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LINEAR THEORY OF IMPULSIVE VELOCITY CORRECTIONS FOR SPACE MISSION GUIDANCE

by Luigi S. Cicolani

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SUMMARY

One aspect of midcourse guidance is the determination of the guidance correction, usually specified by an equation that gives the required velocity change as a function of the current state of the vehicle. With the assumptions that impulsive velocity corrections are made and that the equations of motion can be linearized about a reference orbit, the required midcourse velocity correction will be a linear function of the deviation of the vehicle state from the reference state.

Such linear guidance equations form a particularly simple class which is analyzed in this paper from the viewpoints of linear vector spaces and the theory of linear equations. Existence conditions and two formulations of the guidance equation from the mission constraints are given. Properties of form and time-invariant properties of the guidance equations and relations among the basic notions of the theory are also discussed. Finally, a number of examples from the recent literature are examined.

INTRODUCTION

A mission must satisfy certain objectives which, in turn, specify constraints on the vehicle's orbit. The launch system attempts to place the vehicle on a reference orbit that satisfies the constraints, but misses this objective slightly. For lunar and interplanetary missions the injection errors will usually be unacceptable for mission success and a midcourse navigation and guidance system and one or more midcourse corrections will be required. The navigation system estimates the vehicle state from observations of stars, planets, or the vehicle, and the vehicle guidance system then attempts to correct to a satisfactory orbit (one which meets the mission constraints on orbital motion) by means of a change of velocity. Because the actual state is imperfectly known and the correction imperfectly made, the vehicle is not yet on a satisfactory orbit and further observations and corrections may be necessary.

Only one element of the midcourse guidance and navigation process will be examined here - the equation which gives the required velocity correction as a function of current vehicle′ state.

In the study of midcourse guidance for the lunar and interplanetary missions, it has frequently been adequate to assume that velocity corrections are impulsive and departures from a reference orbit are sufficiently small to allow the equations of motion to be linearized about the reference orbit. The guidance equations that result from these assumptions give the velocity correction as a linear function of the deviation of the current vehicle state from the reference state. Guidance equations of this type will be termed linear impulsive guidance laws herein. A number of such guidance laws designed to satisfy various mission constraints have been reported (e.g., refs. 1-5). All of these examples of guidance laws show some similarity of form, suggesting that they are mathematically related. It is the object of this paper to determine the general properties and logical structure of the class of linear impulsive guidance laws.

The analysis adopts two approaches; in the first we consider the state space. The midcourse velocity correction is an attempt to correct some arbitrary state to a satisfactory state, that is, to a state which defines an orbit that satisfies the mission constraints. Linear impulsive guidance laws occur when the satisfactory states are, collectively, certain types of subspaces of the state space, and the guidance law can be formulated from any basis of the satisfactory states. Further, such guidance laws have properties of form which depend on the dimension of the set of satisfactory states and are invariant with time. The set of satisfactory states is, in turn, defined by the mission constraints on orbital motion, and a basis of the satisfactory states can be generated by a sufficient number of derivatives of the state, evaluated on the reference orbits, with respect to appropriate orbital parameters.

The second approach utilizes the mission constraints directly; from the linearized relation between the constraint parameters and the current state, the general solution for the velocity correction which satisfies the constraints can be obtained. This construction gives the guidance law from derivatives of the constraint parameters with respect to the Cartesian components of the current vehicle state.

Finally, the theory is illustrated with an examination of the recent literature on lunar and interplanetary midcourse guidance.

The first construction is carried out from the viewpoint of linear vector spaces, but only elementary principles from that subject are used (cf. ref. 6), A convenient notion in the second approach is the pseudo-inverse of a matrix (refs. 7, 8). The two constructions give equivalent solutions for the guidance laws, although the application of the second construction will usually require less labor.

SYMBOLS

$A(t)$ various matrices

$A(t_2,t_1)$ transition matrix, relating the deviation state at t_2 to the deviation at t_1

$A_j(t_2,t_1)$	submatrices of $A(t_2,t_1)$, $j = 1, \ldots, 4$
A_Z	azimuth angle
$B(t)$	various matrices
$B_1(t_2,t_1)$, $B_2(t_2,t_1)$, B	matrices relating states in $S(t_2)$ to states in $S(t_1)$
D	declination angle
$F_1(t)$	matrix relating state deviations to deviations in orbital parameters
$G(t)$, $G_1(t)$, $G_2(t)$	guidance law matrix and its submatrices
I	identity matrix
$N(t)$	state space; set of all deviations of position and velocity from a reference position and velocity at time, t
$\bar{n}$	unit vector normal to orbital plane
p	m-tuple of independent orbital parameters $(p_1, \ldots, p_m)$
RA	right ascension
$R(t)$	set of all position deviations occupied by the satisfactory orbits at time, t
$\bar{r}$	position vector
$S(t)$	set of satisfactory deviation states
s	element of $S(t)$
t	time
t_F	reference arrival time
$\bar{u}$	unit vector
$V(t)$, $W(t)$	matrices appearing in construction of guidance laws
$\bar{v}$	velocity vector; also used as a unit vector

$\overline{w}$	vector appearing in construction of guidance law, otherwise used as a unit vector
$X(t)$	vehicle position and velocity, in Cartesian component, relative to a central body
$x(t)$	state vector; six Cartesian components of position and velocity deviations from a reference orbit; sometimes written $\begin{pmatrix}\delta\overline{r}\\ \delta\overline{v}\end{pmatrix}$
α	earth's rotation rate
$\alpha_1, \ldots, \alpha_m$	specified values of parameters $p_1, \ldots, p_m$ of reference orbit
β_1,β_2	constants relating entry flight time and entry range angle
$\delta(\ )$	small deviation from reference value of ( )
μ	gravitational constant of central body
φ	entry range angle
$\underline{\psi},\underline{\psi}_p$	matrices occurring in landing site guidance law
$\overline{(\ )}$	vectors
$(\ )^T$	transpose
$\{\ \}$	set of objects defined by contents of { }

DEFINITIONS

It is useful to begin with definitions of the primitive notions underlying the theory.

The satisfactory orbits are all those that meet the mission constraints on vehicle motion. The reference orbit is one member of the set of satisfactory orbits; the choice of reference orbit may be based on one or more factors such as fuel consumption, flight time, etc.

Since the launch system attempts to inject the vehicle on the reference orbit, it is frequently adequate to assume that the orbits of interest in the midcourse guidance problem are sufficiently close to the reference orbit to allow the equations of motion to be linearized about the reference. With this assumption, the theory need consider only deviations from the reference state.

The state deviation, $x(t)$, refers to a column of the six Cartesian components of position and velocity deviations from the reference state at time, t. It will sometimes be replaced by the convenient form $(\delta\bar{r}, \delta\bar{v})$, in which the position and velocity deviations are stated explicitly. There is no loss of generality in using position and velocity deviations to define the state since these are related to the deviations of any other six independent parameters by means of appropriate transformations. Further, this definition of the deviation state is especially suited to the development.

The state deviations at two different times on an orbit are related by

$$x(t_2) = A(t_2,t_1)x(t_1) \tag{1}$$

where $A(t_2,t_1)$ is the transition matrix obtained by linearizing the equations of motion about the reference orbit. Its columns are derivatives of the state coordinates at t_2 with respect to the state coordinates at t_1 evaluated on the reference orbit.

Certain sets of states pertinent to the discussion are next defined.

The set of all deviation states at time t forms a six-dimensional linear vector space $N(t)$ called the state space. Each element of $N(t)$, when added to the reference state at t, defines an orbit. The state space $N(t)$ is everywhere equivalent to the set of all sextuples and, hence, includes some arbitrarily large deviations from the reference state. Although equation (1) inadequately represents the actual motion when the deviations are large, the theory is consistent based on the model of equation (1). In a practical context this means the linearized theory may be applied only in guidance problems in which equation (1) adequately describes the orbits that occur. Such a restriction is, of course, implicit in the use of linearized equations of motion.

The set of satisfactory states $S(t)$ at time t is the set of all those deviation states in $N(t)$ which define satisfactory orbits. The guidance process is an attempt to correct an arbitrary state in $N(t)$ to a state in $S(t)$. The set of orbits defined by $S(t)$ is fixed at all times; no new satisfactory orbits are added or others deleted as the mission proceeds.

Lastly, the set $R(t)$ is the set of position deviations occupied at time t by the satisfactory orbits. It is evident that any arbitrary state $(\delta\bar{r},\delta\bar{v})$ can be corrected to a state in $S(t)$ by means of an impulsive velocity correction only if $\delta\bar{r}$ is in $R(t)$.

ANALYSIS

Linear Impulsive Guidance Laws

In midcourse guidance of space missions, it has frequently been assumed that the velocity corrections are impulsive, and that the orbits of interest can be described by linearized equations of motion. With these assumptions, the relationship between the state deviation and the velocity correction, that is, the guidance law, is a linear one and is defined over the entire state space. The class of guidance laws with these general properties may be termed linear impulsive guidance laws.

One approach to the analysis is axiomatic; we begin with a definition of the class of linear impulsive guidance laws in terms of the elementary properties it should possess, and then derive other properties of this class as a consequence of the definition.

Definition.- A linear impulsive guidance law is a rule which specifies a unique instantaneous change of velocity, linearly related to the present state deviation, such that the final state is satisfactory. Such a correction is specified for every state; if the present state is satisfactory, then no correction is made.

Suppose such a guidance law is defined at time t, and $x(t)$ is any state deviation. The change of state specified by the guidance law is linearly related to $x(t)$ for every $x(t)$:

$$\Delta x = G(t)x(t)$$

The 6×6 matrix $G(t)$ will be termed the guidance law matrix at t. Since only the velocity is to be changed, then $G(t)$ has the form

$$G(t) = \begin{bmatrix} 0 & 0 \\ G_1(t) & G_2(t) \end{bmatrix}$$

where $G(t)$ has been partitioned into 3×3 submatrices. The velocity correction can also be written as

$$\Delta\overline{v} = G_1(t)\delta\overline{r}(t) + G_2(t)\delta\overline{v}(t)$$

After a correction has been made, the final state is

$$x_f(t) = [I + G(t)]x(t)$$

If the present state is satisfactory, then no correction is required; that is,

$$G(t)x(t) = 0$$

when $x(t)$ is a satisfactory state. Since $[I + G(t)]x(t)$ is a satisfactory state for any $x(t)$, the guidance law matrix satisfies the equation

$$G(t)[I + G(t)] = 0$$

The partitioned form of $G(t)$ can be inserted to reduce the previous equation to

$$\left.\begin{aligned} G_1(t) &= -G_2(t)G_1(t) \\ G_2(t) &= -G_2^2(t) \end{aligned}\right\} \tag{2}$$

The guidance law matrix associated with linear impulsive corrections satisfies equations (2), but not all matrices which satisfy equations (2) are necessarily guidance law matrices. The next section will determine existence conditions and the guidance law matrix in terms of various features of the set of satisfactory states.

Existence Conditions and the General Guidance Law Matrix

This section explores the connections between the guidance law matrix and the set of satisfactory states. It is found that linear impulsive guidance laws occur when the satisfactory states are certain types of subspaces of the state space, and that the guidance law matrix can be given in terms of appropriate states from $S(t)$. Some general properties of the guidance law matrix are given and, in particular, it is found that the submatrix $G_2(t)$ can have only certain mathematical forms.

The satisfactory states.- Suppose the guidance law exists and $G(t)$ is its matrix. Then from the definition of the guidance law, the states in $[I + G(t)]N(t)$ are satisfactory and also

$$G(t)x(t) = 0$$

for any state which is satisfactory. It follows easily that the converse of these two statements is also true. Consequently, if $G(t)$ exists, then the satisfactory states can be defined from the guidance law matrix by either of the following

$$\left.\begin{aligned} S(t) &= [I + G(t)]N(t) \\ S(t) &= \{x(t) : G(t)x(t) = 0\} \end{aligned}\right\} \tag{3}$$

Necessary conditions for the existence of $G(t)$.- The first of equation (3) states that if $G(t)$ exists, then $S(t)$ is the linear vector subspace of $N(t)$ formed by the set of all linear combinations of the columns of $I + G(t)$; that is, the columns of $I + G(t)$ are satisfactory states and span the subspace $S(t)$. The dimension of $S(t)$ is equal to the order of the largest square nonsingular submatrix of $I + G(t)$. In terms of its 3×3 submatrices

$$I + G(t) = \begin{bmatrix} I & 0 \\ G_1(t) & I + G_2(t) \end{bmatrix} \tag{4}$$

and it is clear that the dimension of $S(t)$ is at least 3.

Further, $R(t)$, the set of positions occupied at t by the satisfactory orbits is, equivalently, the set of all linear combinations of the position elements of any set of states which span $S(t)$. In particular, $R(t)$ is given by all linear combinations of the columns in the upper half of $I + G(t)$ and is therefore three-dimensional, since the upper left submatrix I in equation (4) is nonsingular. These statements are collected in the following necessary conditions for the existence of a linear impulsive guidance law matrix.

Lemma 1: If $G(t)$ exists, then $S(t)$ is a linear vector subspace of $N(t)$ such that $R(t)$ is three-dimensional.

Thus, $S(t)$ is a three-, four-, or five-dimensional subspace of $N(t)$. The trivial case in which $S(t)$ is six-dimensional need not be considered, since in this case every state in $N(t)$ is satisfactory, and no correction is ever made. That $R(t)$ must be three-dimensional is physically obvious, since it guarantees that at least one satisfactory state exists corresponding to every position deviation; and in any other case there will be position deviations for which no satisfactory state exists and for which a correction is therefore impossible.

The guidance law matrix.- The necessary conditions for existence given by lemma 1 are also sufficient. The proof of this fundamental fact will follow from lemma 2 and theorem 1 below. Lemma 2 is a formality; theorem 1 shows that the guidance law matrix can be given from any basis of the satisfactory states and will be useful in the later text.

Lemma 2: Let $S(t)$ be an n-dimensional subspace of $N(t)$ and let $s_1 = (\bar{r}_1,\bar{v}_1), \ldots, s_n = (\bar{r}_n,\bar{v}_n)$ be any basis of $S(t)$. Then $R(t)$ is three-dimensional if and only if $3 \leq n \leq 6$ and three of the vectors $\bar{r}_1, \ldots, \bar{r}_n$ are independent.

A basis of a linear vector space is any set of linearly independent elements in the space which will generate every element in the space. The number of elements in a basis is equal to the dimension of the space. Further, the space can be generated as the set of all linear combinations of the basis elements. Thus, $S(t)$ is the set of all linear combinations of $s_1, \ldots, s_n$, and $R(t)$, the set of all positions occupied by the satisfactory states, is the set of all linear combinations of $\bar{r}_1, \ldots, \bar{r}_n$. Hence, $R(t)$ is three-dimensional if and only if $n \geq 3$ and three of the vectors $\bar{r}_1, \ldots, \bar{r}_n$ are independent.

Theorem 1: If $S(t)$ is an n-dimensional subspace of $N(t)$, $n = 3, 4, 5$, and $s_1 = (\bar{r}_1, \bar{v}_1), \ldots, s_n = (\bar{r}_n, \bar{v}_n)$ form a basis of $S(t)$ such that $\bar{r}_1, \bar{r}_2, \bar{r}_3$ are independent (renumber the states if necessary), then a linear impulsive guidance law exists and its matrix is given as follows: Let W, V be the 3×3 matrices whose columns are $\bar{r}_1, \bar{r}_2, \bar{r}_3$, and $\bar{v}_1, \bar{v}_2, \bar{v}_3$, respectively. Then

$$G_1(t) = -G_2(t)VW^{-1}$$

$$G_2(t) = \begin{cases} -I & \text{for } n = 3 \\ -I + \bar{u}\bar{u}^T & \text{for } n = 4 \\ -I + \bar{u}_1\bar{u}_1^T + \bar{u}_2\bar{u}_2^T & \text{for } n = 5 \end{cases}$$

where (i) $G(t)$ specifies the minimum velocity correction

(ii) $\bar{u}$ is a unit vector in the direction of the (nonzero) vector

$$\bar{v}_4 - VW^{-1}\bar{r}_4$$

(iii) $\bar{u}_1, \bar{u}_2$ are any orthonormal pair of vectors in the plane of the two (nonzero, independent) vectors

$$\bar{v}_j - VW^{-1}\bar{r}_j, \quad j = 4, 5$$

The proof of theorem 1 is carried out in appendix A. Note that the matrix W is nonsingular since its columns are assumed independent. The form of $G(t)$ is not the most general linear impulsive guidance law matrix when $S(t)$ is four- or five-dimensional but corresponds to the condition that the magnitude of the velocity correction be minimized. The general result is noted in appendix A but is not of interest here except in discussing the uniqueness of the guidance law matrix.

In view of lemmas 1 and 2, if $G(t)$ exists, then the conditions of theorem 1 can always be satisfied; therefore, the guidance law matrix can always be given in one of the forms of that theorem.

It should be noted in theorem 1 that guidance law matrices exhibit certain properties of form which depend only on the dimensions of $S(t)$. The submatrix $G_2(t)$ can have only certain mathematical forms. If we write

$$\bar{w}_j = \bar{v}_j - VW^{-1}\bar{r}_j, \quad 3 < j \leq n$$

then $G_2(t)$ can be described as the reflection-projection matrix with projection to the space orthogonal to the $n-3$ independent vectors $\bar{w}_j$. An alternate view is that the mission constraints are independent of the $\bar{w}_j$ components of the velocity deviation. Thus, $G_2(t)\delta\bar{v}$ cancels out the part of $\delta\bar{v}$ which is orthogonal to the $\bar{w}_j$ and leaves unchanged the components of $\delta\bar{v}$ along $\bar{w}_j$

Such a restriction on the form of $G_2(t)$ is expected in view of equations (2). However, a larger class of matrices than the reflection-projection matrices given in theorem 1 will satisfy equations (2); the remaining matrices do not reflect the condition that the velocity correction is to be minimized. That is, since the mission constraints are independent of the $\bar{w}_j$ components of the final velocity deviation, then these components can be changed arbitrarily without affecting the mission constraints, and the minimum correction is the one that makes no change in these components.

Further, the submatrix $G_1(t)$ is restricted to the extent that the range space of $G_1(t)$ is included in the range space of $G_2(t)$; that is, $G_1(t)\delta\bar{r}$ will lie in the space orthogonal to the $\bar{w}_j$. These properties are partly summarized by the following singularities, which follow easily from theorem 1:

$$\bar{u}^T G_1(t) = \bar{u}^T G_2(t) = G_2(t)\bar{u} = \Delta\bar{v} \cdot \bar{u} = 0 \qquad \text{for} \quad n = 4$$

$$\bar{u}_j^T G_1(t) = \bar{u}_j^T G_2(t) = G_2(t)\bar{u}_j = \Delta\bar{v} \cdot \bar{v}_j = 0 \quad j = 1, 2 \qquad \text{for} \quad n = 5$$

<u>Necessary and sufficient conditions for the existence of</u> $G(t)$.- Finally, lemma 2 and theorem 1 together state the converse of lemma 1; that is, if $S(t)$ is a linear vector subspace of $N(t)$ such that $R(t)$ is three-dimensional, then $G(t)$ exists. These results combine to give a necessary and sufficient condition for the existence of the guidance law.

<u>Theorem 2</u>: A linear impulsive guidance law exists at t if and only if $S(t)$ is a linear vector subspace of $N(t)$ such that $R(t)$ is three-dimensional.

<u>Uniqueness of the guidance law matrix</u>.- The uniqueness of the guidance law matrix corresponding to a given set of satisfactory states is investigated in appendix B. If $G(t)$ is assumed to exist, the results state that $G(t)$ is unique, provided $S(t)$ is three-dimensional and is unique when $S(t)$ is four- or five-dimensional if the magnitude of the velocity correction is minimized.

To summarize the results of this section, a linear impulsive guidance law occurs when $S(t)$ is any three-, four-, five-, or, trivially, six-dimensional subspace of $N(t)$ such that there is at least one element of $S(t)$ corresponding to every position deviation. When this condition is satisfied and the velocity correction is assumed to be minimized, then the guidance law matrix is given uniquely by theorem 1 from any basis of $S(t)$.

Invariant Properties and Propagation of the Guidance Law Matrix

Since the guidance law is of continuing interest throughout the mission, this section will consider the propagation of the guidance matrix. We first determine whether the properties discussed in the previous section are invariant with time. It is found that $S(t)$ remains a linear vector space of invariant dimension throughout the mission. Hence, the form of $G_2(t)$ is also invariant for all times when $G(t)$ exists. Secondly, if $G(t_o)$, the guidance law at the start of the mission, is assumed to exist, the relation between $G(t)$ and $G(t_o)$ can be determined for those times at which $G(t)$ exists.

Invariant properties.- The set of satisfactory orbits is fixed throughout the mission. If $x(t_o)$ is on a satisfactory orbit, then from equation (1) the corresponding satisfactory state on the same orbit at time t is

$$x(t) = A(t,t_o)x(t_o)$$

If $G(t_o)$ exists, then $S(t_o)$ is a linear vector space. By taking the union over all states in $S(t_o)$, we obtain

$$S(t) = A(t,t_o)S(t_o) \tag{5}$$

The state transition matrix $A(t,t_o)$ is never singular (ref. 1);[1] hence, if $G(t_o)$ exits, then from equation (5), $S(t)$ is everywhere a linear vector space with the same dimension. Therefore, the form of $G_2(t)$ and other dimension-dependent properties of form noted in theorem 1 are also invariant wherever the guidance law exists. Further, since $S(t)$ is everywhere a linear vector space then the existence conditions of theorem 2 are met if $R(t)$ is three-dimensional. It should be noted that there may be one or more times during a mission at which this condition is not met. In summary,

> Theorem 3: If $G(t_o)$ exists, then (i) $S(t)$ is everywhere a linear vector space with time-invariant dimension, and (ii) $G(t)$ exists if and only if $R(t)$ is a three-dimensional linear vector space.

Existence criteria.- In the following, $G(t_o)$ is assumed to exist, and the relation between $G(t)$ and $G(t_o)$ is obtained. It is first necessary to give the existence condition in terms of $G(t_o)$ and the transition matrix.

The transition matrix may be partitioned into its 3×3 submatrices:

$$A(t_2,t_1) = \begin{bmatrix} A_1(t_2,t_1) & A_2(t_2,t_1) \\ \\ A_3(t_2,t_1) & A_4(t_2,t_1) \end{bmatrix} \tag{6}$$

The satisfactory states at t, from equations (3) and (5), can now be expressed as

$$S(t) = \begin{Bmatrix} B_1(t,t_o) & A_2(t,t_o)[I + G_2(t_o)] \\ \\ B_2(t,t_o) & A_4(t,t_o)[I + G_2(t_o)] \end{Bmatrix} N(t_o) \tag{7}$$

---

[1]If $A(t,t_o)$ were singular, distinct states would exist at t_o which would generate identical states at t under free motion and, hence, distinct orbits which become identical orbits.

The following notation has been adopted in equation (7)

$$\left.\begin{aligned} B_1(t,t_o) &= A_1(t,t_o) + A_2(t,t_o)G_1(t_o) \\ B_2(t,t_o) &= A_3(t,t_o) + A_4(t,t_o)G_1(t_o) \end{aligned}\right\} \tag{8}$$

The positions occupied by the satisfactory orbits at t form the linear vector space given by

$$R(t) = \{\delta\bar{r}(t) = B_1(t,t_o)\delta\bar{r}(t_o) + A_2(t,t_o)[I + G_2(t_o)]\delta\bar{v}(t_o);$$

$$\delta\bar{r}(t_o),\delta\bar{v}(t_o) \text{ arbitrary}\} \tag{9}$$

A necessary and sufficient condition for the existence of $G(t)$ is that $R(t)$ be three-dimensional; that is, at least one satisfactory state exists corresponding to every position deviation at t. The matrix $G_2(t_o)$ can have only certain forms as given in theorem 1 of the preceding section. These forms depend only on the dimension of $S(t)$

$$G_2(t_o) = \begin{cases} -I & \text{for } n = 3 \\ -I + \bar{u}\bar{u}^T & \text{for } n = 4 \\ -I + \bar{u}_1\bar{u}_1^T + \bar{u}_2\bar{u}_2^T & \text{for } n = 5 \end{cases}$$

where n is the dimension of $S(t)$ and $\bar{u}$, $\bar{u}_1$, $\bar{u}_2$ are the unit vectors of theorem 1.

Existence criteria are obtained by substituting these forms of $G_2(t_o)$ into equation (9) and determining the conditions for which $R(t)$ is three-dimensional. Equation (9) gives $R(t)$ as the set of all linear combinations of the six columns of $B_1(t,t_o)$ and $A_2(t,t_o)[I + G_2(t_o)]$; hence, $R(t)$ is three-dimensional when three of the six columns are independent. The results are readily obtained and are summarized by the following corollary.

<u>Corollary 1</u>: If $G(t_o)$ exists and $\dim[S(t)] = n$, $n = 3, 4, 5$ then $G(t)$ exists if and only if one of the following is satisfied:

(a) $B_1(t,t_o)$ is nonsingular

(b) $n = 4$, $B_1(t,t_o)$ has rank 2, and $A_2(t,t_o)\bar{u}$ is not in the range space of $B_1(t,t_o)$

(c) $n = 5$, $B_1(t,t_o)$ has rank 2, and one of the vectors $A_2(t,t_o)\bar{u}_1$, $A_2(t,t_o)\bar{u}_2$ is not in the range space of $B_1(t,t_o)$

(d) $n = 5$, $B_1(t,t_o)$ has rank 1, and the two vectors, $A_2(t,t_o)\bar{u}_1$, $A_2(t,t_o)\bar{u}_2$ are neither in the range space of $B_1(t,t_o)$ nor dependent

A propagation formula.- The guidance law matrix $G(t)$ exists as specified in corollary 1. When $G(t)$ exists, it can be related to $G(t_o)$ through the transition matrix. This relation is obtained below for case (a) of corollary 1 in which $B_1(t,t_o)$ is nonsingular. The remaining cases are neglected here because the results have not been obtained in a useful form.

Equation (7) gives $S(t)$ as the set of all linear combinations of the columns of a matrix. To obtain the propagation formula, substitute the appropriate form of $G_2(t_o)$ according to the dimension of $S(t)$ into the matrix in equation (7). If n is the dimension of $S(t)$, the resulting matrix must have n independent columns. For example, if $n = 4$ the matrix becomes

$$\begin{bmatrix} B_1(t,t_o) & A_2(t,t_o)\bar{u}\bar{u}^T \\ B_2(t,t_o) & A_4(t,t_o)\bar{u}\bar{u}^T \end{bmatrix}$$

There is exactly one independent column among the right-hand three columns and this together with the left-hand three columns provides four independent states in $S(t)$. If, next, $B_1(t,t_o)$ is assumed nonsingular then the conditions of theorem 1 are met; the matrix above provides four independent states in $S(t)$ of which the first three provide three independent position deviations. Consequently, the guidance law matrix follows immediately from theorem 1. First, define the matrix

$$B = A_4(t,t_o) - B_2(t,t_o)B_1^{-1}(t,t_o)A_2(t,t_o) \tag{10a}$$

This matrix is nonsingular (appendix C), and if $A_2(t,t_o)$ is also nonsingular, B can be reduced to (appendix C):

$$B = [A_2(t,t_o)^{-1}]^T \quad B_1^{-1}(t,t_o)A_2(t,t_o) \tag{10b}$$

Finally, the guidance law matrix itself is given by the following corollary to theorems 1 and 3:

Corollary 2: If $G(t_o)$ exists and $B_1(t,t_o)$ is nonsingular, then $G(t)$ exists and is given as follows:

Let $S(t)$ be n-dimensional; $n = 3, 4, 5$. Then

$$G_1(t) = -G_2(t)B_2(t,t_o)B_1^{-1}(t,t_o)$$

$$G_2(t) = \begin{cases} -I & \text{for } n = 3 \\ -I + \bar{u}(t)\bar{u}^T(t) & \text{for } n = 4 \\ -I + \bar{u}_1(t)\bar{u}_1^T(t) + \bar{u}_2(t)\bar{u}_2^T(t) & \text{for } n = 5 \end{cases}$$

where (i) the form of $G_2(t)$ assumes that the velocity correction is minimized when $n = 4, 5$

(ii) For $n = 4$, $\bar{u}(t)$ is a unit vector in the direction of $B\bar{u}(t_o)$

(iii) For $n = 5$, $\bar{u}_1(t)$, $\bar{u}_2(t)$ are any orthonormal pair of vectors in the plane of the two (independent)[2] vectors $B\bar{u}_1(t_o)$, $B\bar{u}_2(t_o)$

(iv) $\bar{u}(t_o)$, $\bar{u}_1(t_o)$, $\bar{u}_2(t_o)$ are the unit vectors associated with $G_2(t_o)$ in theorem 1 and $B_1(t,t_o)$, $B_2(t,t_o)$, B are matrices defined in equations (8) and (10)

Derivation of Guidance Laws From Constraints

The preceding analysis began with the primitive notion of satisfactory states and a definition of the linear impulsive guidance law and then obtained the theoretical results as consequences of the definition. In practice, guidance laws are obtained by applying constraints to the vehicle motion in order to meet mission requirements. This process is sufficiently general that the notion of guidance constraints must play an important role in the theory.

In this section the connection of guidance constraints with the preceding theory is examined, and a uniform method for deriving linear impulsive guidance law matrices from the guidance constraints is given.

Guidance constraints.- The guidance constraints are a number of restrictions on the vehicle motion to be satisfied in order to meet the mission requirements. The guidance law is itself a statement of these constraints. In equation (3),

$$S(t) = \{x(t): \quad G(t)x(t) = 0\} \tag{3}$$

That is, those states which satisfy the mission requirements are given by equation (3) as all states for which the linear function $G(t)x(t)$ is zero. The number of independent scalar constraints imposed by the guidance law on the vehicle motion is then the number of linearly independent equations in

$$G(t)x(t) = 0$$

or, equivalently, the rank of $G(t)$. It is readily determined from equations (2) that the rank of $G(t)$ is equal to the rank of $G_2(t)$, which, from theorem 1, is specified by the dimension of $S(t)$.

Lemma 3: Let $G(t)$ be the matrix of a linear impulsive guidance law defined at t. Then $S(t)$ is an n-dimensional subspace of $N(t)$, $n = 3, 4, 5$, and the guidance law imposes $6-n$ independent scalar constraints on the state.

[2]Since u_1, u_2 are independent and B is nonsingular, $B\bar{u}_1$, $B\bar{u}_2$ are independent.

It may be noted that a linear impulsive guidance law cannot impose more than three constraints. Further, since n is invariant throughout the mission (theorem 3), the number of constraints imposed by $G(t)$ is also invariant.

Derivation of the guidance law from constraints.- In practice, guidance laws are obtained by applying constraints to the vehicle motion in order to satisfy the mission. This process is generally applicable and, with suitable restrictions, leads directly to linear impulsive guidance laws.

Let

$$p^T = (p_1, \ldots, p_6) \tag{11}$$

be any six independent scalar parameters of orbital motion. This means that the set of all p, $\{p\}$, is in one-to-one correspondence with the set of all orbits. The six Cartesian coordinates of position and velocity at some time t, $X(t)$, are often used, but, in general, the parameters may be chosen for convenience. In this case, $X(t)$ can always be given in terms of p by virtue of their one-to-one correspondence with the set of all orbits:

$$X(t) = F(p_1, \ldots, p_6; t) \tag{12}$$

The guidance constraints are a number of restrictions on the vehicle motion generated by the mission requirements. Suppose the guidance constraints have been expressed as constraints on independent orbital parameters and that these are included among the six parameters of (11); in particular, suppose the mission requirements are met by all orbits for which the parameters $p_{n+1}, \ldots, p_6$, $n \leq 6$, have the specified values $\alpha_{n+1}, \ldots, \alpha_6$. (If $n = 6$, then no parameter is specified, and we have the trivial case in which all orbits satisfy the guidance constraints.) The satisfactory orbits are now defined by

$$p_s^T = (p_1, \ldots, p_n, \alpha_{n+1}, \ldots, \alpha_6); \quad p_1, \ldots, p_n \quad \text{arbitrary} \tag{13}$$

A reference orbit is chosen from among the p_s of equation (13). The remaining orbits can be referred to the reference orbit by:

$$\delta_p = p - p_{ref} = (\delta p_1, \ldots, \delta p_6) \tag{14}$$

and the guidance constraints become

$$\delta p_{n+1} = \ldots = \delta p_6 = 0 \tag{15}$$

Assume, next, that the orbits of interest are sufficiently near the reference orbit that they may be adequately represented by the first two terms of a Taylor series expansion of equation (12) about the reference orbit. The deviation state at any time can then be expressed as:

$$x(t) = X(t) - X(t)_{ref} = \begin{bmatrix} \frac{\partial\bar{r}(t)}{\partial p_1} & \cdots & \frac{\partial\bar{r}(t)}{\partial p_6} \\ \frac{\partial\bar{v}(t)}{\partial p_1} & \cdots & \frac{\partial\bar{v}(t)}{\partial p_6} \end{bmatrix} \delta p = F_1(t)\delta p \tag{16}$$

The matrix, $F_1(t)$, is nonsingular by virtue of the one-to-one correspondence of $N(t)$ and $\{\delta p\}$ with the set of all (linearized) orbits and, hence, with each other. That is, the columns of $F_1(t)$ are six independent states in $N(t)$. The satisfactory states are all those deviation states that satisfy (15):

$$S(t) = \left\{ x(t) = \begin{bmatrix} \frac{\partial\bar{r}(t)}{\partial p_1} & \cdots & \frac{\partial\bar{r}(t)}{\partial p_n} \\ \frac{\partial\bar{v}(t)}{\partial p_1} & \cdots & \frac{\partial\bar{v}(t)}{\partial p_n} \end{bmatrix} \begin{pmatrix} \delta p_1 \\ \\ \delta p_n \end{pmatrix}; \quad \begin{matrix} \delta p_1, \ldots, \delta p_n \\ \text{arbitrary} \end{matrix} \right\} \tag{17}$$

Equation (17) gives $S(t)$ as the n-dimensional linear vector subspace of $N(t)$ for which the n states

$$s_1 = \left[\frac{\partial\bar{r}(t)}{\partial p_1}, \frac{\partial\bar{v}(t)}{\partial p_1}\right], \ldots, s_n = \left[\frac{\partial\bar{r}(t)}{\partial p_n}, \frac{\partial\bar{v}(t)}{\partial p_n}\right] \tag{18}$$

are a basis. The existence conditions of theorem 2 for a linear impulsive guidance law can now be satisfied if and only if $R(t)$ is three-dimensional. This occurs if and only if (i) n is not less than 3, and (ii) three of the vectors

$$\frac{\partial\bar{r}(t)}{\partial p_k}, \quad k = 1, \ldots, n$$

are independent. Assuming that these conditions are met, then the conditions of theorem 1 are fully satisfied by the n states of equation (18) and the guidance law matrix can now be written directly from theorem 1. The following theorem gives the results:

> Theorem 4: Let $p^T = (p_1, \ldots, p_6)$ be six independent scalar parameters of orbital motion; let $X_o(t)$ be the state on a reference orbit, and let the equations relating $X(t)$ to p be linearized about the reference orbit:
>
> $$x(t) = \left[\frac{\partial\bar{x}(t)}{\partial p_1} \cdots \frac{\partial\bar{x}(t)}{\partial p_6}\right]\delta p$$

Then (i) a linear impulsive guidance law satisfying the constraints

$$\delta p_j = 0 \quad j = n + 1, \ldots, 6$$

exists if and only if $n \geq 3$ and

$$\operatorname{rank} \left[\frac{\partial \bar{r}(t)}{\partial p_1} \cdots \frac{\partial \bar{r}(t)}{\partial p_n}\right] = 3$$

(ii) If the guidance law exists, then the guidance law matrix which specifies the minimum velocity correction is given as follows. Let $\partial\bar{r}/\partial p_k$; $k = 1, 2, 3$ be three independent vectors (renumber the parameters if necessary) and define

$$W(t) = \left[\frac{\partial \bar{r}(t)}{\partial p_1} \; \frac{\partial \bar{r}(t)}{\partial p_2} \; \frac{\partial \bar{r}(t)}{\partial p_3}\right]$$

$$V(t) = \left[\frac{\partial \bar{v}(t)}{\partial p_1} \; \frac{\partial \bar{v}(t)}{\partial p_2} \; \frac{\partial \bar{v}(t)}{\partial p_3}\right]$$

then

$$G_1(t) = -G_2(t)V(t)W(t)^{-1}$$

$$G_2(t) = \begin{cases} -I & \text{for } n = 3 \\ -I + \bar{u}\bar{u}^T & \text{for } n = 4 \\ -I + \bar{u}_1\bar{u}_1^T + \bar{u}_2\bar{u}_2^T & \text{for } n = 5 \end{cases}$$

where (i) all derivatives are evaluated on the reference orbit

(ii) for $n = 4$, $\bar{u}$ is a unit vector in the direction of $(\partial\bar{v}/\partial p_4) - VW^{-1}(\partial\bar{r}/\partial p_4)$

(iii) for $n = 5$, $\bar{u}_1$, $\bar{u}_2$ are any orthonormal pair of vectors in the plane of $(\partial\bar{v}/\partial p_k) - VW^{-1}(\partial r/\partial p_k)$; $k = 4, 5$

The results in theorem 4 parallel those of theorem 1 except that the guidance law matrix is given in terms of appropriate derivatives of the state. The derivatives appearing in equation (18) and in the guidance law matrix form a basis of $S(t)$, but the guidance law is unique and independent of the basis of $S(t)$ that is used; that is, the derivatives appearing in the guidance law matrix can be taken with respect to any n independent scalar parameters of orbital motion which are also independent of the constraint parameters.

It may be noted that a linear impulsive guidance law does not occur if more than three constraints are required as, for example, in the rendezvous constraints $[\delta\bar{r}(t_f) = 0,\ \delta\bar{v}(t_f) = 0]$.

An Alternate Construction of Guidance Law From Constraints

Thus far, linear impulsive guidance laws have been treated by considering the set of satisfactory states; theorem 1 shows that $G(t)$ can be given from any basis of $S(t)$. To relate the guidance law to the guidance constraints, according to the previous section, if there are m $(m \leq 3)$ independent guidance constraints, then a basis of $S(t)$ is given by derivatives of the state with respect to any $6-m$ independent parameters which are also independent of the constraint parameters. The guidance law matrix can then be formulated from these derivatives.

An alternate construction of the guidance law matrix from the guidance constraints is obtained by considering directly the conditions under which the constraints can be satisfied. The resulting definition of the guidance law will usually be simpler to apply than theorem 4.

Let $p^T = (p_1, \ldots, p_m)$ be m independent scalar parameters of orbital motion related to the present state $X(t)$ by

$$p = F[X(t)] \tag{19a}$$

and suppose that the guidance constraints are that p is the specified vector

$$p^T = (\alpha_1, \ldots, \alpha_m)$$

Choose a reference orbit $X_o(t)$ which satisfies these constraints and assume that the orbits of interest are sufficiently close to the reference orbit that the parameters p may be approximated by the first two terms of a Taylor series expansion of equation (19a) about the reference orbit. Subtracting the reference equations from (19a), obtain

$$\delta p = A(t)\delta\bar{r}(t) + B(t)\delta\bar{v}(t) \tag{19b}$$

Here, $A(t)$, $B(t)$ are $m \times 3$ matrices in which the jth rows are the derivatives, evaluated at the reference orbit, of p_j with respect to the Cartesian coordinates of position and velocity, respectively. Since the m parameters of p are independent, then

$$\text{rank}\ [A\ B] = m \tag{20}$$

The guidance constraints are

$$\delta p_j = 0 \qquad j = 1, \ldots, m \tag{21}$$

and we define the existence of a guidance correction which satisfies these constraints as follows:

> A guidance correction, $\Delta\bar{v}$, exists at t if for every deviation state $(\delta\bar{r},\delta\bar{v})$ there exists a $\Delta\bar{v}$ which satisfies the m independent equations:
>
> $$0 = A(t)\delta\bar{r} + B(t)(\delta\bar{v} + \Delta\bar{v}) \tag{22}$$

<u>Existence conditions</u>.- An existence condition for the guidance correction is readily established.

> <u>Theorem 5</u>: Equation (22) has a solution (i.e., a guidance correction exists at t) if and only if
>
> $$\text{rank}\ (B) = m \leq 3$$

<u>Proof</u> - Equation (22) can be rewritten as:

$$B(t)\ \Delta\bar{v} = -[A(t)\ B(t)]\delta x(t) \tag{23}$$

where A, B are $m \times 3$ matrices. We seek the conditions for which these m inhomogeneous linear equations have a solution, $\Delta\bar{v}$, for arbitrary δx. The Kronecker-Capelli theorem (ref. 6) states that equation (23) is compatible (has at least one solution) if and only if the rank of B equals the rank of the augmented matrix, B^a, where the augmented matrix is formed by attaching the $m \times 1$ column

$$\bar{b} = [A\ B]\delta x$$

to the coefficient matrix, B;

$$B^a = [B\ \bar{b}]$$

Thus, the Kronecker-Capelli theorem is satisfied if and only if $\bar{b}$ is a linear combination of the columns of B. But $\bar{b}$ can be any linear combination of the columns of $[A\ B]$ since δx is arbitrary; hence, the columns of $[A\ B]$ are required to be linear combinations of the columns of B. Thus,

$$\text{rank}\ (B) \geq \text{rank}\ [A\ B] = m$$

and, since the columns of B are contained in the columns of $[A\ B]$,

$$\text{rank}\ (B) \leq \text{rank}\ [A\ B]$$

Thus, the Kronecker-Capelli theorem is satisfied if and only if

$$\text{rank}\ (B) = \text{rank}\ [A\ B] = m \tag{24}$$

Note that B is an $m \times 3$ matrix whose maximum rank is minimum $\{m,3\}$ and equation (24) cannot be satisfied if $m > 3$.

<u>The guidance law matrix</u>.- Suppose that the guidance correction exists. Then the general solution of equation (22) can be given directly from reference 7 (theorem 2, corollary 1);

$$\Delta\bar{v} = -B^{+}A\delta\bar{r} - B^{+}B\delta\bar{v} + \bar{z} \tag{25a}$$

where B^+ indicates the pseudo-inverse[3] of B and the vector $\bar{z}$ is any arbitrary vector orthogonal to the domain (row space) of B. Now, the first two terms on the right-hand side of equation (25a) lie in the domain of B; in fact, B^+B is the matrix which projects on to the (m-dimensional) domain of B. Thus, $\bar{z}$ is orthogonal to the first two terms, and if we adopt the condition that $\bar{z}$ is to be chosen such that $\Delta\bar{v}$ is a minimum, then $\bar{z}$ will be zero. The desired solution of equation (23) is then

$$\Delta\bar{v} = -B^+A\delta\bar{r} - B^+B\delta\bar{v} \tag{25b}$$

The corresponding guidance law matrix can now be identified:

$$G_1(t) = -B^+A$$

$$G_2(t) = -B^+B$$

It is readily verified that equation (25b) satisfies the definition of a linear impulsive guidance law. Further, that definition can be satisfied by more general choices of $\bar{z}$ than $\bar{z} = 0$, as noted in appendix A, but the general case is not of interest in this paper.

Thus far, it has been shown that if a guidance correction exists, a matrix can be written for a linear impulsive guidance law. Conversely, if that matrix can be written, then equation (23) has at least one solution for every deviation state, δx, and the guidance correction can be said to exist. In summary,

<u>Theorem 6</u>: Let $p^T = (p_1, \ldots, p_m)$ be m independent scalar parameters of orbital motion; let $X_o(t)$ be the state on a reference orbit, and let the equations relating p to the state be linearized about the reference orbit

$$\delta p = A(t)\delta\bar{r}(t) + B(t)\delta\bar{v}(t)$$

Then (i) a linear impulsive guidance law satisfying the constraints $\delta p = 0$ exists if and only if

$$\text{rank } B(t) = m \leq 3$$

(ii) If a guidance law exists, then the guidance law matrix which specifies the minimum velocity correction is

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[3]The pseudo-inverse is a generalization to arbitrary matrices of the inverse of nonsingular square matrices. References 7 and 8 may be consulted and some pertinent examples of pseudo-inverses are given in the next section of this paper. The pseudo-inverse, B^+, of B is the unique solution of the four equations:

$$BB^+B = B, \quad BB^+ = (BB^+)^T$$

$$B^+BB^+ = B^+, \quad B^+B = (B^+B)^T$$

Note that if B is a nonsingular square matrix, then B^+ is the ordinary inverse, B^{-1}.

$$G_1(t) = -B^+(t)A(t)$$

$$G_2(t) = -B^+(t)B(t)$$

Theorem 6 will usually be easier to apply than theorem 4. It should be noted that only the 6m (scalar) derivatives of the constraint parameters with respect to the Cartesian coordinates of the state are required for the construction of $G(t)$. Theorem 4 requires the completion of the m constraint parameters to a set of six independent parameters and then defines $G(t)$ from $6(6-m)$ derivatives of the state coordinates with respect to the new parameters.

It may be noted that $G(t)$ can be expressed in terms of the matrices $A(t_o)$, $B(t_o)$ for some fixed time, t_o, and the transition matrix, $A(t,t_o)$ from the following relations

$$A(t) = A(t_o)A_4^T - B(t_o)A_3^T$$

$$B(t) = B(t_o)A_1^T - A(t_o)A_2^T$$

where $A_1, \ldots, A_4$ are the submatrices of $A(t,t_o)$ as in equation (6).

Remarks on theorems 4 and 6.- Theorems 4 and 6 are necessarily equivalent; it can be demonstrated independently that the existence conditions and guidance law matrices of these two theorems are equivalent. The method is outlined briefly below.

Suppose $p_1, \ldots, p_6$ are six independent scalar parameters of orbital motion related to the state deviation by the linearized equations:

$$\left.\begin{aligned} \delta p &= A(t)\delta x(t) \\ \delta x(t) &= B(t)\delta p \end{aligned}\right\} \qquad (26)$$

where A, B are nonsingular 6×6 matrices and $B = A^{-1}$. Suppose that the guidance constraints are

$$\delta p_j = 0 \qquad j = 1, \ldots, m \leq 3$$

and partition A, B as follows:

$$A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} = \begin{bmatrix} (m \times 3) & (m \times 3) \\ (6\text{-}m \times 3) & (6\text{-}m \times 3) \end{bmatrix}$$

$$B = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} = \begin{bmatrix} (3 \times m) & (6\text{-}m \times 3) \\ (3 \times m) & (6\text{-}m \times 3) \end{bmatrix}$$

Theorem 6 constructs the guidance law from A_1 and A_2, and theorem 4 gives $G(t)$ from B_2 and B_4. Noting that $AB = BA = I_6$, we have the two relations

$$A_1B_2 + A_2B_4 = 0 \quad (m \times 6\text{-}m) \tag{27a}$$

$$B_1A_2 + B_2A_4 = 0 \quad (3 \times 3) \tag{27b}$$

With these two relations it can be shown that (proof omitted) for $m \leq 3$:

$$\text{rank } (A_2) = m \text{ IFF rank } (B_2) = 3$$

The left- and right-hand sides above are, respectively, the existence conditions of theorems 6 and 4; hence, the two are equivalent.

Next, assume that the existence conditions are satisfied and let W be the matrix whose columns are the three independent columns of B_2 and V the corresponding columns of B_4. Further, let $\overline{r}_j$, $\overline{v}_j$ be any remaining column of B_2 and its corresponding column from B_4. Then equation (27a) yields

$$\left.\begin{aligned} -A_2{}^+A_1 &= (A_2{}^+A_2)VW^{-1} \\ -A_2{}^+A_1\overline{r}_j &= (A_2{}^+A_2)\overline{v}_j \end{aligned}\right\} \tag{28a}$$

Substituting (28a) into the second equation yields

$$[A_2{}^+A_2][\overline{v}_j - VW^{-1}\overline{r}_j] = 0 \tag{28b}$$

From theorem 6, $G_2(t)$ is the m-dimensional reflection-projection matrix projecting to the space orthogonal to the 3-m independent vectors, $\overline{v}_j - VW^{-1}\overline{r}_j$. From equation (28b) it readily follows that the result for $G_2(t)$ is identical in the two theorems. From this result and equations (28a) it follows immediately that the expressions for $G_1(t)$ are also identical in the two theorems.

Examples

Some examples of guidance law matrices from the lunar and interplanetary mission studies are given below. These will serve to illustrate the applications of theorems 4 and 6 and the properties of form for matrices of linear impulsive guidance laws.

Example 1: Fixed time of arrival guidance (e.g., ref. 1).- The guidance constraints require an interception of the reference orbit at some fixed time, t_F. The linearized relation between the state at two different times is given by equation (1). With the transition matrix partitioned as in equation (6), the position deviation at t_F is:

$$\delta\overline{r}(t_F) = A_1(t_F,t)\delta\overline{r}(t) + A_2(t_F,t)\delta\overline{v}(t) \tag{29}$$

The linearized guidance constraints are:

$$\delta p = \delta\overline{r}(t_F) = 0$$

Theorem 6 can be applied directly to obtain:

(a) $G(t)$ exists if and only if $A_2(t_F,t)$ is nonsingular[4]

(b) If $G(t)$ exists, then

$$G_1(t) = -A_2^{-1}(t_F,t)A_1(t_F,t)$$

$$G_2(t) = -I$$

This is the principal example in the class for which $S(t)$ is three-dimensional. Other examples can be given by applying any three independent scalar constraints to the vehicle motion.

A number of variable arrival time guidance laws occur to illustrate the class for which $S(t)$ is four-dimensional. Some of these are based on the principle of nulling only part of the miss (position deviation at arrival) rather than using fixed-time-of-arrival guidance to correct the total miss (e.g., refs. 2, 3, 4). If such a guidance law can be used, the fuel requirements of fixed-time-of-arrival guidance will usually be relieved, as in some applications of reference 2 which have employed a single midcourse correction, or in the simulations of reference 3 where the downrange miss cannot be estimated very well until late in the mission.

Example 2: Interplanetary variable arrival time guidance (ref. 4).- The guidance attempts to obtain a desired position relative to the target planet. The reference orbit has the desired relative position, $\overline{\Delta}_o$, at the reference arrival time t_F. Let $\overline{V}_R$ be the velocity of the vehicle relative to the planet at t_F. Then the relative position of the vehicle at times near t_F on any nearby orbit can be approximated by the (linearized) expression

$$\overline{\Delta}(t_F + \delta t) = \overline{\Delta}_o + \delta\overline{r}(t_F) + \overline{V}_R\delta t \tag{30}$$

This expression assumes that the relative velocity is approximately fixed on all nearby orbits (acceleration of the vehicle and planet in the time interval $(t_F,t_F + \delta t)$ has been neglected). The constraints are satisfied if $\delta\overline{r}(t_F)$ is chosen so that

$$\overline{\Delta}(t_F + \delta t_A) = \overline{\Delta}_o$$

for some delay in the arrival time, δt_A. For all orbits on which this is true, equation (30) gives the relation

$$\delta\overline{r}(t_F) = -\overline{V}_R\delta t_A \tag{31a}$$

that is, the constraints are satisfied by all orbits for which $\delta\overline{r}(t_F)$ is parallel to $\overline{V}_R$. The converse is also true. Equivalently, the guidance seeks

[4]The matrix $A_2(t_F,t)$ is singular in Keplerian orbital motion when t and t_F either are identical or correspond to points of the reference orbit which differ in true anomaly by $n\pi$. Such instances can occur in interplanetary flights, as in the return legs of the orbits of reference 3.

to cancel the lateral miss (position deviation at t_F perpendicular to $\overline{V}_R$), but the downrange miss (along $\overline{V}_R$) can be arbitrary since it only determines the delay in arrival time.

Let $\overline{v}$ be a unit vector in the direction of $\overline{V}_R$; then the guidance constraints can be written as

$$\delta p = (I - \overline{v}\overline{v}^T)\delta\overline{r}(t_F) = 0 \tag{31b}$$

Equation (29) gives the relation between the guidance constraints and the state deviations:

$$(I - \overline{v}\overline{v}^T)\delta\overline{r}(t_F) = (I - \overline{v}\overline{v}^T)[A_1(t_F,t)\delta\overline{r}(t) + A_2(t_F,t)\delta\overline{v}(t)]$$

Theorem 6 can be applied directly to give:

(a) A guidance law exists IFF rank $(I - \overline{v}\overline{v}^T)A_2(t_F,t) = 2$

(b) If a guidance law exists, then (time arguments dropped):

$$G_1(t) = -[(I - \overline{v}\overline{v}^T)A_2]^+(I - \overline{v}\overline{v}^T)A_1$$

$$G_2(t) = -[(I - \overline{v}\overline{v}^T)A_2]^+(I - \overline{v}\overline{v}^T)A_2$$

If A_2 is nonsingular, then (a) is automatically satisfied and the expressions in (b) can be reduced as follows:

$$[(I - \overline{v}\overline{v}^T)A_2]^+ = (I - \overline{w}\overline{w}^T)A_2^{-1}$$

where

$$\overline{w} = \frac{A_2^{-1}\overline{v}}{|A_2^{-1}\overline{v}|}$$

$$(I - \overline{w}\overline{w}^T)A_2^{-1}(I - \overline{v}\overline{v}^T) = (I - \overline{w}\overline{w}^T)A_2^{-1}$$

from which the guidance law matrix becomes

$$\left.\begin{aligned} G_1(t) &= -(I - \overline{w}\overline{w}^T)A_2^{-1}(t_F,t)A_1(t_F,t) \\ G_2(t) &= -I + \overline{w}\overline{w}^T \end{aligned}\right\} \tag{32}$$

This result is similar to the fixed-arrival-time guidance except that the $\overline{w}$ component of velocity is unchanged since it affects only terminal position deviations along $\overline{V}_R$. Equations (32) preserve the downrange miss due to injection errors, and if this miss is large, there will be large deviations from the reference orbit near the end of the mission.

Example 3: Terminal phase pericenter guidance (ref. 3)[5].- Pericenter guidance was derived for use in the terminal phase of interplanetary missions. It operates within the sphere of influence of the target planet and seeks to obtain the reference value of the radius of pericenter and to place the pericenter position in the plane of the reference orbit relative to the target planet. Reference 3 derives the guidance law matrix from derivatives of the two-body equations of motion. An alternate formulation is given here in terms of the transition matrix.

The linearized equations governing the pericenter position are of the form

$$\delta\bar{r}_p = R(t)\delta\bar{r}(t) + V(t)\delta\bar{v}(t) \tag{33}$$

where $(\delta\bar{r},\delta\bar{v})$ is the state deviation from the reference orbit relative to the target planet. In particular, if t_F is the reference arrival time corresponding to pericenter on the reference orbit, then write equation (33) as:

$$\delta\bar{r}_p = R_o\delta\bar{r}(t_F) + V_o\delta\bar{v}(t_F) \tag{34}$$

where R_o, V_o relate $\delta\bar{r}_p$ to deviations from the reference orbit at t_F.

The matrices R_o, V_o are readily determined as follows. Define the orthonormal vectors $\bar{u}$, $\bar{v}$, $\bar{n}$ which lie, respectively, along the reference pericenter position, along the velocity vector at pericenter, and along the normal to reference orbital plane. The time t_F corresponds to pericenter on the reference orbit so that the reference state at t_F can be written

$$\bar{r}(t_F) = r_{p_o}\bar{u}$$

$$\bar{V}(t_F) = V_{p_o}\bar{v}$$

Next, consider, successively, small deviations $\delta\bar{r}(t_F)$, $\delta\bar{v}(t_F)$ along the three directions $\bar{u}$, $\bar{v}$, $\bar{n}$ and obtain the corresponding deviations in pericenter position, $\delta\bar{r}_p$. Summing these deviations (algebra omitted), obtain

$$\delta\bar{r}_p = (\bar{u}\bar{u}^T + \bar{n}\bar{n}^T)\delta\bar{r}(t_F) - \left(\frac{r_{p_o}}{V_{p_o}}\bar{v}\bar{u}^T\right)\delta\bar{v}(t_F) \tag{35}$$

The guidance constraints require that the pericenter position lie in the reference orbital plane, or

$$\bar{n}^T\delta\bar{r}_p = 0$$

and that the value of the radius of pericenter be the same as the reference value;

$$|\bar{r}(t_F) + \delta\bar{r}_p| - |r(t_F)| = 0$$

[5]The term, periapse guidance, used in reference 3, is lexicographically doubtful and has been changed to pericenter guidance in this paper.

This can be linearized to the form:

$$\bar{u}^T \delta \bar{r}_p = 0$$

Inserting these in equation (35), the linearized guidance constraints are:

$$\begin{pmatrix} \bar{u}^T \\ \bar{n}^T \end{pmatrix} \delta \bar{r}_p = \begin{pmatrix} \bar{u}^T \\ \bar{n}^T \end{pmatrix} \delta \bar{r}(t_F) = 0 \tag{36}$$

Equation (36) states that the constraints for pericenter guidance are equivalent to cancelling out only the lateral miss at t_F. These are exactly the constraints for example 2, and the resulting guidance law should therefore have the same form as equation (32).

Equation (29) may be substituted for $\delta \bar{r}(t_F)$ in equation (36), and the linearized equations for the constraint parameters become:

$$\begin{pmatrix} \bar{u}^T \\ \bar{n}^T \end{pmatrix} \delta r_p = \begin{pmatrix} \bar{u}^T \\ \bar{n}^T \end{pmatrix} [A_1(t_F,t)\delta \bar{r}(t) + A_2(t_F,t)\delta \bar{v}(t)]$$

Theorem 6 can now be applied to obtain the guidance law matrix:

(a) A guidance law exists IFF rank $\left[\begin{pmatrix} \bar{u}^T \\ \bar{n}^T \end{pmatrix} A_2(t_F,t)\right] = 2$

(b) If a guidance law exists, then (time arguments dropped),

$$G_1(t) = -\left[\begin{pmatrix} \bar{u}^T \\ \bar{n}^T \end{pmatrix} A_2\right]^+ \begin{pmatrix} \bar{u}^T \\ \bar{n}^T \end{pmatrix} A_1$$

$$G_2(t) = -\left[\begin{pmatrix} \bar{u}^T \\ \bar{n}^T \end{pmatrix} A_2\right]^+ \begin{pmatrix} \bar{u}^T \\ \bar{n}^T \end{pmatrix} A_2$$

For this case A_2 is generally nonsingular,[6] whence (a) is generally satisfied and the expressions in (b) can be reduced as follows:

$$\left[\begin{pmatrix} \bar{u}^T \\ \bar{n}^T \end{pmatrix} A_2\right]^+ = (I - \bar{w}\bar{w}^T)A_2^{-1}(\bar{u}\bar{n})$$

[6]For Keplerian motion, the orbit relative to the target planet is hyperbolic in which case $A_2(t_F,t)$ will be nonsingular except at $t = t_F$.

where

$$\bar{w} = \frac{A_2^{-1}\bar{v}}{|A_2^{-1}\bar{v}|}$$

$$(I - \bar{w}\bar{w}^T)A_2^{-1}(\bar{u}\bar{n})\begin{pmatrix}\bar{u}^T \\ \bar{n}^T\end{pmatrix} = (I - \bar{w}\bar{w}^T)A_2^{-1}$$

From these, the guidance law matrix becomes, as in equations (32);

$$\left.\begin{aligned} G_1(t) &= -(I - \bar{w}\bar{w}^T)A_2^{-1}A_1 \\ G_2(t) &= -I + \bar{w}\bar{w}^T \end{aligned}\right\} \tag{37}$$

Example 4: Fixed-landing-site guidance (ref. 5).- This guidance law has been applied in studies of the return phase of the lunar mission. The primary objective of guidance in this phase is to make a safe entry and, if possible, to land at a fixed landing site. Fixed-time-of-arrival guidance had been used in previous studies, with interception of the reference orbit at time t_F corresponding to vacuum perigee on the reference orbit. Although adequate for the range of initial errors studied in reference 5, the FTA guidance law does not explicitly recognize the objectives of the midcourse guidance and for comparison the fixed-landing-site guidance law was derived to satisfy the constraints:

(a) Vacuum perigee altitude is fixed at the value of the center of the safe entry corridor.

(b) The vehicle lands at a geographically fixed landing site without crossrange maneuvering during the entry flight.

The second constraint is adopted in view of the limited crossrange maneuvering capability of Apollo-class vehicles, compared to the downrange maneuver capability.

The guidance law matrix will be determined by defining the satisfactory deviation states at any time t and applying theorem 1. The state deviations at t are defined by their corresponding deviations at t_F from equation (1):

$$\delta x(t) = A(t,t_F)\delta x(t_F) \tag{1}$$

The deviations at t_F, in turn, can be defined by deviations in vacuum perigee position, velocity, and time; that is,

$$\delta x(t_F) = H\begin{pmatrix}\delta\bar{r}_p \\ \delta\bar{v}_p \\ \delta t_p\end{pmatrix} \tag{38}$$

To a first-order approximation

$$\delta t_p = t_p - t_F$$

$$\bar{r}(t_F) \cong \bar{r}(t_p) + (t_F - t_p)\bar{V}(t_p) \cong \bar{r}_{p_o} + \delta\bar{r}_p - \bar{V}_{p_o}\delta t_p$$

$$\bar{V}(t_F) \cong \bar{V}(t_p) + (t_F - t_p)\dot{\bar{V}}(t_p) = \bar{V}_{p_o} + \delta\bar{V}_p + \frac{\mu}{r_{p_o}^3}\bar{r}_p\delta t_p$$

where the subscript, o, refers to values on the reference orbit. From these results, we have the 6×7 matrix:

$$H = \begin{bmatrix} I_6 & \begin{matrix} -\bar{V}_{p_o} \\ \\ \dfrac{\mu}{r_{p_o}^3}\bar{r}_{p_o} \end{matrix} \end{bmatrix} \tag{39}$$

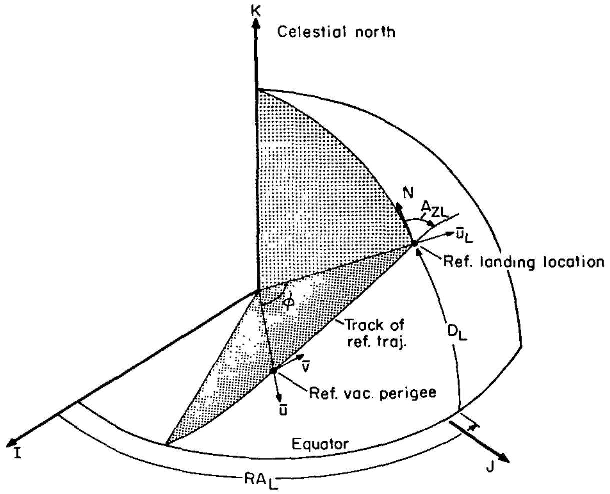
The previous equations have transferred the problem of defining the satisfactory deviation states, $\delta x_s(t)$, into one of finding perigee deviations, $\delta\bar{r}_p$, $\delta\bar{v}_p$, δt_p, which satisfy the constraints. These are obtained as follows: Let $\bar{u}$, $\bar{v}$, $\bar{n}$ be unit vectors in the direction of vacuum perigee position, velocity, and the normal to the orbital plane. Then

$$\left.\begin{aligned} \bar{r}_p &= r_p\bar{u} \\ \bar{V}_p &= V_p\bar{v} \end{aligned}\right\} \tag{40a}$$

and $\bar{u}$, $\bar{v}$, $\bar{n}$ can be given in terms of the landing site location at the time of landing, and entry flight-path parameters by the following transformations to inertial coordinates:

$$(\overline{uvn}) = (\bar{u}_L\overline{EN})\begin{bmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi\sin A_{ZL} & \cos\varphi\sin A_{ZL} & -\cos A_{ZL} \\ -\sin\varphi\cos A_{ZL} & \cos\varphi\cos A_{ZL} & -\sin A_{ZL} \end{bmatrix} \tag{40b}$$

$$(\bar{u}_L\overline{EN}) = (IJK)\begin{bmatrix} \cos D_L\sin RA_L & -\sin RA_L & -\sin D_L\sin RA_L \\ \cos D_L\cos RA_L & \cos RA_L & -\sin D_L\sin RA_L \\ \sin D_L & 0 & \cos D_L \end{bmatrix} \tag{40c}$$



Sketch (a).- Geometric parameters for fixed-landing site guidance

The parameters are illustrated in sketch (a). The vectors $\bar{u}_L$, $\bar{E}$, $\bar{N}$ are the directions of the landing site and local East and North at the landing site; their components in the inertial frame are given by equation (40c) and are functions of the landing site latitude, D_L, and right ascension, $RA_L(t_L)$. For a geographically fixed landing site, the second transformation is a function of the time of landing only:

$$RA_L(t_L) = RA_L(t_{L_o}) + \alpha(t_L - t_{L_o}) \tag{40d}$$

where α is the earth's rotation rate.

The remaining parameters, φ, A_{ZL} are the total range angle from vacuum perigee to landing, and the azimuth of the plane of the entry flight at the landing site latitude. Note that in the first transformation the entry flight motion is assumed to take place in a single inertial plane and, therefore, already contains the constraint that no crossrange maneuvering occurs during the entry flight.

For all entry flights with entry speed near the escape speed, an approximate linear relation between flight time and total range angle is available:

$$t_L - t_p = \beta_1\varphi + \beta_2 \tag{40e}$$

where β_1, β_2 are constants (ref. 5).

The satisfactory perigee conditions are found from equations (40a) to (40e) by imposing the constraints that r_p is fixed at the value corresponding to the center of the entry corridor, r_{p_o}, and that D_L, $RA_L(t_L)$ correspond to the desired landing site. The remaining parameters which may be varied arbitrarily to define the satisfactory perigee conditions are then V_p, φ, A_{ZL}, and t_L. Assuming small deviations in these parameters about their values on the reference orbit, we obtain the satisfactory deviations in vacuum perigee conditions from the appropriate derivatives of equations (40a) to (40e)

$$\begin{pmatrix} \delta\bar{r}_p \\ \delta\bar{V}_p \\ \delta t_p \end{pmatrix}_s = \psi_p \begin{pmatrix} \delta v_p \\ \delta\varphi \\ \delta A_{ZL} \\ \delta t_L \end{pmatrix} \tag{41a}$$

where

$$\psi_p = \begin{bmatrix} \bar{o} & -r_{p_o}\bar{v} & r_{p_o}\sin\varphi_o\bar{n} & \alpha r_{p_o}K x\bar{u} \\ \bar{v} & V_{p_o}\bar{u} & -V_{p_o}\cos\varphi_o\bar{n} & \alpha V_{p_o}K x\bar{v} \\ o & -\beta_1 & 0 & 1 \end{bmatrix} \tag{41b}$$

where $\bar{u}$, $\bar{v}$, $\bar{n}$ are now unit vectors taken on the reference orbit and the symbol, x, indicates the vector cross product. Finally, equations (1), (32), (41) are combined to give the satisfactory deviation states at t:

$$\delta x_s(t) = A(t,t_F)\psi \begin{pmatrix} \delta v_p \\ \delta\varphi \\ \delta A_{ZL} \\ \delta t_L \end{pmatrix} \tag{42}$$

where

$$\psi = H\psi_p$$

Since $A(t,t_F)$ is a nonsingular transformation of the state space, then the dimension of $S(t)$ is the same as the rank of ψ, which can be shown to be four. Hence, the four-columns of $A(t,t_F)\psi$ form a basis of $S(t)$ and theorem 1 can be applied directly. Partition $A(t,t_F)$ as in equation (6); then

$$\text{(i)} \quad G(t) \text{ exists IFF rank } [A_1 \; A_2]\psi = 3 \tag{43a}$$

Assuming that (43a) is satisfied, let W be any three independent columns of $[A_1 \; A_2]\psi$ and V the corresponding columns of $[A_3 \; A_4]\psi$. Further, let $\bar{r}_4$, $\bar{v}_4$ be the remaining column of these two matrices, respectively. Then

$$\left.\begin{aligned} \text{(ii)} \quad G_1(t) &= -G_2(t)VW^{-1} \\ G_2(t) &= -I + \bar{u}\bar{u}^T \end{aligned}\right\} \tag{43b}$$

where $\bar{u}$ is a unit vector in the direction of $\bar{v}_4 - VW^{-1}\bar{r}_4$.

These results are not in a satisfactory final form, but further discussion is beyond the scope of this paper.

The guidance laws in the class for which $S(t)$ is five-dimensional result from imposing a single scalar constraint on the vehicle motion. For instance, in example 3, if the constraint is reduced to obtaining the desired pericenter radius without any adjustment of the plane of motion, the linearized guidance constraints become

$$o = \bar{u}^T\delta\bar{r}_p = \bar{u}^T A_1(t_F,t)\delta\bar{r}(t) + \bar{u}^T A_2(t_F,t)\delta\bar{v}(t)$$

for which the guidance law matrix becomes (algebra omitted):

$$G_1(t) = -\frac{\bar{s}\bar{u}^T A_1(t_F,t)}{|A_2^T\bar{u}|}$$

$$G_2(t) = -\bar{s}\bar{s}^T$$

where

$$\bar{s} = \frac{A_2^T(t_F,t)\bar{u}}{|A_2^T\bar{u}|}$$

After the first correction, the corrected orbit tends to diverge from the reference orbit because the deviation of the actual orbital plane from the reference plane of motion is preserved; consequently, the linearized equation for $\bar{u}^T\delta\bar{r}_p$ becomes a poor approximation later in the flight. This difficulty is relieved when equations (37) define the guidance law matrix.

CONCLUDING REMARKS

This paper has investigated relationships among some of the basic notions of the linearized theory of midcourse guidance in space missions; the satisfactory orbits, linear impulsive guidance laws, and guidance constraints.

A satisfactory orbit is one that satisfies the mission constraints on orbital motion; the set of such orbits is fixed for a mission, and the corresponding set of states at any time, t, is a subset of the state space called the satisfactory states. The guidance process is one of correcting the current state of the vehicle to a satisfactory state by means of a change of velocity.

The guidance law is the rule that specifies the velocity correction, Δx, that should be made in order to correct the vehicle state, $X(t)$, to a satisfactory state:

$$\Delta x = G(X(t))$$

A simple class of such laws, termed linear impulsive guidance laws, was studied in this paper. These guidance laws yield velocity corrections that are linear functions of the state deviation from the reference state, and are defined on the entire state space:

$$\Delta x = G(t)x(t)$$

where $G(t)$ is the matrix of the guidance law. In the study of midcourse guidance in space missions linear impulsive guidance laws arise when the mission imposes three or fewer constraints on the vehicle orbital motion, and when the assumptions are made that velocity corrections are impulsive and the orbits of interest are sufficiently close to a reference satisfactory orbit to allow the equations of motion to be linearized about the reference orbit.

It was demonstrated that linear impulsive guidance laws can be defined from the satisfactory states (the converse is also true). Linear impulsive guidance laws occur whenever $S(t)$ is a three-, four-, five-, or, trivially, six-dimensional linear vector subspace of the deviation state space such that there is at least one satisfactory state corresponding to every position deviation (theorem 2). In this case the guidance law matrix can be given from any set of basis elements of the satisfactory states (theorem 1). The guidance law matrix corresponding to a given set of satisfactory states is unique when $S(t)$ is three-dimensional, and otherwise becomes unique if the minimum velocity correction is specified. Theorem 1 shows that the guidance law matrix has several properties of form which depend only on the dimension of $S(t)$; if $S(t)$ is n-dimensional, then $G_2(t)$ is a reflection-projection matrix projecting to a (6-n)-dimensional space which is also the range space of $G_1(t)$.

Some time-dependent properties of the guidance law matrix were considered. Since the set of satisfactory orbits is fixed, the satisfactory states at two different times in a mission are related by a nonsingular transformation (eq. (5)). Consequently, $S(t)$ has a time invariant dimension, and for those times at which the guidance law exists, the dimension dependent properties of form of $G(t)$ are also invariant. A propagation formula relating the guidance law matrix at two different times in the mission was also derived.

It was demonstrated that the satisfactory states and the guidance law can be defined from the guidance constraints. In practice, a mission requires that the vehicle motion satisfy various constraints. If these are expressed in terms of independent orbital parameters

$$p_j = \alpha_j, \quad j = 1, \ldots, m$$

where α_j are the specified values and m is the number of constraints, then the satisfactory orbits are defined as all orbits that satisfy these constraints. If $\{p_1, \ldots, p_m\}$ are completed to a set of six independent orbital parameters and a satisfactory reference orbit is selected, then a basis of the satisfactory states, $S(t)$, is given by the derivatives:

$$\frac{\partial x(t)}{\partial p_j}, \quad j = m + 1, \ldots, 6$$

where the derivatives are evaluated on the reference orbit. Thus, theorem 1 can be applied to define the existence conditions and the guidance law matrix in terms of this basis (theorem 4).

Finally, an alternate construction of the guidance law matrix is given from the constraints by considering the conditions for which a velocity correction will satisfy the constraints. If the constraint parameters are expressed in terms of m independent orbital parameters and a reference orbit is selected, then for all orbits near the reference orbit, the deviations of the constraint parameters are related to the state deviations by the (linearized) equation:

$$\delta p = A(t)\delta\bar{r}(t) + B(t)\delta\bar{v}(t)$$

A correction is said to exist if for every state, δx, a velocity change, $\Delta\overline{v}$, can be found to satisfy the constraint equations:

$$0 = A(t)\delta\overline{r} + B(t)(\delta\overline{r} + \Delta\overline{v})$$

These are m-inhomogeneous linear equations for which an existence condition (theorem 5) and the general solution for $\Delta\overline{v}$ (eq. (24)) are readily given. A linear impulsive guidance law matrix can then be given from the general solution (theorem 6).

The constructions of theorems 4 and 6 are different; theorem 4 gives $G(t)$ in terms of the derivatives, $\partial x(t)/\partial p_j$, while theorem 6 uses the derivatives $\partial p_k/\partial x$ and theorem 6 will usually be easier to apply in specific midcourse guidance problems. Nevertheless, the two results are equivalent.

Ames Research Center
National Aeronautics and Space Administration
Moffett Field, Calif., Dec. 8, 1965

APPENDIX A

PROOF OF THEOREM 1

The results quoted in theorem 1 are included in the proof of the following abbreviated statement.

Theorem: If $S(t)$ is an n-dimensional subspace of $N(t)$, $n = 3, 4, 5, 6$, and $s_1 = (\bar{r}_1,\bar{v}_1), \ldots, s_n = (\bar{r}_n,\bar{v}_n)$ are n independent states in $S(t)$ such that $\bar{r}_1, \bar{r}_2, \bar{r}_3$ are independent, then a linear impulsive guidance law exists.

By hypothesis, $S(t)$ can be a three-, four-, five-, or, trivially, six-dimensional subspace of $N(t)$ and these cases will be treated separately below.

Case a: $n = 3$

Let s_1, s_2, s_3 be three independent states in $S(t)$

$$s_1 = (\bar{r}_1,\bar{v}_1),\ s_2 = (\bar{r}_2,\bar{v}_2),\ s_3 = (\bar{r}_3,\bar{v}_3)$$

and assume $\bar{r}_1, \bar{r}_2, \bar{r}_3$ are independent vectors. Next, define W and V to be 3×3 matrices whose columns are $\bar{r}_1, \bar{r}_2, \bar{r}_3$ and $\bar{v}_1, \bar{v}_2, \bar{v}_3$, respectively; note that W is nonsingular since its columns are independent.

Let $(\delta\bar{r},\delta\bar{v})$ be any state deviation. To correct this state to a satisfactory state first determine the satisfactory state (s) which has the position deviation $\delta\bar{r}$; that is, determine all those states $(\delta\bar{r},\delta\bar{v}_s)$ that are linear combinations of the form

$$\begin{pmatrix}\delta\bar{r}\\ \delta\bar{v}_s\end{pmatrix} = \alpha_1 s_1 + \alpha_2 s_2 + \alpha_3 s_3 \tag{A1}$$

where the scalars α_i are chosen such that

$$\delta\bar{r} = \alpha_1\bar{r}_1 + \alpha_2\bar{r}_2 + \alpha_3\bar{r}_3 = W\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix} \tag{A2}$$

The constants of combination have a unique solution which is obtained by inverting equation (A2):

$$\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix} = W^{-1}\delta\bar{r}$$

The corresponding velocity deviation on the satisfactory orbit which passes through $\delta\bar{r}$ is, from equation (A1):

$$\left.\begin{aligned} \delta\overline{v}_s &= \alpha_1\overline{v}_1 + \alpha_2\overline{v}_2 + \alpha_3\overline{v}_3 \\ \text{or} \qquad \delta\overline{v}_s &= VW^{-1}\delta\overline{r} \end{aligned}\right\} \tag{A3}$$

The vector, $\delta\overline{v}_s$, depends uniquely on $\delta\overline{r}$, and there is exactly one satisfactory state $(\delta\overline{r},\delta\overline{v}_s)$ corresponding to each position deviation, $\delta\overline{r}$. The union of all such states is:

$$S(t) = \{x(t) = (\delta\overline{r},VW^{-1}\delta\overline{r});\ \delta\overline{r}\ \ \text{arbitrary}\} \tag{A4}$$

the velocity change required to correct the arbitrary state $(\delta\overline{r},\delta\overline{v})$ to the satisfactory state $(\delta\overline{r},\delta\overline{v}_s)$ is then

$$\Delta\overline{v} = \delta\overline{v}_s - \delta\overline{v} = VW^{-1}\delta\overline{r} - \delta\overline{v} \tag{A5}$$

and the guidance law matrix can now be written as:

$$\left.\begin{aligned} G_1(t) &= VW^{-1} \\ G_2(t) &= -I \end{aligned}\right\} \tag{A6}$$

The guidance law obtained specifies a change of state to a satisfactory state for every arbitrary state in $N(t)$ by means of a unique impulsive velocity correction linearly related to the original state. Further, no correction is made if the original state is satisfactory. In short, equation (A5) satisfies the definition of a linear impulsive guidance law.

Case b: $n = 4$

Let s_1, s_2, s_3, s_4 be any basis of $S(t)$ and assume that $\overline{r}_1$, $\overline{r}_2$, $\overline{r}_3$ are independent. Define the 3×3 matrices, V, W as above.

Next, let $(\delta\overline{r},\delta\overline{v})$ be any state in $N(t)$ and determine all satisfactory states corresponding to $\delta\overline{r}$; that is, determine all linear combinations

$$\begin{pmatrix}\delta\overline{r}\\ \delta\overline{v}_s\end{pmatrix} = \alpha_1 s_1 + \alpha_2 s_2 + \alpha_3 s_3 + \alpha_4 s_4 \tag{A7}$$

such that

$$\delta\overline{r} = \alpha_1\overline{r}_1 + \alpha_2\overline{r}_2 + \alpha_3\overline{r}_3 + \alpha_4\overline{r}_4$$

or

$$\delta\overline{r} = W\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix} + \alpha_4\overline{r}_4 \tag{A8}$$

The solution of (A8) is not unique, but α_1, α_2, α_3 can be given in terms of the arbitrary parameter, α_4, by inverting equation (A8):

$$\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix} = W^{-1}\delta\overline{r} - \alpha_4 W^{-1}\overline{r}_4;\ \alpha \quad \text{arbitrary}$$

The corresponding velocity deviations on the satisfactory orbits which pass through $\delta\overline{r}$ are, from equation (A7):

$$\delta\overline{v}_s = \alpha_1\overline{v}_1 + \alpha_2\overline{v}_2 + \alpha_3\overline{v}_3 + \alpha_4\overline{v}_4$$

or

$$\delta\overline{v}_s = VW^{-1}\delta\overline{r} + \alpha_4[\overline{v}_4 - VW^{-1}\overline{r}_4];\ \alpha_4 \quad \text{arbitrary}$$

The vector in brackets cannot be zero, since otherwise we obtain the consequence that s_4 is a linear combination of s_1, s_2, s_3 in contradiction to the hypothesis that $s_1, \ldots, s_4$ were independent.[1] Since the vector in brackets is nonzero, it may be replaced by a unit vector in the same direction and, since α_4 is arbitrary, $\delta\overline{v}_s$ can now be rewritten as

$$\delta\overline{v}_s = VW^{-1}\delta\overline{r} + \alpha\overline{u};\ \alpha \quad \text{arbitrary} \tag{A9}$$

where

$$\overline{u} = (\overline{v}_4 - VW^{-1}\overline{r}_4)/|\overline{v}_4 - VW^{-1}\overline{r}_4|$$

In this case there are arbitrarily many satisfactory states corresponding to each position deviation, $\delta\overline{r}$. The union of all such states is then

$$S(t) = \{x(t) = (\delta\overline{r}, VW^{-1}\delta\overline{r} + \alpha\overline{u});\ \delta\overline{r},\ \alpha \quad \text{arbitrary}\} \tag{A10}$$

The velocity change required to correct the state $(\delta\overline{r},\delta\overline{v})$ to a satisfactory state is:

$$\Delta\overline{v} = \delta\overline{v}_s - \delta\overline{v} = VW^{-1}\delta\overline{r} + \alpha\overline{u} - \delta\overline{v};\ \alpha \quad \text{arbitrary}$$

Since α is arbitrary in this result, the velocity correction is not uniquely specified as a function of the state alone, and a condition that properly specifies the value of α must be adopted. In equation (A9) α is arbitrary and, therefore, a satisfactory state is achieved independent of the $\overline{u}$-component of the final velocity. Equivalently, the mission constraints are independent of the velocity deviation in the direction $\overline{u}$. Consequently, it is

---

[1]If $\overline{v}_4 - VW^{-1}\overline{r}_4$ is zero, then the state s_4 can be given as

$$s_4 = \begin{pmatrix}\overline{r}_4\\ \overline{v}_4\end{pmatrix} = \begin{pmatrix}W\\ V\end{pmatrix} W^{-1}\overline{r}_4$$

That is, s_4 would be a linear combination of s_1, s_2, s_3 with the three elements of the vector $W^{-1}\overline{r}_4$ as the constants of combination.

pointless to make any change in the u-component of velocity deviation. Assuming that no such change occurs, α must satisfy the condition[2]

$$0 = \bar{u}^T \Delta \bar{v}$$

from which

$$\alpha = \bar{u}^T(-VW^{-1}\delta\bar{r} + \delta\bar{v})$$

This gives the result

$$\Delta\bar{v} = (I - \bar{u}\bar{u}^T)VW^{-1}\delta\bar{r} - (I - \bar{u}\bar{u}^T)\delta\bar{v} \tag{A11}$$

Equation (A11) satisfies the definition of a linear impulsive guidance law. Its corresponding matrix is then:

$$\left.\begin{aligned} G_1(t) &= (I - \bar{u}\bar{u}^T)VW^{-1} \\ G_2(t) &= -I + \bar{u}\bar{u}^T \\ |\bar{u}| &= 1 \end{aligned}\right\} \tag{A12}$$

Case c: $n = 5$

The proof of the theorem will not be carried out for this case. The steps require only a minor extension of the proof for Case b above.

Case d: $n = 6$

This is the trivial case in which every state in $N(t)$ is satisfactory and no correction is ever made; that is,

$$G_1 = G_2 = 0$$

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[2]The condition adopted is equivalent to choosing α such that the magnitude of the velocity correction is minimized. This condition is not required by the definition of the guidance law, but is included as a possibility. The most general choice of α which satisfies the definition is the form:

$$\alpha = (\bar{u} + \bar{p})^T(-VW^{-1}\delta\bar{r} + \delta\bar{v})$$

where $\bar{p}$ is any vector perpendicular to $\bar{u}$. The corresponding guidance law matrix will then be

$$G_1(t) = -G_2(t)VW^{-1}$$

$$G_2(t) = -I + \bar{u}(\bar{u} + \bar{p})^T$$

This general result points out that the guidance law corresponding to some particular four-dimensional set of satisfactory states $S(t)$ is not unique until a condition on α is adopted. However, the general result is not of any suspected practical interest and will not be investigated further.

In summary, theorem 1 of the text is now proved. Equations (A6) and (A12) together with the analogous result for Case c give the expressions for the guidance law matrix quoted in the text.

APPENDIX B

UNIQUENESS OF THE GUIDANCE LAW MATRIX

It is easily shown that the guidance law matrix for the minimum correction to a given set of satisfactory states is unique. Let $S(t)$ be a given set of satisfactory states and assume that the guidance law exists at t. Let $G(t)$, $G^*(t)$ be two guidance law matrices which satisfy the definition in the first section and correct any state in $N(t)$ to a state in the given set $S(t)$. Since $(I + G^*)x(t)$ is a satisfactory state for any $x(t)$, we have

$$G(I + G^*) = 0$$

$$G^*(1 + G^*) = 0$$

or

$$\left.\begin{aligned} G_1 &= -G_2G_1^* \\ G_2 &= -G_2G_2^* \\ G_1^* &= -G_2^*G_1^* \end{aligned}\right\} \qquad \text{(B1)}$$

The set $S(t)$ may be three-, four-, or five-dimensional, and the remainder of the proof is carried out according to the dimension of $S(t)$.

(a) dim $[S(t)] = 3$. Theorem 1 gives G_2, G_2^* uniquely for this case, as

$$G_2 = G_2^* = -I$$

Then, from the first of equations (B1)

$$G_1 = G_1^*$$

that is, $G = G^*$, and the guidance law matrix is unique.

(b) dim $[S(t)] = 4$. Assuming the condition that the guidance law minimizes the magnitude of the velocity correction, theorem 1 gives G_2, G_2^* in the form:

$$G_2 = -I + \bar{u}\bar{u}^T$$

$$G_2^* = -I + \bar{u}^*\bar{u}^{*T}$$

$$|\bar{u}| = |\bar{u}^*| = 1$$

Inserting these forms in the second equation of (B1), obtain

$$0 = [\bar{u}^* - (\bar{u} \cdot \bar{u}^*)\bar{u}]\bar{u}^{*T}$$

which, since u, u^* are unit vectors, requires that

$$\bar{u}^* = \pm\bar{u}$$

In either case,

$$G_2 = G_2^*$$

and from the first and third equations of (B1)

$$G_1 = -G_2G_1^* = -G_2^*G_1^* = G_1^*$$

that is, $G = G^*$, and the guidance law matrix is unique.

(c) dim [S(t)] = 5. If the guidance law is assumed to minimize the magnitude of the velocity correction, theorem 1 gives G_2, G_2^* in the form

$$G_2 = -I + \bar{u}_1\bar{u}_1^T + \bar{u}_2\bar{u}_2^T$$

$$G_2^* = -I + \bar{u}_1^*\bar{u}_1^{*T} + \bar{u}_2^*\bar{u}_2^{*T}$$

where $\bar{u}_1,\bar{u}_2$ and $\bar{u}_1^*,\bar{u}_2^*$ are orthonormal pairs of vectors. It is convenient to replace these with the equivalent forms:

$$G_2 = -\bar{u}_3\bar{u}_3^T$$

$$G_2^* = -\bar{u}_3^*\bar{u}_3^{*T}$$

where $\bar{u}_1,\bar{u}_2,\bar{u}_3$ and $\bar{u}_1^*,\bar{u}_2^*,\bar{u}_3^*$ are orthonormal triads of vectors. Inserting these forms in the second of equations (B1), obtain

$$0 = \bar{u}_3[\bar{u}_3^T - (\bar{u}_3 \cdot \bar{u}_3^*)\bar{u}_3^{*T}]$$

Following the remaining steps of Case b above, obtain

$$\bar{u}_3^* = \pm\bar{u}_3$$

$$G_2^* = G_2$$

$$G_1^* = G_1$$

that is, $G = G^*$ and the guidance law matrix is unique.

APPENDIX C

PROPERTIES OF THE MATRIX B

$$B = A_4(t,t_o) - B_2(t,t_o)B_1^{-1}(t,t_o)A_2(t,t_o) \tag{C1}$$

The matrix, $B_1(t,t_o)$, has been assumed nonsingular and the matrices of equation (C1) are defined by equations (6) and (8) of the text. The time arguments are dropped in the following.

B is a Nonsingular Matrix

Consider the four independent states given by the columns of

$$\begin{bmatrix} I & \bar{o} \\ G_1(t_o) & \bar{u} \end{bmatrix} \tag{C2}$$

where $\bar{u}$ is any nonzero vector, I is the 3×3 identity matrix, and $G_1(t_o)$ is the appropriate submatrix from the guidance law matrix at t_o. The state transition matrix, $A(t,t_o)$, is never singular, and, hence, the four columns of

$$\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \begin{bmatrix} I & \bar{o} \\ G_1(t_o) & \bar{u} \end{bmatrix} = \begin{bmatrix} B_1 & A_2\bar{u} \\ B_2 & A_4\bar{u} \end{bmatrix} \tag{C3}$$

must also be independent. Suppose, next, that $B\bar{u} = 0$; that is, from equation (C1):

$$A_4\bar{u} = B_2B_1^{-1}A_2\bar{u}$$

Note the identity, $I = B_1B_1^{-1}$, from which:

$$A_2\bar{u} = B_1B_1^{-1}A_2\bar{u}$$

Combining these last two equations yields:

$$\begin{pmatrix} A_2\bar{u} \\ A_4\bar{u} \end{pmatrix} = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} B^{-1}A_2\bar{u}$$

that is, if $B\bar{u} = 0$, then the state $\begin{pmatrix} A_2\bar{u} \\ A_4\bar{u} \end{pmatrix}$ can be given as a linear combination of the columns of $\begin{pmatrix} B_1 \\ B_2 \end{pmatrix}$ where the constants of combination are the three components of the vector $B_1^{-1}A_2\bar{u}$. However, this contradicts the fact that the four columns in equation (C3) are independent; consequently, $B\bar{u}$ cannot be zero. Since this is true for every nonzero vector, $\bar{u}$, it follows that B is nonsingular.

Reduction of B When A_2 is Nonsingular

The definition of B_1 in equations (8) gives

$$I = B_1B_1^{-1} = (A_1 + A_2G_1)B_1^{-1}$$

which can be rearranged to give

$$I - A_2G_1B_1^{-1} = A_1B_1^{-1}$$

Assuming A_2 is nonsingular, pre- and post-multiply by A_2^{-1} and A_2, respectively, and obtain

$$I - G_1B_1^{-1}A_2 = A_2^{-1}A_1B_1^{-1}A_2 \tag{C4}$$

Next, rearrange equation (C1) after introducing the definition of B_2 from equations (8)

$$B = A_4[I - G_1B_1^{-1}A_2] - A_3B_1^{-1}A_2$$

The result in equation (C4) is introduced:

$$B = [A_4A_2^{-1}A_1 - A_3]B_1^{-1}A_2 \tag{C5}$$

The expression in brackets in equation (C5) is $[A_2^{-1}]^T$, which can be proved from the inversion property of the transition matrix[1] (ref. 1)

$$A^{-1} = \begin{bmatrix} A_4{}^T & -A_2{}^T \\ -A_3{}^T & A_1{}^T \end{bmatrix}$$

When the identity $I = AA^{-1}$ is carried out in terms of its 3×3 submatrices, two of the four equations obtained are

$$A_2A_1{}^T - A_1A_2{}^T = 0$$

$$A_4A_1{}^T - A_3A_2{}^T = I$$

From the first of these it follows that $A_2^{-1}A_1$ is a symmetric matrix. Noting this fact in a rearrangement of the second equation then yields the indicated expression for $(A_2^{-1})^T$. Thus, when A_2 is nonsingular, B reduces to

$$B = (A_2^{-1})^TB_1^{-1}A_2$$

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[1]This inversion property holds for all canonical transformations.

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