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A COMPARATIVE ANALYSIS OF CASCADE AND FEEDBACK COMPENSATION

A Thesis Presented to

the Faculty of the Department of Electrical Engineering

University of Houston

In Partial Fulfillment

of the Requirements for the Degree

Master of Science in Electrical Engineering

by

Donald K. Norling

Manned Spacecraft Center

May 1966

A COMPARATIVE ANALYSIS OF CASCADE AND

FEEDBACK COMPENSATION

APPROVED:

Sydney R. Parker
Chairman

Thomas N. Whitaker

Richard H. Gunderson

W. J. Kettinger

C. V. Kirkpatrick
Dean, Cullen College of Engineering

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To my wife

Marilyn

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ABSTRACT

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This thesis presents a comprehensive analysis and comparison of the relative effects of cascade and feedback compensation upon the steady-state and dynamic performance of feedback control systems. In particular, system sensitivity, steady-state system error and actuating signal, log-modulus response, and pole-zero considerations are investigated for cascade compensation and various forms of feedback compensation. Equations relating equivalent feedback and cascade compensators for a given uncompensated plant and an overall system transfer function are developed. Conditions are specified for the realizability of feedback compensators as R-C networks. Specific advantages and limitations of the various modes of compensation are noted and general insight is provided into the relative suitability of cascade and feedback compensation for a given system and set of performance specifications.

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CHAPTER 1

INTRODUCTION

1-1. INTRODUCTION TO CONTROL SYSTEM COMPENSATION

The control systems that are investigated in this thesis are linear, continuous signal feedback control systems. According to the AIEE proposed definition:

> A feedback control system is a control system which tends to maintain a prescribed relationship of one system variable to another by comparing functions of these variables and using the difference as a means of control.[1]

The subject of control system compensation can be introduced by considering a typical design procedure used in arriving at a control system for a given application. This procedure may be summarized as follows:

1. The requirements for the control system are established by a set of performance specifications.

2. A basic system is assembled to perform the desired control function. This basic system will normally consist of the minimum amount of equipment necessary to accomplish the control function.

3. The basic system is analyzed to determine if the performance specifications are met.

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[1] "AIEE Committee Report, Proposed Symbols and Terms for Feedback Control Systems," Elec. Eng., Vol. 70, pt. 2, pp. 905-909, 1951.

4. If the performance of the basic system is not satisfactory, additional elements are introduced into the basic system to modify its characteristics so that it can provide absolute stability and meet the steady-state and transient performance requirements.

The elements that are introduced into the basic system are referred to as a compensator or compensation network[2] since they compensate for the undesirable characteristics of the original system. If the network is introduced into the forward path of the control system, i.e., in series or cascade with the original system, this is referred to as cascade compensation. If the network is placed in a feedback path around the original system, this is referred to as feedback compensation. The network itself may consist of active elements such as amplifiers or tachometers, may consist entirely of passive components such as resistors and capacitors, or may be a combination of both active and passive elements.

1-2. THESIS OBJECTIVE

The objective of this thesis is to present a comparison of the relative effects of cascade compensation and feedback compensation upon the steady-state and dynamic performance of feedback control systems. The specific performance character-

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[2]The compensation elements may in general be mechanical, hydraulic, electrical, etc., in nature; however, this thesis will be concerned with electrical networks when references are made to specific types of compensators.

istics and relationships that are investigated are the following:

1. The sensitivity of the controlled output of the control system to changes in the basic plant and changes in the compensation networks.

2. The steady-state system error and steady-state actuating signal for compensated systems.

3. The approximate log-modulus response of compensated systems.

4. The effects of compensation on the root-locus and corresponding pole-zero configurations.

The conditions for equivalency between cascade and feedback compensated systems will also be investigated. And finally, the relative advantages and disadvantages of the two modes of compensation will be presented.

1-3. RESULTS OF LITERATURE REVIEW

The subject of cascade compensation has been developed in considerable depth in the literature and this information provides the basis of comparison for feedback compensation.

The subject of feedback compensation has, for the most part, received only casual attention in the literature. A notable exception is the chapter that is devoted to feedback compensation in the textbook by D'Azzo and Houpis.[3] However,

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[3]John J. D'Azzo and Constantine H. Houpis, Feedback Control System Analysis and Synthesis (New York: McGraw-Hill Book Company, 1966), Chap. 14.

even this treatment is relatively superficial when compared with the voluminous data that exists for cascade compensation.

Many of the discussions of feedback compensation in the literature are limited to the special case of tachometric feedback of type 1, third-order systems. Several references have rather comprehensive discussions of the root-locus analysis of tachometric feedback compensation; however, the direct comparison of feedback compensation and cascade compensation is almost totally ignored.

Thaler, Bronzino and Kirk have described a technique for reducing multi-loop feedback compensated systems to equivalent cascade compensated systems.[4] As a design tool, this technique is significant in that it permits the design of feedback compensators by applying the well-known techniques of cascade compensation. However, no general insight into the relative advantages and disadvantages of the two modes of compensation is afforded by this technique.

1-4. SYSTEM DESCRIPTIONS AND NOMENCLATURE

The block-diagram representations of cascade compensation and the general case of feedback compensation are shown in Figures 1-1 and 1-2, respectively. In these figures and throughout the thesis, the letter "G" with qualifying subscript

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[4] G. J. Thaler, J. D. Bronzino and D. E. Kirk, "Feedback Compensation: A Design Technique," AIEE Transactions, Vol. 80, pp. 905-909, 1961.

or superscript will denote elements of a system in the direct path and, similarly, the letter "H" will denote elements in a feedback path. The specific elements shown in Figures 1-1 and 1-2 are defined as follows:

1. G_1 is the plant or original uncompensated (basic) system.

2. G_c is the cascade compensator.

3. G_2 may be either a part of the plant or an additional element added to the direct path during compensation.

4. H_1 is a feedback compensator inserted in the inner feedback path.

5. H_2 is a feedback compensator inserted in the outer feedback path.

R denotes the reference input for the system and C denotes the output controlled variable.

Two special cases of feedback compensation are developed in depth during the course of this investigation. Both of these special cases may be derived from the general case of feedback compensation of Fig. 1-2 by selectively setting certain elements in the general case equal to one (short-circuit) or zero (open-circuit). The first special case is derived by setting H_2 equal to zero and G_2 equal to one. The block-diagram of Fig. 1-2 then reduces to a single feedback path containing H_1 as shown in Fig. 1-3. This special case will be referred to as the "single-loop feedback compensated system."

The second special case is derived by setting G_2 and H_2 both equal to one in Fig. 1-2. The general case of feedback compensation then reduces to the form shown in Fig. 1-4. This special case will be referred to as the "double-loop feedback compensated system."

The figures on page 7 depicting the four cases of compensation will be referred to throughout the thesis to avoid their duplication in each chapter. Special system configurations and additional nomenclature will be developed as the need arises.

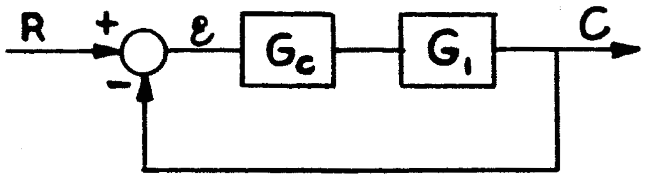

Figure 1-1. Cascade compensated system.

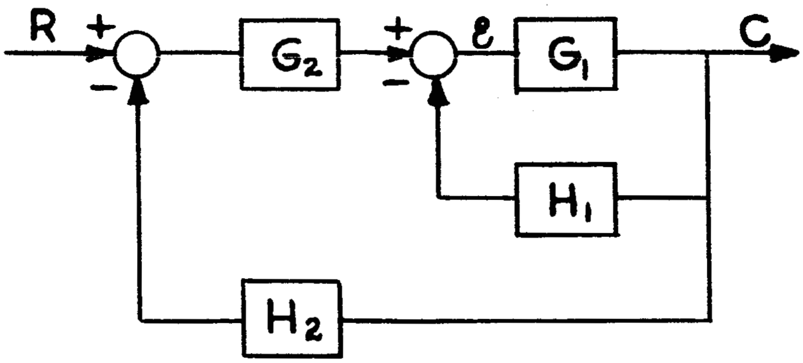

Figure 1-2. General case of feedback compensation.

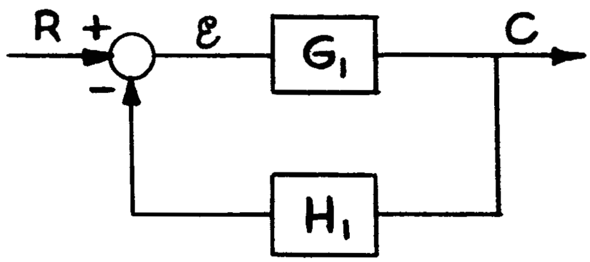

Figure 1-3. Single-loop feedback compensated system.

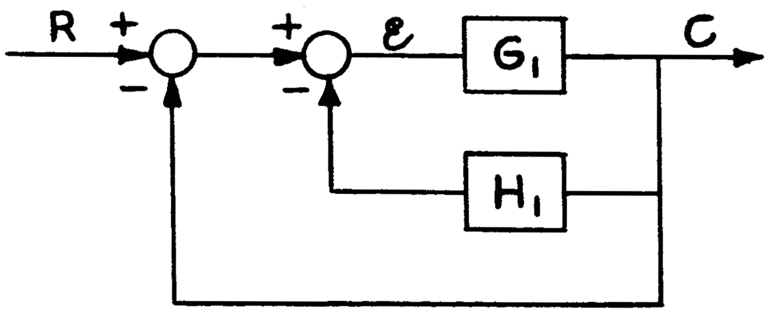

Figure 1-4. Double-loop feedback compensated system.

CHAPTER 2

CLOSED-LOOP EQUIVALENCY FOR COMPENSATED SYSTEMS

The vast majority of existing information on system compensation techniques is concerned with the subject of cascade compensation. However, for each cascade compensation network it is possible to derive a mathematical feedback function that will produce the same overall system transfer function when placed in a feedback path around the uncompensated plant. The form of the feedback function depends upon the feedback configuration, the uncompensated system, and the cascade compensation network to be replaced. The equations relating equivalent feedback and cascade compensation schemes for a given uncompensated plant and an overall system transfer function are developed in this chapter. It remains to be seen whether or not the transfer function so derived can be physically realized in a practical control system. Realizability conditions are therefore considered to determine if a physical passive network can be synthesized that will yield the desired transfer function. Special attention is given to the synthesis of R-C networks for single-loop and double-loop feedback compensated systems.

2-1. EQUATIONS FOR EQUIVALENCY

The equations relating cascade and feedback compensation networks will be developed first for the general case of feedback compensation shown in Fig. 1-2, page 7. The corresponding cascade compensated system is shown on the same page in Fig. 1-1. The general equations for G_2 and H_1 are then simplified for the special cases of single-loop feedback compensation and double-loop feedback compensation by allowing appropriate terms to equal one or zero.

Equations for General Case. The transfer function for the general case of feedback compensation can be expressed as follows:

$$\frac{C}{R} = \frac{G_1G_2}{1 + G_1(H_1 + G_2H_2)} \tag{2-1}$$

The transfer function for the cascade compensated system is given by

$$\frac{C}{R} = \frac{G_1G_c}{1 + G_1G_c} \tag{2-2}$$

For dynamic equivalence, Eq. (2-1) must equal Eq. (2-2). Setting these equations equal and solving for G_2,

$$\frac{G_1G_2}{1 + G_1(H_1 + G_2H_2)} = \frac{G_1G_c}{1 + G_1G_c}$$

$$G_2 + G_1G_2G_c = G_c + G_1G_cH_1 + G_1G_cG_2H_2 \tag{2-3}$$

$$G_2(1 + G_1G_c - G_1G_cH_2) = G_c + G_1G_cH_1$$

$$G_2 = \frac{G_c(1 + G_1H_1)}{1 + G_1G_c(1 - H_2)} \tag{2-4}$$

Solving Eq. (2-3) for G_c

$$G_c(1 + G_1H_1 + G_1G_2H_2 - G_1G_2) = G_2$$

$$G_c = \frac{G_2}{1 + G_1[H_1 + G_2(H_2 - 1)]} \qquad (2\text{-}5)$$

Solving Eq. (2-3) for H_1

$$G_1G_cH_1 = G_2 + G_1G_2G_c - G_c - G_1G_2G_cH_2$$

$$H_1 = \frac{G_2[1 + G_1G_c(1 - H_2)] - G_c}{G_1G_c} \qquad (2\text{-}6)$$

Finally, solving Eq. (2-3) for H_2

$$G_1G_2G_cH_2 = G_2 + G_1G_2G_c - G_c - G_1G_cH_1$$

$$H_2 = \frac{G_c[G_1(G_2 - H_1) - 1] + G_2}{G_1G_2G_c} \qquad (2\text{-}7)$$

Equations (2-4), (2-5), (2-6) and (2-7) relate the various transfer functions of the cascade compensated system and the general case of feedback compensation for equivalence. For a given uncompensated plant, G_1, and cascade compensation network, G_c, three inter-dependent equations must be solved for the parameters of the equivalent feedback system. Two of the parameters can be selected arbitrarily on a trial and error basis and the third parameter calculated from the appropriate equation.

Cascade and Single-loop Equivalency. If H_2 is set equal to zero and G_2 is set equal to one in Fig. 1-2, page 7, the block-diagram reduces to the single-loop feedback compensated system of Fig. 1-3 on the same page. The same substitutions in Eq. (2-6) results in the following expression for H_1 for the single-loop case:

$$H_1 = \frac{1 + G_c(G_1 - 1)}{G_1 G_c} \tag{2-8}$$

Similarly, substitution of $H_2 = 0$ and $G_2 = 1$ into Eq. (2-5) results in the following expression for G_c

$$G_c = \frac{1}{1 + G_1(H_1 - 1)} \tag{2-9}$$

The equivalency that is assured by these equations can be demonstrated by determining the characteristic equation for the system employing a single-loop feedback compensation network defined by Eq. (2-8). The open-loop transfer function for this system is G_1H_1, where

$$G_1H_1 = \frac{1 + G_c(G_1 - 1)}{G_c} \tag{2-10}$$

The characteristic equation for this system is given by the expression $1 + G_1H_1 = 0$ or

$$1 + \frac{1 + G_c(G_1 - 1)}{G_c} = 0 \tag{2-11}$$

Equation (2-11) reduces to the following

$$1 + G_1G_c = 0 \tag{2-12}$$

But Eq. (2-12) is also the characteristic equation for the cascade compensated system, and equivalency is thus seen to exist.

<u>Cascade and Double-loop Equivalency</u>. If H_2 and G_2 are

both set equal to one in Fig. 1-2, page 7, the block-diagram reduces to the double-loop feedback compensated system of Fig. 1-4 on the same page. The same substitutions in Eq. (2-6) results in the following expression for H_1 for the double-loop case:

$$H_1 = \frac{1 - G_c}{G_1 G_c} \qquad (2\text{-}13)$$

Similarly, substitution of $H_2 = G_2 = 1$ into Eq. (2-5) results in the following expression for G_c

$$G_c = \frac{1}{1 + G_1 H_1} \qquad (2\text{-}14)$$

The same results for H_1 and G_c could have been obtained by equating the appropriate open-loop transfer functions (OLTF) for the two systems. The appropriate OLTF is derived by transforming the block-diagram shown in Fig. 1-4 into its equivalent form as illustrated in Fig. 2-1. The system G' in Fig. 2-1 is simply $G_1/(1 + G_1 H_1)$, and this transfer function is the OLTF for the equivalent unity feedback system.

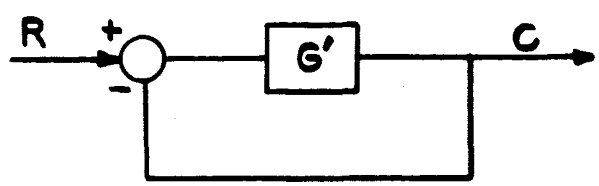

Figure 2-1. Equivalent double-loop feedback compensated system.

Equating G' and the OLTF for the cascade compensated system

$$G_1 G_c = \frac{G_1}{1 + G_1 H_1} \qquad (2\text{-}15)$$

Solving Eq. (2-15) for H_1

$$H_1 = \frac{1 - G_c}{G_1 G_c} \qquad (2\text{-}16)$$

And Eq. (2-16) and Eq. (2-13) are the same.

2-2. NETWORK SYNTHESIS

For a given uncompensated plant, G_1, and cascade compensation network, G_c, Eq. (2-8) and (2-13) can be used to derive the transfer function for equivalent feedback compensation networks that will yield an overall system transfer function identical to the cascade compensated system. The expressions resulting from Eq. (2-8) and (2-13) can then be analyzed to determine whether the transfer functions they represent are physically realizable as a linear passive network. In general, if these equations indicate the requirement for an active element in the compensation network, the cascade compensation approach would be preferred. An exception to this rule could exist in those cases where a tachometer by itself or in combination with some form of passive network could provide the desired transfer function for the feedback compensation network.

The subject of network synthesis is so vast and involved that no attempt will be made to develop the theory or techniques in this paper. The only aspect of network synthesis that will be discussed is the realizability of the transfer functions expressed by Eq. (2-8) and (2-13). Basic conditions for the realizability of passive networks in general will be presented first, followed by a discussion and delineation of conditions for the special case of R-C networks. The reader is referred to several references for the proof of these conditions and the general development of synthesis techniques.[1] The remainder of this chapter is concerned with the application of the realizability conditions to the single-loop and double-loop feedback compensation cases and the interpretation of the results.

Transfer Functions for Passive Networks in General. The characteristics of passive transfer functions in general may be summarized as follows:[2]

1. All poles of the transfer function must lie within the left-half portion of the s-plane.

2. Zeros of the transfer function may lie anywhere within the s-plane. Minimum phase-shift transfer functions have their zeros restricted to the left-half of the s-plane.

3. The highest power of s in the numerator may equal but cannot exceed the highest power of s in the denominator.

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[1]Vincent Del Toro and Sydney R. Parker, Principles of Control Systems Engineering (New York: McGraw-Hill Book Company, Inc., 1960), Chap. 12; Ernst A. Guillemin, Synthesis of Passive Networks (New York: John Wiley & Sons, Inc., 1957); John G. Truxal, Automatic Feedback Control System Synthesis (New York: McGraw-Hill Book Company, Inc., 1955).

[2]Vincent Del Toro and Sydney R. Parker, Principles of Control Systems Engineering, pp. 508-9.

Transfer Functions for R-C Networks. Once the realizability of the transfer function for the feedback compensation network is established, it would be desirable to synthesize the network solely in terms of resistance and capacitance elements. Inductances are normally avoided since the frequencies of interest in control systems are so low that large and heavy inductors would be required. The simplest R-C networks to achieve are the ladder networks; however, the zeros of the transfer function are restricted to the negative real axis of the s-plane for the ladder form. The lattice is the most general network configuration and any transfer function realizable as an R-C network can be synthesized in the lattice form. The characteristics of R-C networks may be summarized as follows:[3]

1. The poles of the transfer function are restricted to the negative real axis of the s-plane.

2. For minimum-phase-shift networks, the zeros of the transfer function are restricted to the left-half s-plane.

   a) R-C ladder network--zeros must lie on the negative real axis.

   b) Parallel-ladder or split-T networks--zeros allowed off the negative real axis.

3. For non-minimum-phase-shift networks, the zeros of the transfer function are permitted in the right-half s-plane.

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[3] Ibid., pp. 509-11.

A lattice network is required for this case.

Practical Considerations. For any given transfer function an infinite number of physical networks can be derived that will satisfy the pole-zero location and gain requirements. However, most of these solutions will be impractical for one or more of the following reasons: the network requires too many elements, the magnitude of the element values are impractical, the steady-state attenuation is excessive, or the network transfer function is overly sensitive to small deviations in the network element values. Even after these factors are considered there may be many practical networks that will satisfy the given transfer function. The final choice of a compensation network may be arbitrary or simply depend on the availability of components and the circuit designer's own preferences.

2-3. APPLICATION OF NETWORK SYNTHESIS CONDITIONS TO FEEDBACK COMPENSATED SYSTEMS

The equivalent feedback compensation transfer functions defined by Equations (2-8) and (2-13) are analyzed for their realizability in terms of the functions G_1 and G_c of the cascade compensated system (See Fig. 2-2.). The functions G_1 and G_c are defined as follows:

$$G_1(s) = \frac{K_1 N_1(s)}{s^N D_1(s)} \qquad (2\text{-}17)$$

$$G_c(s) = \frac{K_cN_c(s)}{s^MD_c(s)} \tag{2-18}$$

In general, the order of s in the denominator of G_1 will be equal to or greater than the order of s in the numerator. The order of s in the denominator of the cascade compensation network, G_c, will be equal to or greater than the order of s in the numerator for a passive network.

Single-loop Feedback Compensation. If equations (2-17) and (2-18) are substituted in Eq. (2-8), H_1 will take the following form:

$$H_1 = \frac{1 + G_c(G_1 - 1)}{G_1G_c} = \frac{1 + (K_cN_c/s^MD_c)[(K_1N_1/s^ND_1) - 1]}{(K_cN_c/s^MD_c)(K_1N_1/s^ND_1)}$$

$$= \frac{s^{N+M}D_1D_c + K_cN_c(K_1N_1 - s^ND_1)}{K_1N_1K_cN_c} = \frac{N'}{D'} \tag{2-19}$$

Analyzing Eq. (2-19) in terms of the network synthesis conditions presented in Section 2-2, the following restrictions must be placed on G_1 and G_c if H_1 is to be realizable as a passive network:

1. The order of s in D' must be equal to or greater than the order of s in N', i.e., $\Theta[D'(s)] \geq \Theta[N'(s)]$.[4] This condition will exist for the following cases:

a) $\Theta[N_1(s)] = \Theta[s^ND_1(s)]$ and $\Theta[N_c(s)] = \Theta[s^MD_c(s)]$

[4] $\Theta[\,]$ denotes the order of the function appearing within the brackets.

b) $\theta[N_1(s)] < \theta[s^N D_1(s)]$ and $\theta[N_c(s)] > \theta[s^M D_c(s)]$
so that $\theta[D'(s)] \geq \theta[N'(s)]$

2. The zeros of G_1 and G_c must lie in the left-half of the s-plane. This restriction results because the zeros of G_1 and G_c are the poles of H_1.

Realization of H_1 as an R-C network imposes the additional condition that the zeros of G_1 and G_c must lie on the negative real axis of the s-plane.

Double-loop Feedback Compensation. If Equations (2-17) and (2-18) are substituted in Eq. (2-13), H_1 will take the following form:

$$H_1 = \frac{1 - G_c}{G_1 G_c} = \frac{1 - (K_c N_c / s^M D_c)}{(K_1 N_1 / s^N D_1)(K_c N_c / s^M D_c)}$$

$$= \frac{s^N D_1 (s^M D_c - K_c N_c)}{K_1 N_1 K_c N_c} = \frac{N''}{D''} \qquad (2\text{-}20)$$

A comparison of Eq. (2-20) and Eq. (2-19) reveals that the same restrictions must be placed on G_1 and G_c for H_1 to be realizable as a passive network in general, or an R-C network in particular, as were specified for the single-loop feedback compensation case.

For the particular case where G_c is a phase-lag or phase-lead network defined as follows:

$$G_c = \frac{s + z}{s + p} \qquad (2\text{-}21)$$

H_1 for the double-loop feedback compensation case may be expressed as

$$H_1 = \frac{1 - G_c}{G_1 G_c} = \frac{1}{G_1} \frac{1 - (s + z)/(s + p)}{(s + z)/(s + p)}$$

$$= \frac{1}{G_1} \frac{(p - z)}{(s + z)} \qquad (2\text{-}22)$$

Notice that the term $(p - z)/(s + z)$ is a simple phase-lag network if $p > z$. However, the transfer function for H_1 may still be very involved depending upon the form of G_1.

2-4. SUMMARY AND CONCLUSIONS

Equations have been presented that relate the transfer functions for cascade compensation networks and equivalent feedback compensation networks in terms of the uncompensated system transfer function. The basis for these equations was closed-loop equivalency for the compensated system. Since two systems having the same transfer function are equivalent both statically and dynamically, the equations relating the transfer functions of the various forms of compensation are applicable for both steady-state system error equivalence (See Chapter 4.) and dynamic equivalence. The equations for the feedback compensation networks for single-loop and double-loop systems were analyzed to determine the conditions under which they could be physically realized as passive networks, with special attention given to R-C networks. It was noted that the transfer functions for the feedback networks are usually rather involved expressions, and more significantly, there are rather severe constraints imposed on G_1 and G_c to permit an equivalent feedback network to be physically realizable with only resistances and capacitances.

CHAPTER 3

COMPARISON OF SYSTEM SENSITIVITIES

The characteristics of the components making up a control system can change as a result of changing environmental conditions, aging of components, etc. Any change in the component characteristics will be reflected by a change in the transfer function for the system, with a resulting effect on the controlled quantity. It has been shown by D'Azzo and Houpis that the degree of accuracy and stability of a control system can be improved by using feedback compensation.[1] The conclusions by D'Azzo and Houpis are based on the comparison of a single-loop feedback compensated system with a unity feedback uncompensated system. It is true that the non-unity feedback system can reduce the effects of system component changes on the controlled quantity when compared with a unity feedback system having the same forward transfer function, G_1. However, the same conclusion is not valid when comparing the non-unity feedback system with a cascade compensated system. The significance of these conclusions will become apparent in the development that follows.

The effects of changes in both the uncompensated system,

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[1]John J. D'Azzo and Constantine H. Houpis, Feedback Control System Analysis and Synthesis (New York: McGraw-Hill Book Company, 1966), pp. 467-470.

G_1, and the compensation elements, G_c and H_1, are evaluated for the following system configurations: (1) open-loop, (2) unity feedback uncompensated, (3) cascade compensated, (4) single-loop feedback compensated, and (5) double-loop feedback compensated. The input signal R and the frequency are considered to be constant.

3-1. OPEN-LOOP SYSTEM SENSITIVITY

The open-loop system is shown in Fig. 3-1. The effect of a change in G_1 can be determined by differentiating

$$C = RG_1 \tag{3-1}$$

giving

$$dC = RdG_1 \tag{3-2}$$

Substituting R from Eq. (3-1) into Eq. (3-2)

$$\frac{dC}{C} = \frac{dG_1}{G_1} \tag{3-3}$$

Therefore, a change in G_1 causes a corresponding change in the output C. The performance specifications of the components of G_1 must then be such that the system accuracy is kept within specified limits.

Figure 3-1. Open-loop system.

Employing the identity $d(\ln u) = u^{-1}du$, Eq. (3-3) can be put into the following form:

$$d\ln C = d\ln G_1 \tag{3-4}$$

$$\frac{d\ln C}{d\ln G_1} = 1 \tag{3-5}$$

Defining C and G_1 as follows:

$$C = |C|\, e^{j\theta_c} \tag{3-6}$$

$$G_1 = |G_1|\, e^{j\theta_{G_1}} \tag{3-7}$$

Substituting Equations (3-6) and (3-7) into Eq. (3-4)

$$d\ln|C| + jd\theta_c = d\ln|G_1| + jd\theta_{G_1} \tag{3-8}$$

Equating real and imaginary parts of Eq. (3-7) results in the following relationship between the differential changes in the magnitudes and phase angles for the system, G_1, and the output, C:

$$d\ln|C| = d\ln|G_1| \tag{3-9}$$

$$d\theta_c = d\theta_{G_1} \tag{3-10}$$

3-2. UNITY FEEDBACK UNCOMPENSATED SYSTEM SENSITIVITY

The unity feedback uncompensated system is shown in Fig. 3-2. Proceeding in the same manner as in 3-1

$$C = R\frac{G_1}{1 + G_1} \tag{3-11}$$

$$dC = R\frac{dG_1}{(1 + G_1)^2} \tag{3-12}$$

Substituting R from Eq. (3-11) into Eq. (3-12)

$$\frac{dC}{C} = \frac{1}{1 + G_1}\left(\frac{dG_1}{G_1}\right) \tag{3-13}$$

$$\frac{d\ln C}{d\ln G_1} = \frac{1}{1 + G_1} \tag{3-14}$$

A comparison of Eq. (3-14) and Eq. (3-5) reveals that the effect of parameter changes in G_1 upon the output C is reduced by the factor $1/(1 + G_1)$ when going from open-loop to closed-loop control.

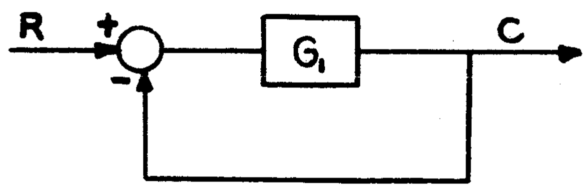

Figure 3-2. Unity feedback uncompensated system.

3-3. CASCADE COMPENSATED SYSTEM SENSITIVITY

The cascade compensated system is shown in Fig. 1-1, page 7. The effect of changes in the uncompensated system, G_1, and the compensation element, G_c, are evaluated below.

Sensitivity to Changes in G_1. First consider that the compensation element is a constant with respect to the changes that are affecting G_1. Proceeding as before,

$$C = R\frac{G_1 G_c}{1 + G_1 G_c} \tag{3-15}$$

$$dC = R\frac{G_c dG_1}{(1 + G_1 G_c)^2} \tag{3-16}$$

Substituting R from Eq. (3-15) into Eq. (3-16),

$$\frac{dC}{C} = \frac{1}{1 + G_1 G_c}\left(\frac{dG_1}{G_1}\right) \tag{3-17}$$

$$\frac{d\ln C}{d\ln G_1} = \frac{1}{1 + G_1 G_c} \tag{3-18}$$

Therefore cascade compensation has reduced the effect of changes in G_1 by the factor $1/(1 + G_1G_c)$ when compared with the open-loop uncompensated system. This also constitutes an improvement over the unity feedback uncompensated system if the magnitude of G_c is greater than one for the frequencies of interest.

Sensitivity to Changes in G_c. Now consider that the uncompensated system, G_1, is a constant and only the components of G_c are effected by changes. From Eq. (3-15),

$$dC = \frac{R(1 + G_1G_c)G_1 - G_1^2G_c}{(1 + G_1G_c)^2}$$

$$= \frac{RG_1dG_c}{(1 + G_1G_c)^2} \tag{3-19}$$

Substituting R from Eq. (3-15) into Eq. (3-19),

$$\frac{dC}{C} = \frac{1}{1 + G_1G_c}\left(\frac{dG_c}{G_c}\right) \tag{3-20}$$

$$\frac{d\ln C}{d\ln G_c} = \frac{1}{1 + G_1G_c} \tag{3-21}$$

A comparison of Eq. (3-21) and Eq. (3-18) reveals that the effect of parameter changes in G_c upon the output C is the same as for changes in G_1, as would be expected.

3-4. SINGLE-LOOP FEEDBACK COMPENSATED SYSTEM SENSITIVITY

The single-loop feedback compensated system is shown in Fig. 1-3, page 7. The effect of changes in G_1 and the compensation element, H_1, are evaluated below.

Sensitivity to Changes in G_1. First consider that H_1 is a constant with respect to the changes that are affecting G_1. Proceeding as before

$$C = R\frac{G_1}{1 + G_1H_1} \tag{3-22}$$

$$dC = R\frac{dG_1}{(1 + G_1H_1)^2} \tag{3-23}$$

Substituting R from Eq. (3-22) into Eq. (3-23)

$$\frac{dC}{C} = \frac{1}{1 + G_1H_1}\left(\frac{dG_1}{G_1}\right) \tag{3-24}$$

$$\frac{d\ln C}{d\ln G_1} = \frac{1}{1 + G_1H_1} \tag{3-25}$$

A comparison of Eq. (3-25) and Eq. (3-18) reveals the fact that single-loop feedback compensation and cascade compensation offer the same reduction in the effect of changes in G_1 upon the controlled quantity, C. That is, if G_c and H_1 are equal, then Eq. (3-25) and Eq. (3-18) are identical.

Sensitivity to Changes in H_1. Now consider that G_1 is a constant and only the components of H_1 are affected by changes. From Eq. (3-22)

$$dC = \frac{- G_1{}^2RdH_1}{(1 + G_1H_1)^2} \tag{3-26}$$

Substituting R from Eq. (3-22) into Eq. (3-26) and multiplying and dividing the resulting equation by H_1 gives

$$\frac{dC}{C} = \frac{- G_1H_1}{1 + G_1H_1}\left(\frac{dH_1}{H_1}\right) \tag{3-27}$$

For those values of frequency where $|G_1H_1| \gg 1$, Eq. (3-27) reduces to the following form

$$\frac{dC}{C} \approx -\frac{dH_1}{H_1} \qquad (3\text{-}28)$$

$$\frac{d\ln C}{d\ln H_1} \approx -1 \qquad (3\text{-}29)$$

A comparison of Eq. (3-29) and Eq. (3-5) shows that a change in the feedback function has approximately a direct effect upon the output in the same manner as for the open-loop system.

3-5. DOUBLE-LOOP FEEDBACK COMPENSATED SYSTEM SENSITIVITY

The double-loop feedback compensated system is shown in Fig. 1-4, page 7. The effect of changes in G_1 and H_1 are again evaluated below.

Sensitivity to Changes in G_1. Again consider that H_1 is a constant. Proceeding as before

$$C = R\frac{G_1}{1 + G_1(H_1 + 1)} \qquad (3\text{-}30)$$

$$dC = R\frac{dG_1}{[1 + G_1(H_1 + 1)]^2} \qquad (3\text{-}31)$$

Substituting R from Eq. (3-30) into Eq. (3-31)

$$\frac{dC}{C} = \frac{1}{1 + G_1(H_1 + 1)}\left(\frac{dG_1}{G_1}\right) \qquad (3\text{-}32)$$

$$\frac{d\ln C}{d\ln G_1} = \frac{1}{1 + G_1(H_1 + 1)} \qquad (3\text{-}33)$$

A comparison of Equations (3-33), (3-25) and (3-18) reveals that double-loop feedback compensation can have a greater effect in reducing output changes due to changes in G_1 than either single-loop feedback or cascade compensation. The degree of improvement depends upon the magnitude of H_1.

Sensitivity to Changes in H_1. Again consider that G_1 is a constant. From Eq. (3-30)

$$dC = \frac{-\,G_1}{[1 + G_1(H_1 + 1)]^2} R dH_1 \tag{3-34}$$

Substituting R from Eq. (3-30) into Eq. (3-34) and multiplying and dividing the resulting equation by H_1 gives

$$\frac{dC}{C} = \frac{-\,G_1H_1}{1 + G_1(H_1 + 1)}\left(\frac{dH_1}{H_1}\right) \tag{3-35}$$

$$\frac{d\ln C}{d\ln H_1} = \frac{-\,G_1H_1}{1 + G_1(H_1 + 1)} \tag{3-36}$$

A comparison of Eq. (3-36) and Eq. (3-27) reveals a potential improvement for the double-loop feedback compensated system.

3-6. SENSITIVITY FUNCTION

The sensitivity of a system's response to a variation in a system parameter can best be expressed by the "sensitivity function,"[2] S_δ^M, which is discussed by D'Azzo and Houpis,[3]

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[2]"Sensitivity and Modal Response for Single-loop and Multiloop Systems," Technical Documentary Report ASD-TDR-62-812, Flight Control Laboratory, ASD, AFSC, Wright-Patterson AFB, Ohio, January, 1963.

[3]D'Azzo and Houpis, op. cit., pp. 469-470.

and is defined as

$$S_{\delta}^{M} = \left[\frac{\text{Change in system response}}{\text{Change in open-loop parameter}}\right] \text{ for specified parameter variations} \qquad (3\text{-}37)$$

Change is defined as the ratio of the differential variation of a function to the function itself. Expressed differently, change may be defined as the differential of the natural logarithm of the function. For each of the system cases in Section 3-1 through Section 3-5, M = C/R in Eq. (3-37). δ refers to G_1 for those cases where the uncompensated plant is changing and refers to G_c or H_1 when the compensation element is the changing quantity.

To demonstrate the application of Eq. (3-37), consider the single-loop feedback compensated system with $\delta = G_1$. Then

$$S_{\delta}^{M} = \frac{(dC/R)/(C/R)}{dG_1/G_1} = \frac{dC/C}{dG_1/G_1}$$

$$= \frac{d\ln C}{d\ln G_1} = \frac{1}{1 + G_1H_1}$$

Similarly, for $\delta = H_1$

$$S_{\delta}^{M} = \frac{(dC/R)/(C/R)}{dG_1/G_1} = \frac{dC/C}{dG_1/G_1}$$

$$= \frac{d\ln C}{d\ln H_1} = \frac{-G_1H_1}{1 + G_1H_1} \approx -1, \text{ for } G_1H_1 \gg 1$$

The sensitivity functions for each of the system cases are tabulated in Table 3-1.

3-7. SUMMARY AND CONCLUSIONS

The effects of changes in the uncompensated system, G_1, and the compensation elements, G_c or H_1, on the controlled quantity, C, have been calculated for several system configurations and parameter variations. The sensitivity of each system's response to system parameter variations has been expressed by the sensitivity function, S_g^M, and the results tabulated in Table 3-1. The sensitivity function never exceeds a value of one, and the smaller its value, the less sensitive the system will be to parameter variations.

Referring to Table 3-1, it is noted that for variations in G_1, cascade compensation and single-loop feedback compensation can provide the same reduction in the sensitivity function for values of G_c and H_1 greater than one. For variations in the compensation network itself, the cascade compensated system's sensitivity function, $1/(1 + G_1G_c)$, will normally be less than the corresponding sensitivity function for the single-loop feedback compensated system, $- G_1H_1/(1 + G_1H_1)$. However, since the signal in the forward path is normally going from a low to a high energy level while the opposite is true for the feedback path, it will often be more practical to provide the power requirement in the forward path and then design the feedback compensation network to give the desired output accuracy and stability.

Double-loop feedback compensation offers a potential reduction in the sensitivity functions for changes in both

TABLE 3-1

SENSITIVITY FUNCTIONS

SYSTEM	CHANGING PARAMETER OF SYSTEM*	SENSITIVITY FUNCTION S_{δ}^{M}
Open-loop (uncompensated)	G_1	1
Unity feedback (uncompensated)	G_1	$1/(1 + G_1)$
Cascade compensated	G_1	$1/(1 + G_1G_c)$
	G_c	$1/(1 + G_1G_c)$
Single-loop feedback compensated	G_1	$1/(1 + G_1H_1)$
	H_1	$- G_1H_1/(1 + G_1H_1) \approx - 1$
Double-loop feedback compensated	G_1	$1/[1 + G_1(H_1 + 1)]$
	H_1	$- G_1H_1/[1 + G_1(H_1 + 1)]$

*The system input, R, is constant. For those cases where G_1 is the changing parameter, H_1 and G_c are constant with respect to the changes that are affecting G_1. Conversely, when H_1 and G_c are the changing parameters, G_1 is constant.

the plant and the compensation network when compared with the single-loop case. When compared with the cascade compensated system, the double-loop system offers a potential improvement in the sensitivity function for changes in the plant. The actual improvement will depend upon the magnitude of the compensation function for each case over the frequency range of interest.

CHAPTER 4

COMPARISON OF STEADY-STATE PERFORMANCE

This chapter is concerned with the steady-state performance that can be achieved with feedback compensation as compared with cascade compensation. The functions of interest are the steady-state system error and the steady-state actuating signal. For this analysis the system error, θ_e, is defined as the difference between the input to the system and the system response or output. The actuating signal, $\mathcal{E}$, is defined as the difference between the input signal and the feedback signal as they appear at the input to the compensated plant. These functions are well known for the cascade compensation case but they have received very little attention in the literature for the case of feedback compensation. The results of the analysis presented will give valuable insight into the relative suitability of the two modes of compensation for a given plant and a given set of performance specifications.

4-1. CASCADE COMPENSATION

Steady-state conditions are presented for the system shown in Fig. 1-1, page 7. The first function of interest is the steady-state system error.

Steady-state System Error. The system error, θ_e, is defined as follows:

$$\theta_e = R - C = R(1 - C/R) \tag{4-1}$$

The system transfer function is

$$\frac{C}{R} = \frac{G_1G_c}{1 + G_1G_c} \tag{4-2}$$

Substitution of Eq. (4-2) into Eq. (4-1) gives

$$\theta_e = R\left[1 - \frac{G_1G_c}{1 + G_1G_c}\right] = \frac{R}{1 + G_1G_c} \tag{4-3}$$

From Eq. (4-3), the steady-state system error is

$$\theta_e(t)_{ss} = \lim_{t\to\infty} \theta_e(t)$$

$$= \lim_{s\to 0} sR(s) \frac{1}{1 + G_1(s)G_c(s)} \tag{4-4}$$

The input, expressed in a general Laplace transform for step, ramp, parabolic and other algebraic inputs, is given by

$$R(s) = r_\alpha/s^\alpha \tag{4-5}$$

where

$$\begin{aligned} R(s) &= r_1/s \quad \text{for a step input} \\ &= r_2/s^2 \quad \text{for a ramp input} \\ &= r_3/s^3 \quad \text{for a parabolic input} \end{aligned}$$

Substitution of Eq. (4-5) into Eq. (4-4) gives

$$\theta_e(t)_{ss} = \lim_{s\to 0} \frac{r_\alpha}{s^{\alpha-1}}\left[\frac{1}{1 + G_1(s)G_c(s)}\right] \tag{4-6}$$

The Laplace transform of the uncompensated system, G_1, is defined in the following factored form:

$$G_1 = \frac{K_1(1 + s\tau_a)(1 + s\tau_b)\cdots}{s^N(1 + s\tau_1)(1 + s\tau_2)\cdots} = \frac{K_1N_1}{s^ND_1} \tag{4-7}$$

where

$$N_1 = (1 + s\tau_a)(1 + s\tau_b)\cdots$$

$$D_1 = (1 + s\tau_1)(1 + s\tau_2)\cdots$$

In eq. (4-7), N denotes the type (number of pure integrations) of the uncompensated system and will in general be equal to, or greater than, zero.

Similarly, the compensation element, G_c, is defined in the following generalized form:

$$G_c = \frac{K_c(1 + s\tau_a')(1 + s\tau_b')\cdots}{S^M(1 + s\tau_1')(1 + s\tau_2')\cdots} = \frac{K_c N_c}{S^M D_c} \qquad (4\text{-}8)$$

where

$$N_c = (1 + s\tau_a')(1 + s\tau_b')\cdots$$

$$D_c = (1 + s\tau_1')(1 + s\tau_2')\cdots$$

In Eq. (4-8), M may take on any value (positive or negative) depending on the form of the compensation element. If G_c is of a form having a positive power of s in the numerator, then M in Eq. (4-8) will be negative, and G_c will be referred to as a negative type M element.

Substitution of Eq. (4-7) and Eq. (4-8) into Eq. (4-6) gives

$$\theta_e(t)_{ss} = \lim_{s \to 0} \frac{r_\alpha}{s^{\alpha-1}} \left[\frac{s^{N+M} D_1 D_c}{s^{N+M} D_1 D_c + K_1 N_1 D_c N_c} \right] \qquad (4\text{-}9)$$

Equation (4-9) may be simplified by noting that

$$\lim_{s \to 0} N_1 = \lim_{s \to 0} D_1 = \lim_{s \to 0} N_c = \lim_{s \to 0} D_c = 1 \qquad (4\text{-}10)$$

Equation (4-9) then reduces to the following general expression for the steady-state system error:

$$\theta_e(t)_{ss} = \lim_{S \to o} \frac{r_\alpha}{s^{\alpha-1}} \left(\frac{s^{N+M}}{s^{N+M} + K_1 K_c} \right) \qquad (4\text{-}11)$$

Note that N + M in Eq. (4-11) denotes the type of the compensated system, where it is understood that if M is negative and $|M| > N$, a negative system type will result.

The steady-state system error can be developed in a systematic manner by considering the value of Eq. (4-11) for three values of the function N + M.

1. $N + M = 0$

$$\theta_e(t)_{ss} = \lim_{S \to o} \frac{r_\alpha}{s^{\alpha-1}} \left(\frac{1}{1 + K_1 K_c} \right) \qquad (4\text{-}12)$$

$$= \frac{K_1}{1 + K_1 K_c} \quad \text{for } \alpha = 1$$

$$= \infty \quad \text{for } \alpha > 1$$

2. $N + M > 0$

$$\theta_e(t)_{ss} = 0 \quad \text{for } N + M > (\alpha - 1)$$

$$= r_\alpha / K_1 K_c \quad \text{for } N + M = (\alpha - 1)$$

$$= \infty \quad \text{for } N + M < (\alpha - 1)$$

3. $N + M < 0$

For negative values of the function N + M, Eq. (4-11) may be converted to a more convenient form by multiplying numerator and denominator by $s^{|N+M|}$ giving

$$\theta_e(t)_{ss} = \lim_{S \to o} \frac{r_\alpha}{s^{\alpha-1}} \left(\frac{1}{1 + K_1 K_c s^{|N+M|}} \right), \quad M < 0 \qquad (4\text{-}13)$$

$$= r_\alpha \text{ for } \alpha = 1$$

$$= \infty \text{ for } \alpha > 1$$

To demonstrate the application of the equations derived above, consider the case of a step input, where $\alpha = 1$, and Eq. (4-11) becomes

$$\theta_e(t)_{ss} = \lim_{s \to 0} \frac{r_1 s^{N+M}}{s^{N+M} + K_1 K_c} \qquad (4\text{-}14)$$

Equation (4-14) reduces to one of three values depending on the value of N + M, i.e.,

$$\begin{aligned} \theta_e(t)_{ss} &= 0 && \text{for } N + M \geq 1 \\ &= r_1/(1 + K_1 K_c) && \text{for } N + M = 0 \\ &= \infty && \text{for } N + M < 0 \end{aligned}$$

The term K_1K_c in the preceding equations corresponds to what is usually referred to as the error coefficient for a system. For a step input, K_1K_c is the position or step error coefficient. For a ramp input, K_1K_c is the velocity or ramp error coefficient. And for a parabolic input, K_1K_c is the acceleration or parabolic error coefficient.

The steady-state system errors for several values of N and M are tabulated in Table 4-1 for a step, ramp, and parabolic input. Results can of course be obtained for higher-order inputs by substituting the appropriate value of α into Eq. (4-11).

Steady-state Actuating Signal. Referring to Fig. 1-1, it is obvious that the steady-state actuating signal, $\varepsilon(t)_{ss}$, and the steady-state system error, $\theta_e(t)_{ss}$, are equal for the cascade compensated system, i.e., $\varepsilon = \theta_e = R - C$. Therefore,

TABLE 4-1

STEADY-STATE SYSTEM ERROR, $\theta_e(t)_{ss}$, AND STEADY-STATE ACTUATING SIGNAL, $\mathcal{E}(t)_{ss}$, FOR CASCADE COMPENSATED SYSTEMS $[\theta_e(t)_{ss} = \mathcal{E}(t)_{ss}]$

TYPE		COMPENSATED SYSTEM TYPE	STEP INPUT	RAMP INPUT	PARABOLIC INPUT
N	M	N+M	(α= 1)	(α= 2)	(α= 3)
0 P	0 -P	0	$\frac{r_1}{1+K_1K_c}$	∞	∞
0 1 P+1	1 0 -P	1	0	$\frac{r_2}{K_1K_c}$	∞
1 2 0 P+2	1 0 2 -P	2	0	0	$\frac{r_3}{K_1K_c}$
2 1 3 0 P+3	1 2 0 3 -P	3	0	0	0
Q+1	P	>3	0	0	0
0 1 P	-1 -2 -Q	<0	r_1	∞	∞

NOTE: P denotes any integer ≥ 1. Q denotes any integer such that $P < Q$.

the results in Table 4-1 are valid for both of these steady-state functions.

4-2. FEEDBACK COMPENSATION

The steady-state system error is developed for the general case of feedback compensation illustrated by Fig. 1-2, page 7. The general expression for this case is then applied to the special cases of single-loop and double-loop feedback compensation. The steady-state actuating signals for these special cases are then developed.

Steady-state System Error--General Case. The system error, θ_e, is defined as follows:

$$\theta_e = R - C = R(1 - C/R) \tag{4-15}$$

The system transfer function is

$$\frac{C}{R} = \frac{G_1G_2}{1 + G_1(H_1 + G_2H_2)} \tag{4-16}$$

Substitution of Eq. (4-16) into Eq. (4-15) gives

$$\theta_e = R\left[1 - \frac{G_1G_2}{1 + G_1(H_1 + G_2H_2)}\right]$$

$$= R\left[\frac{1 + G_1(H_1 + G_2H_2 - G_2)}{1 + G_1(H_1 + G_2H_2)}\right] \tag{4-17}$$

From Eq. (4-17), the steady-state system error is

$$\theta_e(t) = \lim_{s \to o} R(s)\left[\frac{1 + G_1(H_1 + G_2H_2 - G_2)}{1 + G_1(H_1 + G_2H_2)}\right] \tag{4-18}$$

Again expressing the input in generalized form

$$R(s) = r_\alpha/s^\alpha \tag{4-19}$$

Substitution of Eq. (4-19) into Eq. (4-18) gives the expression for the steady-state system error for the general case of feedback compensation

$$\theta_e(t)_{ss} = \lim_{s \to 0} \frac{r_\alpha}{s^{\alpha-1}} \left[\frac{1 + G_1(H_1 + G_2H_2 - G_2)}{1 + G_1(H_1 + G_2H_2)} \right] \qquad (4\text{-}20)$$

Steady-state System Error--Single-loop Feedback Compensation. The block-diagram for the single-loop feedback compensated system is shown in Fig. 1-3, page 7. The steady-state system error for this system is derived from Eq. (4-20) by setting $H_2 = 0$ and $G_2 = 1$. If these substitutions are made, Eq. (4-20) reduces to the following expression:

$$\theta_e(t)_{ss} = \lim_{s \to 0} \frac{r_\alpha}{s^{\alpha-1}} \left[\frac{1 + G_1(H_1 - 1)}{1 + G_1H_1} \right] \qquad (4\text{-}21)$$

G_1 and H_1 are defined by the same expressions as for the cascade compensated case, i.e., G_1 is defined by Eq. (4-7) and H_1 is the same as G_c, defined by Eq. (4-8). Substitution of these equations into Eq. (4-21) gives

$$\theta_e(t)_{ss} = \lim_{s \to 0} \frac{r_\alpha}{s^{\alpha-1}} \left[\frac{s^{N+M}D_1D_c + K_1N_1(K_cN_c - s^MD_c)}{s^{N+M}D_1D_c + K_1N_1K_cN_c} \right] \qquad (4\text{-}22)$$

Equation (4-22) may be simplified by making the substitutions of Eq. (4-10). The resulting equation is the general expression for the steady-state system error

$$\theta_e(t)_{ss} = \lim_{s \to 0} \frac{r_\alpha}{s^{\alpha-1}} \left[\frac{s^{N+M} + K_1(K_c - s^M)}{s^{N+M} + K_1K_c} \right] \qquad (4\text{-}23)$$

Equation (4-23) is evaluated for a step input ($\alpha = 1$)

and for higher-order inputs ($\alpha > 1$) as follows:

1. For a step input, $\alpha = 1$ and Eq. (4-23) takes the form

$$\theta_e(t)_{ss} = \lim_{S\to 0} r_1 \left[\frac{s^{N+M} + K_1(K_c - s^M)}{s^{N+M} + K_1K_c}\right] \qquad (4\text{-}24)$$

The evaluation of Eq. (4-24) is more involved than for the comparable cascade compensation equation [See Eq. (4-12).]. Notice that Eq. (4-12) reduces to only one of three values depending on the value of the sum of N and M. Equation (4-24) reduces to one of five values depending on the separate values of N and M. As in the cascade compensation case, N is assumed to be equal to, or greater than, zero, whereas M may take on any value. The steady-state system error for the various possible combinations of M and N are tabulated in Table 4-2. These results were obtained by substitution of the appropriate values of M and N into Eq. (4-24). For negative values of M, Eq. (4-24) can be converted to a more convenient form by multiplying the numerator and denominator by the factor $s^{|M|}$ as follows:

$$\theta_e(t)_{ss} = \lim_{S\to 0} r_1 \left[\frac{s^{N+M} + K_1(K_c - s^M)}{s^{N+M} + K_1K_c}\right] \cdot \frac{s^{|M|}}{s^{|M|}}$$

$$= \lim_{S\to 0} r_1 \left[\frac{s^N + K_1(K_c s^{|M|} - 1)}{s^N + K_1K_c s^{|M|}}\right], \; M < 0 \qquad (4\text{-}25)$$

Equation (4-25) is valid only for negative values of M.

2. For $\alpha > 1$, i.e., for inputs of an order greater than a step function, the steady-state system error approaches

TABLE 4-2

STEADY-STATE SYSTEM ERROR, $\theta_e(t)_{ss}$, AND STEADY-STATE ACTUATING SIGNAL, $\mathcal{E}(t)_{ss}$, FOR SINGLE-LOOP FEEDBACK COMPENSATED SYSTEMS

TYPE		STEP INPUT ($\alpha = 1$)		HIGHER-ORDER INPUTS ($\alpha > 1$)
N	M	$\theta_e(t)_{ss}$	$\mathcal{E}(t)_{ss}$	$\theta_e(t)_{ss}$*
0	0	$r_1\left[\frac{1+K_1(K_c-1)}{1+K_1K_c}\right]$	$\frac{r_1}{1+K_1K_c}$	∞
>0	0	$r_1\frac{K_c-1}{K_c}$	0	∞
≥0	>0	r_1	0	∞
0	-1	$r_1(1-K_1)$	r_1	∞
>0	<0	$-\infty$	--	∞

*The actuating signals for higher-order inputs ($\alpha > 1$) are not shown since the corresponding error signals are infinite.

infinity for any value of M and N. Referring to Eq. (4-23), it is obvious that $\theta_e(t)_{ss} \rightarrow \infty$ for $M \geq 0$ since $N \geq 0$ (by definition) and the numerator of Eq. (4-23) is always finite, whereas the denominator approaches zero due to the factor $s^{\alpha-1}$. (The limit of $s^{\alpha-1}$ as $s \rightarrow 0$ is zero for $\alpha > 1$.) The same result is obtained for negative values of M. Eq. (4-23) is again converted to a more convenient form by multiplying numerator and denominator by $s^{|M|}$. Thus

$$\theta_e(t)_{ss} = \lim_{s \to 0} \frac{r_\alpha}{s^{\alpha-1}} \left[\frac{s^N + K_1(K_c s^{|M|} - 1)}{s^N + K_1 K_c s^{|M|}} \right], \quad M < 0 \qquad (4\text{-}26)$$

By reasoning similar to that used for the case where $M \geq 0$, it follows that Eq. (4-26) approaches infinity for $\alpha > 1$.

A number of significant conclusions can be drawn from Table 4-2 concerning the steady-state response of a single-loop feedback compensated system.

1. Since the steady-state system error is infinite for any input function of a higher order than a step, a single-loop feedback compensated system can never function as a follow-up device.[1] The steady-state system error will be

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[1]A step input refers to a step change in the reference function, whatever form the reference function may take. For example, if the reference input is velocity (zero for $t < 0$ and a constant value for $t > 0$), this is a step function input in velocity and the single-loop feedback compensated system can produce a finite steady-state system error <u>in velocity</u>. However, if the reference input is considered to be position, then the step input in velocity corresponds to a ramp input in position and the steady-state system error <u>in position</u> is infinite. This fact is evident when the

finite and the system can behave as a regulator (constant output for constant input) only for the following values of M and N with the additional restrictions on K_1 and K_c as noted:

a) $N = 0$, $M = 0$

$$\theta_e(t)_{ss} = r_1 \left[\frac{1 + K_1(K_c - 1)}{1 + K_1 K_c} \right] \qquad (4\text{-}27)$$

For $K_c = 1$, $\theta_e(t)_{ss} = r_1/(1 + K_1)$ and the error can be made small for $K_1 \gg 1$. For $K_1 = 1$, $\theta_e(t)_{ss} = r_1 K_c/(1 + K_c) = r_1/(1 + 1/K_c)$ and the error can be made small for $K_c \ll 1$.

b) $N > 0$, $M = 0$

$$\theta_e(t)_{ss} = r_1 \left[\frac{K_c - 1}{K_c} \right] \qquad (4\text{-}28)$$

Notice that the error is independent of K_1 and can be made equal to zero for $K_c = 1$. Therefore, it is possible to reduce the error to zero by proper selection of the gain or attenuation constant of H_1. K_1 can then be adjusted independently to place the roots of the characteristic equation for the closed-loop system in the proper location for the desired system

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finite error in velocity is considered in terms of the position functions. The input position and output position will be ramp functions having different slopes due to the velocity error. At steady-state ($t \rightarrow \infty$), the error between the input and output position will thus be infinite.

transient response. A change in K_1 will not affect the steady-state system error. An example of a system that takes advantage of this principle is presented in Section A-1 of the Appendix.

c) $N = 0$, $M = -1$

$$\theta_e(t)_{ss} = r_1(1 - K_1) \tag{4-29}$$

For $0 < K_1 < 1$, $\theta_e(t)_{ss} < r_1$ and for the special case where $K_1 = 1$, the error is zero. Notice that the error is independent of K_c. K_1 can thus be adjusted to give a small error and K_c adjusted independently to give the desired transient response.

2. The steady-state system error for $M > 0$ is equal to r_1 regardless of the value of N (the type of the uncompensated system). This situation exists because the steady-state output is always zero when the compensating element is type 1 or greater.

$$C(t)_{ss} = \lim_{s \to 0} sC(s)$$

$$\lim_{s \to 0} sR(s)\frac{C}{R}(s) = \lim_{s \to 0} r_1\left(\frac{G_1}{1 + G_1H_1}\right)$$

$$\lim_{s \to 0} r_1\left[\frac{s^M K_1 N_1 D_c}{s^{N+M} D_1 D_c + K_1 N_1 K_c N_c}\right] \tag{4-30}$$

Equation (4-30) is equal to zero for $M \geq 1$ so that

$$\theta_e(t)_{ss} = r(t)_{ss} - C(t)_{ss}$$

$$= r_1 - 0 = r_1$$

3. The steady-state system error approaches infinity

for a type 1 or greater uncompensated system ($N > 0$) for any negative value of M ($M < 0$).

Block-diagram manipulation gives insight to a physical interpretation of the results of the steady-state system error analysis. An example of block-diagram manipulation and interpretation of a single-loop feedback compensated system is given in Section A-2 of the Appendix.

Steady-state Actuating Signal--Single-loop Feedback Compensation. The steady-state actuating signal for the single-loop feedback compensated system (See Fig. 1-3, page 7.) is defined as follows:

$$\mathcal{E} = R - CH_1 = R\left(1 - \frac{C}{R} \cdot H_1\right) \tag{4-31}$$

The system transfer function is

$$\frac{C}{R} = \frac{G_1}{1 + G_1H_1} \tag{4-32}$$

Substitution of Eq. (4-32) into Eq. (4-31) gives

$$\mathcal{E} = R\left(1 - \frac{G_1}{1 + G_1H_1} \cdot H_1\right) = \frac{R}{1 + G_1H_1} \tag{4-33}$$

But Eq. (4-33) is the same form as Eq. (4-3) for the cascade compensated steady-state system error (or steady-state actuating signal). Therefore, the actuating signal for single-loop feedback compensation can be determined by substituting $H_1(s)$ for $G_c(s)$ in the appropriate equations derived in Section 4-1. Since the generalized equations for $H_1(s)$ and

$G_c(s)$ are the same, the results for single-loop feedback compensation can be obtained directly from Table 4-1. The actuating signals for a step input are tabulated along with the error signals in Table 4-2. The actuating signals for higher-order inputs ($\alpha > 1$) are not shown in this table since the corresponding error signals are infinite.

<u>Steady-state System Error--Double-loop Feedback Compensation</u>. The block-diagram for the double-loop feedback compensated system is shown in Fig. 1-4, page 7. The steady-state system error for this system is derived from Eq. (4-20) by setting $H_2 = 1$ and $G_2 = 1$. If these substitutions are made, Eq. (4-20) reduces to the following expression:

$$\theta_e(t)_{ss} = \lim_{s \to 0} \frac{r_\alpha}{s^{\alpha-1}} \left[\frac{1 + G_1 H_1}{1 + G_1(H_1 + 1)} \right] \qquad (4\text{-}34)$$

G_1 is defined by Eq. (4-7) and H_1 is defined by Eq. (4-8). Substitution of these equations into Eq. (4-34) gives

$$\theta_e(t)_{ss} = \lim_{s \to 0} \frac{r_\alpha}{s^{\alpha-1}} \left[\frac{s^{N+M} D_1 D_c + K_1 N_1 K_c N_c}{s^{N+M} D_1 D_c + K_1 N_1 (K_c N_c + s^M D_c)} \right] \qquad (4\text{-}35)$$

Equation (4-35) may be simplified by making the substitutions of Eq. (4-10). The resulting equation is the general expression for the steady-state system error.

$$\theta_e(t)_{ss} = \lim_{s \to 0} \frac{r_\alpha}{s^{\alpha-1}} \left[\frac{s^{N+M} + K_1 K_c}{s^{N+M} + K_1 (K_c + s^M)} \right] \qquad (4\text{-}36)$$

As will be shown, the steady-state system error for the double-loop feedback compensation case can be finite for input functions of an order greater than one ($\alpha > 1$). The most convenient method for developing the system error is to categorize the compensated systems first in terms of the value of M, the type of the compensation element; secondly in terms of the value of N, the type of the uncompensated plant; and finally in terms of α, the order of the input function. The indicated substitutions for M and N are made in Eq. (4-36) to arrive at each $\theta_e(t)_{ss}$ in the development which follows.

1. M = 0

$$\theta_e(t)_{ss} = \lim_{s \to 0} \frac{r_\alpha}{s^{\alpha-1}} \left[\frac{s^N + K_1K_c}{s^N + K_1(K_c + 1)} \right]$$

a) For N = 0,

$$\theta_e(t)_{ss} = \lim_{s \to 0} \frac{r_\alpha}{s^{\alpha-1}} \left[\frac{1 + K_1K_c}{1 + K_1(K_c + 1)} \right] \tag{4-37}$$

$$= r_1 \left[\frac{1 + K_1K_c}{1 + K_1(K_c + 1)} \right] \quad \text{for } \alpha = 1$$

$$= \infty \text{ for } \alpha > 1$$

b) For N > 0,

$$\theta_e(t)_{ss} = \lim_{s \to 0} \frac{r_\alpha}{s^{\alpha-1}} \left(\frac{K_c}{K_c + 1} \right) \tag{4-38}$$

$$= r_1 \left(\frac{K_c}{K_c + 1} \right) \quad \text{for } \alpha = 1$$

$$= \infty \text{ for } \alpha > 1$$

2. M > 0

For any value of N ($N \geq 0$ by definition),

$$\theta_e(t)_{ss} = \lim_{S\to 0} \frac{r_\alpha}{s^{\alpha-1}}(1) \tag{4-39}$$

$$= r_1 \text{ for } \alpha = 1$$

$$= \infty \text{ for } \alpha > 1$$

3. $M < 0$

$$\theta_e(t)_{ss} = \lim_{S\to 0} \frac{r_\alpha}{s^{\alpha-1}}\left[\frac{s^{N+M} + K_1K_c}{s^{N+M} + K_1(K_c + s^M)}\right] \cdot \frac{s^{|M|}}{s^{|M|}}$$

$$= \lim_{S\to 0} \frac{r_\alpha}{s^{\alpha-1}}\left[\frac{s^N + K_1K_c s^{|M|}}{s^N + K_1(K_c s^{|M|} + 1)}\right] \tag{4-40}$$

a) For $N = 0$,

$$\theta_e(t)_{ss} = \lim_{S\to 0} \frac{r_\alpha}{s^{\alpha-1}}\left(\frac{1}{1 + K_1}\right) \tag{4-41}$$

$$= r_1/(1 + K_1) \text{ for } \alpha = 1$$

$$= \infty \text{ for } \alpha = 1$$

b) For $0 < N < |M|$,

$$\theta_e(t)_{ss} = \lim_{S\to 0} \frac{r_\alpha s^N}{s^{\alpha-1}}\left[\frac{1 + K_1K_c s^{|M|-N}}{s^N + K_1(K_c s^{|M|} + 1)}\right]$$

$$= \lim_{S\to 0} \frac{r_\alpha s^N}{s^{\alpha-1}} \cdot \frac{1}{K_1} \tag{4-42}$$

$$\theta_e(t)_{ss} = 0 \text{ for } N > (\alpha - 1)$$

$$= r_\alpha/K_1 \text{ for } N = (\alpha - 1)$$

$$= \infty \text{ for } N < (\alpha - 1)$$

c) For $N > |M|$,

$$\theta_e(t)_{ss} = \lim_{S\to 0} \frac{r_\alpha s^{|M|}}{s^{\alpha-1}}\left[\frac{s^{N-|M|} + K_1K_c}{s^N + K_1(K_c s^{|M|} + 1)}\right]$$

$$= \lim_{S \to 0} \frac{r_\alpha s^{|M|}}{s^{\alpha - 1}} (K_c) \tag{4-43}$$

$$= 0 \text{ for } |M| > (\alpha - 1)$$

$$\theta_e(t)_{ss} = K_c \alpha \text{ for } |M| = (\alpha - 1)$$

$$= \infty \text{ for } |M| < (\alpha - 1)$$

d) For $N = |M|$,

$$\theta_e(t)_{ss} = \lim_{S \to 0} \frac{r_\alpha s^N}{s^{\alpha - 1}} \left[\frac{1 + K_1 K_c}{s^N + K_1 (K_c s^{|M|} + 1)} \right]$$

$$= \lim_{S \to 0} \frac{r_\alpha s^{|M|}}{s^{\alpha - 1}} \left[\frac{1 + K_1 K_c}{s^N + K_1 (K_c s^{|M|} + 1)} \right] \tag{4-44}$$

$$\theta_e(t)_{ss} = 0 \text{ for } N > (\alpha - 1),\ |M| > (\alpha - 1)$$

$$= r_\alpha \left(\frac{1 + K_1 K_c}{K_1} \right) \text{ for } N = |M| = (\alpha - 1)$$

$$= \infty \text{ for } N < (\alpha - 1),\ |M| < (\alpha - 1)$$

The steady-state system errors are tabulated in Table 4-3 for step, ramp, and parabolic input functions.

Referring to Table 4-3 and the preceding development, several significant conclusions can be drawn from the results of the steady-state system analysis. These conclusions are summarized below.

1. The steady-state system error for input functions of a higher order than a step ($\alpha > 1$) is finite only if (a) the compensation element has a positive power of s in the numerator ($M < 0$), (b) the type of the uncompensated system is one or greater ($N > 0$), and (c) the lesser value of $|M|$ and N is

TABLE 4-3

STEADY-STATE SYSTEM ERROR, $\theta_e(t)_{ss}$, AND STEADY-STATE ACTUATING SIGNAL, $\mathcal{E}(t)_{ss}$, FOR DOUBLE-LOOP FEEDBACK COMPENSATED SYSTEMS

TYPE		STEP INPUT ($\alpha = 1$)		RAMP INPUT ($\alpha = 2$)		PARABOLIC INPUT ($\alpha = 3$)	
N	M	$\theta_e(t)_{ss}$	$\mathcal{E}(t)_{ss}$	$\theta_e(t)_{ss}$	$\mathcal{E}(t)_{ss}$*	$\theta_e(t)_{ss}$	$\mathcal{E}(t)_{ss}$*
0	0	$r_1\left[\frac{1+K_1K_c}{1+K_1(K_c+1)}\right]$	$\frac{r_1}{1+K_1(K_c+1)}$	∞	--	∞	--
0	>0	r_1	0	∞	--	∞	--
0	<0	$r_1/(1+K_1)$	$r_1/(1+K_1)$	∞	--	∞	--
1	0	$r_1 \cdot \frac{K_c}{K_c+1}$	0	∞	--	∞	--
1	>0	r_1	0	∞	--	∞	--
1	-1	0	0	$r_2 \cdot \frac{1+K_1K_c}{K_1}$	r_2/K_1	∞	--
1	<-1	0	0	r_2/K_1	r_2/K_1	∞	--
2	0	$r_1 \frac{K_c}{K_c+1}$	0	∞	--	∞	--
2	>0	r_1	0	∞	--	∞	--
2	-1	0	0	r_2K_c	0	∞	--
2	-2	0	0	0	0	∞	--
2	<-2	0	0	0	0	$r_3 \cdot \frac{1+K_1K_c}{K_1}$	r_3/K_1
>2	0	$r_1 \cdot \frac{K_c}{K_c+1}$	0	∞	--	r_3/K_1	r_3/K_1
>2	>0	r_1	0	∞	--	∞	--
>2	-1	0	0	r_2K_c	0	∞	--
>2	-2	0	0	0	--	∞	--
>2	<-2	0	0	0	--	r_3K_c	0

*$\mathcal{E}(t)_{ss}$ is not tabulated for those cases where the corresponding $\theta_e(t)_{ss}$ is infinite.

equal to, or greater than, the order of the input function minus one ($\alpha - 1$). The error will be equal to zero for those cases where the lesser value of $|M|$ and N is equal to or greater than α. The significance of these conditions in terms of the physical system is presented in Section A-3 of the Appendix.

2. For $N = M = 0$ and $\alpha = 1$ or $N = -M$ and $\alpha = N + 1$,

$$\theta_e(t)_{ss} = r_\alpha \left[\frac{1 + K_1 K_c}{1 + K_1(K_c + 1)} \right]$$

The error can be made small by making $K_c \ll 1$ and $K_1 \gg 1$ so that

$$\theta_e(t)_{ss} \approx r_\alpha \left(\frac{1 + K_1 K_c}{K_1} \right) = r_\alpha \left(\frac{1}{K_1} + K_c \right)$$

3. For $M < 0$, $N = 0$ and $\alpha = 1$,

$$\theta_e(t)_{ss} = r_1/(1 + K_1)$$

For $M < 0$, $|M| > N > 0$ and $\alpha = N + 1$

$$\theta_e(t)_{ss} = r_\alpha/K_1$$

The error for both these cases is independent of K_c and can be made small for large values of K_1. The steady-state system error can be set by adjusting K_1, and the dominant roots can be set for the desired transient response by adjusting K_c.

4. For $M < 0$, $|M| < N$ and $\alpha = |M| + 1$

$$\theta_e(t)_{ss} = r_\alpha K_c$$

The error is independent of K_1 and can be made small for $K_c \ll 1$.

5. For $N > 0$, $M = 0$ and $\alpha = 1$,

$$\theta_e(t)_{ss} = r_1 K_c/(1 + K_c)$$

The error is independent of K_1 and can be made small for $K_c \ll 1$.

Steady-state Actuating Signal--Double-loop Feedback Compensation. The steady-state actuating signal for the double-loop feedback compensated system (See Fig. 1-4, page 7.) is defined as follows:

$$\mathcal{E} = C/G_1$$

$$\frac{\mathcal{E}}{R} = \frac{1}{G_1} \cdot \frac{C}{R} \qquad (4\text{-}45)$$

The system transfer function is

$$\frac{C}{R} = \frac{G_1}{1 + G_1(H_1 + 1)} \qquad (4\text{-}46)$$

Substitution of Eq. (4-46) into Eq. (4-45) gives

$$\frac{\mathcal{E}}{R} = \frac{1}{1 + G_1(H_1 + 1)} \qquad (4\text{-}47)$$

From Eq. (4-47) the steady-state actuating error is

$$\mathcal{E}(t)_{ss} = \lim_{S \to 0} R(s) \frac{1}{1 + G_1(s)[H_1(s) + 1]} \qquad (4\text{-}48)$$

Again expressing R(s) in the general form of Eq. (4-5), and $G_1(s)$ and $H_1(s)$ in the general forms of Eq. (4-7) and Eq. (4-8), respectively, Eq. (4-48) becomes

$$\mathcal{E}(t)_{ss} = \lim_{S \to 0} \frac{r_\alpha}{s^{\alpha-1}} \cdot \frac{s^{N+M} D_1 D_c}{s^{N+M} D_1 D_c + K_1 N_1 (K_c N_c + s^M D_c)} \qquad (4\text{-}49)$$

Equation (4-49) may be simplified by making the substitutions of Eq. (4-10). The resulting equation is the general expression for the steady-state actuating signal.

$$\mathcal{E}(t)_{ss} = \lim_{S \to 0} \frac{r_\alpha}{s^{\alpha-1}} \left[\frac{s^{N+M}}{s^{N+M} + K_1(K_c + s^M)} \right] \qquad (4\text{-}50)$$

The steady-state actuating signal is developed by categorizing the compensated systems first in terms of M, secondly in terms of N, and finally in terms of α. The indicated substitutions for M and N are made in Eq. (4-50) to arrive at each $\mathcal{E}(t)_{ss}$ in the development which follows.

1. M = 0

$$\mathcal{E}(t)_{ss} = \lim_{S \to 0} \frac{r_\alpha}{s^{\alpha-1}} \left[\frac{s^{N}}{s^{N} + K_1(K_c + 1)} \right]$$

a) For N = 0,

$$\mathcal{E}(t)_{ss} = \lim_{S \to 0} \frac{r_\alpha}{s^{\alpha-1}} \left| \frac{1}{1 + K_1(K_c + 1)} \right| \qquad (4\text{-}51)$$

$$= \frac{r_1}{1 + K_1(K_c + 1)} \quad \text{for } \alpha = 1$$

$$= \infty \quad \text{for } \alpha > 1$$

b) For N > 0,

$$\mathcal{E}(t)_{ss} = 0 \quad \text{for } N > (\alpha - 1)$$

$$= r_\alpha / K_1(K_c + 1) \quad \text{for } N = (\alpha - 1)$$

$$= \infty \quad \text{for } N < (\alpha - 1)$$

2. M > 0

$$\mathcal{E}(t)_{ss} = 0 \quad \text{for } (N + M) > (\alpha - 1)$$

$$= r_\alpha/K_1K_c \quad \text{for } (N + M) = (\alpha - 1)$$

$$= \infty \quad \text{for } (N + M) < (\alpha - 1)$$

3. $M < 0$

$$\mathcal{E}(t)_{ss} = \lim_{s\to 0} \frac{r_\alpha}{s^{\alpha-1}}\left[\frac{s^{N+M}}{s^{N+M} + K_1(K_c + s^M)}\right] \cdot \frac{s^{|M|}}{s^{|M|}}$$

$$= \lim_{s\to 0} \frac{r_\alpha}{s^{\alpha-1}}\left[\frac{s^N}{s^N + K_1(K_c s^{|M|} + 1)}\right] \qquad (4\text{-}52)$$

a) For $N = 0$,

$$\mathcal{E}(t)_{ss} = \lim_{s\to 0} \frac{r_\alpha}{s^{\alpha-1}}\left[\frac{1}{1 + K_1(K_c s^{|M|} + 1)}\right] \qquad (4\text{-}53)$$

$$= r_1/1 + K_1 \quad \text{for } \alpha = 1$$

$$= \infty \quad \text{for } \alpha > 1$$

b) For $N > 0$

$$\mathcal{E}(t)_{ss} = 0 \quad \text{for } N > (\alpha - 1)$$

$$= r_\alpha/K_1 \quad \text{for } N = (\alpha - 1)$$

$$= \infty \quad \text{for } N < (\alpha - 1)$$

The steady-state actuating signals for step, ramp, and parabolic inputs are tabulated in Table 4-3 for cases where the corresponding steady-state error is finite.

4-3. SUMMARY AND CONCLUSIONS

The discussions and developments presented in this chapter have brought to light the important fact that the choice of feedback versus cascade compensation must be considered in terms of the steady-state system error as well as the dynamic behavior. Finite system error is impossible to achieve for ramp, parabolic, or higher-order input signals into a single-loop feedback compensated system. The use of this form of compensation is therefore limited to regulator applications where the input is a reference-level type of step function. (See Footnote on page 43.) The double-loop feedback compensated system maintains a direct correspondence between the system input and output functions because of the unity feedback path. The steady-state system error for this configuration can be made finite, in fact zero, for a ramp or higher-order input function by proper choice of the compensation network.

There are several additional conclusions which may be deduced from Tables 4-1, 4-2 and 4-3 by interpreting the results in these tables in terms of the block-diagrams of the physical systems represented. For example, for a double-loop feedback compensated system with $N = 0$, $M < 0$, it is noted from Table 4-3 that the steady-state actuating signal and the steady-state system error are both equal to $r_1/(1 + K_1)$. From a physical standpoint this result is expected since the inner-loop for the system is open-circuited at steady-state $(s \rightarrow 0)$ and the system would therefore reduce to a simple type 0 unity feedback system for which the steady-state system error and actuating signal are equal, i.e., $r_1/(1 + K_1)$. The single-loop feedback compensated system with $N = 0$, $M = 0$, reduces to the same unity feedback system at steady-state when $K_c = 1$ (See Table 4-2). Carrying this physical interpretation further, the double-loop feedback compensated system for $N > 0$, $M < 0$ (See Table 4-3) will reduce to a unity feedback system of type N as $s \to 0$ ($H_1 \to 0$). The steady-state system error and actuating signal are therefore zero for a step input, because a step input into a type 1 or greater unity feedback system produces both zero steady-state system error and actuating signal.

CHAPTER 5

LOG-MODULUS ANALYSIS OF COMPENSATED SYSTEMS

The log-modulus plot is an effective method for graphically comparing the relative effects of cascade and feedback compensation in terms of frequency response. This method is normally applied to the open-loop transfer function for the cascade compensation case, and the closed-loop system response is then determined from zero db. crossing points and the phase margin.[1] The approach in this chapter is to develop a systematic method for approximating the magnitude of closed-loop transfer functions. The closed-loop transfer functions for cascade compensated systems and various forms of feedback compensated systems are then analyzed to establish the relative effects of the two modes of compensation.

The validity of the approximation that is used in arriving at the magnitude of the closed-loop transfer function is demonstrated for the simple unity feedback system of Fig. 5-1, page 62. This proof applies to the compensated systems that are investigated in this chapter as well.

5-1. AN APPROXIMATION FOR THE MAGNITUDE OF A CLOSED-LOOP TRANSFER FUNCTION

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[1] Vincent Del Toro and Sydney R. Parker, Principles of Control Systems Engineering (New York: McGraw-Hill Book Company, Inc., 1960), Chap. 9.

The closed-loop transfer function for the system shown in Fig. 5-1 is

$$\frac{C}{R} = \frac{G_1}{1 + G_1} \tag{5-1}$$

The magnitude of C/R is given by

$$\left|\frac{C}{R}\right| = \frac{|G_1|}{|1 + G_1|} \tag{5-2}$$

The denominator of Eq. (5-2) may be expressed as follows:

$$|1 + G_1| = \left\{[1 + \mathrm{Re}(G_1)]^2 + [\mathrm{Im}(G_1)]^2\right\}^{1/2} \tag{5-3}$$

where $\mathrm{Re}(G_1)$ is the real part of G_1 and $\mathrm{Im}(G_1)$ is the imaginary part of G_1. The approximate value of $|1 + G_1|$ will depend on the value of $|G_1|$ as follows:

1. When $|G_1| \ll 1$,

   $$\mathrm{Re}(G_1) \ll 1 \text{ and } \mathrm{Im}(G_1) \ll 1$$

   Therefore, referring to Eq. (5-3),

   $$|1 + G_1| \approx 1$$

2. When $|G_1| \gg 1$, one of the following must be true:

   a) $\mathrm{Re}(G_1) \gg 1$ and $|G_1| \approx \mathrm{Re}(G_1)$

   b) $\mathrm{Im}(G_1) \gg 1$ and $|G_1| \approx \mathrm{Im}(G_1)$

   c) $\left.\begin{array}{l}\mathrm{Re}(G_1) \gg 1 \\ \mathrm{Im}(G_1) \gg 1\end{array}\right\}$ and $|G_1| = \left\{[\mathrm{Re}(G_1)]^2 + [\mathrm{Im}(G_1)]^2\right\}^{1/2}$

   For any of these cases, $|1 + G_1| \approx |G_1|$

Therefore, the following approximation is seen to be valid:

$$\left|\frac{C}{R}\right| \approx \frac{|G_1|}{|G_1|} = 1 \text{ for } |G_1| \gg 1 \qquad (5\text{-}4a)$$

$$\left|\frac{C}{R}\right| \approx |G_1| \qquad \text{for } |G_1| \ll 1 \qquad (5\text{-}4b)$$

5-2. LOG-MODULUS REPRESENTATIONS FOR COMPENSATED SYSTEMS

The magnitude approximation technique developed in Section 5-1 is employed in the analysis and comparison of cascade and feedback compensated systems. The first system to be considered is the cascade compensated case.

Cascade Compensation. The transfer function for the cascade compensated system shown in Fig. 1-1, page 7, is given by

$$\frac{C}{R} = \frac{G_1 G_c}{1 + G_1 G_c} \qquad (5\text{-}5)$$

The magnitude of this transfer function may be expressed as follows:

$$\left|\frac{C}{R}\right| \approx 1 \qquad \text{for } |G_1 G_c| \gg 1 \qquad (5\text{-}6a)$$

$$\approx |G_1 G_c| \text{ for } |G_1 G_c| \ll 1 \qquad (5\text{-}6b)$$

The straight-line log-modulus plot of Eq. (5-6) is constructed by plotting $|G_1 G_c|$ in db. units versus log w on semilog paper as illustrated in Fig. 5-2, page 62. The frequency where the plot crosses zero db. will be referred to as w_c. At w_c, $|G_1 G_c| = 1$. Usually $|G_1 G_c|$ will be greater than one, i.e., +db., for $w < w_c$ and will be less than one, i.e., -db., for $w > w_c$. This is the case when the order of s in the numerator of G_1G_c is less than the order of s in the denominator, such that as $w \rightarrow 0$, $G_1G_c \rightarrow \infty$ and as $w \rightarrow \infty$, $G_1G_c \rightarrow 0$.

The magnitude approximation for the cascade compensated system transfer function can be represented in block-diagram form as shown in Fig. 5-3.

Feedback Compensation. The magnitude approximation technique is applied to the single-loop and double-loop feedback compensated systems. The information derived from these systems is then used in developing the magnitude approximation technique for the general case of feedback compensation.

The single-loop feedback compensated system is shown in Fig. 1-3, page 7. The system transfer function is given by

$$\frac{C}{R} = \frac{G_1}{1 + G_1H_1} \tag{5-7}$$

The magnitude of this transfer function may be expressed as follows:

$$\left|\frac{C}{R}\right| \approx \frac{1}{|H_1|} \quad \text{for } |G_1H_1| \gg 1 \text{ or } |G_1| \gg \frac{1}{|H_1|} \tag{5-8a}$$

$$\left|\frac{C}{R}\right| \approx |G_1| \quad \text{for } |G_1H_1| \ll 1 \text{ or } |G_1| \ll \frac{1}{|H_1|} \tag{5-8b}$$

Notice that $|G_1H_1| = 1$ for $|G_1| = \frac{1}{|H_1|}$.

The straight-line log-modulus plot of Eq. (5-8) is constructed by plotting $|G_1|$ and $|H_1|^{-1}$ in db. units versus log w on semi-log paper as illustrated in Fig. 5-4, page 64. The

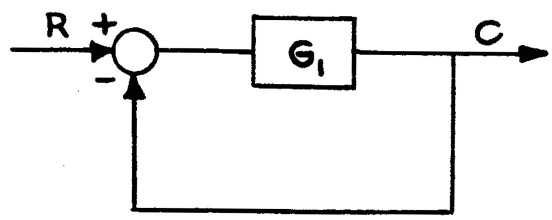


Figure 5-1. Unity feedback system (uncompensated).

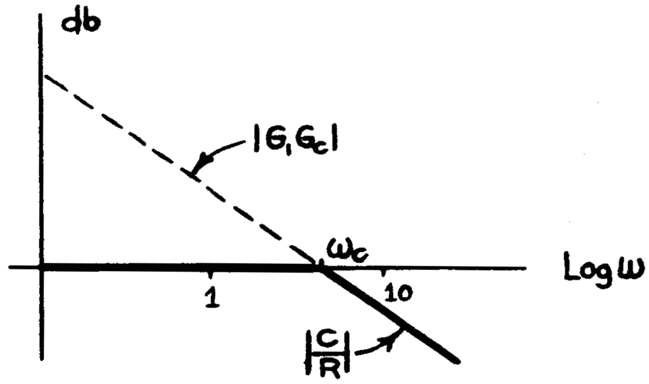


Figure 5-2. Straight-line log-modulus plot of |C/R| for a cascade compensated system.

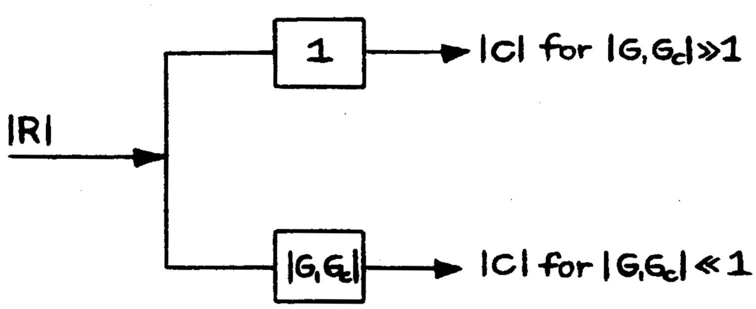


Figure 5-3. Magnitude approximation for a cascade compensated system transfer function.

frequency where $|G_1| = |H_1|^{-1}$ is determined by the intersection of these two plots. If we call this frequency, w_o, the same reasoning applies to the plot of $\left|\frac{C}{R}\right|$ with respect to this frequency as was specified for w_c in the previous cascade compensation case.

The single-loop feedback compensated system may be converted to an equivalent cascade compensated system with an H^{-1} block preceding the summation point as shown in Fig. 5-5. The magnitude approximation can therefore be represented as in Fig. 5-6.

The double-loop feedback compensated system is shown in Fig. 1-4, page 7. The system transfer function is given by

$$\frac{C}{R} = \frac{G_1}{1 + G_1(H_1 + 1)} \tag{5-9}$$

The magnitude of this transfer function may be expressed as

$$\left|\frac{C}{R}\right| \approx \frac{1}{|H_1+1|} \quad \text{for } |G_1(H_1 + 1)| \gg 1 \text{ or } |G_1| \gg \frac{1}{|H_1+1|} \tag{5-10a}$$

$$\approx |G_1| \quad \text{for } |G_1(H_1 + 1)| \ll 1 \text{ or } |G_1| \ll \frac{1}{|H_1+1|} \tag{5-10b}$$

The straight-line log-modulus plot of $\left|\frac{C}{R}\right|$ may be constructed from Eq. (5-10) by plotting $|G_1|$ and $|H_1 + 1|^{-1}$; however, a more systematic approach is to convert the double-loop system of Fig. 1-4 into its equivalent single-loop form shown in Fig. 5-7, page 66. The approximate magnitude of C/R is then evaluated in two steps as follows:

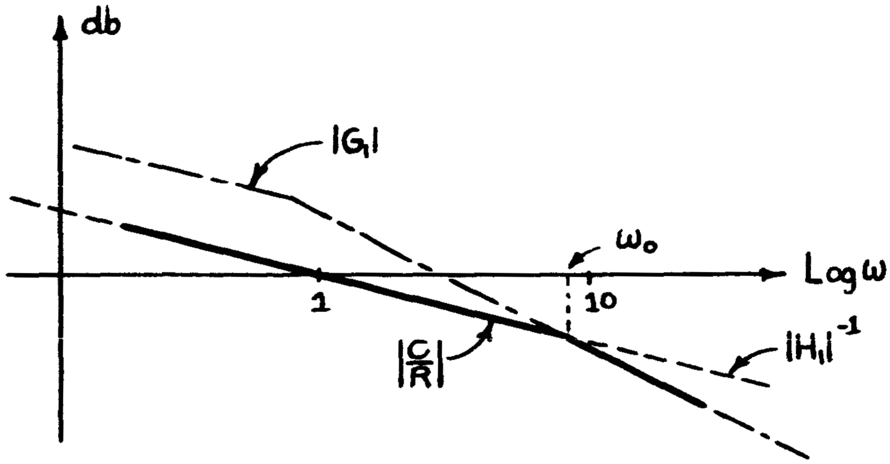


Figure 5-4. Straight-line log-modulus plot of |C/R| for a single-loop feedback compensated system.

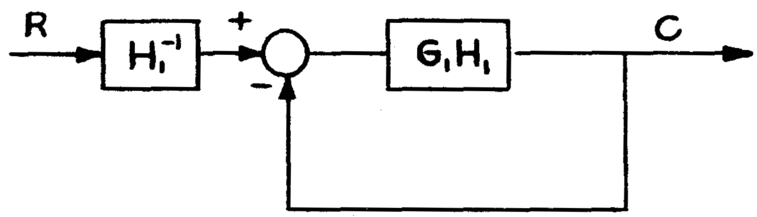


Figure 5-5. Equivalent block-diagram form for a single-loop feedback compensated system.

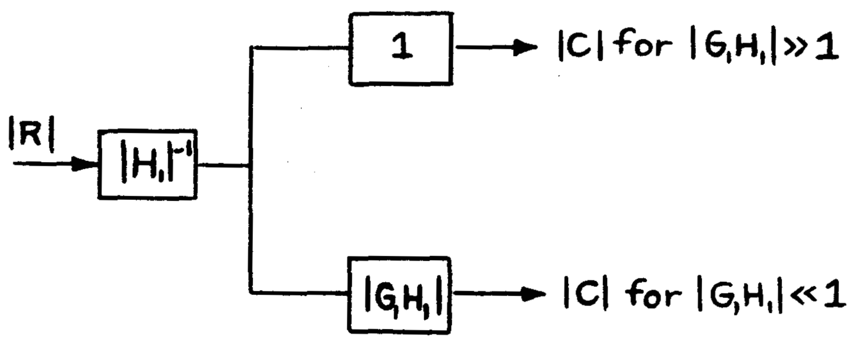


Figure 5-6. Magnitude approximation for a single-loop feedback compensated system transfer function.

1. Fig. 5-7 is in the same form as Fig. 5-1, and

$$\frac{C}{R} = \frac{G'}{1 + G'}$$

Thus

$$\left|\frac{C}{R}\right| \approx 1 \quad \text{for } |G'| \gg 1 \qquad (5\text{-}11a)$$

$$\approx |G'| \quad \text{for } |G'| \ll 1 \qquad (5\text{-}11b)$$

2. Since G' is the transfer function for a single-loop feedback compensated system [See Eq. (5-8).]

$$|G'| = \left|\frac{G_1}{1 + G_1H_1}\right| \approx \frac{1}{|H_1|} \quad \text{for } |G_1H_1| \gg 1 \qquad (5\text{-}12a)$$

$$\approx |G_1| \quad \text{for } |G_1H_1| \ll 1 \qquad (5\text{-}12b)$$

The straight-line log-modulus plot of $\left|\frac{C}{R}\right|$ may be constructed now by plotting $|G_1|$ and $|H_1|^{-1}$ in db. units versus log w as illustrated in Fig. 5-8. Notice that if w_o is less than the frequency w_c' where the plot of $|G_1|$ crosses the 0 db. axis, then $|C/R|$ is equal to one for $w < w_c'$ and equal to $|G_1|$ for $w > w_c'$. The function $|H_1|^{-1}$ is not involved in the solution for $|C/R|$ in this case.

The general case for feedback compensation is illustrated in Fig. 1-2, page 7. The system transfer function is

$$\frac{C}{R} = \frac{G_1G_2}{1 + G_1(H_1 + G_2H_2)} \qquad (5\text{-}13)$$

The magnitude approximation for C/R may be derived in a systematic manner by converting Fig. 1-2 into its equivalent single-loop feedback form shown in Fig. 5-9. $|C/R|$ is then

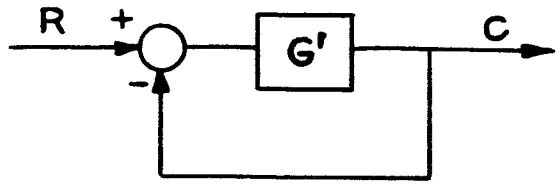
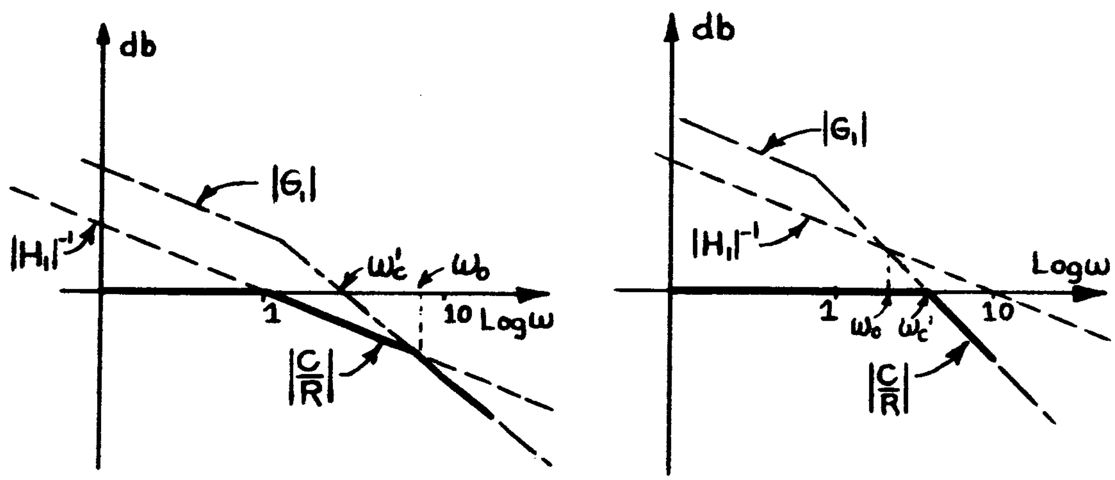


Figure 5-7. Equivalent block-diagram form for a double-loop feedback compensated system.



a) $W_o > W_c'$ b) $W_o < W_c'$

Figure 5-8. Straight-line log-modulus plot of |C/R| for a double-loop feedback compensated system.

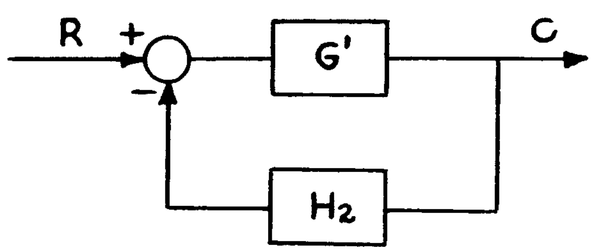


Figure 5-9. Equivalent block-diagram form for general case of feedback compensation.

evaluated in two steps as follows:

1. Fig. 5-8 is in the same form as the single-loop feedback compensated system of Fig. 1-3, and

$$\frac{C}{R} = \frac{G'}{1 + G'H_2}$$

Thus

$$\left|\frac{C}{R}\right| \approx \frac{1}{|H_2|} \quad \text{for } |G'H_2| \gg 1 \qquad (5\text{-}14a)$$

$$\left|\frac{C}{R}\right| \approx |G'| \quad \text{for } |G'H_2| \ll 1 \qquad (5\text{-}14b)$$

2. Since G' is the transfer function for a single-loop feedback compensated system preceded by the function G_2

$$|G'| = \left|\frac{G_1G_2}{1 + G_1H_1}\right| \approx \left|\frac{G_2}{H_1}\right| \quad \text{for } |G_1H_1| \gg 1 \qquad (5\text{-}15a)$$

$$\approx |G_1G_2| \quad \text{for } |G_1H_1| \ll 1 \qquad (5\text{-}15b)$$

Referring to Equations (5-14) and (5-15), to determine $|C/R|$ it is necessary to plot only $|H_2|^{-1}$, $|G_2/H_1|$ and $|G_1G_2|$. The plots of $|G_2/H_1|$ and $|G_1G_2|$ will determine $|G'|$ and then the plots of $|G'|$ and $|H_2|^{-1}$ will fix $|C/R|$ for the general case of feedback compensation.

Example. To illustrate the principles that have been developed in this chapter, consider an example where $G_1 = 10/s^2(s + 1)$ and $G_c = H_1 = s$. The magnitude approximation for the system transfer function will be developed for the cascade compensated system and the single-loop and double-loop feedback compensated systems.

For the cascade compensated system, the following term is plotted in db. units in Fig. 5-10:

$$|G_1G_c| = \left|\frac{10}{s(s+1)}\right| \tag{5-16}$$

$|C/R|$ for the cascade compensated system is constructed according to Eq. (5-6) and appears as the heavy line labled ① in Fig. 5-10. The closed-loop system transfer function can be derived as follows:

$$\frac{C}{R} = \frac{G_1G_c}{1 + G_1G_c} = \frac{10/s(s+1)}{1 + 10/s(s+1)}$$

$$= \frac{10}{s^2 + s + 10} = \frac{w_n^2}{s^2 + 2\delta w_n s + w_n^2} \tag{5-17}$$

Notice that the straight-line log-modulus plot of Eq. (5-17) yields the same result as was obtained from the plot of Eq. (5-16), with $w_n = \sqrt{10}$ corresponding to w_c.

For the single-loop feedback compensated system, the following terms are plotted in db. units in Fig. 5-10:

$$|G_1| = \left|\frac{10}{s^2(s+1)}\right| \tag{5-18a}$$

$$|H_1|^{-1} = |s|^{-1} \tag{5-18b}$$

$|C/R|$ for the single-loop feedback compensated system is constructed according to Eq. (5-8) and appears as the heavy line labled ② in Fig. 5-10. The closed-loop system transfer function can be derived as follows:

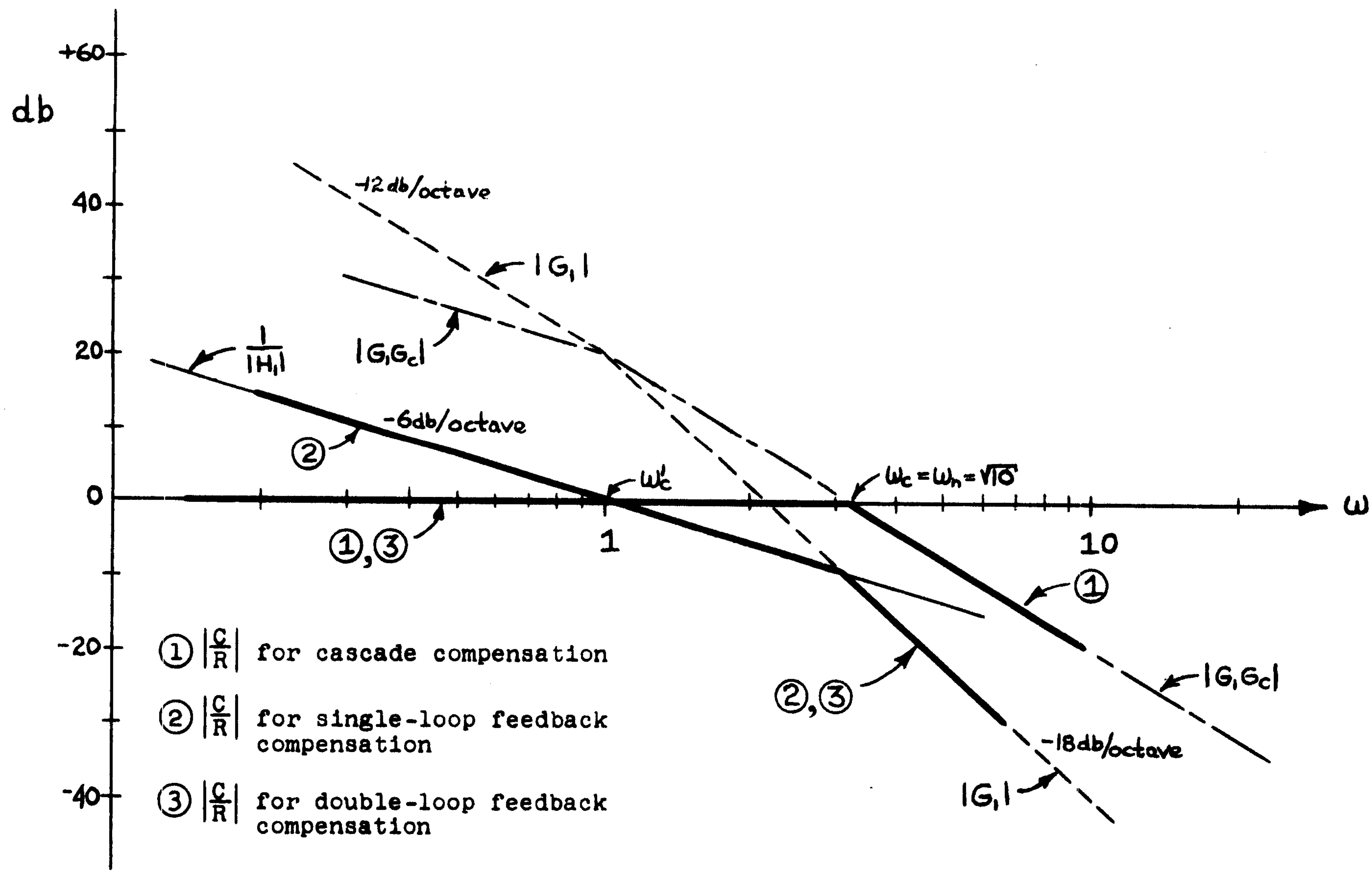


Figure 5-10. Example of log-modulus approximations for magnitudes of compensated system transfer functions.

$G_1 = 10/s^2(s + 1)$ $\qquad G_c = H_1 = s$

$$\frac{C}{R} = \frac{G_1}{1 + G_1H_1} = \frac{10/s^2(s + 1)}{1 + 10/s(s + 1)}$$

$$= \frac{10}{s(s^2 + s + 10)} = \frac{w_n^2}{s(s^2 + 2\delta w_n s + w_n^2)} \qquad (5\text{-}19)$$

Again, notice that the straight-line log-modulus plot of Eq. (5-19) yields the same result as was obtained from the plot of Eq. (5-18).

For the double-loop feedback compensated system, the same terms apply that are plotted in Fig. 5-10 for the single-loop feedback compensated system, i.e., Eq. (5-18). $|C/R|$ for the double-loop case is constructed according to Equations (5-11) and (5-12) and appears as the heavy line labled ③ in Fig. 5-10. The closed-loop system transfer function can be derived as follows:

$$\frac{C}{R} = \frac{G_1}{1 + G_1(H_1 + 1)} = \frac{10/s^2(s + 1)}{1 + 10(s + 1)/s^2(s + 1)}$$

$$= \frac{10}{(s + 1)(s^2 + 10)} \qquad (5\text{-}20)$$

Finally, notice again that the straight-line log-modulus plot of Eq. (5-20) yields the same result as was obtained from the plot of Eq. (5-18).

5-3. SUMMARY AND CONCLUSIONS

The log-modulus plot has been shown to be a convenient technique for graphically displaying the approximate closed-loop frequency response for compensated systems. Several important characteristics of compensated systems become apparent from the log-modulus plot analysis. It is noted that the cascade compensated system transfer function is unity for lower frequencies and falls off according to the product G_1G_c for higher frequencies, providing that $G_1(jw)G_c(jw) \rightarrow 0$ as $w \rightarrow \infty$. The transfer function for the single-loop feedback compensated system depends on the inverse of the compensation network for lower frequencies and falls off in accordance with the uncompensated system transfer function, G_1, at higher frequencies. The transfer function for the double-loop feedback compensated system will either have a positive slope (if H_1 has a positive slope) or be unity as in the cascade case for low frequencies. At intermediate frequencies, the transfer function can assume several forms depending upon the form of the compensation network and the uncompensated system, G_1. At higher frequencies, the transfer function falls off in accordance with G_1, as in the single-loop feedback compensation case.

Each form of compensation has been shown to affect the closed-loop system response in its own characteristic manner. These results must be considered in the selection of a compensation network and the system configuration for this network for a given plant and desired system response.

CHAPTER 6

POLE-ZERO ANALYSIS OF COMPENSATED SYSTEMS

The characteristic equation for a control system is determined by setting the denominator of the system's closed-loop transfer function equal to zero. The roots of the resulting equation are the poles of the closed-loop transfer function. The zeros are the roots of the numerator of the closed-loop transfer function. The transient response of a control system depends upon the pole-zero configuration of the closed-loop transfer function for the system. For a given plant, cascade compensation and the various forms of feedback compensation will affect the pole-zero configuration of the closed-loop transfer function in a different manner.

For a second-order system, the roots of the characteristic equation may be calculated by solving a quadratic equation. For higher-order systems several techniques are available for calculating the roots of the characteristic equation. The technique that is employed in this chapter is the root-locus method developed by Walter R. Evans.[1] The theory behind this method will not be developed in this chapter, but is readily available to the reader in almost any recent

[1]Walter R. Evans, Control System Dynamics (New York: McGraw-Hill Book Company, Inc., 1954).

control systems textbook. Basically, the root-locus method is a graphical technique for determining the roots of a characteristic equation in terms of a system parameter that varies from zero to infinity. The parameter that is usually varied is the open-loop gain of the system. An important point to note is that the root-locus method uses the open-loop transfer function of a system to yield precise information about the closed-loop transient response of the system.

The first part of this chapter compares the relative effects of cascade compensation and feedback compensation on the pole-zero configuration of the open-loop and closed-loop transfer functions. An example is presented to illustrate these effects in terms of the root-locus plot. The latter part of the chapter investigates the effects of zeros on the transient response of a control system.

6-1. POLES AND ZEROS OF COMPENSATED SYSTEMS

The poles and zeros for cascade compensated systems and the single-loop and double-loop forms of feedback compensation are defined in terms of the poles and zeros of the uncompensated plant, G_1, and the compensation network, G_c or H_1. First consider the cascade compensated system.

Cascade Compensation. $G_1(s)$ and $G_c(s)$ for the cascade compensated system shown in Fig. 1-1, page 7, are defined as follows for this analysis:

$$G_1(s) = \frac{N_1(s)}{D_1(s)} \tag{6-1}$$

$$G_c(s) = \frac{N_c(s)}{D_c(s)} \tag{6-2}$$

The adjustable gain parameters occur as factors in N_1 and N_c. The open-loop transfer function for the cascade compensated system is G_1G_c. Substitution of Equations (6-1) and (6-2) into this expression gives

$$G_1G_c = \frac{N_1N_c}{D_1D_c} \tag{6-3}$$

The closed-loop transfer function for this case is

$$\frac{C}{R} = \frac{G_1G_c}{1 + G_1G_c} \tag{6-4}$$

Substitution of Equations (6-1) and (6-2) into Eq. (6-4) gives

$$\frac{C}{R} = \frac{N_1N_c}{D_1D_c + N_1N_c} \tag{6-5}$$

From Eq. (6-5) it is noted that the zeros of C/R occur where $N_1N_c = 0$, and therefore, C/R has zeros where G_1 and G_c have zeros.[2] Referring to Eq. (6-3), the zeros of the open-loop and closed-loop transfer functions are the same.

[2]This conclusion is in general valid only when the uncompensated plant and the compensation network have no common poles or zeros. When the compensation network introduces pole-zero cancellation, this conclusion must be re-interpreted. See page 79 for a discussion of this problem.

The poles of Eq. (6-5) occur where

$$D_1D_c + N_1N_c = 0 \quad (6\text{-}6)$$

[3]

Equation (6-6) is the characteristic equation for the cascade compensated system. This equation is put into root-locus equation form by dividing through by D_1D_c, giving the expression

$$\frac{N_1N_c}{D_1D_c} = -1 \quad (6\text{-}7)$$

The left side of Eq. (6-7) is the open-loop transfer function (G_1G_c) for the cascade compensated system.

Single-loop Feedback Compensation. The single-loop feedback compensated system is shown in Fig. 1-3, page 7. $G_1(s)$ for this system is defined by Eq. (6-1) and $H_1(s)$ is given by

$$H_1(s) = \frac{N_c(s)}{D_c(s)} \quad (6\text{-}8)$$

The adjustable gain parameter again occurs in the numerator, N_c. The open-loop transfer function for this case is the same as for cascade compensation, i.e.,

$$G_1H_1 = \frac{N_1N_c}{D_1D_c} \quad (6\text{-}9)$$

The closed-loop transfer function is given by

$$\frac{C}{R} = \frac{G_1}{1 + G_1H_1} \quad (6\text{-}10)$$

---

[3] Ibid.

Substitution of Equations (6-1) and (6-8) into Eq. (6-10) gives

$$\frac{C}{R} = \frac{N_1 D_c}{D_1 D_c + N_1 N_c} \tag{6-11}$$

From Eq. (6-11) it is noted that the zeros of C/R occur where $N_1 D_c = 0$, and therefore, C/R has zeros where G_1 has zeros and where H_1 has poles.[4] The zeros of C/R are not the same as the zeros of the open-loop transfer function [See Eq. (6-9).] as was the case for cascade compensation. The significance of the difference in zeros will be established in Section 6-2.

The poles of Eq. (6-11) occur where

$$D_1 D_c + N_1 N_c = 0 \tag{6-12}$$ [5]

Equation (6-12) is the characteristic equation for the single-loop feedback compensated system. Notice that this equation is identical to Eq. (6-6) for the cascade compensated system.[6] Equation (6-12) will yield the same root-locus equation as Eq. (6-7), i.e.,

$$\frac{N_1 N_c}{D_1 D_c} = -1$$

Therefore, the root-locus plots will be the same for cascade

[4]*Ibid.*

[5]*Ibid.*

[6]*Ibid.*

and single-loop feedback compensated systems.

Double-loop Feedback Compensation. $G_1(s)$ and $H_1(s)$ for the double-loop feedback compensated system shown in Fig. 1-4, page 7, are defined by Equations (6-1) and (6-8) respectively. The closed-loop transfer function for this case is

$$\frac{C}{R} = \frac{G_1}{1 + G_1(H_1 + 1)} \tag{6-13}$$

Substitution of Equations (6-1) and (6-8) into Eq. (6-13) gives

$$\frac{C}{R} = \frac{N_1 D_c}{D_1 D_c + N_1(N_c + D_c)} \tag{6-14}$$

A comparison of Eq. (6-14) and Eq. (6-11) reveals that the zeros for the double-loop case are the same as for the single-loop case.

The poles of Eq. (6-14) occur where

$$D_1 D_c + N_1(N_c + D_c) = 0 \text{ [7]} \tag{6-15}$$

Equation (6-15) is the characteristic equation for the double-loop feedback compensated system. Notice that the poles of Eq. (6-15) are not the same as for the single-loop case.

Equation (6-15) can be reduced to three different root-locus equations corresponding to three different forms of block-diagram manipulation of the double-loop feedback

[7] Ibid.

compensated system shown in Fig. 1-4, page 7. These root-locus equations are derived as follows:

1. If Eq. (6-15) is divided through by D_1D_c, the following root-locus equation results:

$$\frac{N_1(N_c + D_c)}{D_1D_c} = -1 \qquad (6\text{-}16)$$

Notice that Eq. (6-16) corresponds to the system block-diagram form shown in Fig. 6-1(a), page 80, which has the open-loop transfer function

$$G_1(H_1 + 1) = \frac{N_1(N_c + D_c)}{D_1D_c} \qquad (6\text{-}17)$$

2. If Eq. (6-15) is divided through by the expression $D_1D_c + N_1N_c$, the following root-locus equation results:

$$\frac{N_1D_c}{D_1D_c + N_1N_c} = -1 \qquad (6\text{-}18)$$

Equation (6-18) corresponds to the system block-diagram form shown in Fig. 6-1(b), page 80, which has the open-loop transfer function

$$\frac{G_1}{1 + G_1H_1} = \frac{N_1D_c}{D_1D_c + N_1N_c} \qquad (6\text{-}19)$$

Notice that Eq. (6-19) is the closed-loop transfer function for the single-loop feedback compensated system.

3. If Eq. (6-15) is divided through by the expression $D_1D_c + N_1D_c$, the following root-locus equation results:

$$\frac{N_1N_c}{D_1D_c + N_1D_c} = -1$$

Equation (6-20) corresponds to the system block-diagram form shown in Fig. 6-1(c) which has the open-loop transfer function

$$\frac{G_1H_1}{1 + G_1} = \frac{N_1N_c}{D_1D_c + N_1D_c} \qquad (6\text{-}21)$$

The usefulness of each of the previous forms for the root-locus equation will depend on which system parameter needs to be isolated as the variable for the root-locus plot.

Effects of Pole-zero Cancellation. For each of the compensated systems that have been discussed in the previous paragraphs, certain statements were made concerning the pole-zero configurations for the closed-loop transfer functions that do not necessarily apply when the compensation network introduces pole-zero cancellation. These statements were indicated by a reference to the Footnote on page 74. Since compensation networks are often selected to produce a pole-zero cancellation, it is necessary to investigate this situation and re-evaluate the statements in question. An example will serve to clarify the problem.

Consider an uncompensated plant having the transfer function

$$G_1 = \frac{K_1(s + 1)}{(s + 2)(s + 3)} = \frac{N_1(s)}{D_1(s)} \qquad (6\text{-}22)$$

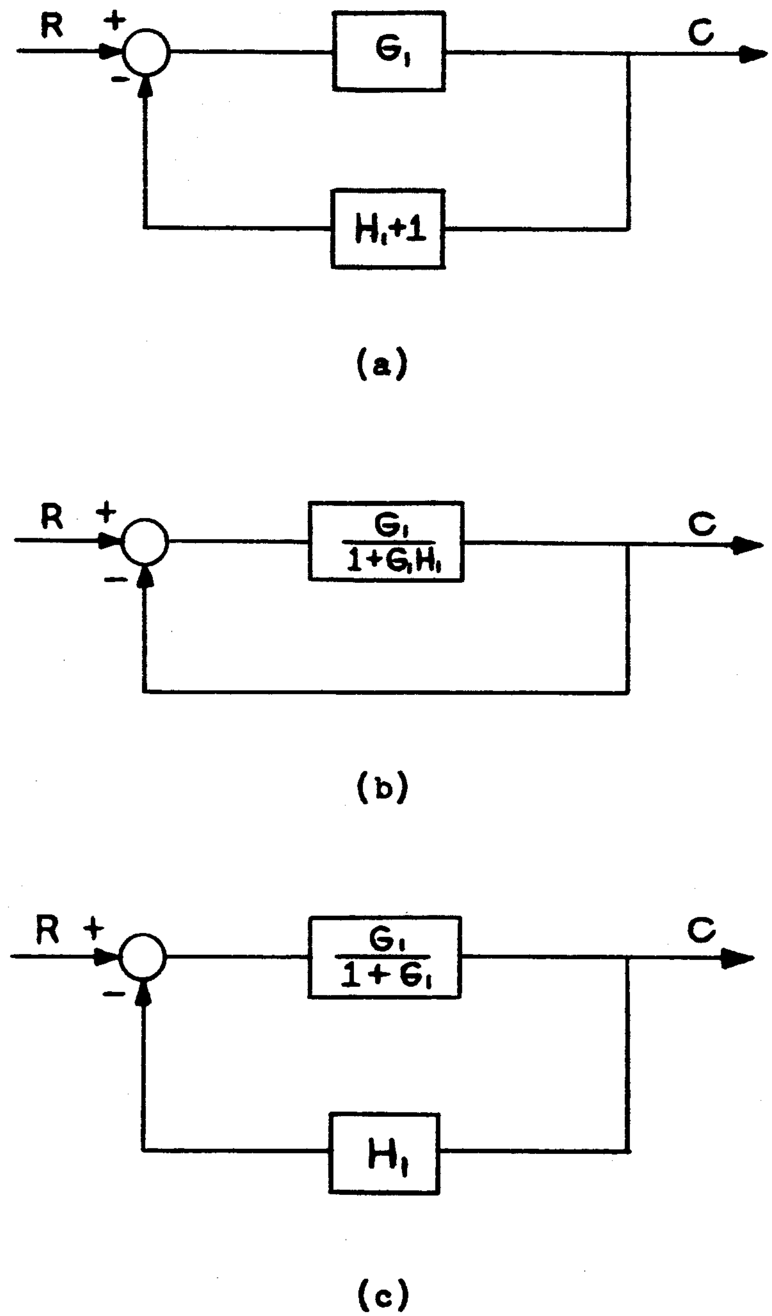


Figure 6-1. Block-diagram forms for the double-loop feedback compensated system.

The closed-loop transfer functions are calculated below for each form of compensation that has been considered previously and for two cases of pole-zero cancellation: (1) Zero of compensation network cancels pole of plant, and (2) Pole of compensation network cancels zero of plant. Exceptions will be noted to the previous statements that have been referenced to the Footnote on page 74.

1. The compensation network is given by

$$G_c = H_1 = \frac{K_c(s + 2)}{(s + 4)} = \frac{N_c(s)}{D_c(s)} \tag{6-23}$$

The open-loop transfer function for cascade or single-loop feedback compensation is

$$G_1G_c = G_1H_1 = \frac{K_1K_c(s + 1)}{(s + 3)(s + 4)} \tag{6-24}$$

The zero of the compensation network has canceled one of the poles of the plant.

The closed-loop transfer function for the cascade compensated system is

$$\frac{C}{R} = \frac{G_1G_c}{1 + G_1G_c} = \frac{K_1K_c(s + 1)}{(s + 3)(s + 4) + K_1K_c(s + 1)} \tag{6-25}$$

Notice that the zero of Eq. (6-25) is the zero of G_1 but <u>not</u> the zero of G_c. Also, the poles of Eq. (6-25) do <u>not</u> occur where $D_1D_c + N_1N_c = 0$.

For the single-loop feedback compensated system

$$\frac{C}{R} = \frac{G_1}{1 + G_1H_1} = \frac{K_1(s + 1)(s + 4)}{(s + 2)[(s + 3)(s + 4) + K_1K_c(s + 1)]} \tag{6-26}$$

Notice that the characteristic equation for this case is not the same as for the cascade case.

For the double-loop feedback compensated system

$$\frac{C}{R} = \frac{G_1}{1 + G_1(H_1+1)}$$

$$= \frac{K_1(s+1)(s+4)}{(s+2)(s+3)(s+4)+K_1(K_c+1)(s+1)[s+(2K_c+4)/(K_c+1)]} \quad (6\text{-}27)$$

Notice that the poles of Eq. (6-27) do not occur where $D_1D_c + N_1(N_c + D_c) = 0$.

2. The compensation network is given by

$$G_c = H_1 = \frac{K_c(s + 4)}{(s + 1)} = \frac{N_c(s)}{D_c(s)} \quad (6\text{-}28)$$

The open-loop transfer function for cascade or single-loop feedback compensation is

$$G_1G_c = G_1H_1 = \frac{K_1K_c(s + 4)}{(s + 2)(s + 3)} \quad (6\text{-}29)$$

The pole of the compensation network has canceled the zero of the plant.

The closed-loop transfer function for the cascade compensated system is

$$\frac{C}{R} = \frac{G_1G_c}{1 + G_1G_c} = \frac{K_1K_c(s + 4)}{(s + 2)(s + 3) + K_1K_c(s + 4)} \quad (6\text{-}30)$$

Notice that the zero of Eq. (6-30) is the zero of G_c but not the zero of G_1. Also, the poles of Eq. (6-30) do not occur where $D_1D_c + N_1N_c = 0$.

For the single-loop feedback compensated system

$$\frac{C}{R} = \frac{G_1}{1 + G_1H_1} = \frac{K_1(s + 1)}{(s + 2)(s + 3) + K_1K_c(s + 4)} \qquad (6\text{-}31)$$

Notice that the zero of Eq. (6-31) is the zero of G_1 but <u>not</u> the pole of H_1. Also, the poles of Eq. (6-31) do <u>not</u> occur where $D_1D_c + N_1N_c = 0$.

For the double-loop feedback compensated system

$$\frac{C}{R} = \frac{G_1}{1 + G_1(H_1 + 1)}$$

$$= \frac{K_1(s+1)^2}{(s+1)\left\{(s+2)(s+3) + K_1[(K_c+1)s + 4K_c+1]\right\}}$$

$$= \frac{K_1(s+1)}{(s+2)(s+3) + K_1(K_c+1)[s + (2K_c+4)/(K_c+1)]} \qquad (6\text{-}32)$$

Notice that the poles of Eq. (6-32) do <u>not</u> occur where $D_1D_c + N_1(N_c + D_c) = 0$.

<u>Example--Tachometer Plus Phase-lag Compensation</u>. As an example of the principles that have been developed in this section, consider the system having the transfer function

$$G_1 = \frac{K_1}{s(s + 1)} \qquad (6\text{-}33)$$

The root-locus for the uncompensated system is sketched in Fig. 6-2, page 86.[8]

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[8]The root-loci and the pole-zero configurations that are presented for this example are not drawn to scale but serve

The compensation network for this example consists of a tachometer in series with a phase-lag network and has the transfer function

$$G_c = H_1 = \frac{K_c s}{(s + 4)} \qquad (6\text{-}34)$$

The root-loci and the pole-zero configurations of the closed-loop transfer functions are developed below for the cases of cascade compensation and single-loop and double-loop feedback compensation.

1. Cascade compensation. The open-loop transfer function for the cascade compensated system is given by

$$G_1 G_c = \frac{K_1 K_c s}{s(s + 1)(s + 4)} = \frac{K_1 K_c}{(s + 1)(s + 4)} \qquad (6\text{-}35)$$

The root-locus corresponding to Eq. (6-35) is shown in Fig. 6-3, page 86, for K_1 as the variable parameter. The closed-loop transfer function for the cascade compensated system is given by

$$\frac{C}{R} = \frac{G_1 G_c}{1 + G_1 G_c} = \frac{K_1 K_c s}{s[(s + 1)(s + 4) + K_1 K_c]}$$

$$= \frac{K_1 K_c}{(s + 1)(s + 4) + K_1 K_c} \qquad (6\text{-}36)$$

Notice that the zero at the origin and the pole at the origin cancel each other in Eq. (6-36). The pole-zero configuration

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to illustrate the relative effects of the various forms of compensation on the original system.

corresponding to Eq. (6-36) is shown in Fig. 6-4 for an arbitrary value of K_1, indicated in Fig. 6-3 as K_1'.

2. Single-loop feedback compensation. The open-loop transfer function for this system is the same as that of the cascade compensated system, i.e.,

$$G_1H_1 = \frac{K_1K_c}{(s + 1)(s + 4)} \tag{6-37}$$

The root-locus corresponding to Eq. (6-37) is shown in Fig. 6-3. The closed-loop transfer function for the single-loop feedback compensated system is

$$\frac{C}{R} = \frac{G_1}{1 + G_1H_1} = \frac{K_1(s + 4)}{s[(s + 1)(s + 4) + K_1K_c]} \tag{6-38}$$

Notice that the pole at the origin was canceled in the open-loop transfer function but appears in the closed-loop transfer function. The pole-zero configuration corresponding to Eq. (6-38) is shown in Fig. 6-5 for K_1', an arbitrary value of K_1.

3. Double-loop feedback compensation. The closed-loop transfer function for this system is

$$\frac{C}{R} = \frac{G_1}{1 + G_1(H_1 + 1)}$$

$$= \frac{K_1(s + 4)}{s(s + 1)(s + 4) + K_1[s(K_c + 1) + 4]} \tag{6-39}$$

The numerator of Eq. (6-39) is a third-order equation in s, and will therefore yield three roots (poles) for a given value

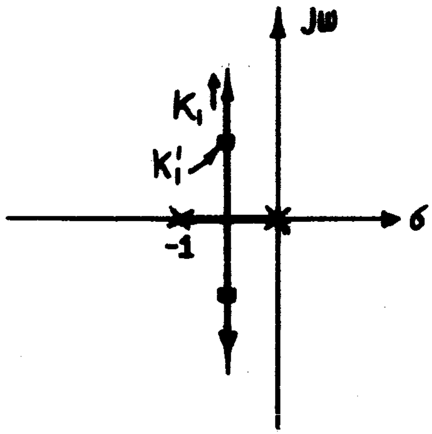


Figure 6-2. Root-locus for uncompensated system. [$G_1 = K_1/s(s + 1)$].

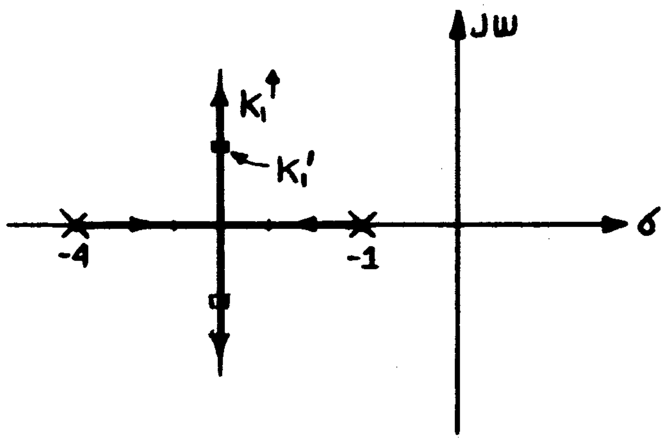


Figure 6-3. Root-locus for cascade compensated system and single-loop feedback compensated system. [$G_1G_c = G_1H_1 = K_1K_c/(s + 1)(s + 4)$].

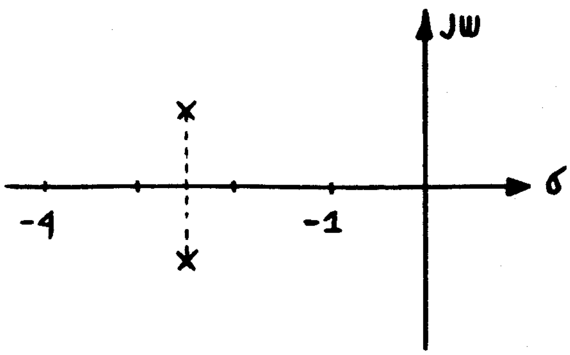


Figure 6-4. Pole-zero configuration for cascade compensation.

$$\frac{C}{R} = \frac{K_1K_c}{(s + 1)(s + 4) + K_1K_c}$$

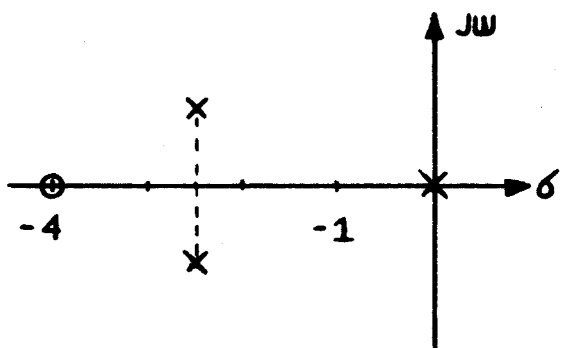


Figure 6-5. Pole-zero configuration for single-loop feedback compensation.

$$\frac{C}{R} = \frac{K_1(s + 4)}{s[(s + 1)(s + 4) + K_1K_c]}$$

of K_1 and K_c. The root-loci and corresponding closed-loop pole-zero configurations are determined below for the three system cases depicted in Fig. 6-1.

a) For the open-loop transfer function

$$G_1(H_1 + 1) = \frac{K_1(K_c + 1)(s + 4/K_c + 1)}{s(s + 1)(s + 4)} \qquad (6\text{-}40)$$

The root-loci will assume one of three forms, depending on the value of K_c. These root-loci and the corresponding pole-zero configurations for an arbitrary value of K_1, K_1', are shown in Fig. 6-6. Notice that when $K_c = 3$, there is a pole-zero cancellation in Eq. (6-40) but Eq. (6-39), the closed-loop transfer function, yields a pole at $s = 1$. For $K_c > 3$, a dominant pole occurs on the negative real axis.

b) For the open-loop transfer function

$$\frac{G_1}{1 + G_1H_1} = \frac{K_1(s + 4)}{s[(s + 1)(s + 4) + K_1K_c]} \qquad (6\text{-}41)$$

Notice that Eq. (6-41) is identical to Eq. (6-38), the closed-loop transfer function for the single-loop feedback compensated system. The pole-zero configuration for the open-loop transfer function for the double-loop system is therefore specified by Fig. 6-5 for K_1', the arbitrary value of K_1. The resulting root-locus for the double-loop system with K_c as the

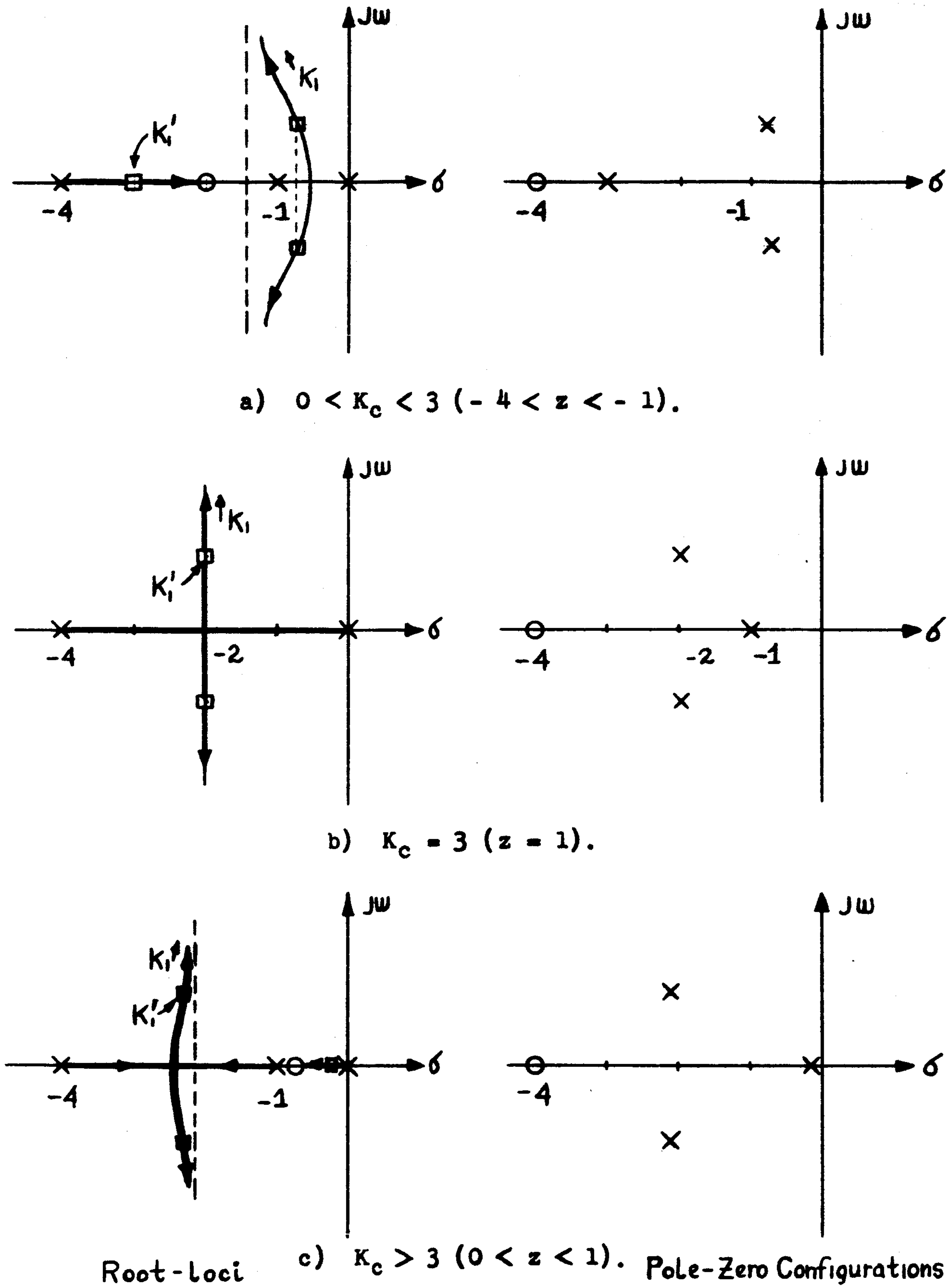


Figure 6-6. Root-loci and pole-zero configurations for double-loop feedback compensated system having open-loop transfer function of $G_1(H_1 + 1)$.

variable parameter is shown in Fig. 6-7(a). If K_1 is chosen to correspond to a set of real poles in Fig. 6-3, page 86, the root-locus for the double-loop system as shown in Fig. 6-7(b) will result. The pole-zero configurations for the closed-loop systems are also shown in Fig. 6-7 for an arbitrary value of K_c, K_c'.

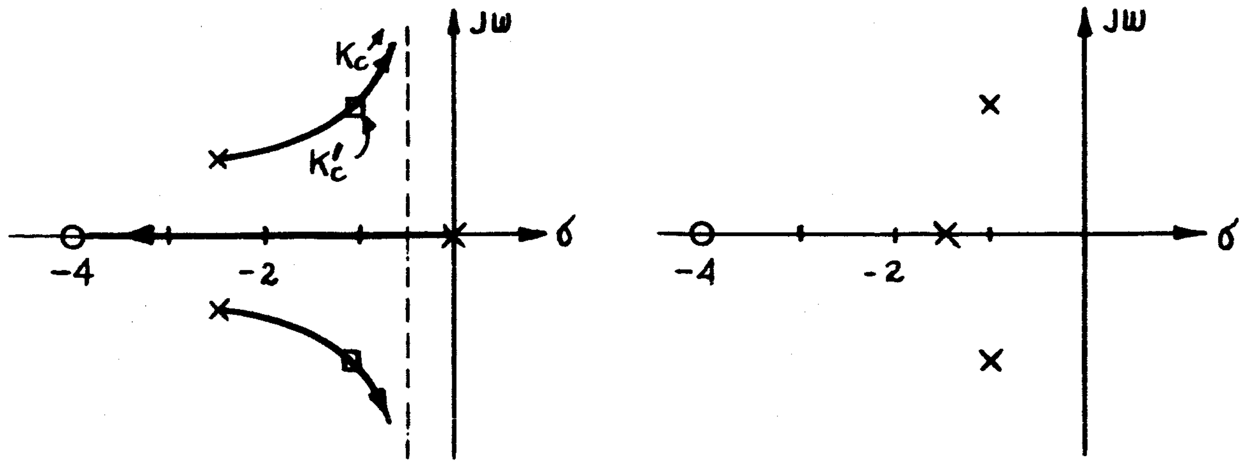
c) For the open-loop transfer function

$$\frac{G_1H_1}{1 + G_1} = \frac{K_1K_cs}{(s + 4)[s(s + 1) + K_1]} \qquad (6\text{-}42)$$

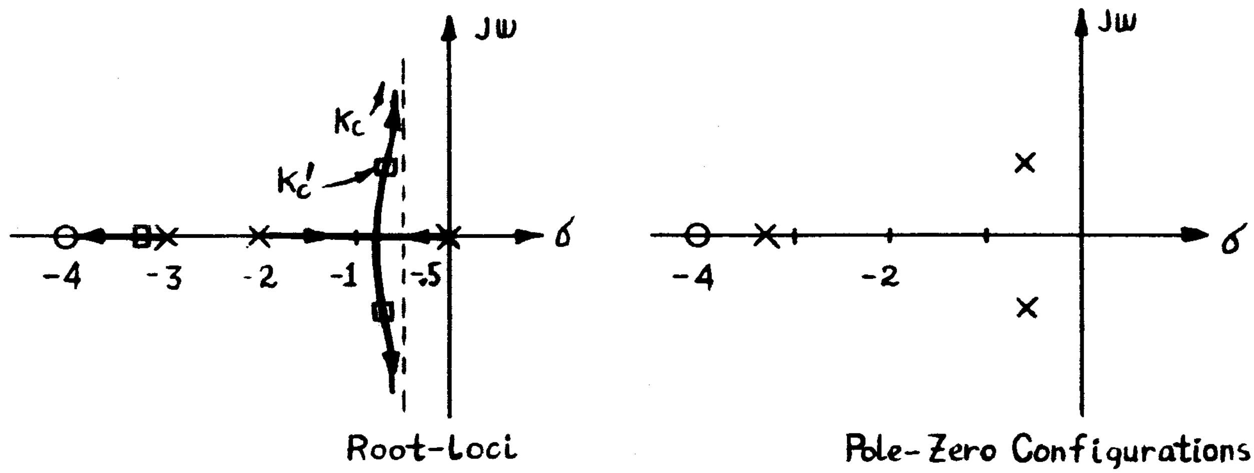
Notice that the bracketed term in the denominator of Eq. (6-42) is the characteristic equation for the closed-loop system consisting of the uncompensated plant, G_1, with a unity feedback path. The root-locus for this system is shown in Fig. 6-2, page 86. For K_1', an arbitrary value of K_1, the resulting root-locus for the double-loop feedback compensated system is shown in Fig. 6-8. An arbitrary value of the varying parameter, K_c', has been selected to produce the corresponding pole-zero configuration shown in the same figure.

6-2. EFFECT OF ZEROS ON TRANSIENT RESPONSE

It has been shown in the previous section that a cascade compensated system and a single-loop feedback compensated system may have the same open-loop transfer function and the



a) K_1 chosen to give complex poles for $K_c = 0$.



b) K_1 chosen to give real poles for $K_c = 0$.

Figure 6-7. Root-loci and pole-zero configurations for double-loop feedback compensated system having open-loop transfer function of $G_1/(1 + G_1H_1)$.

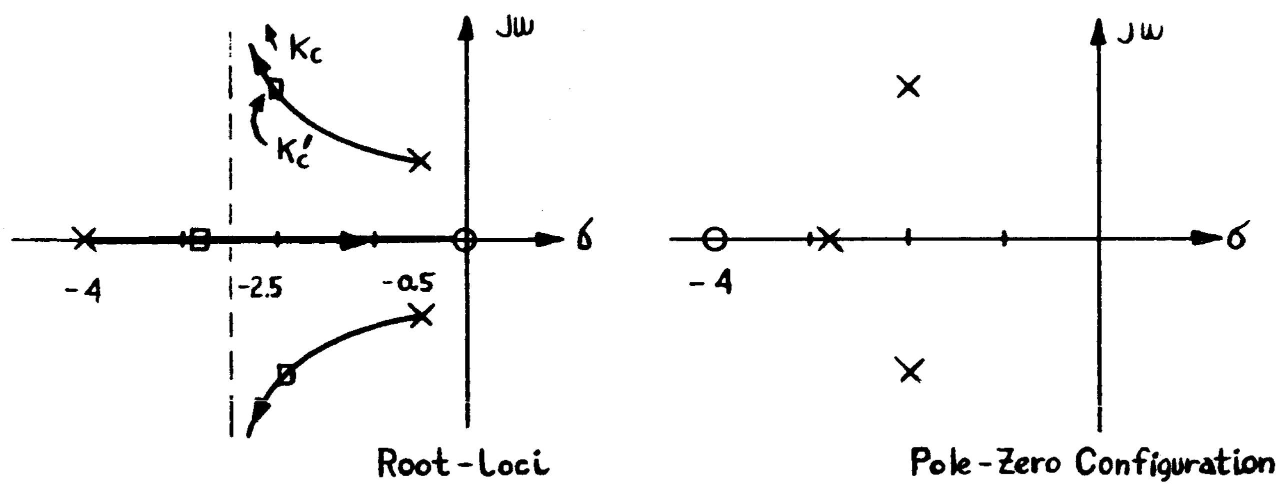


Figure 6-8. Root-loci and pole-zero configurations for double-loop feedback compensated system having open-loop transfer function of $G_1H_1/(1 + G_1)$.

same characteristic equation, but different zeros. In particular, one system may have finite zeros while the other system does not. The effect of a zero on the transient response of a system has been discussed by Del Toro and Parker for a second-order system.[9] The results of this discussion can be extended to higher-order systems providing these systems can be approximated by an equivalent second-order system over the frequency range of interest.

Consider a second-order system having the normalized closed-loop transfer function

$$\frac{C}{R}(s) = \frac{w_n^2}{s^2 + 2\delta w_n s + w_n^2} \qquad (6\text{-}43)$$

w_n is the natural frequency of the system and δ is the system damping ratio. The system response to a step-input of magnitude r_1 is

$$C(s) = \frac{r_1}{s} \cdot \frac{w_n^2}{s^2 + 2\delta w_n s + w_n^2} \qquad (6\text{-}44)$$

If a zero at $- z_1$ is added to the transfer function of Eq. (6-43), the system's response to the step-input becomes

$$C(s) = \frac{r_1}{s} \cdot \frac{1 + s/z_1}{(s/w_n)^2 + (2\delta/w_n)s + 1}$$

$$= \frac{r_1}{s} \cdot \frac{w_n^2}{s^2 + 2\delta w_n s + w_n^2} \cdot \frac{s + z_1}{z_1} \qquad (6\text{-}45)$$

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[9]Vincent Del Toro and Sydney R. Parker, Principles of Control Systems Engineering (New York: McGraw-Hill Book Company, Inc., 1960), pp. 434-5.

A comparison of Eq. (6-45) and (6-44) reveals that the addition of the zero cannot alter either the damping ratio or the natural frequency of the system because the characteristic equation of the system is unchanged. However, the presence of the zero may affect the amplitude of the transient response of the system, depending upon the relative magnitude of z_1 compared with δw_n. According to Del Toro and Parker:

> ... if z_1 is large compared with the values of s which are predominant in characterizing the time solution (i.e., that portion of the frequency spectrum up to w_n of the prevailing complex roots), then the influence is very small because $(s + z_1)/z_1$ is not appreciably larger than unity. On the other hand, in those situations where the magnitude of z_1 is small compared with the w_n of the predominant complex roots, the effect may be quite significant depending upon the value of δ.[10]

The precise manner in which a zero affects the value of the maximum percent overshoot for a system is shown in Fig. 6-9. As noted by Del Toro and Parker, the maximum overshoot is not appreciably affected by the presence of a zero when $z_1/\delta w_n \geq 10$.

<u>Example</u>. Consider a plant with the transfer function $G_1 = K/s(s + 2)$ and the compensation network $G_c = H_1 = (s + 4)$. The closed-loop transfer function for the cascade compensated system is

$$\frac{C}{R} = \frac{G_1 G_c}{1 + G_1 G_c} = \frac{K(s + 4)}{s(s + 2) + K(s + 4)} \qquad (6\text{-}46)$$

For the single-loop feedback compensated system

[10]<u>Ibid</u>.

$$\frac{C}{R} = \frac{G_1}{1 + G_1H_1} = \frac{K}{s(s + 2) + K(s + 4)} \qquad (6\text{-}47)$$

A comparison of Eq. (6-47) and (6-46) shows that the damping ratio, δ, and the natural frequency, w_n, of the two systems are the same but the magnitude of the transient response will

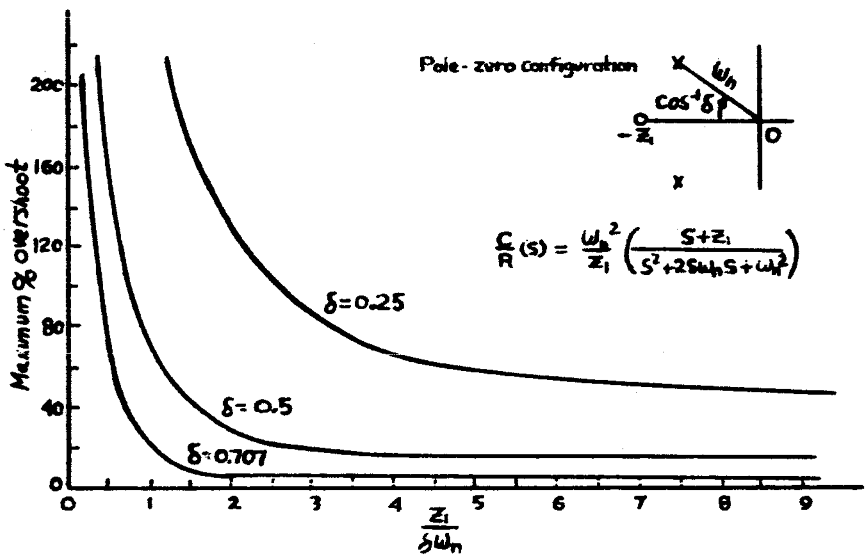

Figure 6-9. Variation of maximum percent overshoot with a zero of C/R(s).[11]

be greater for the cascade system because of the presence of the zero. A check to determine whether the difference is of significance is made by calculating the function $z/\delta w_n$. The factors δ and w_n are determined by comparing the characteristic equation for the two systems, $s^2 + s(2 + K) + 4K = 0$, with the normalized form for the second-order system, $s^2 + 2\delta w_n s + w_n^2 = 0$. It is seen that $w_n = 2\sqrt{K}$ and $\delta = (2 + K)/4\sqrt{K}$. Calculating $z/\delta w_n$ and calling this expression X,

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[11]*Ibid.*, p. 435.

$$\frac{z}{w_n} = \frac{4}{(2 + K)/4\sqrt{K}} = X \tag{6-48}$$

Solving Eq. (6-48) for K

$$K^2 + (4 - 256/X^2)K + 4 = 0$$

$$K = \frac{-(4 - 256/X^2) \pm \sqrt{(4 - 256/X^2)^2 - 16}}{2} \tag{6-49}$$

K must be real and positive. Therefore, from Eq. (6-49)

$$(4 - 256/X^2)^2 \geq 16$$

$$4 - 256/X^2 \leq -4 \tag{6-50}$$

Solving Eq. (6-50) for X

$$X = z/\delta w_n \leq 4\sqrt{2}$$

Since the condition $z/\delta w_n \geq 10$ has not been met, the zero will have an appreciable effect on the transient response of the cascade compensated system. The overshoot for this system must therefore be determined from Fig. 6-9.

6-3. SUMMARY AND CONCLUSIONS

The pole-zero configurations for cascade and feedback compensated systems have been compared and similarities and differences noted. In particular, for identical compensation networks and no pole-zero cancellation, it was noted that the poles of the closed-loop transfer function are the same for cascade and single-loop feedback compensation. Also, the zeros of the cascade compensated system are the same as the zeros of the open-loop transfer function, whereas the zeros of the single-loop feedback compensated system are the same as the zeros of the plant and the poles of the compensation network. The zeros of the closed-loop transfer function for the double-loop and single-loop feedback compensated systems are identical. It was also noted that the characteristic equation for the double-loop feedback compensated system can be put into three different root-locus equation forms that correspond to three equivalent single-loop forms of the original system and the three corresponding open-loop transfer functions.

When the compensation network introduces pole-zero cancellation in either the open-loop or closed-loop transfer functions for a system, the general statements concerning the system pole-zero configuration must be re-interpreted. It has been shown that the effects resulting from pole-zero cancellation when the zeros involved are in the plant and the poles involved are in the compensation network differ from the effects when the poles are in the compensation network and the zeros are in the plant.

The presence of zeros in the closed-loop transfer function for a system have been shown to increase the maximum percent overshoot of the system response. The significance of the increase depends upon the magnitude of the zero compared to the product of the system damping ratio and natural frequency (δw_n).

CHAPTER 7

SUMMARY AND CONCLUSIONS

7-1. SUMMARY

As stated in the introduction, the objective of this thesis has been to analyze and compare the effects of cascade and feedback compensation upon the steady-state and dynamic performance of feedback control systems. This objective has been observed throughout the thesis, with significant results being summarized in the closing section of each chapter.

Several relative merits of cascade and feedback compensation have been disclosed by the investigations in this thesis. These factors are summarized in the following section. Some of the analyses have not resulted in conspicuous advantages or limitations, but rather have indicated the characteristic effects of the various modes of compensation, or have introduced supporting information. For example, Chapter 2 revealed the problems involved in synthesizing a passive feedback compensator to replace the corresponding cascade compensator in a given system. Similarly, Chapter 5 introduced a convenient technique for approximating the magnitude of closed-loop transfer functions for compensated systems and analyzed the effects of compensation in terms of this technique. And finally, Chapter 6 presented the relative effects of cascade and feedback compensation upon the pole-zero configuration for a given system and discussed the significance of these effects. All of these results provide insight for selecting a particular compensation mode and compensator type for a given plant and given performance specifications.

7-2. RELATIVE MERITS OF COMPENSATION MODES

The relative advantages and disadvantages of the various modes of compensation are summarized below. The reader is referred to the appropriate chapter for a detailed discussion of these points.

System Sensitivity. The sensitivity function for a system is defined in Section 6 of Chapter 3. The smaller the value of this function, the less the control system output is affected by changes in a given parameter. The sensitivity function is in general less for the cascade compensated system for changes in the compensation network. The double-loop feedback compensated system offers a potential reduction in the sensitivity function for changes in the plant when compared with both the single-loop and cascade compensation cases. Another consideration in favor of feedback compensation in general is the fact that it may be more practical to design the feedback compensator to give the desired output accuracy and stability, regardless of whether or not the sensitivity function is less for the cascade compensated system.

Steady-state Performance. Some very significant conclusions were developed in Chapter 4 concerning the steady-state error and steady-state actuating signals for compensated systems. For example, this investigation revealed that finite system error is impossible to achieve for ramp or higher-order inputs into a single-loop feedback compensated system. This limitation restricts the single-loop system to regulator applications. No such restriction exists for cascade and double-loop feedback compensated systems. The steady-state errors and actuating signals are listed in Tables 4-1, 4-2 and 4-3 for each of the compensation modes and various input functions. An evaluation of these tables reveals that it is difficult to make general statements concerning the relative magnitudes of the steady-state functions for each of the compensation modes. In fact each of the modes of compensation can offer reduced steady-state system errors and actuating signals for specific combinations of plants and compensators. In other words, each given system should be analyzed independently in terms of the analyses of Chapter 4 and the information presented in the aforementioned tables.

Additional Considerations. In addition to the factors that have been discussed in the previous paragraphs, there are several other considerations that should be taken into account when making a choice between cascade and feedback compensation[1]:

1. The design procedures for cascade compensation tend to be more direct than those for feedback compensation.

2. Because of the physical form of the control system, a particular type of compensation may not be possible or at least not be practical.

3. The type of signal seen by the compensator must be considered. For example, the design of a feedback compensator may be more difficult than a cascade compensator when the feedback signal is modulated on a carrier.

4. Some control systems require the isolation of the dynamics of one part of the system from other parts of the complete system. This isolation can be accomplished by introducing an inner feedback loop around the part of the system that requires isolation.

5. The signal normally goes from a low to a high energy level in the forward path, whereas the opposite is usually the case for the feedback path. An amplifier is therefore generally required for a cascade compensator but will often not be necessary for feedback compensation. Also, the capacitors and other components for the cascade compensator may be larger and heavier than for the corresponding components in the feedback compensator.

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[1]John J. D'Azzo and Constantine H. Houpis, Feedback Control System Analysis and Synthesis, (New York: McGraw-Hill Book Company, 1966), pp. 465-467.

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APPENDIX

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APPENDIX

SPECIAL STEADY-STATE CONSIDERATIONS FOR FEEDBACK COMPENSATED SYSTEMS

A-1. A SPECIAL CASE OF A SINGLE-LOOP FEEDBACK COMPENSATED SYSTEM (N = 1, M = 0)

Consider a type 1 uncompensated system

$$G_1 = \frac{K_1}{s(1 + s\tau_a)(1 + s\tau_b)} \tag{A-1}$$

and the feedback compensation network

$$H_1 = \frac{K_c(1 + s\tau_a)}{(1 + s\tau_c)} \tag{A-2}$$

The open-loop transfer function for the single-loop feedback compensated system of Fig. A-1, page 106, is

$$G_1H_1 = \frac{K_1(1 + s\tau_a)}{s(1 + s\tau_a)(1 + s\tau_b)(1 + s\tau_c)}$$

$$= \frac{K_1}{\tau_b\tau_c}\cdot\frac{1}{s(s + 1/\tau_b)(s + 1/\tau_c)} \tag{A-3}$$

The compensation network is selected so that its zero will cancel a pole of the uncompensated system. The root locus plots for the uncompensated system and the compensated system are shown in Fig. A-2, page 106.

From Table 4-2 on page 42, the steady-state system error for a step input is

$$\theta_e(t)_{ss} = r_1 \frac{K_c - 1}{K_c} \qquad \text{(A-4)}$$

The error is not a function of K_1. If K_c is made equal to one by proper choice of preamplifier or attenuation network in Eq. (A-2), the steady-state system error is reduced to zero. Now K_1 can be adjusted independently to place the dominant roots in Fig. A-2(b) in the proper location for the desired transient response. Since the system error is independent of K_1, the error will not be affected by setting the roots.

A-2. BLOCK-DIAGRAM MANIPULATION AND INTERPRETATION OF STEADY-STATE SYSTEM ERROR FOR SINGLE-LOOP FEEDBACK COMPENSATION

Consider the single-loop feedback compensated system of Fig. A-1, with G_1 being a type 1 or greater system ($N \geq 1$) and H_1 being type 0 ($M = 0$). From Table 4-2 on page 42, the steady-state system error for a ramp input into this system is seen to be infinite. At first thought, this result might appear to be incorrect. The fact that the error is indeed infinite may be seen by transforming the block-diagram of Fig. A-1 into the equivalent form shown in Fig. A-3.

Since H_1 is a type 0 element, it reduces to K_c at steady state. The input function R and the function $R' = R/K_c$ are sketched in Fig. A-4, page 108. These functions diverge and the difference between them approaches infinity as $t \rightarrow \infty$.

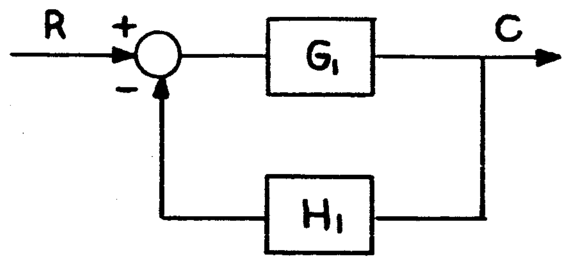
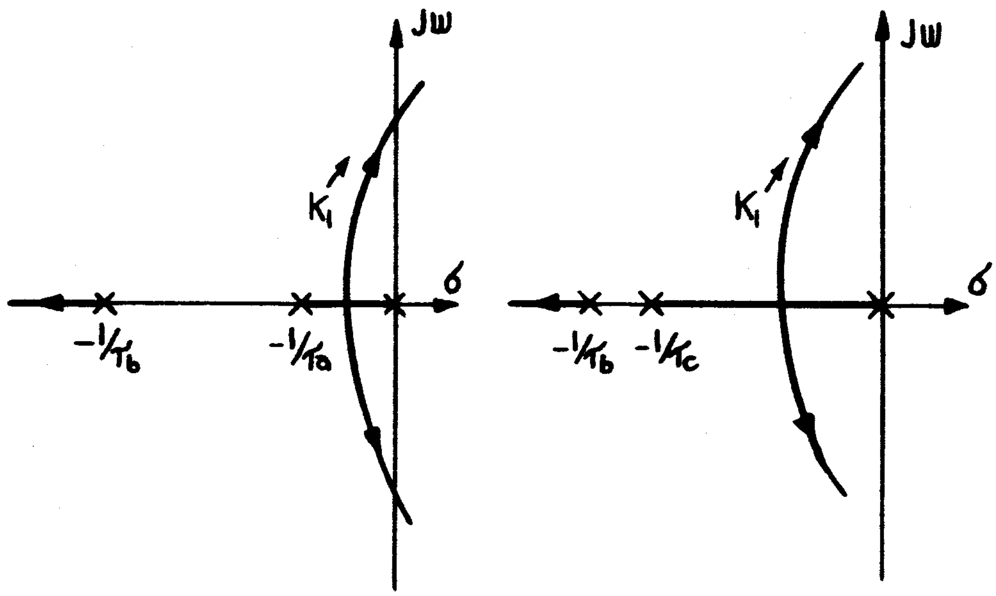


Figure A-1. Single-loop feedback compensated system.



a) Uncompensated system. b) Compensated system.

Figure A-2. System root-loci.

$G_1 = K_1/s(1 + s\tau_a)(1 + s\tau_b)$

$H_1 = K_c(1 + s\tau_a)/(1 + s\tau_c)$

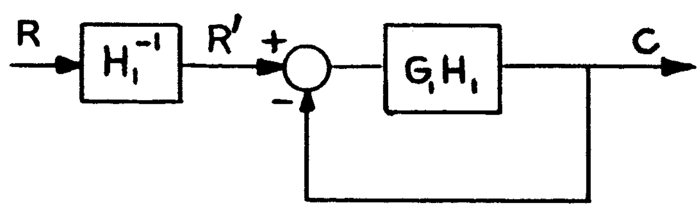


Figure A-3. Equivalent block-diagram for a single-loop feedback compensated system.

Since the error is finite for a ramp input into a type 1 or greater unity feedback system, the error between R' and C is finite (in fact zero for $N \geq 2$). The error for the single-loop feedback compensated system is defined as R-C and is therefore infinite for steady-state conditions.

A-3. BLOCK-DIAGRAM MANIPULATION AND INTERPRETATION OF STEADY-STATE SYSTEM ERROR FOR DOUBLE-LOOP FEEDBACK COMPENSATION

The physical significance of the steady-state system errors for a double-loop feedback compensated system may be made more apparent by analyzing the system obtained by transforming the system of Fig. A-5(a) into the equivalent unity feedback system of Fig. A-5(b).

The inner-loop of the double-loop system has been replaced by its transfer function G', where

$$G' = \frac{G_1}{1 + G_1 H_1} \qquad \text{(A-5)}$$

The steady-state system error for a unity feedback system was developed in Chapter 4, Section 1, and these errors are tabulated in Table 4-1 on page 38 in terms of the compensated system type (N + M) and the order (α) of the input function. Referring to Table 4-1, the error is seen to be infinite when the system type is less than the order of the input function minus one; the error is a constant when the system type equals the order of the input function minus one; and the

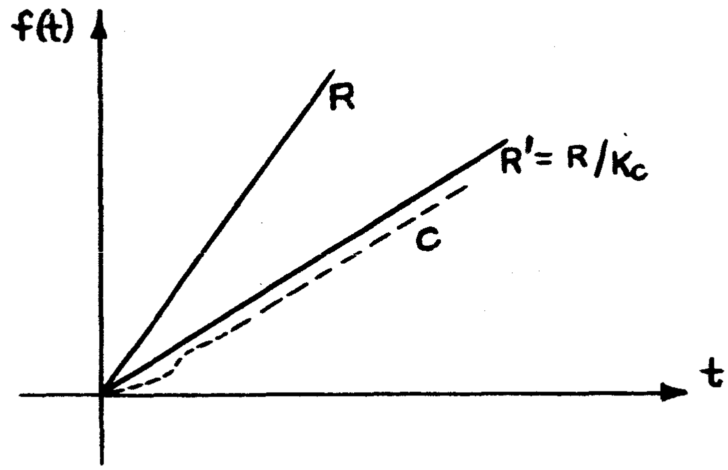
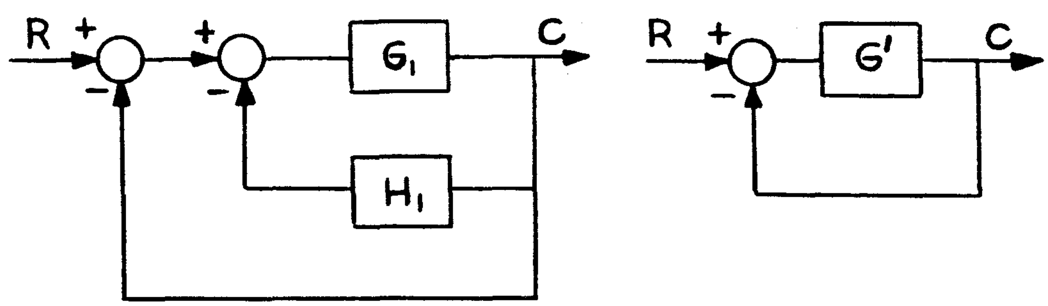


Figure A-4. Parameters R, R', and C versus t. (See Figure A-3, page 106.)



a) Double-loop feedback compensated system.

b) Equivalent unity feedback system.

Figure A-5. Block-diagram transformation of a double-loop feedback compensated system.

error is zero when the system type is greater than the order of the input function minus one.

The error for the system of Fig. A-5(b) is a function of the type of G' just as the error for the cascade compensated system is a function of the type of G_1G_c. It is therefore necessary to define the type of G' in terms of the type of the uncompensated system, G_1, and the feedback compensation element H_1. Expressing G' in its generalized form [See Eq. (4-30).]

$$G' = \frac{s^M K_1 N_1 D_c}{s^{N+M} D_1 D_c + K_1 N_1 K_c N_c} \qquad (A-6)$$

Equation (A-6) reveals that G' is a type 0 system for $M \geq 0$. For negative values of M, Eq. (A-6) may be put into a more convenient form by multiplying the numerator and denominator by $s^{|M|}$. After this operation is performed, Eq. (A-6) becomes

$$G' = \frac{K_1 N_1 D_c}{s^N D_1 D_c + s^{|M|} K_1 N_1 K_c N_c}, \quad M < 0 \qquad (A-7)$$

For $|M| < N$, Eq. (A-7) can be expressed as

$$G' = \frac{K_1 N_1 D_c}{s^{|M|}(s^{N-|M|} D_1 D_c + K_1 N_1 K_c N_c)}, \quad |M| < N \qquad (A-8)$$

Therefore G' is a type $|M|$ system for $M < 0$ and $|M| < N$.

Similarly, for $|M| > N$, Eq. (A-7) becomes

$$G' = \frac{K_1 N_1 D_c}{s^N(D_1 D_c + s^{|M|-N} K_1 N_1 K_c N_c)}, \quad |M| > N \qquad (A-9)$$

Therefore G' is a type N system for $M < 0$ and $|M| > N$.

Relating the type of G' to the system type in Table 4-1, page 38, and the steady-state system error for the double-loop feedback compensated system with the errors in Table 4-1, the physical significance of the restrictions on M and N for finite error that were stated on pages 50 and 52 for the double-loop feedback compensated case becomes apparent.