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TRAJECTORY OPTIMIZATION BY EXPLICIT NUMERICAL METHODS

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ABSTRACT

The problem of trajectory optimization is considered from the standpoint of numerical analysis. A numerical solution is obtained for an assumed thrust angle history, and an explicit numerical solution is obtained for the linearized equations of motion for neighboring solutions. With the explicit solution available, it is not difficult to determine whether or not the assumed solution is optimum. The first and second variations together provide a straightforward iteration method of approaching the optimum solution. It is of particular interest that the procedure remains unaltered even by the introduction of discontinuities and intermediate constraints.

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TECHNICAL AND SCIENTIFIC STAFF
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DEFINITION OF SYMBOLS

Symbol

Definition

X =	$\begin{bmatrix} x \\ y \\ \dot{x} \\ \dot{y} \end{bmatrix}$	
\dot{X}		time derivative of X
$\left. \begin{array}{l} f_1 \\ f_2 \\ g_1 \\ g_2 \\ h_1 \\ h_2 \\ m_1 \\ m_2 \end{array} \right\}$		impulse response functions defined by equations (4) and (12)
r		radial distance
r_c		required radial distance at cutoff time
r_n		actual radial distance at cutoff time
t		time
t_c		cutoff time
v		velocity
v_c		velocity at cutoff time
x,y		Cartesian coordinates of position in a space fixed system

DEFINITION OF SYMBOLS (Continued)

<u>Symbol</u>	<u>Definition</u>
\dot{x}, \dot{y}	time derivatives of x and y, respectively
z	an arbitrary function of X and t
z_1	the value of z at time t_1
z_c	required value of z at t_1
Δr	deviation in cutoff radial distance
Δt	deviation in cutoff time
Δv	deviation in cutoff velocity
$\Delta\lambda_1, \Delta\lambda_2, \Delta\lambda_3$	corrections for λ_1, λ_2 and λ_3 , respectively
$\Delta\chi$	variation in thrust angle
$\Delta\bar{\chi}$	correction to $\bar{\chi}$
$\lambda_1, \lambda_2, \lambda_3$	coefficients determined such that $h_1 + \lambda_1 f_1 + \lambda_2 g_1 + \lambda_3 m_1 = 0$
$\bar{\lambda}_1, \bar{\lambda}_2, \bar{\lambda}_3$	first approximations to λ_1, λ_2 and λ_3 , respectively
χ	thrust angle measured from the y-axis
$\bar{\chi}$	first approximation for χ

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TRAJECTORY OPTIMIZATION BY EXPLICIT NUMERICAL METHODS

SUMMARY

The problem of trajectory optimization is considered from the standpoint of numerical analysis. A numerical solution is obtained for an assumed thrust angle history, and an explicit numerical solution is obtained for the linearized equations of motion for neighboring solutions. With the explicit solution available, it is not difficult to determine whether or not the assumed solution is optimum. The first and second variations together provide a straightforward iteration method of approaching the optimum solution. It is of particular interest that the procedure remains unaltered even by the introduction of discontinuities and intermediate constraints.

OPTIMIZATION WITH END POINT CONSTRAINTS

The problem of trajectory optimization can be considerably simplified in concept if advantage is taken of available numerical methods from the beginning. To illustrate the point, the problem of minimizing the burning time required to meet desired end conditions will be considered. The differential equations describing the motion of the vehicle are defined in the following vector equation:

$$\dot{X} = F(X, \lambda, t). \quad (1)$$

The problem is to minimize the time required to fulfill the following conditions at cutoff:

$$\begin{aligned} r(t_c) &= r_c \\ \theta(t_c) &= \theta_c \\ v(t_c) &= v_c \end{aligned} \quad (2)$$

For some function $\bar{\chi}(t)$ with a velocity cutoff constraint, the following numerical solution is obtained:

$$\left. \begin{aligned} r(t_c) &= r_n \\ \theta(t_c) &= \theta_n \\ v(t_c) &= v_c \end{aligned} \right\} \quad (3)$$

The linearized equations of motion can be solved explicitly for the same initial conditions and cutoff conditions to give a solution of the following form*:

$$\left. \begin{aligned} \Delta r &= r_n - r_c + \int_{t_0}^{t_n} [f_1 \Delta X + f_2 \Delta X^2 + \dots] dt \\ \Delta \theta &= \theta_n - \theta_c + \int_{t_0}^{t_n} [g_1 \Delta X + g_2 \Delta X^2 + \dots] dt \\ \Delta t &= \int_{t_0}^{t_n} [h_1 \Delta X + h_2 \Delta X^2 + \dots] dt \end{aligned} \right\} \quad (4)$$

where

$$\left. \begin{aligned} \Delta r &= r - r_c \\ \Delta \theta &= \theta - \theta_c \\ \Delta t &= t - t_n \\ \Delta X &= X(t) - \bar{X}(t) \end{aligned} \right\} \quad (5)$$

*The derivation of these equations is described in detail in Reference 2.

The above equations can be combined in the following expression:

$$\begin{aligned} \Delta t + \lambda_1 \Delta r + \lambda_2 \Delta \theta = & \int_{t_0}^{t_n} \left[(h_1 + \lambda_1 f_1 + \lambda_2 g_1) \Delta X \right. \\ & \left. + (h_2 + \lambda_1 f_2 + \lambda_2 g_2) \Delta X^2 + \dots \right] dt + \lambda_1 (r_n - r_c) \\ & + \lambda_2 (\theta_n - \theta_c). \end{aligned}$$

This expression is general at this point and imposes no constraint on r , θ or $\Delta X(t)$. Setting $r = r_c$ and $\theta = \theta_c$ gives the following expression for Δt :

$$\begin{aligned} \Delta t = & \int_{t_0}^{t_n} \left[(h_1 + \lambda_1 f_1 + \lambda_2 g_1) \Delta X + (h_2 + \lambda_1 f_2 + \lambda_2 g_2) \Delta X^2 + \dots \right] dt \\ & + \lambda_1 (r_n - r_c) + \lambda_2 (\theta_n - \theta_c). \end{aligned} \tag{6}$$

If $h_1 + \lambda_1 f_1 + \lambda_2 g_1 = 0$ and $(h_2 + \lambda_1 f_2 + \lambda_2 g_2) > 0$, any change in the function $\Delta X(t)$ over any interval will increase the value of Δt in a neighborhood. If this condition exists, the function $\bar{X}(t)$ yields a minimum value of t locally. If $h_1 + \lambda_1 f_1 + \lambda_2 g_1 \neq 0$, consider $\bar{\lambda}_1$ and $\bar{\lambda}_2$ such that

$$h_1 + \bar{\lambda}_1 f_1 + \bar{\lambda}_2 g_1 = \epsilon.$$

The values of $\bar{\lambda}_1$ and $\bar{\lambda}_2$ can be obtained by the method of least squares or any other comparable method. Setting $\lambda_1 = \bar{\lambda}_1 + \Delta \lambda_1$, $\lambda_2 = \bar{\lambda}_2 + \Delta \lambda_2$ and $\Delta X = \bar{\Delta X} + \delta X$ gives the following expression for Δt .

$$\Delta t = \int_{t_0}^{t_n} \left\{ \epsilon(t) + \Delta\lambda_1 f_1 + \Delta\lambda_2 g_1 + 2(h_2 + \bar{\lambda}_1 f_2 + \bar{\lambda}_2 g_2) \Delta\bar{x} \right\} \delta x + \dots \int dt$$

$$+ \lambda_1(r_n - r_c) + \lambda_2(\theta_n - \theta_c). \quad (7)$$

Setting the weighting function for δx equal to zero gives the following equation:

$$2(h_2 + \bar{\lambda}_1 f_2 + \bar{\lambda}_2 g_2) \Delta\bar{x} = -\epsilon(t) - f_1 \Delta\lambda_1 - g_1 \Delta\lambda_2. \quad (8)$$

For simplification, the following definitions are employed:

$$\left. \begin{aligned} h_2^* &= (h_2 + \bar{\lambda}_1 f_2 + \bar{\lambda}_2 g_2) \\ \bar{\epsilon}(t) &= + \frac{\epsilon(t)}{2h_2^*} \\ \bar{f}_1 &= - \frac{f_1}{2h_2^*} \\ \bar{g}_1 &= - \frac{g_1}{2h_2^*} \end{aligned} \right\} \quad (9)$$

The expression for $\Delta\bar{x}$ is the following:

$$\Delta\bar{x} = \bar{f}_1 \Delta\lambda_1 + \bar{g}_1 \Delta\lambda_2 - \bar{\epsilon}. \quad (10)$$

The terms $\Delta\lambda_1$ and $\Delta\lambda_2$ can be determined by setting $\Delta r = 0$ and $\Delta\theta = 0$ in equations (4) and keeping only linear terms. This results in the following system of equations:

$$\left. \begin{aligned} \Delta\lambda_1 \int_{t_0}^{t_n} f_1 \bar{f}_1 dt + \Delta\lambda_2 \int_{t_0}^{t_n} f_1 \bar{g}_1 dt &= \int_{t_0}^{t_n} f_1 \bar{\epsilon} dt + r_c - r_n \\ \Delta\lambda_1 \int_{t_0}^{t_n} g_1 \bar{f}_1 dt + \Delta\lambda_2 \int_{t_0}^{t_n} g_1 \bar{g}_1 dt &= \int_{t_0}^{t_n} g_1 \bar{\epsilon} dt + \theta_c - \theta_n \end{aligned} \right\} \quad (11)$$

Solving this system for $\Delta\lambda_1$ and $\Delta\lambda_2$ provides the necessary quantities to determine $\Delta\bar{X}$ from equation (10) at each time point along the trajectory. With the new function $\chi_1 = \bar{X} + \Delta\bar{X}$, another iteration can be performed to determine χ_2 . This can be continued until $r_n - r_c$, $\theta_n - \theta_c$ and $\epsilon(t)$ all meet the required tolerances. Convergence of this iteration process ensures that the first order weighting function in equation (6) is zero. Similar operations can be applied to the second order term in the event that it is not everywhere positive.

INTERMEDIATE CONSTRAINTS

Assume that at some other time, t_1 , which may or may not be a function of X , an additional constraint is imposed requiring that some other function $z(t_1) = z_c$ and that for $\bar{X}(t)$, $z(t_1) = z_1$. The expression for Δz can be written as

$$\Delta z = z_1 - z_c + \int_{t_0}^{t_1} \left[m_1 \Delta\bar{X} + m_2 \Delta\bar{X}^2 + \dots \right] dt. \quad (12)$$

If $t_1 > t_n$, $m_1 = 0$ and $m_2 = 0$ for $t_n < t \leq t_1$. Equation (12) can then be written as

$$\Delta z = z_1 - z_c + \int_{t_0}^{t_n} \left[m_1 \Delta\bar{X} + m_2 \Delta\bar{X}^2 + \dots \right] dt. \quad (13)$$

If $t_0 < t_1 \leq t_n$, define $m_1 = m_2 = 0$ for $t_1 < t \leq t_n$. Then Δz can also be represented by equation (13). This equation is identical in form to expressions for Δr and $\Delta \theta$ appearing in equations (4). The same analysis used before can be applied to the constraints that

$$r(t_c) = r_c$$

$$\theta(t_c) = \theta_c$$

$$v(t_c) = v_c$$

$$z(t_1) = z_c.$$

The resulting expression is obtained for $\Delta \bar{x}$:

$$\Delta \bar{x} = \bar{f}_1 \Delta \lambda_1 + \bar{g}_1 \Delta \lambda_2 + \bar{m}_1 \Delta \lambda_3 + \bar{\epsilon} \quad (14)$$

where

$$\left. \begin{aligned} \bar{f}_1 &= -\frac{f_1}{2h_2^*}, & h_2^* &= h_2 + \bar{\lambda}_1 f_2 + \bar{\lambda}_2 g_2 + \bar{\lambda}_3 m_2 \\ \bar{g}_1 &= -\frac{g_1}{2h_2^*} \\ \bar{m}_1 &= -\frac{m_1}{2h_2^*} \\ \bar{\epsilon} &= \frac{\epsilon}{2h_2^*} \end{aligned} \right\} \quad (15)$$

and $\bar{\lambda}_1$, $\bar{\lambda}_2$ and $\bar{\lambda}_3$ have been determined as before. The values $\Delta \lambda_1$, $\Delta \lambda_2$ and $\Delta \lambda_3$ are solutions to the following system of equations:

$$\Delta \lambda_1 \int_{t_0}^{t_n} f_1 \bar{f}_1 dt + \Delta \lambda_2 \int_{t_0}^{t_n} f_1 \bar{g}_1 dt + \Delta \lambda_3 \int_{t_0}^{t_n} f_1 \bar{m}_1 dt = \int_{t_0}^{t_n} f_1 \bar{\epsilon} dt + r_c - r_n$$

$$\Delta\lambda_1 \int_{t_0}^{t_n} g_1 \bar{f}_1 dt + \Delta\lambda_2 \int_{t_0}^{t_n} g_1 \bar{g}_1 dt + \Delta\lambda_3 \int_{t_0}^{t_n} g_1 \bar{m}_1 dt = \int_{t_0}^{t_n} g_1 \bar{\epsilon} dt + \theta_c - \theta_n$$

$$\Delta\lambda_1 \int_{t_0}^{t_n} m_1 \bar{f}_1 dt + \Delta\lambda_2 \int_{t_0}^{t_n} m_1 \bar{g}_1 dt + \Delta\lambda_3 \int_{t_0}^{t_n} m_1 \bar{m}_1 dt = \int_{t_0}^{t_n} m_1 \bar{\epsilon} dt + z_c - z_1$$

The fact that the form of this system remains the same whether the constraints are applied to the end conditions or at some other point along the trajectory makes this method readily adaptable to computer operations and an extremely versatile multistage optimization program could easily be developed with a wide choice of constraints imposed at arbitrary points along the trajectory.

APPENDIX

An example to illustrate the method has been completed* from the following initial conditions:

$$\begin{aligned} x_0 &= 236.25546 \text{ km} & \dot{x}_0 &= 2800.5746 \text{ m/sec} \\ y_0 &= 6456.66782 \text{ km} & \dot{y}_0 &= 635.94317 \text{ m/sec} \\ t_0 &= 184.99481 \text{ sec.} \end{aligned}$$

The motion is assumed to be described by the following two-dimensional system of differential equations:

$$\ddot{x} = \frac{F}{m} \sin \chi + \ddot{x}_g,$$

$$\ddot{y} = \frac{F}{m} \cos \chi + \ddot{y}_g,$$

$$\ddot{x}_g = \frac{x}{r} g, \quad \ddot{y}_g = \frac{y}{r} g,$$

$$g = -\frac{g_0 r_0^2}{r^2}, \quad g_0 = 9.81 \text{ m/sec}^2, \quad r_0 = 6370 \text{ km}$$

$$\frac{F}{m} = \frac{10.787315}{1.2867664 - .261282644\tau_1}, \quad \tau_1 = \frac{t - 184.99481}{100}$$

$$184.99481 \leq t < 412.85503$$

$$\frac{F}{m} = \frac{8.92405155}{.69140715 - .212616813\tau_2}, \quad \tau_2 = \frac{t - 412.85503}{100}$$

$$412.85503 \leq t < 523.87973$$

$$\frac{F}{m} = 0, \quad 523.87973 \leq t < 528.87973.$$

*The computer program used to solve this example was developed in cooperation with James Hilliard, R-COMP, MSFC, and is described in Reference 4.

$$\frac{F}{m} = \frac{1.96132994}{.36021199 - .046948359\tau_3}, \quad \tau_3 = \frac{t - 428.87973}{100}$$

$$t \geq 428.87973.$$

F/m is in m/sec^2 and t is in seconds.

The problem is to determine the function χ which will minimize the burning time required to reach the following conditions:

$$r = r_c = 6555.2 \text{ km}$$

$$v = v_c = 7792.5746 \text{ m/sec}$$

$$\theta = \theta_c = 90^\circ.$$

In addition to the above constraints, an intermediate constraint has been included which requires that the spent stage which separates at $t = 412.85503$ seconds must, moving only under the influence of gravity, pass through the prespecified point (x_1, y_1) where

$$x_1 = 2588.9871 \text{ km}$$

$$y_1 = 5971.13760 \text{ km.}$$

The following function was initially assumed for χ :

$$\chi = 75.111 + 1.1982 \left(\frac{t - t_0}{100} \right),$$

where χ is in degrees and t in seconds.

The equations of motion were solved numerically by fourth order Runge Kutta at five-second time steps until the stage dropping off at $t = 412.85503$ seconds fulfilled the condition that $x = x_1$ and the upper stage fulfilled the condition that $v = v_c$. The weighting functions, f_1 , g_1 , m_1 and k_1 , were computed numerically by the method outlined in Chapter I of Reference 2. Corrections to the function χ were determined from equations (14) and (15), and another iteration was begun. After the seventh iteration, the following conditions were fulfilled:

$$r(t_c) = r_c \pm 0.1m$$

$$v(t_c) = v_c \pm 0.01 \text{ m/sec}$$

$$\theta(t_c) = \theta_c \pm 10^{-5} \text{ radians}$$

$$x(t_1) = x_1 \pm 0.1m$$

$$y(t_1) = y_1 \pm 0.1m$$

$$\epsilon(t_i) = 0 \pm 10^{-6}/\text{radian.}$$

Figure 1 shows the resulting trajectory compared with the solution that resulted when the intermediate constraint was removed. The total burning time with the additional constraint required 768.60 seconds compared with 690.24 seconds without this constraint. The additional 78.36 seconds indicates that this is a rather expensive maneuver but illustrates the versatility of the method.

The guidance angle, χ , is shown in Figure 2, where it is compared with the function used for the first iteration. A complete tabulation of χ at every point for which it was computed is included in Table 1 at the end of the appendix. In performing the numerical integration, this table was used with second order interpolation employed to determine the mid-point value required by the Runge-Kutta method.

Figure 3 illustrates the first order weighting functions f_1 , g_1 , m_1 , and \bar{f}_1 . These functions are tabulated for several time points in Table 1. For simplicity they have been rounded off to the fourth decimal place. It can be readily verified that they satisfy the equation $h_1 + \lambda_1 f_1 + \lambda_2 g_1 + \lambda_3 m_1 = 0$ within a tolerance of less than one unit in the last significant figure for the following values of λ_1 , λ_2 and λ_3 .

$$\lambda_1 = .299026 \text{ sec/km}$$

$$\lambda_2 = .702527 \cdot 10^3 \text{ sec/radian}$$

$$\lambda_3 = -.633224 \text{ sec/km.}$$

Figure 4 shows the functions \bar{f}_1 , \bar{g}_1 and \bar{h}_1 which were used to determine the corrections to χ . The second order weighting function, h_2 , is also shown in this illustration, and it can be seen to be positive.

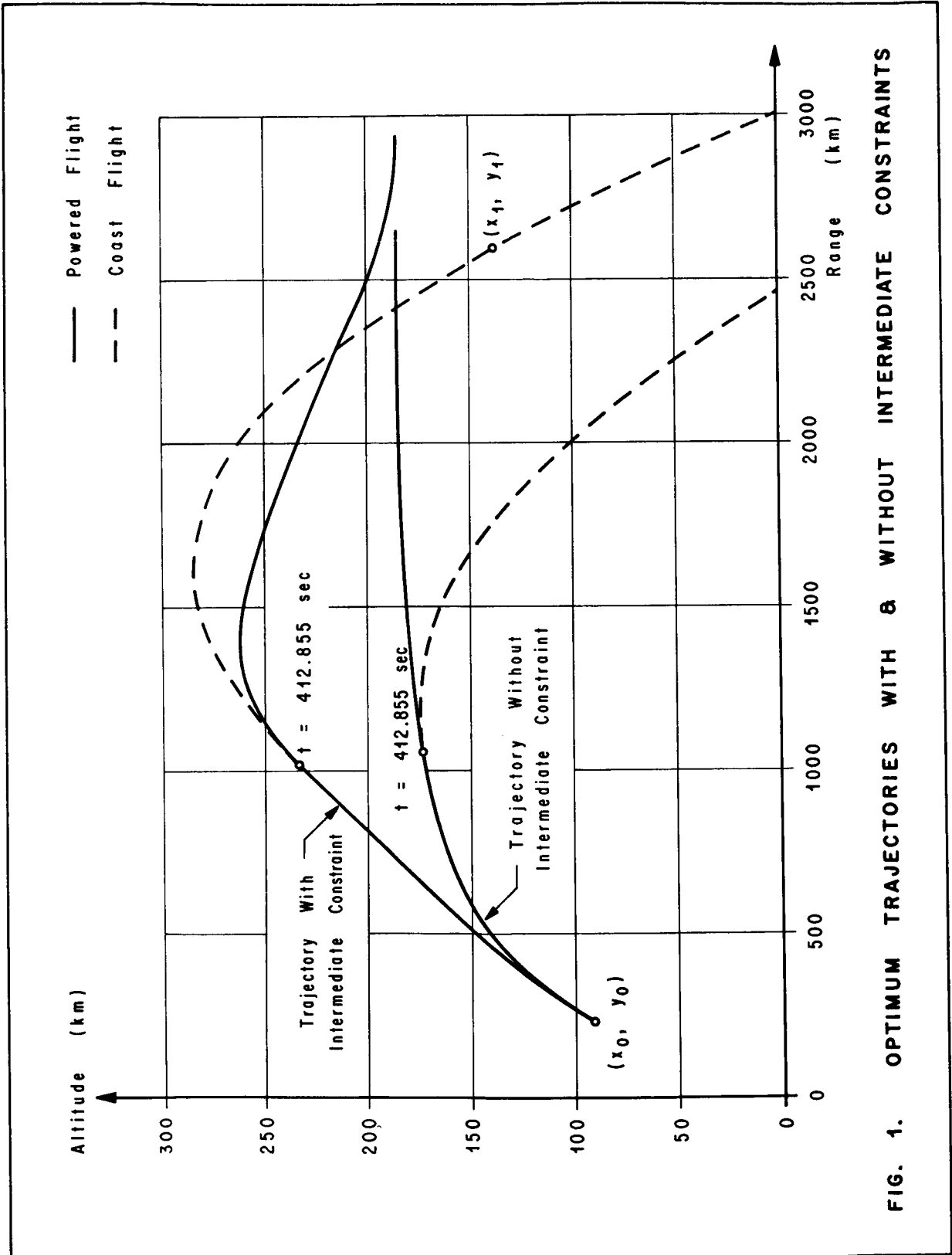


FIG. 1. OPTIMUM TRAJECTORIES WITH & WITHOUT INTERMEDIATE CONSTRAINTS

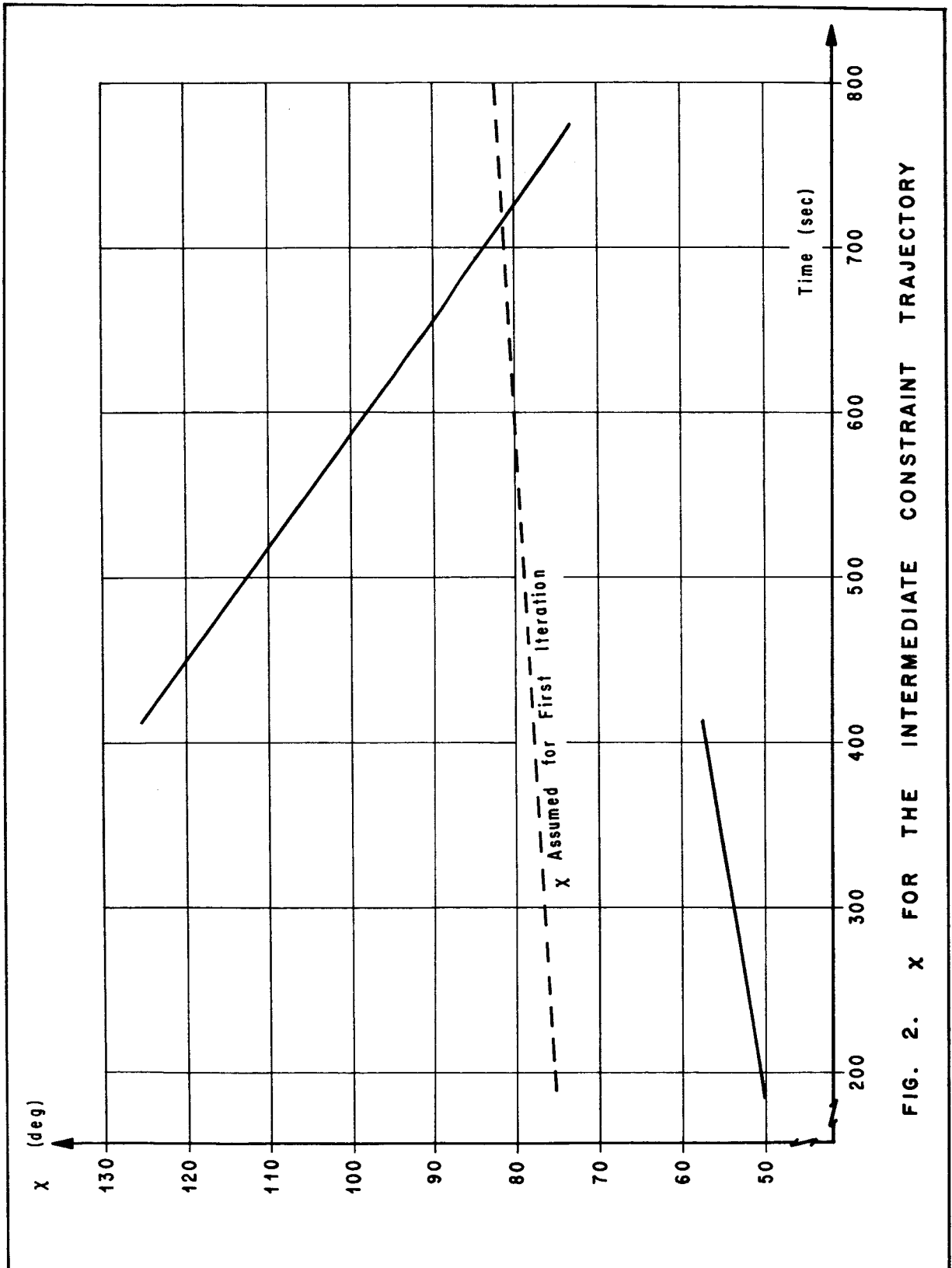


FIG. 2. X FOR THE INTERMEDIATE CONSTRAINT TRAJECTORY

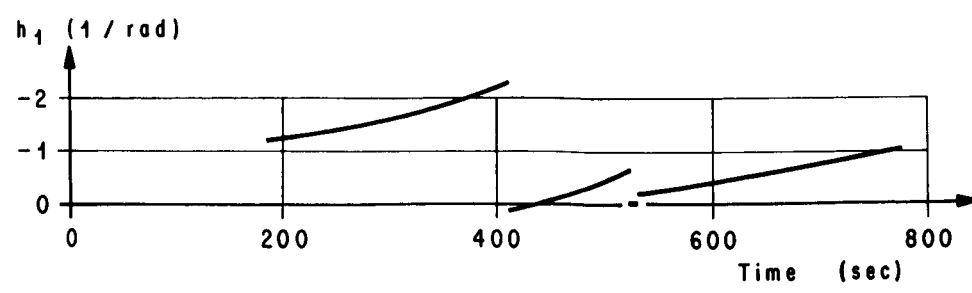
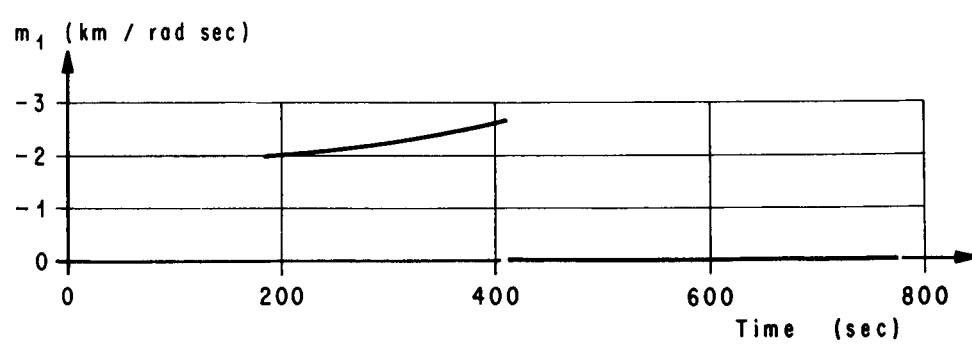
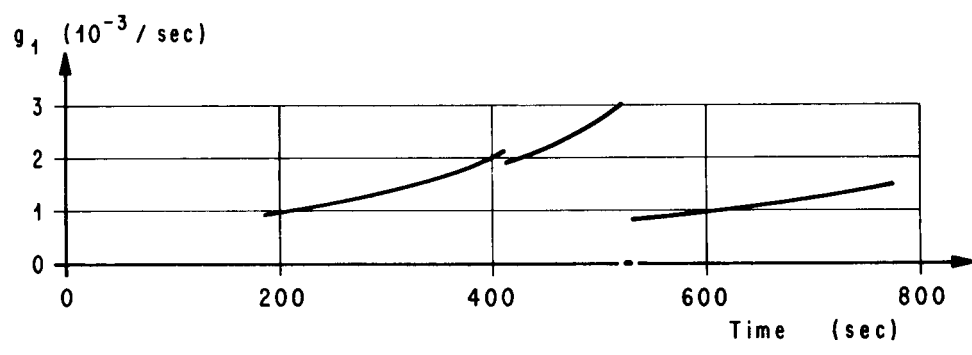
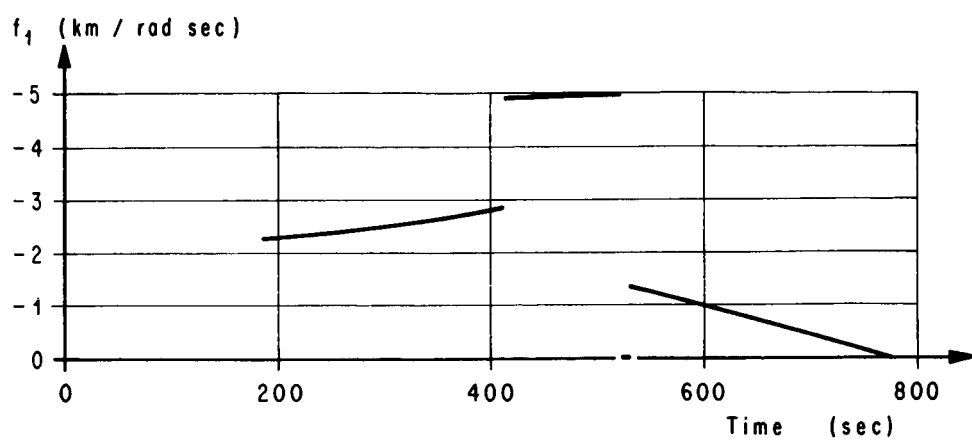


FIG. 3. FIRST ORDER WEIGHTING FUNCTIONS

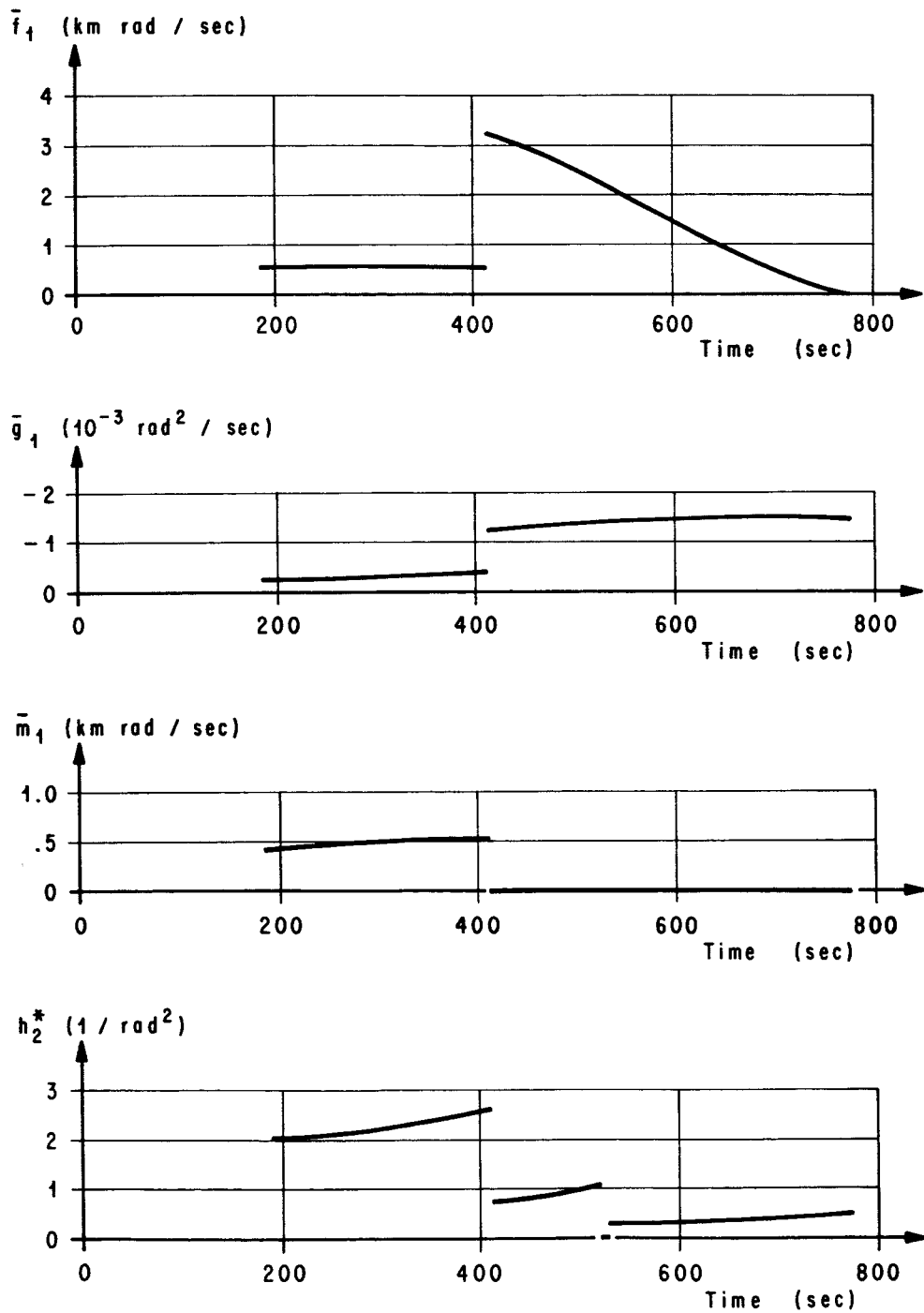


FIG. 4. FUNCTIONS USED TO DETERMINE Δx AND THE SECOND ORDER WEIGHTING FUNCTION h_2^*

TABLE 1

t(sec)	f_1 [km/rad sec]	g_1 [10^{-3} /sec]	m_1 [km/rad sec]	h_1 [1/rad]
185	-2.2642	.9238	-1.9664	-1.2171
200	-2.2895	.9679	-1.9963	-1.2594
250	-2.3827	1.1380	-2.1054	-1.4201
300	-2.4932	1.3542	-2.2331	-1.6199
350	-2.6290	1.6382	-2.3880	-1.8769
400	-2.8032	2.0267	-2.5837	-2.2217
412.9	-2.8568	2.1507	-2.6433	-2.3305
412.9	-4.8807	1.8953	0	.1280
450	-4.9237	2.1708	0	-.0527
500	-4.9454	2.6950	0	-.4146
523.9	-4.9455	3.0395	0	-.6565
523.9	0	0	0	0
528.9	0	0	0	0
528.9	-1.3418	.8472	0	-.1939
550	-1.2401	.8836	0	-.2499
600	-.9809	.9823	0	-.3968
650	-.7032	1.1000	0	-.5625
700	-.4169	1.2384	0	-.7454
750	-.1317	1.3997	0	-.9440
773.6	.0000	1.4886	0	-1.0458

$$\lambda_1 = .299026 \text{ sec/km}$$

$$\lambda_2 = .702527 \cdot 10^3 \text{ sec/rad}$$

$$\lambda_3 = -.633224 \text{ sec/km}$$

TABLE 2

t(sec)	χ(degrees)	t(sec)	χ(degrees)
184.99481	50.0558152	325	54.7621191
185	50.0559990	330	54.9239170
190	50.2312133	335	55.0854054
195	50.4058228	340	55.2465994
200	50.5798353	345	55.4075148
205	50.7532584	350	55.5681676
210	50.9261002	355	55.7285747
215	51.0983688	360	55.8887534
220	51.2700725	365	56.0487214
225	51.4412196	370	56.2084973
230	51.6118189	375	56.3681001
235	51.7818790	380	56.5275498
240	51.9514090	385	56.6868666
245	52.1204181	390	56.8460719
250	52.2889156	395	57.0051875
255	52.4569111	400	57.1642363
260	52.6244145	405	57.3232420
265	52.7914357	410	57.4822289
270	52.9579851	412.855030	57.5733702
275	53.1240733	412.855030	125.234677
280	53.2897111	415	124.935432
285	53.4549096	420	124.236574
290	53.6196802	425	123.535040
295	53.7840346	430	122.830917
300	53.9479848	435	122.124289
305	54.1115432	440	121.415244
310	54.2747226	445	120.703869
315	54.4375359	450	119.990252
320	54.5999968	455	119.274482

TABLE 2 (Continued)

t(sec)	X(degrees)	t(sec)	X(degrees)
460	118.556646	585	100.202874
465	117.836835	590	99.4630176
470	117.115138	595	98.7235515
475	116.391644	600	97.9845716
480	115.666444	605	97.2461745
485	114.939627	610	96.5084576
490	114.211284	615	95.7715192
495	113.481505	620	95.0354585
500	112.750381	625	94.3003754
505	112.018000	630	93.5663709
510	111.284453	635	92.8335468
515	110.549831	640	92.1020062
520	109.814223	645	91.3718527
523.879730	109.242771	650	90.6431911
523.879730	109.242771	655	89.9161274
526.379730	108.874254	660	89.1907682
528.879730	108.505553	665	88.4672214
528.879730	108.505553	670	87.7455956
530	108.340267	675	87.0260004
535	107.602271	680	86.3085466
540	106.863648	685	85.5933453
545	106.124489	690	84.8805090
550	105.384885	695	84.1701506
555	104.644926	700	83.4623837
560	103.904703	705	82.7573229
565	103.164310	710	82.0550831
570	102.423837	715	81.3557796
575	101.683378	720	80.6595283
580	100.943026		

TABLE 2 (concluded)

t(sec)	X(degrees)
725	79.9664454
730	79.2766473
735	78.5902504
740	77.9073712
745	77.2281261
750	76.5526312
755	75.8810022
760	75.2133544
765	74.5498023
770	73.8904599
773.599673	73.4188071

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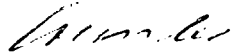
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TRAJECTORY OPTIMIZATION BY EXPLICIT NUMERICAL METHODS

By Lyle R. Dickey

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