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EFFECT OF COULOMB COLLISIONS ON CONTRASTREAMING PLASMAS

B. Buti*

Theoretical Physics Division, National Physical Laboratory,
New Delhi, India

and Laboratory for Theoretical Studies,
Goddard Space Flight Center, Greenbelt, Maryland

and

S. K. Trehan*

Department of Physics and Astrophysics, University of Delhi
New Delhi, India

and Laboratory for Theoretical Studies,
Goddard Space Flight Center, Greenbelt, Maryland

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Present Address:
*National Aeronautics and
Space Administration
Goddard Space Flight Center
Laboratory for Theoretical Studies
Greenbelt, Maryland 20771

ABSTRACT

The instability of contrastreaming plasmas is investigated taking into account the Coulomb Collisions via the Fokker-Planck co-efficients in the Boltzmann equation. The dispersion relation is obtained on the assumption that the Coulomb collisions are weak and solved on the additional assumption that the phase velocity of the wave is much larger than the mean thermal velocity of the particles. It is found that while the temperature has the effect of increasing the maximum wavenumber x_c (which for a cold plasma is equal to $(2)^{1/2}$ in units of ω_p/U, where ω_p is the electron plasma frequency and U is the streaming velocity) below which the plasma is unstable, the collisions have no effect on this wavenumber. However, the growth rate of maximum instability decreases (compared to its value for a cold plasma) on taking into account the thermal motions but increases when the collisions are taken into account.

[redacted]

I. INTRODUCTION

The instability of longitudinal electron oscillations in a plasma where we have two streams of electrons moving with equal and opposite velocities has been discussed extensively by several authors (1,2) when the thermal spread of the electrons as well as the collisions between the particles are ignored. It is well known that such a system is unstable for all wavenumbers $x (= kU/\omega_p$, where k is the wavenumber of the perturbation, U the speed of the stream and ω_p the electron plasma frequency) which are less than a certain critical value $x_c = (2)^{1/2}$. Further the wavenumber for which the instability is maximum is known to be $x_* = (3)^{1/2}/2$. Jackson (3) has considered the problem of two stream instability when the velocities of the two streams are smeared out by an equal amount and finds that x_c increases on taking into account the spread of the distribution function of the electron streams.

Tidman and Weiss (4) calculated the effect of collisions on the two stream instability assuming zero thermal spread of the electrons and the ions. The collisions were taken into account by a simple relaxation model in which the electrons are taken to be scattered into a local Maxwellian distribution in a time τ . They found that collisions increased this instability. This result led them to suspect that the diffusion nature of the Coulomb thermalization of the electron streams is an essential feature of the problem and hence the more exact Fokker-Planck equation for Coulomb scattering should be used. It was later pointed out by May (5) that Tidman and Weiss did not carry out the calculations consistently as they neglected the equilibrium current density and the associated magnetic field which results due to the mass motion of the electrons. May considered two interpenetrating streams so that there is no current in equilibrium. He calculated the effect of collisions using the relaxation model of Gross and Krook (6) and found that the unstable mode ceases to grow after a time of the order of one collision time. However, he had taken into account only the electron-electron collisions claiming that the electron-ion collisions are negligible due to the large ion-electron mass ratio. We will show here that the electron-ion collisions are actually comparable with the electron-electron collisions for the problem at hand.

Tidman (7) considered the effect of collisions on the instability of a stream of particles moving with a velocity $\underset{\sim}{V}$ (with no thermal spread) through cold ions at rest. He took into account the effect of Coulomb collisions using the Fokker-Planck equation and found that as the ordered streaming energy of the electrons is thermalized due to diffusion in velocity space, there results an enchanced Landau damping which competes with the growth of instability. He did not take into account the equilibrium current density which results in his model. It will be shown from our more rigorous calculations that weak collisions cannot even cancel the effect of thermal motions much less quenching the instability.

Comisar (8) and Buti and Jain (9) have studied the effect of weak Coulomb collisions on the damping of longitudinal and transverse plasma oscillations respectively using the Fokker-Planck equation in the form given by Rosenbluth et al (10). They find that to the first order in the collision frequency ν_c, the electron-ion collisions dominate the electron-electron collisions in damping the plasma oscillations.

Following Comisar and Buti and Jain we consider the effect of Coulomb collisions on the stability of contrastreaming hot plasmas in a systematic manner. We assume that the collisions are not too frequent so that we can make an expansion in powers of (ν_c/ω_p) and we calculate the collisional effects to the first order in the collision frequency. In the absence of collisions, the dispersion relation, including the thermal effects, is exactly solved numerically to determine the region of instability and the growth rate of maximum instability. When collisions are taken into account along with the thermal effects, then we use an iterative procedure to solve the dispersion relation and to study the effect of collisions on the critical wavenumber and on the growth rate of maximum instability.

It is, perhaps, in order to remark here that the Fokker-Planck equation is not strictly valid for an unstable plasma, particularly near the electron plasma frequency (11,12). However, the motivation for the present undertaking is to clarify the contributions of the frictional and the diffusion terms in the electron-electron and electron-ion collisions. We find that the contribution of the frictional term is dominant compared to that of the diffusion term in electron-ion collisions whereas in electron-electron collisions the two are of the same order. Further, the contribution of the electron-ion collisions is much larger than that of the electron-electron collisions. It may be reasonable to expect that the qualitative nature of these results will not be altered by using a more exact kinetic equation for an unstable plasma.

II. GENERAL THEORY

Let us consider two streams of hot unbounded plasmas in which the ions form only the neutralizing background but the electrons have non-relativistic streaming velocities. The equilibrium distribution functions, normalized to unity, for the electrons and the ions are assumed to have the form:

$$f_{oa} = (2\pi V_a^2)^{-3/2}\, e^{-(\underset{\sim}{v}-\underset{\sim}{U}_a)^2/2V_a^2} \tag{1}$$

and

$$f_{oi} = (2\pi V_i^2)^{-3/2}\, e^{-v^2/2V_i^2}\,, \tag{2}$$

where $V_a^2 = \Theta_a/m$ and $V_i^2 = \Theta_i/M$; Θ_a and Θ_i denote the temperatures (in energy units) of the electrons and the ions respectively. The subscript a on the electron distribution function refers to either of the streams. The density of electrons on ions in each stream is taken to be N. The distribution functions given by Eqs. (1) and (2) do not satisfy the Fokker-Planck equation:

$$\frac{\partial f}{\partial t} + \underset{\sim}{v}\cdot\underset{\sim}{\nabla} f + \underset{\sim}{F}\cdot\underset{\sim}{\nabla}_v f = -\frac{\partial}{\partial \underset{\sim}{v}}\cdot\left[\langle\underset{\sim}{\Delta}\rangle f - \frac{1}{2}\frac{\partial}{\partial \underset{\sim}{v}}\cdot\langle\underset{\sim}{\Delta}\underset{\sim}{\Delta}\rangle f\right], \tag{3}$$

where $\underset{\sim}{F}$ is the force per unit mass and

$$\langle\underset{\sim}{\Delta}\rangle = N\Gamma \sum_{\alpha=i,e}\left(1+\frac{m}{m_\alpha}\right)\frac{\partial}{\partial \underset{\sim}{v}}\int d\underset{\sim}{v}'\,\frac{f_\alpha(\underset{\sim}{v}')}{|\underset{\sim}{v}-\underset{\sim}{v}'|}\,, \tag{4}$$

$$\langle \underset{\sim}{\Delta}\underset{\sim}{\Delta} \rangle = N\Gamma \sum_{\alpha=i,e} \frac{\partial^2}{\partial \underset{\sim}{v}\, \partial \underset{\sim}{v}} \int d\underset{\sim}{v}' f_\alpha(\underset{\sim}{v}') \, | \underset{\sim}{v} - \underset{\sim}{v}' | \, , \tag{5}$$

$$\Gamma = \frac{4\pi e^4}{m^2} \log \Lambda \, , \quad \Lambda = 4\pi N \lambda_D^3 \, , \tag{6}$$

where λ_D is the Debye length. In fact we find that

$$\frac{d\underset{\sim}{U}_a}{dt} = \frac{4\nu_c \underset{\sim}{U}_a(t)}{(\pi)^{1/2} (U_a/V_0)^3} \left[2\Omega(U_a^2/2V_0^2) + \frac{1}{4} \Omega(U_a^2/V_0^2) \right], \tag{7}$$

where

$$\Omega(x) = \int_0^x u^2 e^{-u^2} du \, , \tag{8}$$

and the effective collision frequency is defined to be

$$\nu_c = (\omega_p / \Lambda) \log \Lambda \tag{9}$$

In writing Eq. (7) we have taken the electron temperatures of each stream to be the same and put $V_a^2 = V_0^2$.

As we shall see later the effect of thermal motions enters the dispersion relation through the parameter $(k\lambda_D)^2 = Ax^2$ with $A = V_0^2/U^2$ and $x = kU/\omega_p$. Now for all situations of physical interest $(k\lambda_D)$ must be much less than unity. Further we know that two-stream instability arises for values of x of order unity. Thus the condition $k\lambda_D \ll 1$ implies that A be less than unity for the unstable situations we are considering here. Since

$$\Omega(U_a^2/V_0^2) \simeq \frac{1}{4}(\pi)^{1/2} \quad \text{for } V_0^2/U_a^2 \ll 1 , \tag{10}$$

we can write

$$\frac{d\underset{\sim}{U}_a}{dt} = \frac{9}{4}\nu_c \underset{\sim}{U}_a(t)\left(\frac{V_0}{U_a}\right)^3 , \tag{11}$$

or in order of magnitude

$$\frac{1}{\omega_p U}\frac{dU}{dt} = \frac{9}{4}\frac{\nu_c}{\omega_p}\left(\frac{V_0}{U}\right)^3 . \tag{12}$$

The quantity (ν_c/ω_p) is always much less than unity for physical situations of interest and we have seen that (V_0/U_a) is also less than unity for the unstable situations under consideration here. Hence U changes in time on a scale which is very much larger than a plasma period. On the other hand we know that in the absence of collisions the two stream instability has e-folding time of the order of a plasma period. Thus for the problem at hand, the distribution function satisfies the Fokker-Planck equation to a good approximation and we may take $\underset{\sim}{U}_a$ in Eq. (1) to be constant. Moreover, the neglect of this term is justified on the ground that this is independent of $\underset{\sim}{x}$ and thus would affect only the $k=0$ mode when we take the Fourier transform of the linearized equations in $\underset{\sim}{x}$- space; and we are not primarily concerned with this mode here. In a similar manner, it can be shown that the Fokker-Planck equation for the ions is also satisfied to an equally good approximation.

For longitudinal oscillations the linearized Fokker-Planck equation for the motion of electrons has the form:

$$\frac{\partial f_a}{\partial t} + \underset{\sim}{v}\cdot\underset{\sim}{\nabla} f_a - \frac{Ne}{m}\underset{\sim}{E}\cdot\underset{\sim}{\nabla}_v f_{oa} = \left(\frac{\partial f_a}{\partial t}\right)_{coll.} , \tag{13}$$

where the right hand side represents the effect of collisions. Again the subscript a in Eq. (13) implies that this equation refers to either of the plasma streams, $a=1$ or 2. The change in f_a due to collisions results from collisions with the ions or electrons in either stream. Thus we can write

$$\left(\frac{\partial f_a}{\partial t}\right)_{coll.} = \sum_b \left[\left(\frac{\partial f_a}{\partial t}\right)_{ib} + \left(\frac{\partial f_a}{\partial t}\right)_{eb}\right] , \tag{14}$$

where $a, b = 1$ or 2. The collision terms are given by

$$\left(\frac{\partial f_a}{\partial t}\right)_{ib} = -\frac{\partial}{\partial \underset{\sim}{v}}\cdot\left(\langle\underset{\sim}{\Delta}\rangle_{oi}^{b} f_a\right) + \frac{1}{2}\frac{\partial^2}{\partial \underset{\sim}{v}\partial \underset{\sim}{v}} : \left(\langle\underset{\sim}{\Delta}\underset{\sim}{\Delta}\rangle_{oi}^{b} f_a\right) \tag{15}$$

and

$$\begin{aligned}\left(\frac{\partial f_a}{\partial t}\right)_{eb} = &-\frac{\partial}{\partial \underset{\sim}{v}}\cdot\left(\langle\underset{\sim}{\Delta}\rangle_{oe}^{b} f_a + \langle\underset{\sim}{\Delta}\rangle_{e}^{b} f_{oa}\right)\\ &+ \frac{1}{2}\frac{\partial^2}{\partial \underset{\sim}{v}\partial \underset{\sim}{v}} : \left(\langle\underset{\sim}{\Delta}\underset{\sim}{\Delta}\rangle_{oe}^{b} f_a + \langle\underset{\sim}{\Delta}\underset{\sim}{\Delta}\rangle_{e}^{b} f_{oa}\right) ,\end{aligned} \tag{16}$$

where

$$\langle\underset{\sim}{\Delta}\rangle_{oi}^{b} = N\Gamma\left(1+\frac{m}{M}\right)\frac{\partial}{\partial \underset{\sim}{v}}\int \frac{f_{oi}(\underset{\sim}{v}')\,d\underset{\sim}{v}'}{|\underset{\sim}{v}-\underset{\sim}{v}'|} , \tag{17}$$

$$\langle \Delta\Delta \rangle^{b}_{oi} = N\Gamma \frac{\partial^2}{\partial v \partial v} \int f_{oi}(v') |v - v'| \, dv' , \tag{18}$$

$$\langle \Delta \rangle^{b}_{oe} = 2N\Gamma \frac{\partial}{\partial v} \int \frac{f_{ob}(v')}{|v - v'|} \, dv' , \tag{19}$$

and

$$\langle \Delta\Delta \rangle^{b}_{oe} = N\Gamma \frac{\partial^2}{\partial v \partial v} \int f_{ob}(v') |v - v'| \, dv' . \tag{20}$$

The expressions for $\langle \Delta \rangle^{b}_{e}$ and $\langle \Delta\Delta \rangle^{b}_{e}$ can be written down from Eqs. (19) and (20) on replacing f_{ob} by f_{b} . The electric field E in Eq. (13) is governed by the Poisson equation:

$$\nabla \cdot E = -4\pi e \sum_{a} \int f_a(x, v, t) \, dv . \tag{21}$$

Equations (13) and (21) from the characteristic value problem which we shall solve by the Fourier-Laplace transform method. We thus define

$$f^{a}_{k}(v, s) = \int_0^\infty dt \int dx \, e^{-st - i k \cdot x} f_a(x, v, t), \tag{22}$$

and

$$E_k(s) = \int_0^\infty dt \int dx \, e^{-st - i k \cdot x} E(x, t), \tag{23}$$

with $\mathrm{Re}\, s > 0$ for the integrals to be convergent. The Fourier-Laplace transforms of Eqs. (13) and (21) are:

$$s f^{a}_{k}(v, s) - g^{a}_{k}(v) + i k \cdot v f^{a}_{k}(v, s) - \frac{Ne}{m} E_k(s) \cdot \frac{\partial f_{oa}}{\partial v} = \left(\frac{\partial f^{a}_{k}}{\partial t} \right)_{coll} . \tag{24}$$

and

$$\underset{\sim}{E}_k(s) = 4\pi i e \frac{\underset{\sim}{k}}{k^2} \sum_a \int f_k^a(\underset{\sim}{v}, s)\, d\underset{\sim}{v} , \tag{25}$$

where $g_k^a(\underset{\sim}{v})$ is the Fourier transform of the initial distribution function i.e.,

$$g_k^a(\underset{\sim}{v}) = \int d\underset{\sim}{x}\, e^{-i \underset{\sim}{k}\cdot\underset{\sim}{x}} f_a(\underset{\sim}{x}, \underset{\sim}{v}, 0) . \tag{26}$$

In order to solve the characteristic value problem posed by Eqs. (24) and (25), it is now convenient to take the Fourier-transform in velocity space; we thus define

$$F_a(\underset{\sim}{k}, \underset{\sim}{\sigma}, s) = \int f_k^a(\underset{\sim}{v}, s)\, e^{-i \underset{\sim}{\sigma}\cdot\underset{\sim}{v}}\, d\underset{\sim}{v} . \tag{27}$$

Equation (24) now takes the form

$$\begin{aligned} &\left(s - \underset{\sim}{k}\cdot\frac{\partial}{\partial \underset{\sim}{\sigma}}\right) F_a(\underset{\sim}{k}, \underset{\sim}{\sigma}, s) - G_a(\underset{\sim}{k}, \underset{\sim}{\sigma}) \\ &- \frac{iNe}{m} \underset{\sim}{\sigma}\cdot\underset{\sim}{E}_k \exp\left\{-i\underset{\sim}{\sigma}\cdot\underset{\sim}{U}_a - \tfrac{1}{2}\sigma^2 V_0^2\right\} = \sum_b \left[\left(\frac{\partial F_a}{\partial t}\right)_{ib} + \left(\frac{\partial F_a}{\partial t}\right)_{eb}\right] . \end{aligned} \tag{28}$$

In order to simplify the collision terms, we first observe that the Fourier-Laplace transform of Eq. (15) leads to

$$\begin{aligned} \left(\frac{\partial F_a}{\partial t}\right)_{ib} = \int d\underset{\sim}{v}'\, e^{-i\underset{\sim}{\sigma}\cdot\underset{\sim}{v}'} \Big[&-i\, \underset{\sim}{\sigma}\cdot\langle \underset{\sim}{\Delta} \rangle_{oi}^{b} \\ &- \tfrac{1}{2} \underset{\sim}{\sigma}\underset{\sim}{\sigma} : \langle \underset{\sim}{\Delta}\underset{\sim}{\Delta} \rangle_{oi}^{b} \Big] f_k^a(\underset{\sim}{x}, \underset{\sim}{v}', s) . \end{aligned} \tag{29}$$

On using the identies

$$\frac{1}{|\underset{\sim}{v} - \underset{\sim}{v}'|} = \frac{1}{2\pi^2} \int d\underset{\sim}{\xi}\, e^{i\underset{\sim}{\xi}\cdot(\underset{\sim}{v} - \underset{\sim}{v}')} \frac{1}{\xi^2}$$

and

$$|\underset{\sim}{v} - \underset{\sim}{v}'| = -\frac{1}{\pi^2} \int d\underset{\sim}{\xi}\, e^{i\underset{\sim}{\xi}\cdot(\underset{\sim}{v} - \underset{\sim}{v}')} \frac{1}{\xi^4} ,$$

we find that

$$\langle \underset{\sim}{\Delta} \rangle_{oi}^{b} = N\Gamma \frac{1}{2\pi^2}\left(1+\frac{m}{M}\right)\int d\underset{\sim}{\xi}\, \frac{i\underset{\sim}{\xi}}{\xi^2} \exp\left\{ i\, \underset{\sim}{\xi}\cdot\underset{\sim}{v} - \frac{1}{2} V_i^2 \xi^2 \right\}, \tag{30a}$$

and

$$\langle \underset{\sim}{\Delta}\,\underset{\sim}{\Delta} \rangle_{oi}^{b} = \frac{N\Gamma}{\pi^2}\int d\underset{\sim}{\xi}\, \frac{\underset{\sim}{\xi}\,\underset{\sim}{\xi}}{\xi^4} \exp\left\{ i\, \underset{\sim}{\xi}\cdot\underset{\sim}{v} - \frac{1}{2} V_i^2 \xi^2 \right\}. \tag{30b}$$

On using the foregoing results in Eq. (29) we obtain

$$\left(\frac{\partial F_a}{\partial t}\right)_{ib} = N\Gamma \int d\underset{\sim}{\eta}\, K_1^{b}(\underset{\sim}{\sigma},\underset{\sim}{\eta})\, F_a(\underset{\sim}{k},\underset{\sim}{\eta},s), \tag{31}$$

where

$$K_1^{b}(\underset{\sim}{\sigma},\underset{\sim}{\eta}) = \frac{1}{2\pi^2}\left[\left(1+\frac{m}{M}\right)\frac{\underset{\sim}{\sigma}\cdot(\underset{\sim}{\sigma}-\underset{\sim}{\eta})}{|\underset{\sim}{\sigma}-\underset{\sim}{\eta}|^2} - \frac{\{\underset{\sim}{\sigma}\cdot(\underset{\sim}{\sigma}-\underset{\sim}{\eta})\}^2}{|\underset{\sim}{\sigma}-\underset{\sim}{\eta}|^4}\right] \exp\left\{-\frac{1}{2}V_i^2(\underset{\sim}{\sigma}-\underset{\sim}{\eta})^2\right\}. \tag{32}$$

Similarly

$$\left(\frac{\partial F_a}{\partial t}\right)_{eb} = N\Gamma \int d\underset{\sim}{\eta}\left[K_2^{b} F_a(\underset{\sim}{k},\underset{\sim}{\eta},s) + K_3^{a} F_b(\underset{\sim}{k},\underset{\sim}{\eta},s)\right], \tag{33}$$

where

$$K_2^{a}(\underset{\sim}{\sigma},\underset{\sim}{\eta}) = \frac{1}{\pi^2}\left[\frac{\underset{\sim}{\sigma}\cdot(\underset{\sim}{\sigma}-\underset{\sim}{\eta})}{|\underset{\sim}{\sigma}-\underset{\sim}{\eta}|^2} - \frac{1}{2}\frac{\{\underset{\sim}{\sigma}\cdot(\underset{\sim}{\sigma}-\underset{\sim}{\eta})\}^2}{|\underset{\sim}{\sigma}-\underset{\sim}{\eta}|^4}\right] \times \exp\left\{-\frac{1}{2}V_0^2(\underset{\sim}{\sigma}-\underset{\sim}{\eta})^2 - i\,\underset{\sim}{U}_a\cdot(\underset{\sim}{\sigma}-\underset{\sim}{\eta})\right\} \tag{34}$$

and

$$K_3^{a}(\underset{\sim}{\sigma},\underset{\sim}{\eta}) = \frac{1}{\pi^2}\left[\frac{\underset{\sim}{\sigma}\cdot\underset{\sim}{\eta}}{\eta^2} - \frac{1}{2}\frac{(\underset{\sim}{\sigma}\cdot\underset{\sim}{\eta})^2}{\eta^4}\right] \times \exp\left\{-\frac{1}{2}V_0^2(\underset{\sim}{\sigma}-\underset{\sim}{\eta})^2 - i\,\underset{\sim}{U}_a\cdot(\underset{\sim}{\sigma}-\underset{\sim}{\eta})\right\}. \tag{35}$$

We shall now take $\underset{\sim}{k}$ to be along the z-axis and also assume that $\underset{\sim}{U}_a = U_a \underset{\sim}{e}_z$ and further we set $N\Gamma = V_0^4/L$, where L represents the mean free path of the particles between collisions. Eq. (28) now reduces to

$$\left(\frac{s}{k} - \frac{\partial}{\partial \sigma_z}\right) F_a(k, \underset{\sim}{\sigma}, s) - \frac{1}{k} G_a(k, \underset{\sim}{\sigma})$$
$$- \frac{iNe}{mk} \underset{\sim}{\sigma} \cdot \underset{\sim}{E}_k \exp\left\{-\tfrac{1}{2}\sigma^2 V_0^2 - i\sigma_z U_a\right\} = \frac{V_0^4}{kL} \sum_b \int d\underset{\sim}{\eta} \left[\left\{K_1^b(\underset{\sim}{\sigma}, \underset{\sim}{\eta})\right.\right. \tag{36}$$
$$\left.\left. + K_2^b(\underset{\sim}{\sigma}, \underset{\sim}{\eta})\right\} F_a(k, \underset{\sim}{\eta}, s) + K_3^a(\underset{\sim}{\sigma}, \underset{\sim}{\eta}) F_b(k, \underset{\sim}{\eta}, s)\right].$$

Equation (36) can be readily integrated to give

$$F_a(k, \underset{\sim}{\sigma}, s) = e^{s\sigma_z/k}\left[-\frac{1}{k}\int_\infty^{\sigma_z} e^{-s\sigma_z'/k} G_a(k, \underset{\sim}{\sigma}') d\sigma_z'\right.$$
$$- \frac{iNe}{mk} \underset{\sim}{E}_k(s) \cdot \int_\infty^{\sigma_z} \underset{\sim}{\sigma}' \exp\left\{-\tfrac{1}{2}\sigma'^2 V_0^2 - i\sigma_z' U_a - s\sigma_z'/k\right\} d\sigma_z' \tag{37}$$
$$- \frac{V_0^4}{kL} \sum_b \int_\infty^{\sigma_z} d\sigma_z' \int d^3\underset{\sim}{\eta}\, e^{-s\sigma_z'/k} \left[\left\{K_1^b(\underset{\sim}{\sigma}', \underset{\sim}{\eta}) + K_2^b(\underset{\sim}{\sigma}', \underset{\sim}{\eta})\right\} F_a(k, \underset{\sim}{\eta}, s)\right.$$
$$\left.\left. + K_3^a(\underset{\sim}{\sigma}', \underset{\sim}{\eta}) F_b(k, \underset{\sim}{\eta}, s)\right]\right],$$

where $\underset{\sim}{\sigma}' = (\sigma_x, \sigma_y, \sigma_z')$.

We shall now assume that the collisions are weak so that $kL \gg 1$. Under this assumption, to the lowest order, Eq. (37) gives

$$F_a(k, \underset{\sim}{\sigma}, s) = e^{s\sigma_z/k}\left[Q_a(k, \underset{\sim}{\sigma}) - \frac{iNe}{mk} \underset{\sim}{E}_k \cdot \underset{\sim}{P}_a(k, \underset{\sim}{\sigma})\right], \tag{38}$$

$$Q_a(k,\underset{\sim}{\sigma}) = -\frac{1}{k}\int_{\infty}^{\sigma_z} d\sigma_z' G_a(k,\underset{\sim}{\sigma}')\, e^{-s\sigma_z'/k}\,, \tag{39}$$

and

$$\underset{\sim}{P}_a(k,\underset{\sim}{\sigma}) = \int_{\infty}^{\sigma_z} d\sigma_z'\,\underset{\sim}{\sigma}'\exp\left\{-s\sigma_z'/k - \tfrac{1}{2}\sigma'^2 V_0^2 - i\sigma_z' U_a\right\}. \tag{40}$$

We now substitute the zero-order solution given by Eq. (38) into Eq. (37) to obtain:

$$\begin{aligned} F_a(k,\underset{\sim}{\sigma},s) = e^{s\sigma_z/k}\Bigg[\!\!\Bigg[& Q_a(k,\underset{\sim}{\sigma}) + \frac{V_0^4}{kL}\sum_b \int_{\sigma_z}^{\infty} d\sigma_z' \int d\underset{\sim}{\eta}\, e^{-s(\sigma_z'-\eta_z)/k} \\ & \times\left\{\left[K_1^b(\underset{\sim}{\sigma}',\underset{\sim}{\eta}) + K_2^b(\underset{\sim}{\sigma}',\underset{\sim}{\eta})\right] Q_a(k,\underset{\sim}{\eta}) + K_3^a(\underset{\sim}{\sigma}',\underset{\sim}{\eta}) Q_b(k,\underset{\sim}{\eta})\right\} \\ & - \frac{iNe}{mk}\underset{\sim}{E}_k(s)\cdot\Bigg\{\underset{\sim}{P}_a(k,\underset{\sim}{\sigma}) + \frac{V_0^4}{kL}\sum_b\int_{\sigma_z}^{\infty} d\sigma_z'\int d\underset{\sim}{\eta}\, e^{-s(\sigma_z'-\eta_z)/k} \\ & \times\left[\left\{K_1^b(\underset{\sim}{\sigma}',\underset{\sim}{\eta}) + K_2^b(\underset{\sim}{\sigma}',\underset{\sim}{\eta})\right\}\underset{\sim}{P}_a(k,\underset{\sim}{\eta}) + K_3^a(\underset{\sim}{\sigma}',\underset{\sim}{\eta})\,\underset{\sim}{P}_b(k,\underset{\sim}{\eta})\right]\Bigg\}\Bigg]\!\!\Bigg]. \end{aligned} \tag{41}$$

From Eq. (25) we have

$$E_k(s) = \frac{4\pi i e}{k}\sum_a F_a(k,0,s)\,. \tag{42}$$

On substituting for F_a in accordance with Eq. (41) into Eq. (42), we readily obtain

$$E_k(s) = \frac{\Phi(k,s)}{\Psi(k,s)}\,, \tag{43}$$

where

$$\begin{aligned} \Phi(k,s) = \frac{4\pi i e}{k}\sum_a \Big[& Q_a(k,0) \\ & + \frac{V_0^4}{kL}\sum_b\int_0^{\infty} d\sigma_z\int d\underset{\sim}{\eta}\,\exp\{-s(\sigma_z-\eta_z)/k\}\times\Big\{\big(K_1^b(\sigma_z,\underset{\sim}{\eta}) \\ & + K_2^b(\sigma_z,\underset{\sim}{\eta})\big) Q_a(k,\underset{\sim}{\eta}) + K_3^a(\sigma_z,\underset{\sim}{\eta})\, Q_b(k,\underset{\sim}{\eta})\Big\}\Big] \end{aligned} \tag{44}$$

and

$$\Psi(k,s) = 1 - \frac{\omega_p^2}{k^2}\sum_a \left[\!\!\left[P_{az}(k,0) + \frac{V_0^4}{kL}\sum_b \int_0^\infty d\sigma_z \int d\eta_z \right.\right.$$

$$\times e^{-\frac{s}{k}(\sigma_z-\eta_z)}\Big[\{K_1^b(\sigma_z,\eta_z) + K_2^b(\sigma_z,\eta_z)\} P_{az}(k,\eta_z) \tag{45}$$

$$\left.\left. + K_3^a(\sigma_z,\eta_z)\, P_{bz}(k,\eta_z)\Big]\right]\!\!\right] .$$

We may note here that $\Phi(k,s)$ depends only on the initial perturbation. If we consider only those perturbations for which $\Phi(k,s)$ is analytic in the complex s-plane, then for the Laplace inversion of Eq. (43), we have to consider only the zeros of $\Psi(k,s)$ which are given by

$$\Psi(k,s) = 0 . \tag{46}$$

This gives us the desired dispersion relation. It is now convenient to define the effective collision frequencey as $\nu_c = V_0/L$. The dispersion relation then reads

$$1 = \frac{\omega_p^2}{k^2}\sum_a \left[\!\!\left[P_{az}(k,0) + \frac{\nu_c V_0^3}{k}\sum_b \int_0^\infty d\sigma_z \int d\eta_z \right.\right.$$

$$\times e^{-s(\sigma_z-\eta_z)/k}\Big[\{K_1^b(\sigma_z,\eta_z) + K_2^b(\sigma_z,\eta_z)\} P_{az}(k,\eta_z) \tag{47}$$

$$\left.\left. + K_3^a(\sigma_z,\eta_z)\, P_{bz}(k,\eta_z)\Big]\right]\!\!\right] .$$

III. EVALUATION OF THE INTEGRALS

We first need to evaluate

$$P_{az}(k,\eta_z) = \int_\infty^{\eta_z} d\eta_z' \, \eta_z' \exp\left(-\frac{s}{k}\eta_z' - \frac{1}{2}V_0^2\eta_z'^2 - i\eta_z' U_a\right), \tag{48}$$

where $\eta' = (\eta_x, \eta_y, \eta_z')$. It is convenient to write this as

$$P_{az}(k,\eta) = -\exp(\alpha_a^2 - \tfrac{1}{2}V_0^2\eta_\perp^2)\int_{\eta_z}^{\infty} d\eta_z'\,\eta_z \exp-(\alpha_a + V_0\eta_z'/(2)^{1/2})^2, \quad (49)$$

where

$$\eta_\perp^2 = \eta^2 - \eta_z^2 \quad \text{and} \quad \alpha_a = (s/k + iU_a)/\{V_0(2)^{1/2}\}. \quad (50)$$

The integral which occurs in Eq. (49) can be expressed in terms of the error function and we find that

$$P_{az}(k,\eta) = -\frac{1}{V_0^2}\exp(\alpha_a^2 - \tfrac{1}{2}V_0^2\eta_\perp^2 - \eta_+^2)\left[1 - 2\alpha_a e^{\eta_+^2} Erf(\eta_+)\right], \quad (51)$$

where $\eta_+ = \alpha_a + V_0\eta_z/(2)^{1/2}$ and

$$Erf(x) = \int_x^\infty e^{-y^2}dy, \quad (52)$$

which for large values of the argument has the asymptotic expansion

$$Erf(x) = \frac{e^{-x^2}}{2x}\left[1 - \frac{1}{2x^2} + \frac{3}{2^2x^4} - \frac{1.3.5}{2^3x^6} + \cdots\right]. \quad (53)$$

On making the legitimate assumption that the phase velocity of the wave is much larger than the mean thermal speed i.e. $s/kV_0 \gg 1$, we obtain:

$$P_{az}(k,0) = \frac{k^2}{(\omega - kU_a)^2}\left[1 + \frac{3k^2V_0^2}{(\omega - kU_a)^2} + \cdots\right], \quad (54)$$

where we have put $s = -i\omega$ for the sake of convenience.

We next wish to evaluate

$$I_b^a = \int_0^\infty d\sigma_z \int d\eta\, e^{-s(\sigma_z - \eta_z)/k} K_1^b(\sigma_z, \eta)\, P_{az}(k,\eta). \quad (55)$$

We may note that K_1^b is independent of U and as such of b. We can, therefore, write $I_1^a = I_2^b = I_b^a$, $b = 1, 2$. In order to compare the relative orders of the diffusion term and the frictional term, we split up Eq. (55) as follows:

$$I_{1f,d}^{a} = \int_0^\infty d\sigma_z \int d\underset{\sim}{\eta}\, e^{-\Delta(\sigma_z - \eta_z)/k} K_{1f,d}(\sigma_z, \underset{\sim}{\eta}) P_{az}(k, \underset{\sim}{\eta}), \quad (56)$$

where I_{1f}^a and I_{1d}^a represent the contribution of the frictional and the diffusion terms respectively to the electron-ion collisions. Further

$$K_{1f}(\sigma, \underset{\sim}{\eta}) = \frac{1}{2\pi^2}\left(1 + \frac{m}{M}\right)\frac{\sigma(\sigma - \eta_z)}{|\sigma \underset{\sim}{e}_z - \underset{\sim}{\eta}|^2} \exp\left\{-\tfrac{1}{2}V_i^2(\sigma \underset{\sim}{e}_z - \underset{\sim}{\eta})^2\right\}, \quad (57)$$

and

$$K_{1d}(\sigma, \underset{\sim}{\eta}) = -\frac{1}{2\pi^2}\frac{\{\sigma(\sigma - \eta_z)\}^2}{|\sigma \underset{\sim}{e}_z - \underset{\sim}{\eta}|^4} \exp\left\{-\tfrac{1}{2}V_i^2(\sigma \underset{\sim}{e}_z - \underset{\sim}{\eta})^2\right\}. \quad (58)$$

On substituting for $K_{1f}(\sigma_z, \underset{\sim}{\eta})$ and $P_{az}(k, \underset{\sim}{\eta})$ in accordance with Eqs. (57) and (51) into Eq. (56), we obtain for I_{1f}^a:

$$I_{1f}^{a} = -\frac{e^{\alpha_a^2}}{2\pi^2 V_0^2}\int_0^\infty d\sigma\, e^{-\Delta\sigma/k}\int d\underset{\sim}{\eta}\, \exp\left\{\frac{\Delta}{k}\eta_z - \tfrac{1}{2}V_0^2\eta_z^2 - \tfrac{1}{2}V_i^2(\sigma\underset{\sim}{e}_z - \underset{\sim}{\eta})^2\right\}$$
$$\times\left(e^{-\eta_+^2} - 2\alpha_a \mathrm{Erf}(\eta_+)\right)\left[(1 + m/M)\frac{\sigma(\sigma - \eta_z)}{(\sigma\underset{\sim}{e}_z - \underset{\sim}{\eta})^2}\right]. \quad (59)$$

It is now convenient to introduce the variable $\underset{\sim}{\xi} = \sigma\underset{\sim}{e}_z - \underset{\sim}{\eta}$.

The integral I_{1f}^a can then be written as:

$$I_{1f}^{a} = -\frac{e^{\alpha_a^2}}{2\pi^2 V_0^2}\int_0^\infty d\sigma \int d\underset{\sim}{\xi}\, \exp\left\{-\tfrac{1}{2}(V_0^2 + V_i^2)\xi_\perp^2 - \tfrac{1}{2}V_i^2\xi_z^2 - \frac{\Delta}{k}\xi_z\right\}$$
$$\times\left\{e^{-\xi_+^2} - 2\alpha_a \mathrm{Erf}(\xi_+)\right\}\left[\left(1 + \frac{m}{M}\right)\frac{\sigma\xi_z}{\xi^2}\right], \quad (60)$$

Where now $\xi_+ = \alpha_a + V_0(\sigma - \xi_z)/(2)^{1/2}$. Let us introduce a system of cylinderical polar co-ordinates $(\xi_\perp, \theta, \xi_z)$; Eq. (60) then reduces to

$$I_{1f}^{a} = -\frac{e^{\alpha_a^2}}{\pi V_0^2}\int_0^\infty d\sigma \int_{-\infty}^{\infty} d\xi_z \exp\left(-\frac{\Delta}{k}\xi_z - \frac{1}{2}V_i^2\xi_z^2\right) \times\left\{e^{-\xi_+^2} - 2\alpha_a \mathrm{Erf}(\xi_+)\right\} S_{1f}^{a}, \tag{61}$$

where S_{1f}^{a} is defined to be

$$S_{1f}^{a} = \int_0^\infty d\xi_\perp\, \xi_\perp\, e^{-\frac{1}{2}(V_0^2+V_i^2)\xi_\perp^2}\left(1+\frac{m}{M}\right)\frac{\sigma\xi_z}{\xi_\perp^2+\xi_z^2}. \tag{62}$$

If we now put $\xi_\perp^2 = t\,\xi_z^2$, this integral can be written as

$$S_{1f}^{a} = \frac{1}{2}\int_1^\infty \frac{d\nu}{\nu}\left(1+\frac{m}{M}\right)\sigma\xi_z \exp\left\{(1-\nu)b^2\xi_z^2\right\}, \tag{63}$$

where $b^2 = (V_0^2 + V_i^2)/2$. The integral I_{1f}^{a} then takes the form

$$I_{1f}^{a} = -\frac{e^{\alpha_a^2}}{2\pi V_0^2}\int_1^\infty \frac{d\nu}{\nu}\int_0^\infty d\sigma \int_{-\infty}^{\infty} d\xi_z \exp\left(-\frac{\Delta}{k}\xi_z + \frac{1}{2}V_0^2\xi_z^2 - \nu b^2\xi_z^2 - \xi_+^2\right) \times\left(1+\frac{m}{M}\right)\sigma\xi_z\left[1 - 2\alpha_a e^{\xi_+^2}\mathrm{Erf}(\xi_+)\right]. \tag{64}$$

Again if we assume that $\xi_+ \gg 1$ 1, we can use the asymptotic expansion of error function and after some reductions we find

$$I_{1f}^{a} = -\frac{e^{\alpha_a^2}}{2\pi V_0^2}\int_1 \frac{d\nu}{\nu}\int_0^\infty d\sigma \frac{1}{b}\exp\left\{-\frac{1}{2}\beta^2 + \frac{1}{4\nu b^2}(V_0\sigma + iU_a)^2\right\} S_{2f}^{a}, \tag{65}$$

where

$$S^a_{2f} = \int_{-\infty}^{\infty} dx \exp\left\{-\nu\left[x - \frac{1}{2\nu b}(V_0^2\sigma + i\,U_a)\right]^2\right\} m\mu\frac{\sigma x}{b}$$

$$\times\left[1 - \frac{a_a}{\beta - qx} + \frac{a_a}{(\beta-qx)^3} - \frac{3a_a}{(\beta-qx)^5} + \frac{15a_a}{(\beta-qx)^7} - \cdots\right], \tag{66}$$

and we have put

$$\beta = \left(\frac{\Omega}{k} + i\,U_a + V_0^2\sigma\right)/V_0\,,\quad q = V_0/b,\quad a_a = (2)^{1/2}\alpha_a, \tag{67}$$
$$\mu = (m+M)/mM.$$

We now observe that β is usually much larger than q, being of the order of the phase velocity to the mean thermal velocity. We can, therefore, carry out the expansions of the quantities appearing in the denominators in S^a_{2f}. The resulting integrations are then elementary. We obtain after some straightforward calculations:

$$S^a_{2f} = \sqrt{\frac{\pi}{\nu}}\,\frac{m\mu\sigma}{2\nu b^2}\left[(V_0^2\sigma + i\,U_a)\left\{1 - a_a\left(\frac{1}{\gamma} - \frac{A_2}{\gamma^3}\right.\right.\right.$$
$$\left.\left.\left. + \frac{3A_2^2}{\gamma^5}\right)\right\} - \frac{a_a V_0}{\gamma^2}\left\{1 - \frac{3A_2}{\gamma^2} + \frac{15}{\gamma^4}A_2^2\right\}\right], \tag{68}$$

where we have retained terms up to γ^{-6} only, and

$$A_2 = 1 - V_0^2/2b^2\nu\,,\quad \gamma = \beta - \frac{q}{2\nu b}(V_0^2\sigma + i\,U_a). \tag{69}$$

On substituting the foregoing results in equation (65) and after some reductions we find

$$I^a_{1f} = -\frac{1}{2(\pi)^{1/2}V_0^3 b}\int_1^{\infty}\frac{d\nu}{\nu^{3/2}}\,e^{-U_a^2/4\nu b^2}\,S^a_{3f}\,, \tag{70}$$

where

$$S_{3f}^{a} = \frac{m\mu}{2\nu b^2}\int_0^\infty dx\, x\, \exp\{-(A_1 x + \tfrac{1}{2}A_2 x^2)\}\Big[(x + \tfrac{iU_a}{V_0})$$
$$\times\Big\{1 - \frac{a_a}{A_1 + A_2 x}\Big[1 - \frac{A_2}{(A_1+A_2x)^2} + \frac{3(1 - A_2 q^2/2\nu)}{(A_1+A_2x)^4}\Big]\Big\}$$
$$- \frac{a_a}{(A_1+A_2x)^2}\Big\{1 - \frac{3A_2}{(A_1+A_2x)^2} + \frac{15A_2^2}{(A_1+A_2x)^4}\Big\}\Big],$$

and

$$A_1 = \frac{1}{V_0}\left(\frac{\Delta}{k} + iU_a\right) - \frac{iV_0U_a}{2\nu b^2}\ . \qquad (72)$$

The various integrals occuring in Eq. (71) can be evaluated in terms of the error function $Erf(y_*)$, where $y_* = A_1/(2A_2)^{1/2}$ and the exponential integral $Ei(-x)$. As y_* is a quantity much larger than unity, being of order Δ/kV_0 one can use the asymptotic expansions of these functions. The calculations are lengthy but straightforward. The values of the various integrals are given in Appendix A. The result is (retaining terms up to order A_1^{-6}):

$$S_{3f}^{a} = \frac{m\mu}{2\nu b^2 A_1^2}\Big[\frac{2}{A_1}\Big(1 - \frac{6A_2}{A_1}\Big) - \frac{3a_a}{A_1^2}\Big(1 - \frac{10A_2}{A_1^2}\Big) \qquad (73)$$
$$+ \frac{iU_a}{V_0}\Big\{1 - \frac{3A_2}{A_1^2} - \frac{a_a}{A_1}\Big(1 - \frac{6A_2}{A_1^2}\Big)\Big\}\Big].$$

We now observe that A_1 can be written as

$$A_1 = a_a\left(1 - \frac{ip_a}{2\nu}\right), \qquad (74)$$

where $p_a = V_0U_a/a_a b^2$ and $p_a/\nu << 1$ 1. We can, therefore, again carry out the expansions of A_1^{-n} in powers of p_a/ν. Retaining terms up to the fifth power in p_a/ν we obtain:

$$
I_{1f}^{a} = \frac{m\mu}{2(\pi)^{1/2} V_0^3 b^3 a_a^2}\Big[\Sigma_3 \frac{1}{a_a}\Big(1 - \frac{3iU_a}{a_a V_0} - \frac{18}{a_a^2}\Big)
$$

$$
+\Sigma_5\Big\{\frac{3}{a_a}\Big(ip_a + \frac{3V_0^2}{a_a^2 b^2}\Big) - \frac{iU_a}{V_0}\Big(-\frac{1}{2}ip_a + 9\frac{i}{a_a^2}p_a - \frac{3}{2}\frac{V_0^2}{a_a^2 b^2}\Big)\Big\} \qquad (75)
$$

$$
-\frac{3}{2}\Sigma_7\Big\{\frac{3}{a_a}p_a^2 + \frac{iU_a}{V_0}\Big(\frac{1}{2}p_a^2 - 3ip_a\frac{V_0^2}{a_a^2 b^2}\Big)\Big\} - \frac{3}{4}\frac{U_a}{V_0}p_a^3\Sigma_9\Big],
$$

where

$$
\Sigma_m = \int_0^1 dy\; y^{m-1} \exp(-U_a^2 y^2/4b^2). \qquad (76)
$$

It is, perhaps, worthwhile to remark again that the only assumption which has been made to evaluate the integral I_{1f}^{a} is that the phase velocity of the wave is much larger than the mean thermal velocity; and this is indeed a legitimate one for the problem at hand.

We now assume that $m/M \ll 1$ and $V_i^2 \ll V_0^2$ which is quite valid in all cases of physical interest; the second requirement, as a matter of fact, is a consequence of the first one. Under these assumptions, we have

$$
b^2 \simeq \frac{1}{2}V_0^2, \qquad p_a \simeq \frac{2U_a}{a_a V_0} \qquad (77)
$$

and

$$
\Sigma_m = \int_0^1 dy\; y^{m-1} \exp(-U_a^2/2V_0^2). \qquad (78)
$$

The integrals Σ_m have been evaluated numerically for suitable values of the parameter $A = V_0^2/U_a^2$. We thus finally obtain:

$$I_{1f}^{a} = -\frac{2ik^3}{(2\pi)^{1/2} V_0^3(\omega - kU_a)^3}\Big[\Sigma_3\Big\{1+\frac{3kU_a}{\omega-kU_a}+\frac{18k^2V_0^2}{(\omega-kU_a)^2}\Big\}$$
$$-\Sigma_5\Big\{\frac{U_a^2}{V_0^2}+\frac{9kU_a}{\omega-kU_a}+\frac{18k^2V_0^2}{(\omega-kU_a)^2}\left(1+U_a^2/V_0^2\right)\Big\} \tag{79}$$
$$+\Sigma_7\frac{3U_a^2}{V_0^2(\omega-kU_a)}\Big\{kU_a+\frac{12k^2V_0^2}{(\omega-kU_a)}\Big\}-\Sigma_9\, 6\frac{U_a^2}{V_0^2}\frac{k^2U_a^2}{(\omega-kU_a)^2}\Big],$$

where we have put $s = -i\omega$ for the sake of convenience. Proceeding on similar lines, we obtain

$$I_{1d}^{a} = \frac{-2ik^3}{(2\pi)^{1/2} V_0^3(\omega - kU_a)}\Big[-\frac{8k^2V_0^2}{(\omega-kU_a)}\Sigma_3 \tag{80}$$
$$+\Sigma_5\, 2kV_0\Big\{\frac{U_a}{V_0}+\frac{4kV_0}{(\omega-kU_a)}\Big\}-\Sigma_7\frac{8k^2U_a^2}{(\omega-kU_a)}\Big].$$

It is to be noted from Eqs. (79) and (80) that in ion-electron collisions it is the frictional term which is the dominant one; the diffusion term tends to be smaller by a factor or order $1/a_a$. Combining Eqs. (79) and (80) we obtain for the contribution of the electron-ion collisions:

$$I_1^{a} = -\frac{2ik^3}{(2\pi)^{1/2} V_0^3(\omega - kU_a)^3}\Big[\Sigma_3\Big\{1+\frac{3kU_a}{\omega-kU_a}+\frac{10k^2V_0^2}{(\omega-kU_a)^2}\Big\}$$
$$-\Sigma_5\Big\{\frac{U_a^2}{V_0^2}+\frac{7kU_a}{\omega-kU_a}+\frac{10k^2V_0^2}{(\omega-kU_a)^2}\left(1+\frac{9U_a^2}{5V_0^2}\right)\Big\} \tag{81}$$
$$+\Sigma_7\frac{U_a^2}{V_0^2(\omega-kU_a)}\Big\{3kU_a+\frac{28k^2V_0^2}{\omega-kU_a}\Big\}-\Sigma_9\, 6\frac{U_a^2}{V_0^2}\frac{k^2U_a^2}{(\omega-kU_a)^2}\Big].$$

Next we wish to evaluate

$$I_{2bf,d}^{a} = \int_0^\infty d\sigma\int d\eta\, e^{-s(\sigma-\eta_z)/k} K_{2f,d}^{b}(\sigma,\eta)\, P_{az}(k,\eta), \tag{82}$$

where

$$K_{2f}^{b}(\sigma,\eta) = \frac{1}{\pi^2}\frac{\sigma(\sigma-\eta_z)}{|\sigma\underset{\sim}{e}_z-\eta|^2}\exp\Big\{-\tfrac{1}{2}V_0^2(\sigma\underset{\sim}{e}_z-\eta)^2-iU_b(\sigma-\eta_z)\Big\} \tag{83}$$

and

$$K^{b}_{2d}(\sigma,\underline{\eta}) = -\frac{1}{2\pi^2}\frac{[\sigma(\sigma-\eta_z)]^2}{|\sigma\underline{e}_z-\underline{\eta}|^4}\exp\left\{-\tfrac{1}{2}V_0^2(\sigma\underline{e}_z-\underline{\eta})^2 - iU_b(\sigma-\eta_z)\right\}. \tag{84}$$

The evaluation of $I^{a}_{2bf,d}$ proceeds on similar lines as I^{a}_{1f} and we find:

$$I^{a}_{2bf} = \frac{1}{(\pi)^{1/2}V_0^6 a_a^3}\Big[T_3\Big(1-3\frac{d}{a_a}+\frac{18}{a_a^2}\Big)+T_5\Big(\frac{9}{a_a^2}+\frac{9}{2}\frac{d}{a_a} -\frac{1}{2}d^2 - 9\frac{d^2}{a_a}\Big)+T_7\frac{d^2}{a_a}\Big(\frac{9}{a_a}+\frac{3}{4}d\Big)+\frac{3}{4}\frac{d^4}{a_a^2}T_9\Big] \tag{85}$$

and

$$I^{a}_{2bd} = \frac{1}{(\pi)^{1/2}V_0^6 a_a^3}\Big[\frac{8}{a_a^2}T_3 - \frac{1}{a_a}\Big(d+\frac{4}{a_a}\Big)T_5 - \frac{2d^2}{a_a^2}T_7\Big], \tag{86}$$

where

$$d = i(U_a - U_b)/V_0\,, \quad a_a = (s+ikU_a)/kV_0 \tag{87}$$

and

$$T_m = \int_0^1 dy\, y^{m-1} e^{-y^2(U_a-U_b)^2/4V_0^2}. \tag{88}$$

Combining Eqs. (85) and (86), we obtain

$$I^{a}_{2b} = \frac{1}{(\pi)^{1/2}V_0^6 a_a^3}\Big[T_3\Big(1-\frac{10}{a_a^2}-\frac{3d}{a_a}\Big)+T_5\Big(\frac{5}{a_a^2}+\frac{7d}{2a_a} +\frac{1}{2}d^2-9\frac{d^2}{a_a^2}\Big)-T_7\frac{d^2}{a_a}\Big(\frac{7}{a_a}+\frac{3}{4}d\Big)+\frac{3}{4a_a^2}d^4T_9\Big]. \tag{89}$$

In particular we find that for collisions between particles of the same stream, this reduces to

$$I^{a}_{2a} = \frac{1}{3(\pi)^{1/2}V_0^6 a_a^3}\Big(1-\frac{7}{a_a^2}\Big). \tag{90}$$

Finally, we wish to evaluate

$$I^{a}_{3bf,d} = \int_0^\infty d\sigma \int d\underset{\sim}{\eta}\, e^{-s(\sigma-\eta_z)/k} K^{b}_{3f,d}(\sigma,\underset{\sim}{\eta}) P_{az}(k,\underset{\sim}{\eta}), \quad (91)$$

where

$$K^{b}_{3f}(\sigma,\underset{\sim}{\eta}) = \frac{1}{\pi^2}\frac{\sigma\eta_z}{\eta^2}\exp\left\{-\tfrac{1}{2}V_0^2(\sigma\underset{\sim}{e}_z-\underset{\sim}{\eta})^2 - iU_b(\sigma-\eta_z)\right\} \quad (92)$$

and

$$K^{b}_{3d}(\sigma,\underset{\sim}{\eta}) = -\frac{1}{2\pi^2}\frac{\sigma^2\eta_z^2}{\eta^4}\exp\left\{-\tfrac{1}{2}V_0^2(\sigma\underset{\sim}{e}_z-\underset{\sim}{\eta})^2 - iU_b(\sigma-\eta_z)\right\}. \quad (93)$$

On substituting for $P_{az}(k,\underset{\sim}{\eta})$ and $K^{b}_{3f}(\sigma,\underset{\sim}{\eta})$ into Eq. (91), we obtain:

$$I^{a}_{3bf} = -\frac{e^{\alpha_a^2}}{2\pi^2 V_0^2}\int_0^\infty d\sigma\, e^{-s\sigma/k}\int d\underset{\sim}{\eta}\left[e^{-\eta_+^2} - 2\alpha_a \mathrm{Erf}(\eta_+)\right] \quad (94)$$
$$\times\frac{2\sigma\eta_z}{\eta^2}\exp\left\{V_0 a_b\eta_z - \tfrac{1}{2}V_0^2\eta_z^2 - V_0^2\eta_\perp^2 - \tfrac{1}{2}V_0^2\sigma^2 + V_0^2\sigma\eta_z - i\sigma U_b\right\},$$

where now it is convenient to define

$$a_a = \left(\frac{s}{k} + iU_a\right)/V_0 \quad , \quad a_b = \left(\frac{s}{k} + iU_b\right)/V_0 \quad (95)$$

and

$$(2)^{1/2}\eta_+ = a_a + V_0\eta_z .$$

We now first carry out the σ-integration. The result is

$$I^{a}_{3bf} = -\frac{e^{\alpha_a^2}}{\pi^2 V_0^4}\int d\underset{\sim}{\eta}\,\frac{\eta_z}{\eta^2}\left\{1 - 2\eta_- e^{\eta_-^2}\mathrm{Erf}(\eta_-)\right\} \quad (96)$$
$$\times\left\{e^{-\eta_+^2} - 2\alpha_a \mathrm{Erf}(\eta_+)\right\}\exp\left\{V_0 a_b\eta_z - \tfrac{1}{2}V_0^2\eta_z^2 - V_0^2\eta_\perp^2\right\},$$

where

$$(2)^{1/2}\eta_- = a_b - V_0\eta_z . \quad (97)$$

Again introducing a system of cylinderical co-ordinates in space and carrying out the angular integration we get

$$I^{a}_{3bf} = -\frac{2e^{\alpha_a^2}}{\pi V_0^4}\int_{-\infty}^{\infty} d\eta_z \left[e^{-\eta_+^2} - 2\,\alpha_a \operatorname{Erf}(\eta_+)\right] \times \exp\left(a_b V_0 \eta_z - \tfrac{1}{2} V_0^2 \eta_z^2\right) S_{4f} \ , \tag{98}$$

where

$$S_{4f} = \int_0^\infty d\eta_\perp\, e^{-V_0^2 \eta_\perp^2} \frac{\eta_\perp \eta_z}{(\eta_\perp^2 + \eta_z^2)} \left\{1 - 2\eta_- e^{\eta_-^2} \operatorname{Erf}(\eta_-)\right\}. \tag{99}$$

After some elementary reductions, Eq. (99) can be put in the form

$$S_{4f} = \int_1^\infty \frac{d\nu}{2V_0 \nu}\, e^{-(\nu-1)x^2} \left[x\,(a_b - x)(2)^{1/2} \times e^{(a_b - x)^2/2} \operatorname{Erf}\left(\frac{a_b - x}{(2)^{1/2}}\right) - x\right] , \tag{100}$$

where we have put $x = V_0 \eta_z$. On substituting this expression for S_{4f} into Eq. (98), we get

$$I^{a}_{3bf} = \frac{1}{\pi V_0^6} \int_1^\infty \frac{d\nu}{\nu} \int_{-\infty}^{\infty} dx\, e^{-\nu x^2 - x d}\, F(x) \ , \tag{101}$$

where d is defined in Eq. (87) and

$$F(x) = \left[1 - (2)^{1/2} a_a \exp \tfrac{1}{2}(a_a + x)^2 \operatorname{Erf}\left(\frac{a_a + x}{(2)^{1/2}}\right)\right] \times \left[(2)^{1/2} x\,(a_b - x)\, e^{(a_b - x)^2/2} \operatorname{Erf}\left(\frac{a_b - x}{(2)^{1/2}}\right) - x\right]. \tag{102}$$

If we carry out the asymptotic expansions of the error functions, we obtain on retaining terms upto a^{-5}:

$$F(x) = \frac{-x^2}{(a_a + x)(a_b - x)^2} + \frac{3x^2}{(a_a + x)(a_b - x)^4} - \frac{a_a x}{(a_a + x)^3 (a_b - x)^2} \ . \tag{103}$$

Since $x/a \ll 1$, we can further expand the denominators in this expression in powers of x/a and obtain:

$$F(x) \simeq -\frac{x}{a_a^2 a_b^2} + \frac{x^2}{a_a a_b^2}\left(-1 + \frac{3}{a_b^2} + \frac{3}{a_a^2} - \frac{2}{a_a a_b}\right) + \frac{x^3}{a_a a_b^2}\left(\frac{1}{a_a} - \frac{2}{a_b}\right) + \frac{x^4}{a_a a_b^2}\left(-\frac{1}{a_a^2} + \frac{2}{a_a a_b} - \frac{3}{a_b^2}\right). \tag{104}$$

We now substitute this expression for $F(x)$ into I_{3bf}^{a} given by Eq. (101). The resulting integrations are then elementary and we obtain:

$$I_{3bf}^{a} = -\frac{2}{(\pi)^{1/2} V_0^6 a_a a_b^2}\Big[T_3\left(\frac{1}{2} + \frac{1}{a_a a_b} - \frac{3}{2a_a^2} - \frac{3}{2a_b^2} - \frac{d}{2a}\right) + T_5\left\{\frac{3}{4}\left(\frac{1}{a_a^2} + \frac{3}{a_b^2} - \frac{2}{a_a a_b}\right) - \frac{3}{4}d\left(\frac{2}{a_b} - \frac{1}{a_a}\right) + \left(\frac{d}{2}\right)^2\left(1 - \frac{3}{a_a^2} - \frac{3}{a_b^2} + \frac{2}{a_a a_b}\right)\right\} + T_7\left\{3\left(\frac{d}{2}\right)^2\left(\frac{1}{a_a^2} + \frac{3}{a_b^2} - \frac{2}{a_a a_b}\right) - \left(\frac{d}{2}\right)^3\left(\frac{2}{a_b} - \frac{1}{a_a}\right)\right\} + \left(\frac{d}{2}\right)^4\left(\frac{1}{a_a^2} + \frac{3}{a_b^2} - \frac{2}{a_a a_b}\right)T_9\Big]. \tag{105}$$

In a similar manner, we obtain:

$$I_{3bd}^{a} = \frac{2}{(\pi)^{1/2} V_0^6 a_a a_b^3}\left[\frac{1}{a_a}T_3 + \left(\frac{3}{2a_b} - \frac{1}{2a_a} - \frac{d}{2}\right)T_5 + \left(\frac{d}{2}\right)^2\left(\frac{3}{a_b} - \frac{1}{a_a}\right)T_7\right]. \tag{106}$$

On combining the foregoing expressions, we obtain:

$$\begin{aligned} I_{3b}^{a} &= I_{3bf}^{a} + I_{3bd}^{a} \\ &= \frac{2}{(\pi)^{1/2} V_0^6 a_a a_b^2}\Big[T_3\left\{-\frac{1}{2} + \frac{3}{2}\left(\frac{1}{a_a^2} + \frac{1}{a_b^2}\right) + \frac{d}{2a_a}\right\} \\ &\quad + T_5\left\{\frac{1}{a_a a_b} - \frac{3}{4}\left(\frac{1}{a_a^2} + \frac{1}{a_b^2}\right) + \frac{d}{2}\left(\frac{2}{a_b} - \frac{3}{2a_a}\right) \right. \\ &\quad \left. + \left(\frac{d}{2}\right)^2\left(-1 + \frac{3}{a_a^2} + \frac{3}{a_b^2} - \frac{2}{a_a a_b}\right)\right\} + T_7\left(\frac{d}{2}\right)^2\left\{\left(\frac{5}{a_a a_b} - \frac{3}{a_a^2} - \frac{6}{a_b^2}\right) \right. \\ &\quad \left. - \frac{d}{2}\left(\frac{1}{a_a} - \frac{2}{a_b}\right)\right\} + T_9\left(\frac{d}{2}\right)^4\left(\frac{2}{a_a a_b} - \frac{1}{a_a^2} - \frac{3}{a_b^2}\right)\Big]. \end{aligned} \tag{107}$$

In particular for collisions in the same stream, this reduces to

$$I^{a}_{3a} = -\frac{1}{3(\pi)^{1/2} V_0^6 a_a^3}\left(1 - \frac{27}{5a_a^2}\right). \tag{108}$$

It is interesting to observe that for collisions in the same stream, the total contribution of the electron-electron collisions is

$$I^{a}_{2a} + I^{a}_{3a} = -\frac{8}{15}\,\frac{1}{(\pi)^{1/2} V_0^6 a_a^5}, \tag{109}$$

and compared with the electron-ion collisions, this is smaller by a factor a_a^{-2} . It turns out that the integral T_m defined by Eq. (88) is about an order of magnitude smaller that the integral Σ_m defined by Eq. (78). Hence the contribution of the electron-electron collisions between one stream and the other is an order of magnitude smaller than the electron-ion collisions. Further, in electron-electron collisions, the diffusion terms tend to be of the same order as the frictional terms.

IV ANALYSIS OF THE DISPERSION RELATION

On carrying out the summations indicated in Eq. (47) the dispersion relation can be written as:

$$1 = \frac{\omega_p^2}{k^2}\left[P_{1z}(k,0) + P_{2z}(k,0) + \frac{\nu_c V_0^3}{k}\left\{2I_1' + 2I_1^2 + I_{21}^2 \right.\right. \tag{110}$$

$$\left.\left. + I_{21}' + I_{22}^2 + I_{22}' + I_{31}' + I_{32}^2 + I_{31}^2 + I_{32}'\right\}\right],$$

where we shall now take $U_1 = -U_2 = U$. We introduce the following dimensionless variables:

$$x = kU/\omega_p\,,\ y = \omega/\omega_p\,,\ A = V_0^2/U^2 \text{ and } B = \nu_c/\omega_p\,. \tag{111}$$

on using the foregoing relations, the dispersion relation can be written as:

$$1 = \frac{2(y^2+x^2)}{(y^2-x^2)^2} + C_1 + C_2 + C_{12}\,, \tag{112}$$

with

$$C_1 = 6Ax^2\frac{y^4+6y^2x^2+x^4}{(y^2-x^2)^4} \tag{113}$$

$$C_2 = -\frac{8iB}{(2\pi)^{1/2}}\frac{x^2y}{(y^2-x^2)^3}\Big[f\,\frac{y^2+3x^2}{x^2} + g\,\frac{y^2+x^2}{y^2-x^2} + h\,\frac{y^4+10y^2x^2+5x^4}{(y^2-x^2)^2} \tag{114}$$

and

$$C_{12} = -\frac{2iB}{(\pi)^{1/2}}\frac{y}{(y^2-x^2)^2}\Big[\frac{1}{y^2-x^2}\Big\{(y^2+3x^2)f_1 + Ax^2\frac{y^4+10y^2x^2+5x^4}{(y^2-x^2)^3}g_1 + 4x^2\frac{y^2+x^2}{(y^2-x^2)^2}h_1\Big\} \tag{115}$$

$$-2\Big\{\frac{1}{2}T_3 - T_5\frac{1}{A} + \frac{Ax^2}{y^2-x^2}f_2 - \frac{4x^2}{y^2-x^2}g_2 + h_2\,\frac{Ax^2(y^2+3x^2)}{(y^2-x^2)^2}\Big\}\Big]\,,$$

where

$$f = \Sigma_3 - \frac{1}{A}\Sigma_5 \,, \tag{116}$$

$$g = 4\left(3\Sigma_3 - 7\Sigma_5 + \frac{3}{A}\Sigma_7\right), \tag{117}$$

$$h = A\left\{\frac{2}{15}(2)^{1/2} + 10(\Sigma_3 - \Sigma_5) - \frac{2}{A}\left(9\Sigma_5 - 14\Sigma_7 + \frac{3}{A}\Sigma_9\right)\right\}, \tag{118}$$

$$f_1 = T_3 - \frac{2}{A}T_5 \,, \tag{119}$$

$$g_1 = 10T_3 - 5T_5 - \frac{36}{A}T_5 + \frac{28}{A}T_7 - \frac{12}{A^2}T_9 \,, \tag{120}$$

$$h_1 = 6T_3 - 7T_5 + \frac{6}{A}T_7 \,, \tag{121}$$

$$f_2 = \frac{3}{2}T_3 + \frac{1}{4}T_5 - \frac{1}{A}T_5 - \frac{2}{A}T_7 + \frac{1}{A^2}T_9 \,, \tag{122}$$

$$g_2 = T_5 - \frac{1}{A}T_7 \tag{123}$$

and

$$h_2 = \frac{3}{2}T_3 - \frac{3}{4}T_5 - \frac{3}{A}T_5 + \frac{6}{A}T_7 - \frac{3}{A^2}T_9 \,. \tag{124}$$

It is worth noticing here that $Ax^2 = (k\lambda_D)^2$, a quantity which is always small compared with unity for all cases of physical interest. We shall now discuss some special cases of the dispersion relation (112).

a) $C_1 = 0,\ C_2 = 0,\ C_{12} = 0$

In this case we obtain the well known cold case of two-stream instability which has been extensively discussed. The dispersion relation leads to the roots

$$y^2 = 1 + x^2 \pm (1 + 4x^2)^{1/2}. \tag{125}$$

The roots for ω are thus either real or purely imaginary; the latter one gives rise to instability which occurs for all values of x which are less than $x_c = (2)^{1/2}$. It can be further seen that the growth rate of instability is maximum for $x = x_* = (3)^{1/2}/2$. The corresponding growth rate of maximum instability is $|y_*^2| = 0.25$. Since the instability occurs only for the root with the negative sign in Eq. (78), we shall henceforth consider only this root and denote it by the subscript '0'. i.e.

$$y_0^2 = 1 + x^2 - (1 + 4x^2)^{1/2}. \tag{126}$$

We now wish to see how x_c and x_* change when $A \neq 0$ and $B \neq 0$.

b) $C_2 = 0,\ C_{12} = 0,\ C_1 \neq 0$.

In this case the dispersion relation reduces to

$$1 = \frac{2(y^2 + x^2)}{(y^2 - x^2)^2} + 6Ax^2\,\frac{y^4 + 6y^2x^2 + x^4}{(y^2 - x^2)^4}. \tag{127}$$

This equation has been solved numerically for y^2 for given values of x and A. The results are given in Table 1 and are plotted in Fig. 1 to illustrate the effect of thermal motions on two stream instability. It is rather interesting to note that in this case the maximum wavenumber $x_c^{(1)}$ upto which the instability occurs is larger than x_c i.e.

$$x_c^{(1)} > x_c = 1.4142. \tag{128}$$

It is to be further noted that while the region of instability is increased, the growth rate of maximum instability is decreased, i.e. $|y_m^2| < 0.25$.

c) The effect of Collisions

In order to compute the effect of collisions, we first observe that the terms C_2 and C_{12} are much smaller than unity so that we can solve Eq. (112) by iteration. Thus we can substitute $y = iy_1$ in C_2 and C_{12} where y_1 is the solution of Eq. (127) and y_1 is real for $x \leq x_c^{(1)}$. Thus C_2 and C_{12} become real in this region. Moreover y_1 is positive for an unstable mode which we are considering here. Since C_2 and C_{12} are proportional to y and $y = 0$ at $x = x_c$, we immediately conclude that the collisions cannot change x_c and thus the region of instability. However to study the influence of collisions on the growth rate of maximum instability, we have solved Eq. (112) for various values of A and B. The results are illustrated in Figures 2 and 3.

It is rather interesting to find that the growth rate of maximum instability, which is reduced by the thermal motions of the particles, is enchanced by the collisions but the effect of thermal motions cannot be quenched by the Coulomb collisions.

V. CONCLUSIONS

In the absence of collisions, the region of instability of contrastreaming plasmas is increased by the thermal effects while the growth rate of maximum instability is decreased. However, on the assumption that the collisions are not too frequent ($\nu_c/\omega_p \ll 1$) one finds that they have only second order effects. The region of instability is not affected by the presence of collisions while the growth rate of maximum instability is increased. The binary collisions that we have considered here do not seem to quench the effects of the thermal motions of the plasma species.

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APPENDIX

We wish to evaluate integrals of the type

$$I_{m,n} = \int_0^\infty dx \; x^m \frac{\exp-(A_1 x + \frac{1}{2} A_2 x^2)}{(A_1 + A_2 x)^n} \quad . \tag{A-1}$$

These integrals can be readily evaluated in terms of the Error and the Exponential integrals defined by

$$Erf(x) = \int_x^\infty e^{-y^2} dy \tag{A-2}$$

and

$$Ei(-x) = -\int_x^\infty \frac{e^{-y}}{y} dy \; . \tag{A-3}$$

For large values of the argument, these functions have the asymptotic expansions (1[illegible])

$$Erf(x) = \frac{e^{-x^2}}{2x} \left[1 - \frac{1}{2x^2} + \frac{3}{2^2 x^4} - \frac{1.3.5}{2^3 x^6} + \cdots \right] \tag{A-4}$$

and

$$Ei(-x) = -\frac{e^{-x}}{x} \left[1 - \frac{1}{x} + \frac{2!}{x^2} - \frac{3!}{x^3} + \cdots \right] . \tag{A-5}$$

After some elementary reductions, we find:

$$I_{0,2} = \frac{1}{A_1 A_2} \left[1 - 2 y_* e^{y_*^2} Erf(y_*) \right] , \tag{A-6}$$

$$I_{0,3} = \frac{1}{2 A_1^2 A_2} \left[1 + y_*^2 e^{y_*^2} Ei(-y_*^2) \right] , \tag{A-7}$$

$$I_{1,0} = \frac{1}{A_2} \left[1 - 2 y_* e^{y_*^2} Erf(y_*) \right] , \tag{A-8}$$

$$I_{1,1} = \frac{e^{y_*^2}}{A_1 A_2} y_* \left[2 \, Erf(y_*) + y_* Ei(-y_*^2) \right] \tag{A-9}$$

and

$$I_{2,0} = \frac{1}{A_1 A_2} 2y_* \left[-y_* + (1 + 2y_*^2) e^{y_*^2} \mathrm{Erf}(y_*) \right], \qquad \text{(A-10)}$$

where $y_* = A_1 / (2A_2)^{1/2}$. These integrals have the asymptotic values:

$$I_{0,2} = \frac{1}{A_1^3} \left(1 - \frac{3A_2}{A_1^2} + \frac{15A_2^2}{A_1^4} - \cdots \right), \qquad \text{(A-11)}$$

$$I_{0,3} = \frac{1}{A_1^4} \left(1 - \frac{4A_2}{A_1^2} + \frac{24A_2^2}{A_1^4} - \cdots \right), \qquad \text{(A-12)}$$

$$I_{1,0} = \frac{1}{A_1^2} \left(1 - \frac{3A_2}{A_1^2} + \frac{15A_2^2}{A_1^4} - \cdots \right), \qquad \text{(A-13)}$$

$$I_{1,1} = \frac{1}{A_1^3} \left(1 - \frac{5A_2}{A_1^2} + \frac{33A_2^2}{A_1^4} - \cdots \right) \qquad \text{(A-14)}$$

and

$$I_{2,0} = \frac{2}{A_1^3} \left(1 - \frac{6A_2}{A_1^2} + 45\frac{A_2^2}{A_1^4} - \cdots \right). \qquad \text{(A-15)}$$

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TABLE 1

A	B	x_c	x_*	$-y^2_*$
0.0	0.0	1.414	0.866	0.250
0.001	0.000	1.416	0.867	0.249
	0.050	1.416	0.867	0.249
	0.075	1.416	0.867	0.249
	0.100	1.416	0.867	0.249
0.01	0.000	1.435	0.874	0.245
	0.050	1.435	0.874	0.245
	0.075	1.435	0.874	0.245
	0.100	1.435	0.874	0.245
0.05	0.000	1.516	0.915	0.227
	0.050	1.516	0.916	0.231
	0.075	1.516	0.917	0.232
	0.100	1.516	0.918	0.234
0.10	0.000	1.612	0.976	0.214
	0.050	1.612	0.982	0.225
	0.075	1.612	0.985	0.230
	0.100	1.612	0.988	0.235

Figure Captions.

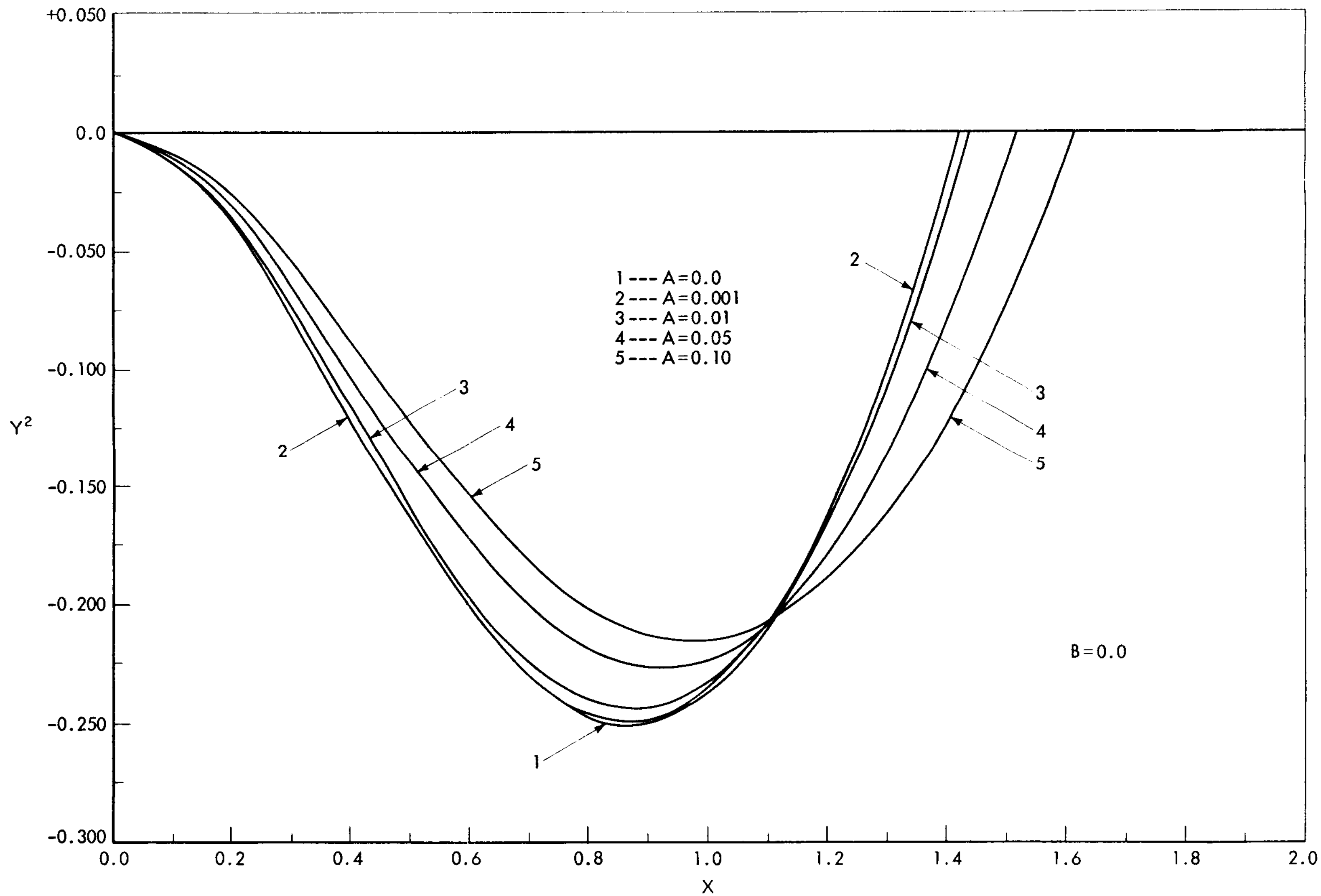
Figure 1. Effect of thermal motions on x_c, x_* and y^2 for $A = 0.001$, 0.01, 0.05 and 0.1.

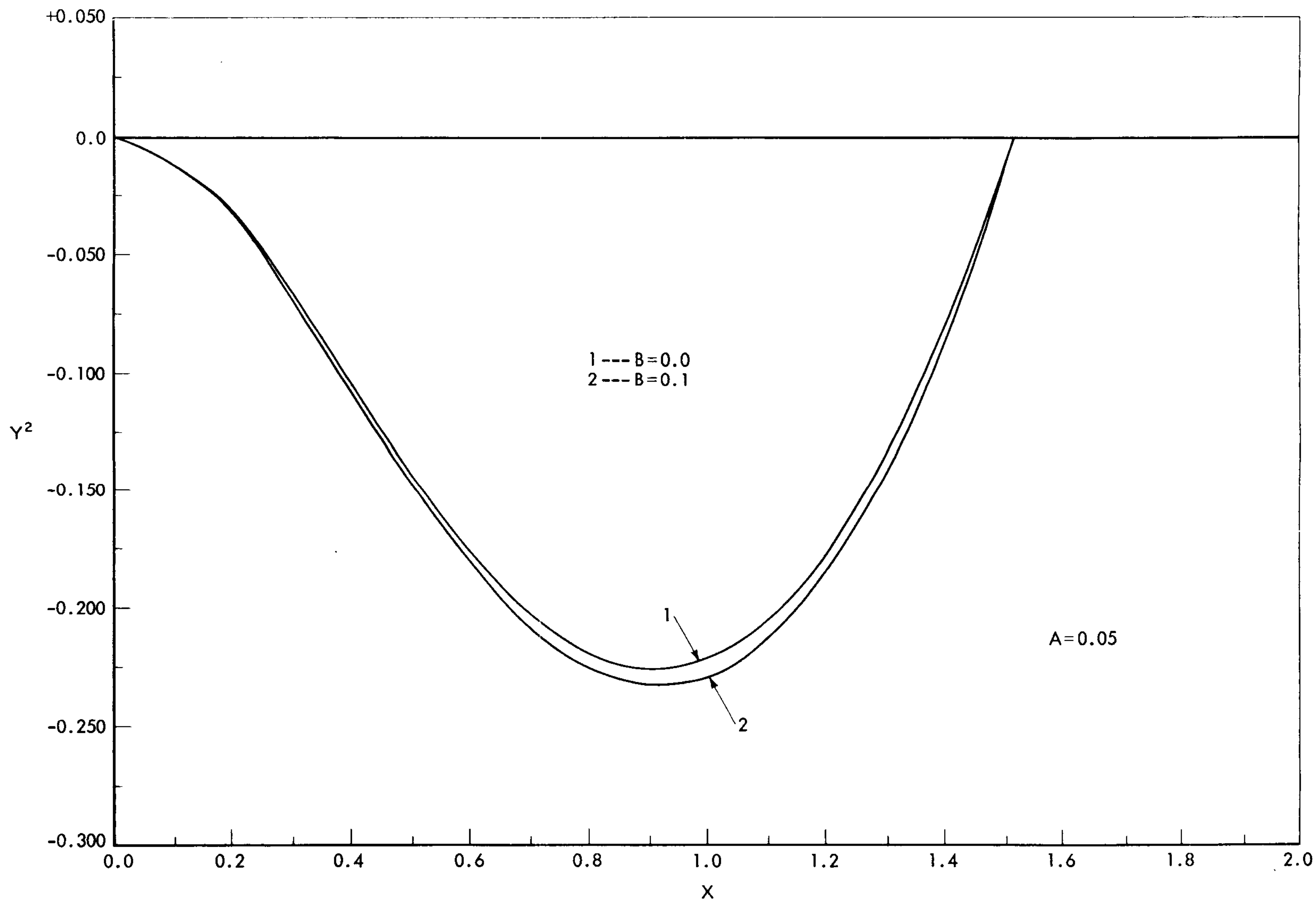
Figure 2. Variation of y^2 with x for $B = 0$ and 0.1 is illustrated for $A = 0.05$.

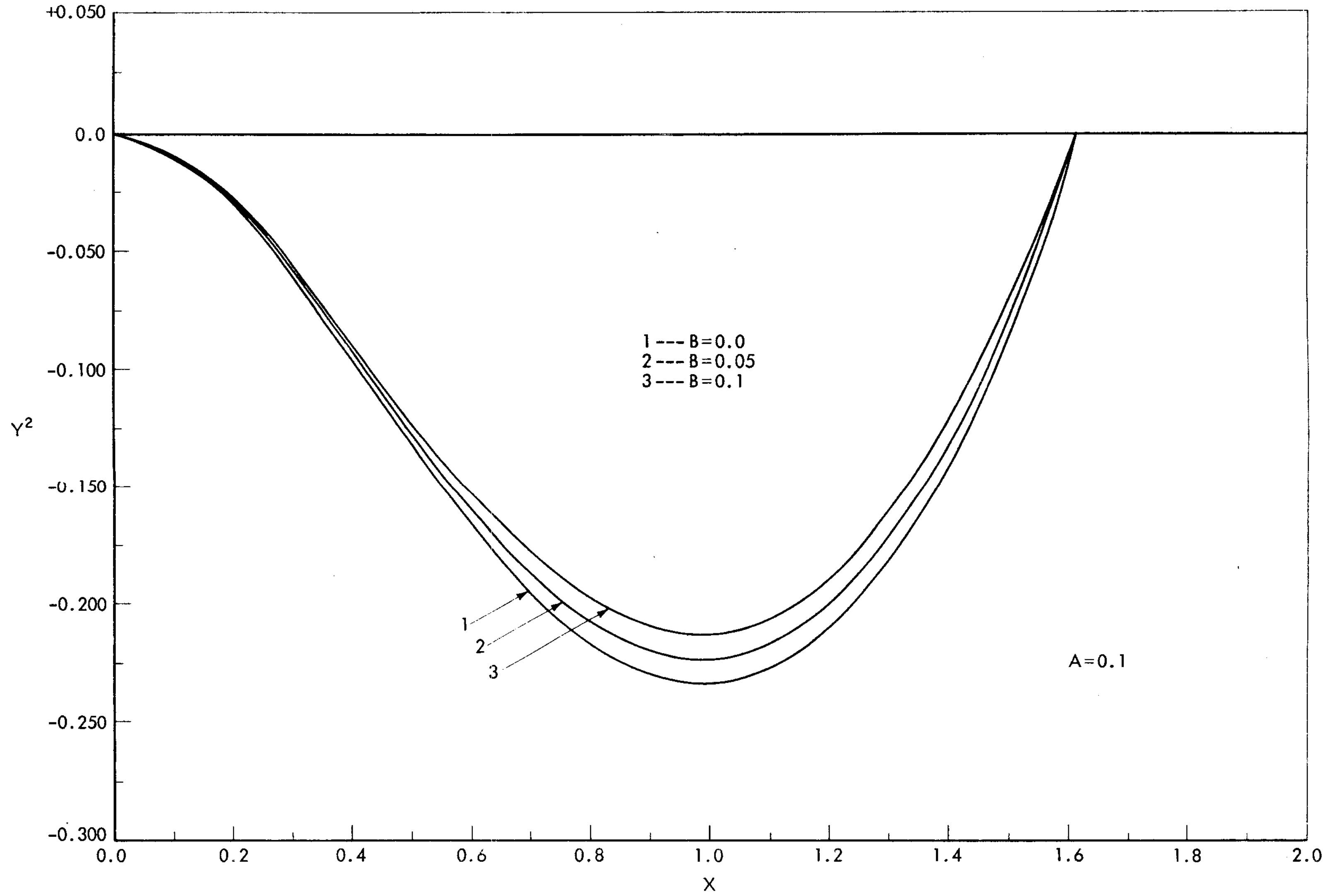
Figure 3. Variation of y^2 with x for $B = 0$, 0.05 and 0.1 is illustrated for $A = 0.1$.

Title for Table 1

The values of x_c, x_* and y_*^2 for various values of $A = V_0^2/U^2$, and $B = \nu_c/\omega_p$.


1 --- A=0.0
2 --- A=0.001
3 --- A=0.01
4 --- A=0.05
5 --- A=0.10
B=0.0
Y^2
X
+0.050
0.0
-0.050
-0.100
-0.150
-0.200
-0.250
-0.300
0.0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
1.6
1.8
2.0


+0.050
0.0
-0.050
-0.100
-0.150
-0.200
-0.250
-0.300
Y^2
0.0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
1.6
1.8
2.0
X
1 --- B=0.0
2 --- B=0.1
1
2
A=0.05


+0.050
0.0
-0.050
-0.100
-0.150
-0.200
-0.250
-0.300
Y^2
0.0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
1.6
1.8
2.0
X
1 --- B=0.0
2 --- B=0.05
3 --- B=0.1
1
2
3
A=0.1