [(m^* h/c^2) \mathbf{v} - (m s T^0_R/c^2) \mathbf{v}_R] . \end{aligned} \tag{4.10c}$$

Note that by contracting (4.10a) with v_j, we have

$$c \mathring{\not{p}}^0 = m c^2 + m G + q \mathring{A}^0 + m (h - s T), \tag{4.11}$$

where $\mathring{\not{p}}^0 = v_j \not{p}^j/c$ and $\mathring{A}^0 = v_j A^j/c$ are respectively the time-like components of $\not{p}^j$ and A^j as seen in the fluid rest-frame.

Thermodynamics is represented in (4.11) by the specific Gibbs function $g = h - sT$. From the intuitive point of view, this is not a surprising result. Each small sample of fluid, because it is constrained to have the same pressure and temperature as the surrounding fluid, seeks a thermodynamic equilibrium characterized by the constraint that virtual displacements from equilibrium must produce no change in either pressure or temperature. But the appropriate thermodynamic potential function in such a case is the Gibbs function. Thus it is not surprising that in (4.11), which gives the total rest-energy of a small sample of the fluid (on a per-particle basis), the thermodynamic energy should be represented by the Gibbs function.

It is possible to use the heat reservoir concept to give an alternative explanation of the fact that it is the Gibbs function that plays the role of thermodynamic potential. For simplicity, we shall first consider the case for which $v_R = v$, and shall work in the common rest-frame of the fluid and reservoir.

We first note that the specific enthalpy h, often called the *heat content* of the gas, is just the quantity of heat that would have to be injected into a sample of unit mass of the fluid in order to expand it against the pressure $\mathcal{P}$ of the surrounding fluid (which, assuming the sample to be very small, would remain constant during the expansion) while bringing its temperature from absolute zero up to the temperature T of the surrounding fluid. This is obvious from the relation $h = u + \mathcal{P}\mathring{\mathcal{V}}$ where u and $\mathring{\mathcal{V}}$ are respectively the internal energy and volume of unit mass of the fluid. Of the total heat energy h injected into the sample of fluid, the amount $\mathcal{P}\mathring{\mathcal{V}}$ is converted into the mechanical work necessary to push back the surrounding fluid as the sample of unit mass is expanded from zero volume to its final volume $\mathring{\mathcal{V}}$, and the amount u remains in the sample as its internal energy. The energy u obviously resides within the fluid sample. The work $\mathcal{P}\mathring{\mathcal{V}}$, however, was performed on the fluid surrounding the sample, and so is a potential energy that is stored *outside* of the sample. Because, however, the heat energy h was injected into the space occupied by the sample in expanding it to its final volume $\mathring{\mathcal{V}}$, and could in principle be extracted from this region of space by reversing the process, it is legitimate to associate the energy h with the *region of space occupied by the fluid sample*. But now we recall that, from the point of view of the heat reservoir model, this region of space is occupied by both the fluid and the reservoir, which are regarded as separate systems. The question thus arises as to how much of the energy h should be associated with the fluid, and how much should be assigned to the reservoir. It is easy to see that this latter energy is just s T (per unit mass of the fluid). The reason for this is that, if an external reservoir at absolute zero of temperature were available, a Carnot engine could extract the energy s T from the fluid reservoir. Thus the energy to be associated with the fluid is $h - sT$, which is just the Gibbs function g. It is to be expected that, in a dynamical theory concentrating on the fluid, rather than on the heat reservoir associated with it, the thermodynamical energy is represented by the fluid energy $g = h - s\,T$ rather than by the total energy h. The relation (4.11) confirms this expectation.

This argument is easily generalized to the case for which $\mathbf{v}_R \neq \mathbf{v}$. In this case, in the fluid rest-frame we may still associate the energy h with the region occupied by unit rest-mass of the fluid. In the observer's frame, however, this energy becomes Γh, and necessarily associated with it is the momentum $(\Gamma h/c^2)\mathbf{v}$. Thus we associate the energy-momentum 4-vector $(h/c^2)\, v^j = [(\Gamma h/c), (\Gamma h/c^2)\,\mathbf{v}]$ with the volume occupied by unit rest-mass of the fluid.

Next, referring to (4.10) of I, we note that, if an external reservoir at absolute zero of temperature were available, a Carnot engine could extract from the fluid reservoir (per unit rest-mass of fluid) an amount of energy and momentum given by the 4-vector $(s\,T^j_R/c) = [(s\,T^0_R/c), (s\,T^0/c^2)\,\mathbf{v}_R]$, where we have used the fact that $T^j_R = T^0_R(1, \mathbf{v}_R/c)$. Thus the energy-momentum to be associated with the fluid above, on a per-particle basis, is $m\,[(h/c^2)\,v^j - s\,T^j_R/c]$. This is just the expression that in (4.10a) represents the thermodynamical contribution to the generalized canonical momentum $\mathfrak{p}^j$.

Finally, it should be noted that we could have inserted $\tilde{a}^j = c\,\mathfrak{I}\,\partial^j s$ rather than $a^j = -c\,s\,\partial^j \mathfrak{I}$ into (1.3), in which case, instead of (4.10a), we would have arrived at the following expression for the generalized canonical momentum[5]:

$$\tilde{\mathfrak{p}}^j = \mu v^j + (q/c)\,A^j + m\,\mathfrak{I}\,\partial^j s. \tag{4.12}$$

Because a^j and $\tilde{a}^j$ differ only by a choice of gauge, which can have no physical significance, whether we use $\mathfrak{p}^j$ or $\tilde{\mathfrak{p}}^j$ in the overall formalism can lead to no physically observable differences. Because, as indicated in (4.8), $\tilde{a}^j$ is orthogonal to v_j, it turns out that $\tilde{\mathfrak{p}}^j$ has certain formal advantages over $\mathfrak{p}^j$. The latter, however, has the intuitive advantage that it lends itself to the simple physical interpretation discussed above.

Hamilton-Jacobi Equation

The generalized Hamilton-Jacobi equation was given in (2.8) in terms of the fluid 4-potential α^j and the corresponding Characteristic Function S, and also in (2.13) in terms of $\tilde{\alpha}^j$ and the corresponding $\tilde{S}$. The 4-vectors α^j and $\tilde{\alpha}^j$ differed only in choice of gauge. In the case of $\tilde{\alpha}^j$, the gauge (the space-like gauge) was chosen so that the orthogonality condition $v_j\,\tilde{\alpha}^j = 0$ was fulfilled. As stated in (2.12), a necessary consequence of this choice is the fact that the norm of $\tilde{\alpha}^j$ is always negative, i.e. $\tilde{\alpha}_j\,\tilde{\alpha}^j \leq 0$. In the case of adiabatic flow, it is evident from (2.4), (4.6), (4.7), and (3.11) that α^j and $\tilde{\alpha}^j$ may be written as follows:

$$(m/c)\,\alpha^j = -\,m\,s\,\partial^j \mathfrak{I} + M\,\partial^j \phi; \tag{4.13a}$$

$$(m/c)\,\tilde{\alpha}^j = m\,\mathfrak{I}\,\partial^j s - \phi\,\partial^j M. \tag{4.13b}$$

Obviously, these two forms of the fluid 4-potential differ only by the gradient of the scalar function $(ms\,\mathfrak{I} - M\phi)$. Because of (3.9) and (4.1a) it follows that $\tilde{\alpha}^j$ satisfies the orthogonality condition $v_j\,\tilde{\alpha}^j = 0$. The form of the fluid 4-potential given in (4.13a) is the sum of the expressions for a^j and b^j that we have used for intuitive purposes. The form $\tilde{\alpha}^j$ given in (4.13b), however, has certain formal advantages arising from the orthogonality of $\tilde{\alpha}^j$ and v_j. Moreover, it has the advantage that it exhibits more explicitly than α^j the fact that if s and M become everywhere constant, i.e. if $\partial^j s = \partial^j M = 0$, then the fluid 4-potential drops out of the picture. (For this case $\tilde{\alpha}^j$ vanishes, whereas α^j becomes the gradient of a scalar.) For these reasons we shall restrict ourselves to $\tilde{\alpha}^j$ and the corresponding Characteristic Function $\tilde{S}$.

Using (4.13b) in (2.13), we see that the Hamilton-Jacobi equation for adiabatic flow becomes

$$(\partial_j \tilde{S} + qA_j/c)\,(\partial^j \tilde{S} + qA^j/c) = (\mu c)^2 + (m\,\mathfrak{I}\,\partial_j s - \phi\,\partial_j M)\,(m\,\mathfrak{I}\,\partial^j s - \phi\,\partial^j M). \tag{4.14}$$

Because of the fact that $\tilde{\alpha}_j\,\tilde{\alpha}^j \le 0$, the effect of nonuniformity in s and M is always to diminish the right side of (4.14).

We now augment (4.14) with the equations of motion of $\mathfrak{I}$, s, ϕ, and M, and the continuity equation (1.9), which is the equation of motion of ρ:

$$d\mathfrak{I}/d\tau = T; \tag{4.15}$$

$$ds/d\tau = 0; \tag{4.16}$$

$$d\phi/d\tau = 0; \tag{4.17}$$

$$dM/d\tau = 0; \tag{4.18}$$

$$d\rho/d\tau = -\rho\,\partial_j v^j. \tag{4.19}$$

The 4-velocity v^j which appears explicitly in (4.19) and implicitly in the time differentiation operator $d/d\tau = v^j \partial_j$ is given by (2.3a):

$$v^j = -\mu^{-1}\,[\partial^j \tilde{S} + (q/c)\,A^j + m\mathfrak{I}\,\partial^j s - \phi\,\partial^j M]. \tag{4.20}$$

We regard G (which appears in μ) and A^j as given space-time functions. If the thermodynamic equation of state of the fluid is given, then h (which appears in μ) and T (which appears in (4.15)) are known functions of ρ and s. (Knowing this functional dependence of T, we may regard (4.15) as the fluid equation of state in terms of $\mathfrak{I}$.) Thus we have expressed the problem in terms of the six scalars $\tilde{S}$, $\mathfrak{I}$, s, M, ϕ, and ρ which are determined by the six scalar equations (4.14)-(4.19). This could be called the <u>scalar formulation</u> of the dynamical problem to distinguish it from the usual formulation based on the 4-vector Euler equation.

The remark that, once the fluid equation of state is known, h and T may be regarded as known functions of ρ and s can be illustrated by the case of a perfect gas. In this case

(Perfect Gas)

$$h = c_p\,T, \tag{4.21a}$$

where

$$T = K\,\rho^{\gamma-1}\,e^{\gamma s/c_p}, \tag{4.21b}$$

where K is an arbitrary constant, c_p is the constant-pressure specific heat, and

$\gamma = c_p/c_v$ is the ratio of constant-pressure and constant-volume specific heats. Using (4.21b) and (4.19), it is easily shown that for adiabatic flow of a perfect gas (4.15) may be replaced by the following equation of motion for $\mathfrak{Z}$, which may also be regarded as the fluid equation of state in terms of $\mathfrak{Z}$:

$$\text{(Perfect Gas)} \qquad d^2\mathfrak{Z}/d\tau^2 + [(\gamma - 1)\, \partial_j\, v^j]\, d\,\mathfrak{Z}/d\tau = 0. \tag{4.22}$$

Three important special cases should be noted. The first of these is isentropic flow which arises if $\partial_j\, s = 0$. In this case s drops completely out of the formalism. The second case is potential flow which, as noted in Section III, arises if either M or ϕ becomes constant, or if either can be expressed as a function of the other. In this case both M and ϕ may be dropped from the formalism. The third special case, isentropic potential flow is simply the simultaneous fulfillment of the above two conditions. In this case the six unknowns of the problem reduce to the three unknowns $\tilde{S}$, $\mathfrak{Z}$, and ρ. Because $\mathfrak{Z}$ appears only in h, it can be replaced by h, so that the three unknowns could be taken to be $\tilde{S}$, h, and ρ. For a perfect gas, the system of equations for isentropic potential flow becomes

(Isentropic Potential Flow)

$$(c\,\partial_j\,\tilde{S} + qA_j)(c\,\partial^j\tilde{S} + qA^j) = (mc^2 + mG + mh)^2\,; \tag{4.23a}$$

$$d\rho/d\tau = -\rho\,\partial_j\, v^j\,; \tag{4.23b}$$

$$\text{(Perfect gas)}\ dh/d\tau = -(\gamma - 1)\,h\,\partial_j\, v^j\,; \tag{4.23c}$$

where

$$v^j/c = -(c\,\partial^j\,\tilde{S} + qA^j)/(mc^2 + mG + mh). \tag{4.23d}$$

Finally, it should be noted that, although in the system of equations (4.14)-(4.19) we regard ρ as one of the independent variables, it is in fact possible to regard only the five variables $\tilde{S}$, $\mathfrak{Z}$, s, ϕ, and M as constituting the complete set of independent variables. From this point of view, the two independent thermodynamic variables are taken to be $\mathfrak{Z}$ and s, and then h and T are regarded as known functions of these (assuming that the thermodynamic properties of the fluid are completely known). The thermodynamic equation of state in terms of $\mathfrak{Z}$ is given by (4.15), where T is regarded as a known function of $\mathfrak{Z}$ and s. This equation would then involve the thermodynamic variables $\mathfrak{Z}$ and s, but not ρ. (The fact that this equation of state involves only two thermodynamic variables is a consequence of the fact that the flow is constrained to be adiabatic.) From (4.22), we see in fact that for a perfect gas it does not even involve s (except for the implicit dependence involved in v^j that is given by (4.20)). Once the problem has been solved and $\tilde{S}$, $\mathfrak{Z}$, s, ϕ, and M are known space-time functions, ρ can be found from (4.19).

V. FOUR-RESERVOIR MODEL OF NONADIABATIC FLOW

In the case of nonadiabatic flow, all entropy-dependent effects (including the effects of π^j and η^j which are always associated with entropy generation) are represented by the thermal 4-potential a^j whose dynamical role is analogous to that of the electromagnetic 4-potential. The "field equation" for a^j was given in (1.4), and may be written

$$(m/c)(\partial^j a^k - \partial^k a^j)v_k + mT\,\partial^j s = \pi^j + \eta^j \tag{5.1}$$

where from (4.2) and (4.3)

$$v_j \; \pi^j = m\,T\,\dot{s} \tag{5.2a}$$

and from (4.3) of II and (5.4) of I

$$v_j \; \eta^j = 0. \tag{5.2b}$$

The 4-vectors π^j and η^j are regarded as known functions of ρ, v^j, s, and T that satisfy the conditions (5.2). The component equations of (5.1) are the four equations whose solution yields the space-time dependence of the four components of a^j.

In addition to this 4-vector treatment of the entropy-dependent forces in the case of nonadiabatic flow, it would be desirable to have a scalar formulation that was a generalization of the one given in the preceding section for the case of adiabatic flow. We arrive at such a generalization by assigning to a^j the functional form

$$a^j = -c \sum_{N=1}^{4} s_{(N)} \, \partial^j \, \mathfrak{Z}_{(N)} \tag{5.3}$$

where

$$\sum_{N=1}^{4} s_{(N)} = s \tag{5.4a}$$

and

$$\dot{\mathfrak{Z}}_{(1)} = \dot{\mathfrak{Z}}_{(2)} = \dot{\mathfrak{Z}}_{(3)} = \dot{\mathfrak{Z}}_{(4)} \tag{5.4b}$$

where $\dot{\mathfrak{Z}}_{(N)} \equiv d\mathfrak{Z}_{(N)}/d\tau$. The form assigned to a^j in (5.3) is an obvious generalization of the form given in (4.6) in the case of adiabatic flow. The conditions given in (5.4) represent four constraints on the eight variables $s_{(N)}$ and $\mathfrak{Z}_{(N)}$

so the right side of (5.3) has four degrees of freedom which suffice to express any arbitrarily given a^j. We need four more equations in addition to (5.4) in order to arrive at a complete specification of the eight variables. The additional four equations are supplied by the following 4-vector equation:

$$m \sum_{N=1}^{4} \dot{s}_{(N)} \, \partial^j \mathfrak{Z}_{(N)} = \pi^j + \eta^j. \tag{5.5}$$

In order to demonstrate that fulfillment of the conditions (5.4) and (5.5) does indeed guarantee that (5.3) satisfies (5.1), we first note that by contracting (5.5) with v_j and using (5.2) and (5.4), we arrive at the following result:

$$\dot{\mathfrak{Z}}_{(N)} \equiv d\mathfrak{Z}_{(N)}/d\tau = T; \; N = 1, 2, 3, 4. \tag{5.6}$$

It is now easily verified that when (5.4) and (5.5) (which implies (5.6)) are satisfied, then (5.3) does indeed satisfy (5.1). Thus we have replaced the four variables a^j and the corresponding four equations (5.1) with the eight scalar functions $s_{(N)}$ and $\mathfrak{Z}_{(N)}$ and the corresponding eight equations (5.4) and (5.5).

In terms of number of variables, this replacement is obviously not advantageous. The fact that the new variables are scalars rather than the components of a 4-vector can, however, be an important advantage. Moreover, it is possible to give the scalar formulation a simple physical interpretation in terms of a thermal interaction of the fluid with four separate heat reservoirs. The basis of this interpretation is the fact that, because of (5.6), it is possible to identify each of the four gradients $c\partial^j \mathfrak{Z}_{(N)}$ with the temperature 4-vector $T^j_{(N)}$ of a scalar heat reservoir in thermal interaction with the fluid:

$$c \, \partial^j \mathfrak{Z}_{(N)} \equiv T^j_{(N)}. \tag{5.7}$$

This is an obvious generalization of (4.9a). The 4-vector equations (5.3) and (5.5) may now be written as follows:

$$(m/c)\, a^j = -m \sum_{N=1}^{4} s_{(N)}\, T^j_{(N)}; \tag{5.8}$$

$$(m/c) \sum_{N=1}^{4} \dot{s}_{(N)}\, T^j_{(N)} = \pi^j + \eta^j. \tag{5.9}$$

There is a simple physical interpretation of (5.9). Referring to (4.22) of I, we note that $m\,\dot{s}_{(N)}\, T^j_{(N)}$ may be interpreted as the energy-momentum (per particle) delivered by the Nth heat reservoir per unit time to the fluid, and $\dot{s}_{(N)}$ is that part of $\dot{s}$, the total fluid entropy increase per unit time, that results from the absorption of heat from the Nth reservoir. Note that it is even possible to account for a purely space-like viscous force on the right side of (5.9) (i.e. $\pi^j = 0$) in terms of this four-reservoir model. In such a case, for example, we might have two reservoirs moving in opposite directions, one absorbing and the other rejecting heat in such a way that no net heat energy is delivered to the fluid (in its own rest-frame), but a net momentum per unit time (i.e. force) would be delivered. This force would be the viscous force. In such a case we would have $\dot{s}_{(1)} > 0$, $\dot{s}_{(2)} < 0$, and $\dot{s} = \dot{s}_{(1)} + \dot{s}_{(2)} = 0$, so the fluid flow would actually be adiabatic even though a viscous force was present. (Adiabatic flow of a viscid fluid is, of course, a mathematical idealization in that physically viscosity is always accompanied by entropy generation.)

Substituting (5.8) into (1.3a), we arrive at the following expression for the generalized canonical momentum in the case of nonadiabatic flow:

$$\mathfrak{p}^j = \mu v^j + (q/c) A^j - m \sum_{N=1}^{4} s_{(N)} T^j_{(N)}. \tag{5.10}$$

Referring to the interpretive discussion given in the preceding section, we note that $m s_{(N)} T^j_{(N)}$ may be interpreted as the energy-momentum (per particle) contained in the Nth reservoir, which could in principle be converted into mechanical form by a Carnot engine operating between the Nth reservoir and a cold reservoir at absolute zero of temperature. By subtracting from the total energy-momentum $\mu v^j + (q/c) A^j$ that part, $m \Sigma s_{(N)} T^j_{(N)}$, which should be associated with the four reservoirs, we are left with the part $\mathfrak{p}^j$, which should be associated with the fluid.

Although the above formulation of the nonadiabatic flow problem lends itself most readily to physical interpretation, there exists an alternative formulation that has certain formal advantages. We arrive at this alternative by writing $\mathfrak{T}_{(N)}$ as

$$\mathfrak{T}_{(N)} = \mathfrak{T}' + F'_{(N)} \tag{5.11a}$$

where

$$\dot{\mathfrak{T}}' \equiv d\mathfrak{T}'/d\tau = T \tag{5.11b}$$

and

$$\dot{F}'_{(N)} \equiv dF'_{(N)}/d\tau = 0, \tag{5.11c}$$

where N = 1, 2, 3, 4. The fulfillment of these conditions will automatically guarantee the fulfillment of (5.4b) and (5.6). Substituting (5.11) into (5.3) and using (5.4a), we may write

$$a^j = -c \left[s\, \partial^j \mathfrak{T} + \sum_{N=1}^{3} s_{(N)}\, \partial^j F_{(N)} \right] \tag{5.12}$$

where

$$\mathfrak{Z} = \mathfrak{Z}' + F'_{(4)}; \quad F_{(N)} = F'_{(N)} - F'_{(4)}, \quad (N = 1, 2, 3,). \tag{5.13}$$

The conditions on $\mathfrak{Z}$ and $F_{(N)}$ that correspond to (5.11) are

$$\dot{\mathfrak{Z}} = T; \tag{5.14a}$$

$$\dot{F}_{(1)} = \dot{F}_{(2)} = \dot{F}_{(3)} = 0. \tag{5.14b}$$

These are four conditions on the eight variables $s, \mathfrak{Z}, F_{(N)}$, and $s_{(N)}$ $(N = 1,2,3)$. The remaining necessary four equations are provided by the following 4-vector equation:

$$m \sum_{N=1}^{3} \dot{s}_{(N)} \partial^j F_{(N)} = \pi^j + \eta^j - m \dot{s} \partial^j \mathfrak{Z}. \tag{5.15}$$

It is easily verified that if (5.14) and (5.15) are satisfied then (5.12) satisfies (5.1).

The condition (5.14a) specifies $\mathfrak{Z}$ only to within an arbitrary additive function that is a constant of motion. We could use this freedom to specify $\mathfrak{Z}$ so that $m \dot{s} \partial^j \mathfrak{Z}$ is the closest possible approximation to $\pi^j + \eta^j$. Then we would interpret the left side of (5.15) as expressing that part of $\pi^j + \eta^j$ that is not describable in terms of a single scalar reservoir model. The formal advantage of (5.12) over (5.3) is that $\pi^j + \eta^j$ is described in terms of three constants of motion $F_{(N)}$ and corresponding entropy contributions $s_{(N)}$ $(N = 1, 2, 3)$ upon which no constraint of the kind given in (5.4a) is imposed. Moreover, the fluid equation of state can be written simply in terms of ρ, $\dot{\mathfrak{Z}}$, and s alone, making it unnecessary to use all four entropy contributions $s_{(N)}$ as would be the case for the formulation based on (5.3).

VI. CONCLUSIONS

Use of the generalized Hamilton-Jacobi equation for a charged fluid in the presence of given electromagnetic and gravitational fields allows the problem to be formulated in terms of a set of independent dynamical and thermodynamical variables that are all scalars. Such a scalar formulation is possible even when viscosity and heat injection into the fluid are taken into account.

REFERENCES

1. See, for example, L. Brand, Vector and Tensor Analysis (Wiley, New York, 1947), p. 227.

2. For discussions of the familiar nonrelativistic Clebsch Transformation of the fluid velocity, see H. Lamb, Hydrodynamics (Cambridge, 1932) 6th ed., p. 248; H. Bateman, Partial Differential Equations of Mathematical Physics (Cambridge, 1932), p. 164; and J. Serrin, Encyclopedia of Physics (Springer, Berlin, 1959) Vol. VIII/1, pp. 169-171.

3. This has been discussed in somewhat greater detail elsewhere: L. A. Schmid, Phys. of Fluids 9 (1966).

4. The elimination of the temperature T in favor of its time integral $\mathfrak{T}$ appears already in Laue's Relativity Text (first edition, 1911): M. von Laue, Die Relativitätstheorie (Vieweg, Braunschweig, 1961) 7th ed., Vol. I, § 23f, p. 181. Laue attributes the first use of the temperature integral (in 1886) to Helmholtz: H. von Helmholtz, Wissenschaftliche Abhandlungen (Barth, Leipzig, 1895) Vol. III, p. 203-248. See especially § 2III, p. 225-226. Helmholtz's use of

the temperature integral was, however, less explicit than Laue's and was not, of course, in the context of a relativistic theory.

In the context of nonrelativistic fluid dynamics, the following authors have made more recent use of the temperature integral: A. H. Taub, Proc. Symposia in Applied Math. 1, 148 (1948); J. W. Herivel, Proc. Cambridge Phil. Soc. 51, 344 (1955); J. Serrin, Encyclopedia of Physics (Springer, Berlin, 1959), Vol. VIII/1, p. 171; C. Eckart, Phys. Fluids 3, 421 (1960).

5. This form of the generalized canonical momentum has already been derived in a nonrelativistic context by C. Eckart, Phys. Fluids 3, 421 (1960).